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Research Report

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Title of Research Report

Towards the Problem of Grouping Coins

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Towards the Problem of Grouping Coins

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Abstract

The aim of this investigation is to show that, for some $2n+1$ gold coins, given that when any coin is removed, it is possible to form two groups of n coins with equal weight, then all $2n+1$ coins have equal weight. The result is then generalized: for some $a+b+1$ gold coins, given that when any coin is removed, it is possible to separate the remaining coins into a coins of total weight A and b coins of total weight B such that $bA = aB$, then all $a+b+1$ coins have equal weight.

Our methods involve writing a matrix equation, using eigenvalues and permutations and applying the rank-nullity theorem.

Keywords: Linear algebra, Rank and kernel of a matrix, Rank-nullity theorem, Modulo operation on a matrix, Leibniz formula, Permutation, Derangement

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1 Preliminaries and definitions

Here we define some of the terminology and notation used. We will only consider vectors in the real vector space.

Notation. The entry on the i-th row and j-th column of a matrix $\mathbf{A}$ is denoted by $A_{i,j}$. A column vector $\mathbf{x}$ is denoted inline by $(x_1, x_2, \dots, x_m)$ and in display style by

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{2n} \end{pmatrix}$$

Definition 1 (Image and kernel of matrix)**.** The *image* or *column space* $\mathrm{im}(\mathbf{A})$ of a matrix $\mathbf{A}$ is the span of its column vectors. The *kernel* or *null space* $\ker(\mathbf{A})$ of the matrix is the set of vectors $\mathbf{x}$ which satisfy $\mathbf{Ax} = \mathbf{0}$.

Theorem 1 (Rank-nullity theorem)**.** The dimensions of the image and kernel of an $m \times n$ matrix $\mathbf{A}$ are related by the *rank-nullity theorem*: $\mathrm{rank}(\mathbf{A}) + \dim(\ker(\mathbf{A})) = n$, where $\mathrm{rank}(\mathbf{A})$ denotes the *rank* of the matrix or the dimension of the image $\dim(\mathrm{im}(\mathbf{A}))$.

Theorem 2. The rank of a matrix is always greater than or equal to the rank of its submatrices.

Remark. The rank of $\mathbf{A}$ will not exceed $\min(m, n)$. The matrix $\mathbf{A}$ can be described as *full-rank* if $\mathrm{rank}(\mathbf{A}) = \min(m, n)$. Otherwise, it is *rank-deficient.* A matrix is full-rank if and only if it is non-singular.

Definition 2 (Eigenvector, eigenvalue and eigenspace)**.** Let $\mathbf{A}$ be a square matrix. $\mathbf{x} \neq \mathbf{0}$ is an *eigenvector* of $\mathbf{A}$ if $\mathbf{Ax} = \lambda\mathbf{x}$ for some λ. λ is the associated *eigenvalue.* The *eigenspace* E_λ is the set of eigenvectors of $\mathbf{A}$ associated with the eigenvalue λ.

Definition 3 (Characteristic polynomial and characteristic equation)**.** The *characteristic polynomial* of a square matrix $\mathbf{A}$ is $p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$. The *characteristic equation* of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ or $p(\lambda) = 0$. The characteristic equation is satisfied by the eigenvalues of $\mathbf{A}$.

Definition 4 (Algebraic multiplicity and geometric multiplicity of eigenvalue)**.** The *algebraic multiplicity* M_λ of an eigenvalue λ is the multiplicity of λ in $p(\lambda) = 0$. The *geometric multiplicity* m_λ is the dimension of the eigenspace E_λ.

Theorem 3. The algebraic multiplicity is always larger than or equal to the geometric multiplicity.

Definition 5 (Permutation, fixed point and derangement)**.** A *permutation* of a set is the rearrangement of the elements of the set. In a permutation, a *fixed point* is an element which appears in its original position. A *derangement* is a permutation which has no fixed points.

Formula (Leibniz formula)**.** Let $\mathbf{A}$ be an $n \times n$ matrix and let S_n be the set of permutations of $\{1, 2, \dots, n\}$. The determinant of $\mathbf{A}$ is given by $\det \mathbf{A} = \sum_{\sigma \in S_n} \left(\mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma_i} \right)$ where $\mathrm{sgn}(\sigma) \in (1, -1)$.

2 Main Problem

The main goal of this project is to show that, for some $2n+1$ gold coins, given that if any coin is removed it is possible to form two groups of n coins with equal weight, then all $2n+1$ coins have equal weight.

In mathematical language, it can be re-formulated as: Let $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_{2n+1}$ be the weights of the coins, where $x_i \geq 0$ for all $1 \leq i \leq n$. After removing any one of these numbers, it is possible to group the remaining numbers such that

$$\sum_{n \text{ terms}} x_i = \sum_{\substack{\text{remaining} \\ n \text{ terms}}} x_j$$

2.1 Rational Case

We observe that $\{x_1, x_2, \ldots, x_{2n+1}\}$ satisfy the condition if and only if $kx_1, kx_2, \ldots, kx_{2n+1}$ and $x_1+k, x_2+k, \ldots, x_{2n+1}+k$, where k is any real constant, also satisfy the condition. Thus if we limit ourselves to rational numbers only, we can multiply $x_1, x_2, \ldots, x_{2n+1}$ by a common constant such that they are all positive integers.

Lemma 1. $x_1, x_2, \ldots, x_{2n+1}$ are either all odd or all even.

Proof. Suppose there are odd and even x_i at the same time. If we remove any odd term, then

$$\sum_{n \text{ terms}} x_i - \sum_{\substack{\text{remaining} \\ n \text{ terms}}} x_j = 0$$

Since $x_1, x_2, \ldots, x_{2n+1}$ are all integers and $a \equiv -a \pmod 2$, we can then deduce

$$\sum_{2n \text{ terms}} x_i \equiv 0 \pmod 2$$

The sum of the terms (excluding the removed number) is even. It follows that there must be an even number of odd terms and an even number of even terms in this group of $2n$ terms. Including the removed term, the total number of odd x_i is odd and the total number of even x_i is even.

If, on the other hand, we remove any even term,

$$\sum_{n \text{ terms}} x_i - \sum_{\substack{\text{remaining} \\ n \text{ terms}}} x_j = 0 \quad \text{and} \quad a \equiv -a \pmod 2$$

$$\sum_{2n \text{ terms}} x_i \equiv 0 \pmod 2$$

The sum of the $2n$ terms is even. It follows that there must be an even number of odd terms and an even number of even terms in this group of $2n$ terms. Including the removed term, the total number of odd x_i is even and the total number of even x_i is odd.

We arrive at a contradiction. Thus, there cannot be odd and even x_i at the same time. We obtain the desired result. ■

We can make a stronger statement about the divisibility of x_i by powers of 2.

Lemma 2. $2^m \mid x_j \implies 2^m \mid x_k$ for all $m \in \mathbb{N}$

Proof. Suppose that for some j, k and m, $2^m \mid x_j$ and $2^m \nmid x_k$. We repeatedly divide every term by 2, until either there is an odd term in the sequence or x_k becomes odd, whichever comes first. In both cases, the number of times the terms are halved will not exceed $n-1$, so x_j will remain even. There are odd and even x_i at the same time. However, **Lemma 1** states that there cannot be odd and even x_i at the same time. We arrive at a contradiction. The desired result follows. ∎

Proposition 1. $x_j = x_k$ for all $1 \leq j, k \leq 2n+1$

Proof. Now we suppose that, without loss of generality, $x_1, x_2, \ldots, x_{2n+1}$ is increasing. This can be done simply by rearranging the terms.

If there exists any $x_j > x_1$, then add $2^t - x_j$ to every term, where t is any positive integer which satisfies $x_j < 2^t$. Let $x'_1, x'_2, \ldots, x'_{2n+1}$ denote the terms after translation. We observe that $x'_1, x'_2, \ldots, x'_{2n+1}$ is also increasing. We have $x'_1 = 2^t - x_j + x_1$, which is smaller than 2^t, and $x'_j = 2^t$.

Since $x'_1, x'_2, \ldots, x'_{2n+1}$ is only a translation of $x_1, x_2, \ldots, x_{2n+1}$, then $x'_1, x'_2, \ldots, x'_{2n+1}$ are also possible solutions, which implies that, by **Lemma 2**, $2^m \mid x'_j \implies 2^m \mid x'_k$ for all $m \in \mathbb{N}$ and for $k = 1, 2, \ldots, 2n+1$.

Given that $x'_j = 2^t$, we can deduce that $2^t \mid x'_j$ and, as a consequence, $2^t \mid x'_1$. However, we also have $x'_1 < 2^t$ and, consequently, $2^t \nmid x'_1$, a contradiction. There must not exist any $x_j \neq x_1$. In other words, $x_j = x_k$ for $1 \leq j, k \leq 2n+1$. ∎

We have thus shown that, if $x_1, x_2, \ldots, x_{2n+1}$ are rational, they must be equal.

We tried to extend this solution to some irrational numbers using field extensions, but even if it would work, it would not bring us to all reals since $\mathbb{R}$ is an infinite extension of $\mathbb{Q}$.

This gives us some insight into how we should approach the general case of this problem: The proof will most likely involve the modulo operation. However, instead of focusing on the properties of the individual x_i, we should approach it from the equation(s) $\sum x_i = \sum x_j$.

2.2 General Case

We make several observations:

(i) We can write $2n+1$ equations, since there are $2n+1$ numbers and each time one number is removed.

(ii) We can rearrange the equations to the form $\sum x_i - \sum x_j + 0 \times x_k = 0$. Each summation has n summands and the left hand side of this equation has a total of $2n+1$ terms.

Thus we can write the system as a matrix equation:

$$\mathbf{Ax} = \mathbf{0}$$

where $\mathbf{A}$ is a $(2n+1) \times (2n+1)$ matrix and $\mathbf{x} = (x_1, x_2, \ldots, x_{2n+1})$. $\mathbf{A}$ has exactly n 1's, n (-1)'s and one 0 on each row. This is because each time one number is removed and two groups of n numbers are formed.

We rearrange the equations such that the j-th equation represents the case when x_j is removed. The entries on the principal diagonal of $\mathbf{A}$ are all zero. For example, in the case that $n = 1$, $\mathbf{A}$ may look like

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Now, our task is to solve for $\mathbf{x}$.

2.2.1 Proof using eigenvalues and permutations

If $\mathbf{x}$ satisfies $\mathbf{Ax} = \mathbf{0}$, then it also satisfies $(\mathbf{A}+\mathbf{I})\mathbf{x} = \mathbf{x}$. In other words, if it exists, $\mathbf{x}$ is an eigenvector of $\mathbf{A}+\mathbf{I}$ corresponding to the eigenvalue $\lambda = 1$. Trivially, $\mathbf{x} = (1, 1, \ldots, 1)$ is one such vector, so $\lambda = 1$ is, in fact, an eigenvalue of $\mathbf{A}+\mathbf{I}$.

We observe that $\mathbf{A}' = \mathbf{A} + \mathbf{I} - \lambda\mathbf{I}$ has values $1-\lambda$ on the diagonal while taking values 1 and -1 elsewhere. Thus, the characteristic polynomial of $\mathbf{A}+\mathbf{I}$ is

$$p(\lambda) = (1-\lambda)^{2n+1} + c_1(1-\lambda)^{2n} + \cdots + c_{2n}(1-\lambda) + c_{2n+1}$$

where $c_1, c_2, \ldots, c_{2n+1}$ are real constants.

Since $\lambda = 1$ is an eigenvalue, we have $p(1) = 0$ and, consequently, $c_{2n+1} = 0$. The characteristic polynomial becomes $p(\lambda) = (1-\lambda)^{2n+1} + c_1(1-\lambda)^{2n} + \cdots + c_{2n}(1-\lambda)$.

$\mathbf{A}$ and $\mathbf{A}'$ are $2n+1 \times 2n+1$ matrices. Recall the formula for determinants

$$\det \mathbf{A}' = \sum_{\sigma \in S_{2n+1}} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{2n+1} A'_{i,\sigma_i} \right)$$

By inspection, for the terms in det $\mathbf{A}'$ where the degree of $(1-\lambda)$ is 1, the corresponding permutation σ has exactly one fixed point. This is because the values $(1-\lambda)$ are located on the diagonal. If k is the number of such permutations, then c_{2n} is the sum of k 1's and -1's. We count the number of such permutations: there are $2n+1$ ways to choose the fixed point and the remaining elements form a derangement of size $2n$.

Lemma 3. The number of derangements of size m is $D(m) = m! \sum_{i=0}^{m} \frac{(-1)^i}{i!}$.

Proof. For $1 \leq k \leq m$ define R_k to be the set of permutations of size m which fix the k-th element. Consider i sets in $R_1, R_2, \ldots, R_m$. The intersection of these sets fixes a particular set of i elements and therefore contains $(m-i)!$ permutations. There are $\binom{m}{i}$ such collections. By the inclusion-exclusion principle, the size of $R_1 \cup R_2 \cup \cdots \cup R_m$ is

$$\begin{aligned}
\left|\bigcup_{k=1}^{m} R_k\right| &= \binom{m}{1}(n-1)! - \binom{m}{2}(m-2)! + \cdots + (-1)^{m+1}\binom{m}{m}(m-m)! \\
&= \sum_{i=1}^{m}(-1)^{i+1}\binom{m}{i}(n-i)! \\
&= m!\sum_{i=1}^{m}\frac{(-1)^{i+1}}{i!}
\end{aligned}$$

Since a derangement is a permutation that has no fixed points,

$$D(m) = m! - \left|\bigcup_{k=1}^{m} R_k\right| = m!\sum_{i=0}^{m}\frac{(-1)^i}{i!}$$

■

Then, by **Lemma 3**, we have

$$\begin{aligned}
D(2n) &= (2n)!\sum_{i=0}^{2n}\frac{(-1)^i}{i!} \\
&= 1 + (2n)!\sum_{i=0}^{2n-1}\frac{(-1)^i}{i!}
\end{aligned}$$

For $i < 2n$, $\frac{(2n)!}{i!}$ is always divisible by $2n$ and is even. It easily follows that $D(2n)$ is odd.

The number of permutations with one fixed point is $(2n+1)D(2n)$, which is odd. c_{2n} is the sum of an odd number of 1's and -1's. Since $1 \equiv -1 \pmod 2$, then $c_{2n} \equiv (2n+1)D(2n) \pmod 2$ and $c_{2n} \neq 0$.

It follows that $p(\lambda)$ is divisible by $1-\lambda$ but not by $(1-\lambda)^2$. The algebraic multiplicity and, by **Theorem 2**, the geometric multiplicity of 1 are both 1.

$\{k(1,1,\ldots,1) \mid k \in \mathbb{R}\}$ is the only set of vectors which satisfies $(\mathbf{A}+\mathbf{I})\mathbf{x} = \mathbf{x}$ and $\mathbf{A}\mathbf{x} = \mathbf{0}$. Thus, all x_i are equal and all coins have the same weight.

2.2.2 Proof using rank-nullity theorem

In this second proof we will attempt to find the rank and nullity of $\mathbf{A}$.

Lemma 4. $\dim(\ker(\mathbf{A})) \geq 1$

Proof. There is at least one non-trivial solution $\mathbf{x} = (1, 1, \ldots, 1)$ to the equation $\mathbf{Ax} = \mathbf{0}$. Noting that $\mathbf{A}k\mathbf{x} = \mathbf{0}$ is true for all k, the desired result follows easily. ■

Let $\mathbf{B}$ be the leading principal $2n \times 2n$ submatrix of $\mathbf{A}$ and let $\mathbf{E}$ be a $2n \times 2n$ matrix with all entries equal to 1. Also let $\mathbf{B}$ mod 2 denote a $2n \times 2n$ matrix whose entries are the least positive remainders of the entries of $\mathbf{B}$ divided by 2. $\mathbf{B}$ mod 2 is a $2n \times 2n$ matrix with 0's on the principal diagonal and 1's elsewhere. Thus, we have $\mathbf{B} \bmod 2 = \mathbf{E} - \mathbf{I}$.

Lemma 5. $\det(\mathbf{E} - \mathbf{I}) = 1 - 2n$

Proof. We calculate the determinant of this matrix by row reduction

$$\begin{aligned}
\det(\mathbf{E} - \mathbf{I}) &= \begin{vmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{vmatrix} \\
&= \begin{vmatrix} 2n-1 & 2n-1 & \ldots & 2n-1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{vmatrix} \\
&= (2n-1) \begin{vmatrix} 1 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{vmatrix} \\
&= (2n-1) \begin{vmatrix} 1 & 1 & \ldots & 1 \\ 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{vmatrix} \\
&= (1-2n)(-1)^{2n-1} \\
&= 1 - 2n
\end{aligned}$$

■

Lemma 6. Let $\mathbf{M}$ be an $m \times m$ matrix with integer entries and let p be a natural number. For all $\mathbf{M}$ and p, $\det(\mathbf{M} \bmod p) \equiv \det \mathbf{M} \pmod{p}$.

Proof. By the Leibniz formula for determinants,

$$\det \mathbf{M} = \sum_{\sigma \in S_m} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{m} M_{i,\sigma_i} \right)$$
$$\det(\mathbf{M} \bmod p) = \sum_{\sigma \in S_m} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{m} (M_{i,\sigma_i} \bmod p) \right)$$

The modulo operation is compatible with addition and multiplication. In other words, if $c_i \equiv d_i \pmod{p}$ for all i, then $\sum_i c_i \equiv \sum_i d_i \pmod{p}$ and $\prod_i c_i \equiv \prod_i d_i \pmod{p}$. Then,

$$\prod_{i=1}^{m} (M_{i,\sigma_i} \bmod p) \equiv \prod_{i=1}^{m} M_{i,\sigma_i} \pmod{p}$$
$$\sum_{\sigma \in S_m} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{m} (M_{i,\sigma_i} \bmod p) \right) \equiv \sum_{\sigma \in S_m} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{m} M_{i,\sigma_i} \right) \pmod{p}$$
$$\det(\mathbf{M} \bmod p) \equiv \det \mathbf{M} \pmod{p}$$

■

Sub $\mathbf{M} = \mathbf{B}$ and $p = 2$, then

$$\det(\mathbf{B} \bmod 2) \equiv \det \mathbf{B} \pmod{2}$$
$$\det \mathbf{B} \equiv 1 - 2n \pmod{2}$$
$$\det \mathbf{B} \neq 0$$

$\mathbf{B}$ is also non-singular, so $\operatorname{rank}(\mathbf{B}) = 2n$.

Lemma 7. $\dim(\ker(\mathbf{A})) \leq 1$

Proof. Since $\mathbf{B}$ is a submatrix of $\mathbf{A}$, $\operatorname{rank}(\mathbf{A}) \geq \operatorname{rank}(\mathbf{B}) = 2n$. By the rank-nullity theorem, $\dim(\ker(\mathbf{A})) = 2n + 1 - \operatorname{rank}(\mathbf{A})$ and, consequently, $\dim(\ker(\mathbf{A})) \leq 1$. ■

Proposition 2. $\ker(\mathbf{A}) = \{k(1, 1, \ldots, 1) \mid k \in \mathbb{R}\}$

Proof. Combining **Lemma 4** and **Lemma 7**, $\dim(\ker(\mathbf{A})) = 1$. The null space of $\mathbf{A}$ only consists of one linearly independent vector. Since $\mathbf{x} = (1, 1, \ldots, 1)$ satisfies $\mathbf{A}\mathbf{x} = \mathbf{0}$, the null space of $\mathbf{A}$ is the set of vectors which are a scalar multiple of $(1, 1, \ldots, 1)$. ■

$\{k(1, 1, \ldots, 1) \mid k \in \mathbb{R}\}$ is the only set of vectors which satisfies $\mathbf{A}\mathbf{x} = \mathbf{0}$. All x_i are equal and all coins have the same weight.

3 Extended Problem

As a generalisation to our main problem, we will also show that for some $a+b+1$ gold coins, given that if any coin is moved away it is possible to form two groups of a coins and b coins with average weight equal, then all $a+b+1$ coins have equal weight.

In mathematical language, it can be re-formulated as: Let $a, b \in \mathbb{N}$ and $x_1, x_2, \ldots, x_{a+b+1}$ be the weights of the coins, where $x_i \geq 0$ for all $1 \leq i \leq a+b+1$. After removing any one of these numbers, it is possible to group the remaining numbers such that

$$b \sum_{a \text{ terms}} x_i = a \sum_{b \text{ terms}} x_j$$

Rearrange the equations to the form $a \sum x_i - b \sum x_j + 0 \times x_k = 0$. There are b summands in the first summation and a summands in the second. The left hand side of this equation has a total of $a+b+1$ terms. There are $a+b+1$ such equations, since each time one number out of $a+b+1$ numbers is removed.

With this we can write the system as a matrix equation:

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

where $\mathbf{A}$ is a $(a+b+1) \times (a+b+1)$ matrix and $\mathbf{x} = (x_1, x_2, \ldots, x_{a+b+1})$. $\mathbf{A}$ has exactly b a's, a $-b$'s and one 0 on each row. Like before, we rearrange the equations so that the diagonal entries of the matrix are all 0. For example, in the case $a = 2$ and $b = 3$, one such possible matrix would look like:

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & -3 & 2 & 2 & -3 \\ -3 & 0 & -3 & 2 & 2 & 2 \\ 2 & 2 & 0 & -3 & -3 & 2 \\ 2 & -3 & 2 & 0 & -3 & 2 \\ -3 & 2 & 2 & -3 & 0 & 2 \\ 2 & 2 & -3 & -3 & 2 & 0 \end{pmatrix}$$

For the sake of tidiness and simplicity, we let $n = a+b+1$ for the following proofs.

3.1 a, b both odd

We first limit ourselves to the case where a and b are odd. For the general case, we made the following observation:

If $\mathbf{x}$ satisfies $\mathbf{Ax} = \mathbf{0}$, then it also satisfies $(\mathbf{A} + \mathbf{I})\mathbf{x} = \mathbf{x}$. In other words, if it exists, $\mathbf{x}$ is an eigenvector of $\mathbf{A} + \mathbf{I}$ corresponding to the eigenvalue $\lambda = 1$. Trivially, $\mathbf{x} = (1, 1, \ldots, 1)$ is one such vector, so $\lambda = 1$ is, in fact, an eigenvalue of $\mathbf{A} + \mathbf{I}$.

This property also applies in this case. Moreover, the characteristic polynomial of $\mathbf{A} + \mathbf{I}$ can be expressed as a polynomial in $(1 - \lambda)$ with degree n. Thus, for some constants $c_1, c_2, \ldots, c_n$, the characteristic polynomial is

$$p(\lambda) = (1-\lambda)^n + c_1(1-\lambda)^{n-1} + \cdots + c_{n-1}(1-\lambda) + c_n$$

Since $\lambda = 1$ is an eigenvalue of $\mathbf{A} + \mathbf{I}$, we have $p(\lambda) = 0$ and, thusly, $c_n = 0$. Recall the formula for determinant:

$$\det(\mathbf{A} + \mathbf{I} - \lambda\mathbf{I}) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} A'_{i,\sigma_i} \right)$$

We count the number of terms $\operatorname{sgn}(\sigma) \prod_i A'_{i,\sigma_i}$ where the degree of $(1 - \lambda)$ is 1. Our previous endeavours in the main problem tells us that the number of such permutations is $nD(n-1)$ or $(a+b+1)D(a+b)$. Since $n-1$ is even, we know that $D(n-1)$ is odd and so is $nD(n-1)$.

$\mathbf{A} + \mathbf{I} - \lambda\mathbf{I}$ takes values $(1-\lambda)$ only on the diagonal and values a and $-b$ elsewhere. Considering the fact that since a, $-b$ and ± 1 are all odd, the value or coefficient of $\operatorname{sgn}(\sigma) \prod_i A'_{i,\sigma_i}$ is odd.

We thus know that c_{n-1} is a sum of an odd number of odd numbers, which means that c_{n-1} is also odd and non-zero. The algebraic multiplicity and geometric multiplicity of the eigenvalue 1 is 1.

The corresponding eigenspace E_1 of $\mathbf{A} + \mathbf{I}$ and the kernel of $\mathbf{A}$ is $\{k(1, 1, \ldots, 1) \mid k \in \mathbb{R}\}$. $x_i = x_j$ for all i, j. All coins have the same weight.

3.2 Any a, b

Let $k = \gcd(a, b), a' = \dfrac{a}{k}, b' = \dfrac{b}{k}$ and $\mathbf{B} = \dfrac{1}{k}\mathbf{A}$. Obviously, a' and b' are coprime. $\mathbf{B}$ is an integer matrix. We observe that if $\mathbf{Ax} = \mathbf{0}$, then $\mathbf{Bx} = \mathbf{0}$ also.

Let $\mathbf{B'}$ be the leading principal $(n-1) \times (n-1)$ submatrix of $\mathbf{B}$. $\mathbf{B'}$ mod $(a' + b')$ is a matrix whose entries take value 0 on the diagonal and a' elsewhere. Let $\mathbf{E}$ be the $(n-1) \times (n-1)$ matrix with all entries 1. Following the same steps as in **Lemma 5**, it can be deduced that $\det(\mathbf{E} - \mathbf{I}) = (n-2)(-1)^{n-2}$.

$$\begin{aligned}\det(\mathbf{B'} \bmod (a' + b')) &= \det(a'(\mathbf{E} - \mathbf{I})) \\ &= (a')^{n-1} \det(\mathbf{E} - \mathbf{I}) \\ &= (a')^{n-1}(n-2)(-1)^{n-2}\end{aligned}$$

Since a' and $a' + b'$ are coprime, $(a')^{n-1}$ is not divisible by $a' + b'$. On the other hand, since $n - 2 = k(a' + b') - 1$, we also know that $n - 2$ is not divisible by $a' + b'$.

$$\begin{aligned}\det(\mathbf{B'} \bmod (a' + b')) &\not\equiv 0 \pmod{(a' + b')} \\ \det(\mathbf{B'}) &\not\equiv 0 \pmod{(a' + b')} \\ \det(\mathbf{B'}) &\neq 0 \\ \det(\mathbf{A'}) &\neq 0\end{aligned}$$

$\mathbf{A'}$ is full rank. Since $\mathbf{A'}$ is a submatrix of $\mathbf{A}$, $\operatorname{rank}(A) \geq \operatorname{rank}(A')$ or $\operatorname{rank}(A) \geq n-1$. We know that $\mathbf{x} = (1, 1, \ldots, 1)$ is a solution to $\mathbf{Ax} = \mathbf{0}$, which implies that $\mathbf{A}$ is rank-deficient and $\operatorname{rank}(A) < n$.

$$\begin{aligned}n - 1 \leq \operatorname{rank}(A) &< n \\ \operatorname{rank}(A) &= n - 1\end{aligned}$$

By the rank-nullity theorem, $\dim(\ker(\mathbf{A})) = 1$. The null space of $\mathbf{A}$ only consists of one linearly independent vector. Since $\mathbf{x} = (1, 1, \ldots, 1)$ satisfies $\mathbf{Ax} = \mathbf{0}$, the null space of $\mathbf{A}$ is the set of vectors which are a scalar multiple of $(1, 1, \ldots, 1)$.

$\{k(1, 1, \ldots, 1) \mid k \in \mathbb{R}\}$ is the only set of vectors which satisfies $\mathbf{Ax} = \mathbf{0}$. $x_i = x_j$ for all i, j. All coins have the same weight.

4 Discussion

It has been proven that all coins have equal weight for the $2n+1$ and $a+b+1$ cases. A further generalization of these cases would be to show that, for some $a+b+c$ gold coins, given that when any c coins are removed, it is possible to form two groups of a coins and b coins with equal average weight, then all $a+b+c$ coins have equal weight. The result has not been proven yet, but here are some of our thoughts:

In this case, we observe that we can write a total of $\binom{a+b+c}{c}$ equations in the form $a\sum x_i - b\sum x_j + 0 \times \sum x_k = 0$ (first summand has b terms, second summand has a terms and third summand has c terms). Writing it as a matrix equation would give us an $\binom{a+b+c}{c} \times a+b+c$ matrix.

This result is obviously true when $a = 1$ and $b = 1$. Our previous efforts show that the result is always true when $c = 1$. We also computed manually that it is true for $(a, b, c) = (1, 2, 2), (2, 2, 2)$ and $(2, 2, 3)$.

For $c > 1$, the matrix would not be square and our previous methods would break down, since determinant and eigenvalues are not defined for non-square matrices.

We attempted to just choose any $a+b+c$ equations and create a new matrix equation, but we could not guarantee that the matrix would have certain properties (diagonal entries all zero) and the system of $a+b+c$ equations might not be equivalent to the system of $\binom{a+b+c}{c}$ equations.

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