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Title of Research Report

On Optimal Favoritism in Asymmetric Competitions

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On Optimal Favoritism in Asymmetric Competitions

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Abstract

Favoritism towards a relatively weak competitor is well adopted as an effective instrument to enhance productive effort supply in asymmetric contests in a variety of economic environments. Examples include affirmative actions in college admissions, preferential treatment to internal job candidates in job applications and domestic firms in government procurements. In this paper, we investigate the effort-maximizing favoritism rule in asymmetric two-player contests with all-pay auction technology, while accommodating fully flexible favoritism rules. Our analysis allows players' values to be their private information or public information.

In the incomplete information scenario, we explicitly characterize the effort-maximizing favoritism rule and the associated equilibrium bidding strategies under plausible assumptions. We find that under hazard rate dominance in terms of players' values distributions, a weaker player always wins if his/her opponent has the same value. However, this is not true merely under first-order dominance. Surprisingly, when compared to a standard all-pay auction without favoritism, we find that the effort-maximizing favoritism rule does not necessarily make the weaker player win with a higher chance and thus may not increase the winner diversity.

In the complete information scenario in which both players' values are fixed, we find that at the optimum, the weaker player with lower value is extremely favored; however his/her winning chance converges to zero. This finding illustrates that the extreme effort-maximizing favoritism rule certainly decreases winner diversity in this scenario of complete information.

Our findings shed light on the optimal design of affirmative action in college admission and preferential treatments of different parties in labor market and international trade.

Keywords: Affirmative action, All-pay auction, Asymmetric contest, Favoritism, Labor market, Incentive compatibility, International trade, Government procurement, Mechanism design, Revelation principle, Sports.

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Table of Contents

1 Introduction

Economic, political, social and athletic competitions, in which the contestants are asymmetric, are abundant. In R&D races, innovators are typically endowed with different technological capacities and expertise. In government procurements, suppliers (say domestic versus foreign firms) can be heterogeneous in their provision efficiencies. In job promotions within hierarchical organizations, contenders usually differ in their competitiveness. In political campaigns, some candidates are more popular or less financially constrained compared to their opponents. In school admissions, applicants come from diverse racial, economic and social and backgrounds. In athletic events, such as golf and horse racing, players essentially differ in their levels of training, field experiences and physical conditions.

It has long been recognized that heterogeneity in contestants' competitiveness can hinder their incentive to exert high effort overall. For example, one notable empirical study by Brown (2011) convincingly shows that average score of golf players falls in the presence of a superstar like Tiger Woods. The intuition behind this finding is clear: with the presence of a dominant opponent, weaker players have less incentive to exert effort due to the slim chance of winning the grand award. Given this, the superstar does not need to exert much effort to win.

This undesirable discouragement effect of player heterogeneity on total effort supply has stimulated enormous academic interest on investigating how to mitigate or overcome it. A well received insight from literature is that leveling the battle field by favoring the weaker player is essential to encourage the weaker and discipline the stronger. In other words, introducing appropriate favoritism, say head start and/or handicap, in the originally asymmetric competition can mitigate the discouragement effect of player heterogeneity.

In this paper, we further investigate the effort-maximizing favoritism rule while adopting an analytical framework of two-player all pay auction with incomplete information or complete information. Differentiating from previous literature, we allow fully nonlinear favoritism rules without restricting to linear instruments such as headstarts, handicaps, or a combination of the two. We will identify the optimal favoritism rules and the induced equilibrium bidding strategies, and study the properties and implications of these rules. In this paper, a favoritism rule is defined as a player's winning effort threshold as a function of the other player's effort. A player wins if and only if his/her effort is above this threshold.

Our main analysis adopts an incomplete information scenario in which players' values of winning the competition are their private information. In this setting, explicitly solving for the players' equilibrium strategies is an intimidating task, if not impossible. This means that searching for the optimal favoritism rule by directly comparing across the explicit equilibria under all rules is nearly impossible. We instead adopt an innovative indirect approach by establishing an effort bound following Myerson's (1981) mechanism design methodology. Then we show this effort bound can be achieved by a particular favoritism rule under the induced equilibrium. Both the favoritism rule and equilibrium are explicitly constructed based on results obtained while applying the Myerson approach.[1] At the optimum, we find that the weaker player in the sense of hazard rate dominance must win his/her stronger opponent if they share the same value. However, this is not true if the dominance is in the sense of first order stochastic dominance. Different players can win the competition if their value is the same but it falls in different ranges. Surprisingly, we find that under optimal favoritism, a weaker player's expected winning chance can be lower than that in a standard all pay auction, which reveals that effort-maximizing favoritism may perversely reduce the winner diversity.

We further investigate the effort-maximizing favoritism rule while assuming players' values are public information. This setting is also quite popular in the literature. We find that the effort-maximizing favoritism rule would favor the weak player to the extreme. However, at the optimum, the weak player would (almost) always lose the competition. This finding reveals that the extreme effort-maximizing favoritism certainly decreases winner diversity in this scenario of complete information. One implication of our finding is that the well received insight of "fully leveling the playing field" (e.g. Nti (2004) and Fu (2006), etc.) is no longer applicable once nonlinear favoritism rules are allowed.

This insight of using favoritism to better incentivize the contestants is corroborated by an established line of theoretical studies, which provide sound economic justification for policies and practices that specify preferential treatment of contestants in asymmetric competitions. The most

[1] Favoritism rule inducing efficient allocation can be similarly constructed.

salient example is affirmative action in school admissions (see Fu (2006), Frank (2012)). In the US, in-state students and minorities are preferred in college admissions in many states. Similar practices are adopted in EU universities that favor EU students versus non-EU students. McAfee and McMillan (1989) provide economic rationale (i.e. cost reduction) for a government's preferential treatment to domestic local enterprises (versus foreign multinational corporations) in public procurements.[2] Epstein et al. (2011) also emphasize on the benefit of public policies that favor small and medium-sized firms in government procurements. Ayres and Cramton (1996) find empirical evidence supporting this rationale. Chan (1996) shows that in promotional competitions within organizations, preferential treatments of the internal employees can better incentivize them if external candidates are much stronger.[3] Preferential policies are often observed in sports. For example, to make the competition more exciting, higher ranked competitors are often handicapped in golf and horse racing (Chowdhury et al. (2019)). Che and Gale (1998) find that the practice of imposing a bidding cap on political lobbying to handicap the financially less constrained politicians can generate a perverse effect of inducing higher expenditure. Che and Gale (2003) establish that with asymmetric contestants, it is optimal to handicap the most efficient to boost their overall performance. This insight echoes the rationale for the research recognition programs, where young researchers obtain head-starts (Kirkegaard (2012)).

Besides studies mentioned above, this line of the literature on favoritism also includes Clark and Riis (2000), Konrad (2002), Gavious et al. (2002), Nti (2004), Sahuguet (2006),Tsoulouhas et al. (2007), Frank et al. (2013), Seel and Wasser (2014), Frank et al. (2018), Zhu (2019), Fu and Wu (2020) among many others. Typically, all these studies are conducted using the analytical framework of Tullock contest and/or all pay auction either with complete or incomplete information. Unlike our paper which allows nonlinear favoritism rules, the favoritism in these existing studies focuses on linear instruments, including head start (an additive bias on a player's performance/effort) and handicap (a multiplicative bias on a player's performance/effort). Our study further strengthens these studies by providing the fully effort-maximizing favoritism rule without imposing any restrictions.

[2]According to Feess et al. (2008), in Germany, there is a clause which allows awarding a public procurement contract to a local firm when its bid is not more than 5% higher than the lowest bid.

[3]Schotter and Weigelt (1992) find experimental evidence that affirmative action programs significantly increase overall effort levels when the cost disadvantage of weaker group is severe.

Favoritism towards weaker competitors can also promote the diversity of winner group, as illustrated by Fu (2006), Pastine and Pastine (2012) and Lee (2013). These studies nevertheless find that typically there is a tension between effort-maximization and winner diversity when the contest organizer is restricted to adopt linear favoritism instruments, including head start and handicap. Our findings further complement these studies by revealing the possibility that effort-maximizing favoritism can perversely reduce the diversity of winner group. This result clearly has important policy implications, since preferential policies including affirmative actions typically aim at boosting winner diversity.

The rest of the paper is organized as follows. In Section 2, we set up the model with incomplete information. Section 3 contains the main analysis. Section 4 is a discussion of environment with complete information. Section 5 provides some concluding remarks.

2 Model setup with incomplete information

We adopt an analytical framework of all pay auctions with incomplete information. There are two players $i = 1, 2$. Bidder i's value of winning the auction is v_i, $i = 1, 2$. Values v_i is private information of bidder i. Bidder i's value distribution is $G_i(\cdot)$ with density $g_i(\cdot) > 0$ on $[0, \bar{v}]$.

We assume the following standard regularity condition on the virtual value functions, which is well adopted in the literature.

Assumption 1. *Virtual value functions* $J_i(v_i) = v_i - \frac{1-G_i(v_i)}{g_i(v_i)}$ *are increasing in* v_i *on* $[0, \bar{v}]$.

Clearly we have the following result.

Lemma 1. $J_1(\bar{v}) = J_2(\bar{v}) = \bar{v}$.

Assumption 2. *(i)* $g_1(0) = g_2(0)$*; (ii)* $g_1(\cdot)$ *and* $g_2(\cdot)$ *single cross in* $(0, \bar{v}]$*. There exists* $\hat{v} \in (0, \bar{v})$ *such that* $g_1(v) < g_2(v)$ *if* $v \in (0, \hat{v})$*, and* $g_1(v) > g_2(v)$ *if* $v \in (\hat{v}, \bar{v}]$*;(iii)* $g_1(\cdot)$ *and* $g_2(\cdot)$ *are continuous.*

Assumption 2 implies that $G_2(\cdot)$ first order stochastically dominates $G_1(\cdot)$.[4] In this sense, bidder

[4]If the cumulative distribution functions cross in $(0, \bar{v})$, then their difference has at lease two internal extreme points, at which the density functions must cross with each other.

1 is a weaker bidder than bidder 2. The assumption of $g_1(0) = g_2(0)$ if for analytical simplification. Our analysis applies when $g_1(0) \leq g_2(0)$.

For example, consider $G_1(v_1) = (v_1)^4$ on $[0, 1]$ and $G_2(v_2) = (v_2)^2$ on $[0, 1]$. We thus have that player 1 is stronger in the sense of first order dominance. Moreover, $g_1(v) = 2v$ and $g_2(v) = 4v^3$, which satisfy Assumption 2. In addition, we have $G_1(v)$ stochastically dominates $G_2(v)$ in the sense of hazard rate, i.e. $\frac{g_2(v)}{1-G_2(v)} \geq \frac{g_1(v)}{1-G_1(v)}$. Note $\frac{g_1(v)}{1-G_1(v)} = \frac{4v^3}{1-v^4} = \frac{2v}{1-v^2}\frac{2v^2}{1+v^2} \leq \frac{2v}{1-v^2} = \frac{g_2(v)}{1-G_2(v)}$.

The players make their bids/effort simultaneously. The player with higher bid wins and pays his bid/effort cost, which equals his/her bid. The ties are broken randomly, unless it will be specified alternatively. Bidder i's bid is denoted by $b_i \geq 0$, $i = 1, 2$. The higher bidder wins and both bidders incur the cost of their bid, which is b_i. In other words, the marginal cost of bid is normalized as 1 for both bidders. In this paper, we use "bid" and "effort" interchangeably. A player's bid is interpreted as their effort supply. A bidder's expected payoff is his/her value multiplied by his winning probability then minus his/her bid/effort. The contest organizer's payoff is simply the total bids of the bidders, i.e. their total effort supply. Everyone is risk neutral.

3 The analysis

In our analysis, we allow a fully flexible favoritism rule as specifies as follows.

Definition 1. *The favoritism rule is specified by bidder* 2*'s winning threshold* $B(b_1) \in [0, \bar{v}]$*, which is an increasing function defined on* $[0, \bar{v}]$*. This rule means that bidder* 1 *placing bid* b_1 *wins if and only if bidder* 2*'s bid* b_2 *is no greater than* $B(b_1)$*.*

The inverse function is defined as $B^{-1}(b_2)$, which means that bidder 2 placing bid b_2 wins if and only if bidder 1's bid b_1 is no greater than $B^{-1}(b_2)$. We consider the set of favoritism rules which induce monotone pure-strategy equilibrium, $b_1(v_1)$ and $b_2(v_2)$, with $b_1(0) = b_2(0) = 0$. Note that this set is not empty. When $B(b_1)$ is an identity function, Amann and Leininger (1996) establish the existence and uniqueness of increasing pure strategy equilibrium. We will show later that a monotone pure strategy equilibrium exists under the identified optimal favoritism rule. Theorem 6 in Athey (2001) can be applied to establish the existence of monotone pure strategy equilibrium.

A mechanism described by winning rule $p_i(v_1, v_2)$ and payment rule $x_i(v_1, v_2)$, $i = 1, 2$ is a *direct* mechanism since it is defined on players' type spaces (here their values). A direct mechanism is truthful if and only if at equilibrium, both bidders reveal their values truthfully. In other words, revealing one's type truthfully maximize his/her expected payoff, given the other player is revealing his/her type truthfully.

The favoritism rule together with the induced monotone pure-strategy equilibrium generates a truthful direct mechanism (formally defined as below): winning rule $p_i(v_1, v_2)$ and effort supply rule $x_i(v_1, v_2)$, $i = 1, 2$:

$$p_1(v_1, v_2) = \begin{cases} 1, & \text{if } b_1(v_1) \geq B^{-1}(b_2(v_2)), \\ 0, & \text{if } b_1(v_1) < B^{-1}(b_2(v_2)), \end{cases} \text{ and } p_2(v_1, v_2) = 1 - p_1(v_1, v_2), \tag{1}$$

$$\text{and } x_1(v_1, v_2) = b_1(v_1), \; x_2(v_1, v_2) = b_2(v_2). \tag{2}$$

On the other hand, our following analysis only relies on the existence of equilibrium in either pure strategy or mixed strategy. Relying on Theorem 6* in Dasgupta and Maskin (1986), it is standard to show the existence of equilibrium in our setting. If the equilibrium is in mixed strategy, $p_i(v_1, v_2)$ in a direct mechanism can be defined as player i's equilibrium winning probability in the all pay auction conditional on their values v_1 and v_2; payments $x_i(v_1, v_2)$ can be defined as their expected equilibrium payments in the all pay auction conditional on their values v_1 and v_2.

Remark 1. *Our setup is different from Myerson (1981) who allows the sum of the players winning probabilities to be strictly smaller than 1. Our goal is not the derive an unrestricted optimal mechanism which maximises the total effort. Our goal is to derive the optimal favouritism rule based on bids within the analytical framework of all-pay auctions.*

3.1 Roadmap

We first identify the effort-maximizing mechanism with the restriction of $\sum_i p_1(v_1, v_2) = 1$. Note that this constraint must be satisfied by any bidding equilibrium and favoritism rule in an all pay auction. Then we identify the favoritism rule together with the induced monotone pure-strategy equilibrium, which generates the same total expected effort. If this procedure goes through, then the identified favoritism rule is the optimal rule, which generates the maximal expected effort.

3.2 Truthful direct mechanism and effort bound

Consider direct mechanisms $\{p_i(v_1, v_2), x_i(v_1, v_2), i = 1, 2\}$ with $\sum_i p_i(v_1, v_2) = 1$. Following Myerson (1981), we can focus on truthful direct mechanisms.

Given the other bidder j reveals his/her value truthfully, the bidder i's expected payoff is the following if his/her value is v_i and s/he reports v_i' :

$$\pi_i(v_i', v_i) = \int_0^{\bar{v}} \left[p_i(v_i', v_j)v_i - x_i(v_i', v_j)\right] g_j(v_j)dv_j. \tag{3}$$

Since we require the mechanism is truthful, this means that reporting truthfully, $v_i' = v_i$, maximizes his/her expected payoff:

$$\pi_i(v_i, v_i) \geq \pi_i(v_i', v_i), \forall v_i', v_i \in [0, \bar{v}]. \tag{4}$$

Since $v_i' = v_i$ maximizes $\pi_i(v_i', v_i)$ for given v_i, we must have

$$\frac{\partial \pi_i(v_i', v_i)}{\partial v_i'}|_{v_i'=v_i} = 0. \tag{5}$$

This is called first order condition for maximizations.

We now look at the property of maximized optimal payoff $\pi_i^*(v_i) = \pi_i(v_i, v_i) = \pi_i(v_i', v_i)|_{v_i'=v_i}$.

$$\begin{aligned}
\frac{d\pi_i^*(v_i)}{dv_i} &= \frac{d\pi_i(v_i, v_i)}{dv_i} = \frac{\partial \pi_i(v_i', v_i)}{\partial v_i'}|_{v_i'=v_i} + \frac{\partial \pi_i(v_i', v_i)}{\partial v_i}|_{v_i'=v_i} \\
&= \frac{\partial \pi_i(v_i', v_i)}{\partial v_i}|_{v_i'=v_i} = \left\{\int_0^{\bar{v}} \left[p_i(v_i', v_j)\right] g_j(v_j)dv_j\right\}|_{v_i'=v_i} \\
&= \int_0^{\bar{v}} \left[p_i(v_i, v_j)\right] g_j(v_j)dv_j,
\end{aligned} \tag{6}$$

which is bidder i's expected winning probability upon his/her truthful revelation of his/her value.

Let

$$P_i(v_i) = \int_0^{\bar{v}} [p_i(v_i, v_j)]\, g_j(v_j) dv_j. \tag{7}$$

Alternatively, we can write

$$P_i(t) = \int_0^{\bar{v}} [p_i(t, v_j)]\, g_j(v_j) dv_j, \forall t \in [0, \bar{v}]. \tag{8}$$

Therefore,

$$\frac{d\pi_i^*(t)}{dt} = P_i(t). \tag{9}$$

This result is called envelop theorem in the mechanism design literature, which says that the slope of the bidders' expected payoff as a function of his/her own value is his/her expected winning probability.

Recall that we consider favoritism rule which induces monotone pure-strategy equilibrium, $b_1(v_1)$ and $b_2(v_2)$, with $b_1(0) = b_2(0) = 0$. This means that we must have $\pi_i^*(0) = \pi_i(0, 0) = 0$ at equilibrium.

Using $\pi_i^*(0)$ and condition (6), we can fully pin down the bidders' expected payoff as follows if his/her value is v_i:

$$\begin{aligned} \pi_i^*(v_i) - \pi_i^*(0) &= \int_0^{v_i} \left[\frac{d\pi_i^*(t)}{dt}\right] dt = \int_0^{v_i} P_i(t) dt, \\ \text{i.e., } \pi_i^*(v_i) &= \int_0^{v_i} P_i(t) dt. \end{aligned} \tag{10}$$

Since bidder i's value v_i follows distribution $G_i(\cdot)$, his/her expected payoff is

$$\int_0^{\bar{v}} \pi_i^*(v_i) g_i(v_i) dv_i = \int_0^{\bar{v}} \int_0^{v_i} P_i(t) dt g_i(v_i) dv_i = \int_0^{\bar{v}} \int_0^{v_i} P_i(t) g_i(v_i) dt dv_i.$$

Switching the order of integration, we have

$$\begin{aligned}
& \int_0^{\bar{v}} \pi_i^*(v_i) g_i(v_i) dv_i \\
= \; & \int_0^{\bar{v}} \int_t^{\bar{v}} g_i(v_i) P_i(t) dv_i dt = \int_0^{\bar{v}} \left[\int_t^{\bar{v}} g_i(v_i) dv_i \right] P_i(t) dt \\
= \; & \int_0^{\bar{v}} \left[1 - G_i(t) \right] P_i(t) dt = \int_0^{\bar{v}} \left[\frac{1 - G_i(t)}{g_i(t)} \right] P_i(t) g_i(t) dt \\
= \; & \int_0^{\bar{v}} \left[\frac{1 - G_i(v_i)}{g_i(v_i)} \right] P_i(v_i) g_i(v_i) dv_i = \int_0^{\bar{v}} \left[\frac{1 - G_i(v_i)}{g_i(v_i)} \right] \left[\int_0^{\bar{v}} p_i(v_i, v_j) g_j(v_j) dv_j \right] g_i(v_i) dv_i \\
= \; & \int_0^{\bar{v}} \int_0^{\bar{v}} \left[\frac{1 - G_i(v_i)}{g_i(v_i)} \right] p_i(v_i, v_j) g_j(v_j) g_i(v_i) dv_i dv_j. \qquad (11)
\end{aligned}$$

We next look at the expression for the seller's expected payoff, and investigate how it relates to the selling probabilities.

$$TE = \int_0^{\bar{v}} \int_0^{\bar{v}} [x_1(v_1, v_2) + x_2(v_1, v_2)] g_1(v_1) g_2(v_2) dv_1 dv_2. \qquad (12)$$

By definition (3), we have

$$\pi_i^*(v_i) = \pi_i(v_i, v_i) = \int_0^{\bar{v}} \left[p_i(v_i, v_j) v_i - x_i(v_i, v_j) \right] g_j(v_j) dv_j.$$

Thus,

$$\int_0^{\bar{v}} \pi_i^*(v_i) g_i(v_i) dv_i = \int_0^{\bar{v}} \pi_i(v_i, v_i) g_i(v_i) dv_i = \int_0^{\bar{v}} \int_0^{\bar{v}} \left[p_i(v_i, v_j) v_i - x_i(v_i, v_j) \right] g_j(v_j) dv_j g_i(v_i) dv_i. \qquad (13)$$

Using (12) and (13), we have

$$
\begin{aligned}
& TE + \int_0^{\bar{v}} \pi_1^*(v_1) g_1(v_1) dv_1 + \int_0^{\bar{v}} \pi_2^*(v_2) g_2(v_2) dv_2 \\
= & \int_0^{\bar{v}} \int_0^{\bar{v}} [x_1(v_1,v_2) + x_2(v_1,v_2)] g_1(v_1) g_2(v_2) dv_1 dv_2 \\
& + \int_0^{\bar{v}} \int_0^{\bar{v}} \left[p_1(v_1,v_2)v_1 - x_1(v_1,v_2)\right] g_2(v_2) dv_2 g_1(v_1) dv_1 \\
& + \int_0^{\bar{v}} \int_0^{\bar{v}} \left[p_2(v_1,v_2)v_2 - x_2(v_1,v_2)\right] g_1(v_1) dv_1 g_2(v_2) dv_2 \\
= & \int_0^{\bar{v}} \int_0^{\bar{v}} \left[p_1(v_1,v_2)v_1 + p_2(v_1,v_2)v_2\right] g_1(v_1) dv_1 g_2(v_2) dv_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
TE = & \int_0^{\bar{v}} \int_0^{\bar{v}} \left[p_1(v_1,v_2)v_1 + p_2(v_1,v_2)v_2\right] g_1(v_1) dv_1 g_2(v_2) dv_2 \\
& - \left[\int_0^{\bar{v}} \pi_1^*(v_1) g_1(v_1) dv_1 + \int_0^{\bar{v}} \pi_2^*(v_2) g_2(v_2) dv_2\right].
\end{aligned}
$$

Using (11), we further have

$$
\begin{aligned}
TE = & \int_0^{\bar{v}} \int_0^{\bar{v}} \left[p_1(v_1,v_2)v_1 + p_2(v_1,v_2)v_2\right] g_1(v_1) dv_1 g_2(v_2) dv_2 \\
& - \int_0^{\bar{v}} \int_0^{\bar{v}} \left[\frac{1 - G_1(v_1)}{g_1(v_1)}\right] p_1(v_1,v_2) g_2(v_2) g_1(v_1) dv_1 dv_2 \\
& - \int_0^{\bar{v}} \int_0^{\bar{v}} \left[\frac{1 - G_2(v_2)}{g_2(v_2)}\right] p_2(v_1,v_2) g_1(v_1) g_2(v_2) dv_1 dv_2 \\
= & \int_0^{\bar{v}} \int_0^{\bar{v}} \left\{ \begin{array}{c} p_1(v_1,v_2)\left[v_1 - \frac{1-G_1(v_1)}{g_1(v_1)}\right] \\ +p_2(v_1,v_2)\left[v_2 - \frac{1-G_2(v_2)}{g_2(v_2)}\right] \end{array} \right\} g_1(v_1) g_2(v_2) dv_1 dv_2. \qquad (14)
\end{aligned}
$$

By definition of $J_i(v_i) = v_i - \frac{1-G_i(v_i)}{g_i(v_i)}, i = 1, 2$. We have

$$TE = \int_0^{\bar{v}} \int_0^{\bar{v}} \{p_1(v_1, v_2)J_1(v_1) + p_2(v_1, v_2)J_2(v_2)\} g_1(v_1)g_2(v_2)dv_1dv_2. \tag{15}$$

Recall the restriction of $p_1(v_1, v_2) + p_2(v_1, v_2) = 1$, we have

$$TE = \int_0^{\bar{v}} \int_0^{\bar{v}} \{p_1(v_1, v_2)[J_1(v_1) - J_2(v_2)] + J_2(v_2)\} g_1(v_1)g_2(v_2)dv_1dv_2. \tag{16}$$

Define

$$p_1^*(v_1, v_2) = \begin{cases} 1, & \text{if } J_1(v_1) - J_2(v_2) \geq 0, \\ 0, & \text{if } J_1(v_1) - J_2(v_2) < 0, \end{cases} \text{ and } p_2^*(v_1, v_2) = 1 - p_1^*(v_1, v_2). \tag{17}$$

Then the following TE^* provides an effort bound in an all pay auction with an arbitrary favoritism rule $B(b_1)$ we consider.

$$TE^* = \int_0^{\bar{v}} \int_0^{\bar{v}} \{p_1^*(v_1, v_2)[J_1(v_1) - J_2(v_2)] + J_2(v_2)\} g_1(v_1)g_2(v_2)dv_1dv_2. \tag{18}$$

Theorem 1. *In our all pay auction, the total expected effort inducible under any favoritism rule cannot go beyond* TE^*.

Remark 2. *If we can find a particular favoritism rule* $B^*(b_1)$*, which induces at equilibrium total expected effort* TE^* *in the original all pay auction, then rule* $B^*(b_1)$ *must be optimal.*

Remark 3. *Total expected effort must be* TE^* *if at equilibrium winning probabilities* $p_1^*(v_1, v_2)$ *is induced and* $\pi_i^*(0) = 0$*, i.e. a bidder with value* 0 *has zero payoff.*

3.3 Optimal favoritism rule and bidding equilibrium in all pay auction

We first identify an all pay effort supply rule, which supports winning probabilities $p_i^*(v_1, v_2), i = 1, 2$ such that they together constitute a truthful direct mechanism.

Recall (10) and by definition

$$\pi_i^*(v_i) = \pi_i(v_i, v_i) = \int_0^{\bar{v}} \left[p_i(v_i, v_j)v_i - x_i(v_i, v_j)\right] g_j(v_j) dv_j.$$

We have for $p_i^*(v_1, v_2), i = 1, 2$, we have

$$\int_0^{v_i} \int_0^{\bar{v}} p_i^*(t, v_j) g_j(v_j) dv_j dt = \int_0^{\bar{v}} \left[p_i^*(v_i, v_j)v_i - x_i(v_i, v_j)\right] g_j(v_j) dv_j$$

$$= \int_0^{\bar{v}} p_i^*(v_i, v_j) v_i g_j(v_j) dv_j - \int_0^{\bar{v}} x_i(v_i, v_j) g_j(v_j) dv_j.$$

Let $b_i^*(v_i) = \int_0^{\bar{v}} x_i(v_i, v_j) g_j(v_j) dv_j$. Recall $P_i^*(v_i) = \int_0^{\bar{v}} p_i^*(v_i, v_j) g_j(v_j) dv_j$.

Then

$$b_i^*(v_i) = v_i P_i^*(v_i) - \int_0^{v_i} P_i^*(t) dt. \tag{19}$$

Lemma 2. *(i) Under Assumption 2, we have $J_1(0) = J_2(0)$. (ii) $P_i^{*\prime}(v_i) \geq 0$, $\forall v_i \geq 0$; $P_i^{*\prime}(v_i) > 0$, $\forall v_i > 0$.*

Proof: $J_1(0) = J_2(0) = -\frac{1}{g_i(0)}$. Define $\hat{v}_j(v_i) \in [0, \bar{v}]$ by $J_i(v_i) = J_j(\hat{v}_j(v_i))$. Note $\hat{v}_j(v_i)$ increases with v_i. $P_i^*(v_i) = \int_0^{\hat{v}_j(v_i)} g_j(v_j) dv_j = G_j(\hat{v}_j(v_i))$. Thus, $P_i^{*\prime}(v_i) = g_j(\hat{v}_j(v_i)) \hat{v}_j'(v_i) > 0$. □

Lemma 3. $b_i^*(0) = 0$, $b_i^{*\prime}(0) = 0, b_i^{*\prime}(v_i) > 0, \forall v_i > 0$.

Proof: It is clear that $b_i^*(0) = 0$.

$$b_i^{*\prime}(v_i) = P_i^*(v_i) + v_i P_i^{*\prime}(v_i) - P_i^*(v_i) = v_i P_i^{*\prime}(v_i). \tag{20}$$

□

We are now ready to define the favoritism rule: Bidder 1 wins if and only if

$$J_1(b_1^{*-1}(b_1)) \geq J_2(b_2^{*-1}(b_2)).$$

In other words,

$$(B^*)^{-1}(b_2) = b_1^* \circ (J_1^{-1}) \circ J_2 \circ b_2^{*-1}(b_2), \text{ and } B^*(b_1) = b_2^* \circ (J_2^{-1}) \circ J_1 \circ b_1^{*-1}(b_1). \tag{21}$$

We next establish the following result.

Proposition 1. *Under favoritism rule $(B^*)^{-1}(b_2)$ or equivalently $B^*(b_1)$, it is an equilibrium for bidder i to adopt bidding strategy $b_i^*(v_i)$.*

Proof: Suppose bidder 2 adopts strategy $b_2^*(v_2)$. Note that bidder 1 has no incentive to bid above $b_1^*(\bar{v})$, since bidding $b_1^*(\bar{v})$ makes sure he wins. We consider bidder 1's expected payoff if s/he bids $b_1 = b_1^*(v_1'), v_1' \in [0, \bar{v}]$, and his/her value is v_1:

$$\begin{aligned}
\pi_1(b_1; v_1) &= v_1 \Pr(v_2 | b_2^*(v_2) \leq B^*(b_1)) - b_1 = v_1 \Pr(v_2 | v_2 \leq b_2^{*-1} \circ B^*(b_1)) - b_1 \\
&= v_1 \Pr(v_2 | v_2 \leq (J_2^{-1}) \circ J_1 \circ b_1^{*-1}(b_1)) - b_1 = v_1 \Pr(v_2 | J_2(v_2) \leq J_1(v_1')) - b_1^*(v_1') \\
&= v_1 P_1^*(v_1') - b_1^*(v_1').
\end{aligned}$$

Let

$$\tilde{\pi}_1(v_1'; v_1) = v_1 P_1^*(v_1') - b_1^*(v_1').$$

We want to show for given v_1, $\tilde{\pi}_1(v_1'; v_1)$ is maximized at $v_1' = v_1$. For this purpose, we want to show the following results:

$$\frac{\partial \tilde{\pi}_1(v_1'; v_1)}{\partial v_1'}|_{v_1' = v_1} = 0; \frac{\partial \tilde{\pi}_1(v_1'; v_1)}{\partial v_1'}|_{v_1' > v_1} < 0; \frac{\partial \tilde{\pi}_1(v_1'; v_1)}{\partial v_1'}|_{v_1' < v_1} > 0. \tag{22}$$

Note

$$\frac{\partial \tilde{\pi}_1(v_1'; v_1)}{\partial v_1'} = v_1 P_1^{*\prime}(v_1') - b_1^{*\prime}(v_1').$$

Using (20), we have

$$\frac{\partial \tilde{\pi}_1(v_1'; v_1)}{\partial v_1'} = \left[v_1 - v_1'\right] P_1^{*\prime}(v_1').$$

Thus (22) holds, which means $\tilde{\pi}_1(v_1'; v_1)$ is maximized at $v_1' = v_1$. In other words, bidding $b_1 = b_1^*(v_1)$ is optimal for bidder 1. Similarly, we can show that given bidder 1 adopts strategy $b_1^*(v_1)$, it is optimal for bidder 2 to bid $b_2 = b_2^*(v_2)$ if his/her value is v_2. □

We next establish the following result.

Proposition 2. *Under favoritism rule $(B^*)^{-1}(b_2)$ or equivalently $B^*(b_1)$, and equilibrium bidding strategy $b_i^*(v_i)$, (i) bidder i wins if and only if $J_i(v_i) \leq J_j(v_j)$. In other words, winning rule $p_i^*(v_i, v_j), i = 1, 2$ is implemented; (ii) $\pi_i^*(0) = 0$.*

Proof: Suppose bidder 1's value is v_1 and bidder 2's value is v_2. Then at equilibrium, bidder 1 bids $b_1^*(v_1)$ and bidder 2 bids $b_2^*(v_2)$. Bidder 1 wins if and only if $b_2^*(v_2) \leq B^*(b_1^*(v_1))$, which is $J_2(v_2) \leq J_1(v_1)$.

Bidders bid zero when their value is zero. Even when they win, their value is zero. Therefore, $\pi_i^*(0) = 0$. □

Based on Remarks 2 and 3, we have the following result according to our roadmap.

Theorem 2. *Favoritism rule $(B^*)^{-1}(b_2)$ or equivalently $B^*(b_1)$ is optimal in the original all pay auction setting. Under this rule, the bidding equilibrium $b_i^*(v_i), i = 1, 2$ generates total expected effort of TE^* in (18).*

Remark 4. *We assumed that $g_1(0) = g_2(0)$ in Assumption 2(i). This assumption is not necessary for the above construction to work. We only need $g_1(0) \leq g_2(0)$. One can verify that with $g_1(0) \leq g_2(0)$, under the above constructed favoritism rule, the identified $b_1^*(v_1)$ and $b_2^*(v_2)$ still constitute a bidding equilibrium, which achieves the revenue bound TE^* in (18).*

3.4 Does the weaker necessarily win with higher chance under the optimal favoritism?

One question naturally arises: Compared to the a scenario of standard all pay auction without favoritism, does the optimal favoritism necessarily give higher expected winning chance to the weaker player? Surprisingly, the answer to this question is negative. We illustrate this by the following example. Let

$$G_1(v_1) = (v_1)^2, \forall v_1 \in [0,1] \text{ and } G_2(v_2) = v_2, \forall v_2 \in [0,1].$$

We thus have

$$g_1(v_1) = 2v_1, \forall v_1 \in [0,1] \text{ and } g_2(v_2) = 1, \forall v_2 \in [0,1].$$

Note that we have $G_1(v)$ stochastically dominates $G_2(v)$ in the sense of hazard rate, i.e. $\frac{g_2(v)}{1-G_2(v)} \geq \frac{g_1(v)}{1-G_1(v)}$, since $\frac{1-G_1(v)}{g_1(v)} = \frac{1-v^2}{2v} = (1-v)\frac{1+v}{2v} \geq 1-v = \frac{1-G_2(v)}{g_2(v)}$. We thus have Assumptions 1 and 2 hold except 2(i). We still have $g_1(0) < g_2(0)$, which ensures that our procedure goes through as we have clarified in the model setup.

Remark 5. *Since we have $J_1(v_1) < J_2(v_2)$ for $v_1 = v_2 = v$, we have that the effort-maximizing favoritism rule mast favor the weaker player 2 compared to the efficient allocation rule, which always makes the player with higher value to win.*

3.4.1 Players' winning chances under optimal favoritism

Under the optimal favoritism, the winning rule of (17) can be rewritten as

$$p_1^*(v_1,v_2) = \begin{cases} 1, & \text{if } v_2 \leq \frac{3(v_1)^2-1}{4v_1} + \frac{1}{2} \text{ and } v_1 \in [\frac{1}{3},1], \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } p_2^*(v_1,v_2) = 1 - p_1^*(v_1,v_2).$$

Player 1's expected winning probability under the optimal favoritism is:

$$\begin{aligned} P_1^* &= \int_{\frac{1}{3}}^{1} G_2(\frac{3(v_1)^2-1}{4v_1} + \frac{1}{2})(2v_1)dv_1 = \int_{\frac{1}{3}}^{1} (\frac{3(v_1)^2-1}{4v_1} + \frac{1}{2})(2v_1)dv_1 \\ &= \int_{\frac{1}{3}}^{1} (\frac{3(v_1)^2-1}{2} + v_1)dv_1 = \left\{ \frac{1}{2}\left[(v_1)^3 - v_1\right] + \frac{(v_1)^2}{2} \right\} |_{1/3}^{1} \\ &= \frac{1}{2} - \{\frac{1}{2}\left[(\frac{1}{3})^3 - \frac{1}{3}\right] + \frac{(\frac{1}{3})^2}{2}\} = \frac{16}{27}. \end{aligned}$$

It follows that player 2's expected winning chance is:

$$P_2^* = \frac{11}{27}.$$

3.4.2 Players' winning chances in a standard all pay auction with no favoritism

To pin down the expected winning chances of the players in a standard all pay auction with no favoritism does not require us to explicitly solve for the equilibrium bidding strategy. What we need to pin down is player 2's winning threshold $k(v_1)$ for each value of player 1. According to Amann and Leininger (1996), this function $k(v_1)$ is the unique solution of the following differential equation with boundary condition $k(1) = 1$:[5]

$$k'(v_1) = \frac{k(v_1)g_1(v_1)}{v_1 g_2(k(v_1))} = 2k(v_1).$$

Therefore, the solution is:

$$k(v_1) = \exp(2v_1 - 2).$$

We now are ready to calculate player 1's expected winning probability in a standard all pay auction:

$$\begin{aligned}
\check{P}_1^* &= \int_0^1 G_2(\exp(2v_1 - 2))(2v_1)dv_1 = \int_0^1 (\exp(2v_1 - 2))(2v_1)dv_1 \\
&= \frac{1}{e^2}\int_0^1 v_1 d\exp(2v_1) = \frac{1}{e^2}\left\{v_1\exp(2v_1)|_0^1 - \int_0^1 \exp(2v_1)dv_1\right\} \\
&= \frac{1}{e^2}\left\{v_1\exp(2v_1) - \frac{\exp(2v_1)}{2}\right\}|_0^1 = \frac{1}{e^2}\left\{\frac{\exp(2)}{2} - \frac{1}{2}\right\} \\
&= \frac{1}{2}[1 + \frac{1}{e^2}].
\end{aligned}$$

[5]Please refer to page 9 in Amann and Leininger (1996) for detail.

It follows that player 2's expected winning chance is:

$$\check{P}_2^* = \frac{1}{2}[1 - \frac{1}{e^2}].$$

3.4.3 Comparison of equilibrium winning chances

We next show that in our example, the winning chance of weaker player (player 2) is lower under the optimal favoritism, i.e. $P_2^* < \check{P}_2^*$. For this purpose, we need to show $27 < 5e^2$, which clearly holds as $e \approx 2.718$.

Remark 6. *This result indicates that the optimal favoritism rule does not necessarily leads to a higher winning chance to the weaker player, although it is intended to favor the weaker player. The discriminatory policy forces the stronger player to bid more aggressively, which makes him/her winning with a higher chance at equilibrium. This observation will be confirmed again when we study the complete information setting in Section 4.1.*

3.5 Further implications of the optimal favoritism rule

We next investigate the implications of optimal favoritism rule defined by: Bidder 1 wins if and only if

$$J_1(v_1) \geq J_2(v_2).$$

In particular, we want to study whether it is possible that a particular bidder is always favored in terms of winning chances, provided the two bidders have the same value.

Recall that we have shown $J_i(v_i)$ starts from the same point under our assumption,[6] and ends at the same point regardless.

With two symmetric bidders, we have $G_1(v) = G_2(v)$, which means $J_1(v) = J_2(v)$ and $b_1^*(v) = b_2^*(v)$. We thus have the following result.

Theorem 3. *With two symmetric bidders, at optimum, there is no favoritism. The bidder with higher value bids higher and wins.*

[6] This property however is not needed for the following analysis.

Recall we assume that $G_1(v)$ stochastically dominates $G_2(v)$ in the sense of first order dominance, i.e. $G_2(v) \geq G_1(v)$, which is implied by Assumption 2. The following condition of hazard rate dominance further strengthens this assumption.

Condition 1. *$G_1(v)$ stochastically dominates $G_2(v)$ in the sense of hazard rate, i.e.* $\frac{g_2(v)}{1-G_2(v)} \geq \frac{g_1(v)}{1-G_1(v)}$.

Under Condition 1, we can rank virtual value functions $J_1(v)$ and $J_2(v)$ as shown in the following lemma.

Lemma 4. *Under Condition 1, we have $J_1(v) \leq J_2(v)$.*

Theorem 4. *Under Condition 1, bidder 2, the weaker player, always wins when both bidders have the same value.*

Next, we show that it is possible that $J_1(v_1)$ and $J_2(v_2)$ cross in $(0, v)$, even when $G_1(v)$ stochastically first-order dominates $G_2(v)$.

Condition 2. *$g_1(v) < g_2(v)$ in a small neighborhood of 0, and $g_1(v) \gg g_2(v)$ in a small neighborhood of $\bar{v}$.*

Remark 7. *Note Condition 2 is consistent with single-crossing density functions as required by Assumption 2, which entails first-order dominance of cumulative distribution functions.*

Lemma 5. *Under Condition 2, $J_1(v) < J_2(v)$ in a small neighborhood of 0, and $J_1(v) > J_2(v)$ in a small neighborhood of $\bar{v}$; Moreover, $J_1(v)$ and $J_2(v)$ must cross at least once in $(0, v)$.*

Theorem 5. *Under Condition 2, in different value ranges, at optimum, different bidders win the competition if the bidders have the same value.*

3.6 Favoritism achieving efficient allocation

We now look the favoritism rule that achieves the efficient allocation, i.e. always making the player with higher value to win. This winning rule can be written as

$$\tilde{p}_1^*(v_1, v_2) = \begin{cases} 1, & \text{if } v_1 - v_2 \geq 0, \\ 0, & \text{if } v_1 - v_2 < 0, \end{cases} \text{ and } \tilde{p}_2^*(v_1, v_2) = 1 - \tilde{p}_1^*(v_1, v_2). \tag{23}$$

The corresponding expected winning probabilities are:

$$\tilde{P}_i^*(v_i) = \int_0^{\bar{v}} \left[\tilde{p}_i^*(v_i, v_j)\right] g_j(v_j) dv_j. \tag{24}$$

Then let

$$\tilde{b}_i^*(v_i) = v_i \tilde{P}_i^*(v_i) - \int_0^{v_i} \tilde{P}_i^*(t) dt. \tag{25}$$

We are now ready to define the favoritism rule: Bidder 1 wins if and only if

$$\tilde{b}_1^{*-1}(b_1) \geq \tilde{b}_2^{*-1}(b_2).$$

In other words,

$$(\tilde{B}^*)^{-1}(b_2) = \tilde{b}_1^* \circ \tilde{b}_2^{*-1}(b_2), \text{ and } \tilde{B}^*(b_1) = \tilde{b}_2^* \circ \tilde{b}_1^{*-1}(b_1). \tag{26}$$

Following the same procedure as in Section 3.3, we can show that $\tilde{b}_1^*(v_1)$ and $\tilde{b}_2^*(v_2)$ constitute an increasing pure strategy Bayesian equilibrium under favoritism rule $(\tilde{B}^*)^{-1}(b_2)$ or $\tilde{B}^*(b_1)$. To save space, we omit the proofs. The following is an example which illustrates this procedure of deriving the favoritism rule and equilibrium bidding strategies.

3.6.1 Example

We now use an example to illustrate how to write down the efficient favoritism rule and the induced equilibrium bidding strategies of the players. To this end, we assume

$$G_1(v_1) = (v_1)^2, \forall v_1 \in [0,1] \text{ and } G_2(v_2) = v_2, \forall v_2 \in [0,1].$$

We thus have

$$g_1(v_1) = 2v_1, \forall v_1 \in [0,1] \text{ and } g_2(v_2) = 1, \forall v_2 \in [0,1].$$

The efficient winning rule of (23) can be rewritten as

$$\tilde{p}_1^*(v_1, v_2) = \begin{cases} 1, & \text{if } v_1 - v_2 \geq 0, \\ 0, & \text{if } v_1 - v_2 < 0, \end{cases} \text{ and } \tilde{p}_2^*(v_1, v_2) = 1 - \tilde{p}_1^*(v_1, v_2).$$

It follows that

$$\tilde{P}_1^*(v_1) = \int_0^1 p_1^*(v_1, v_2) g_2(v_2) dv_2 = \int_0^{v_1} g_2(v_2) dv_2 = G_2(v_1) = v_1,$$

and

$$P_2^*(v_2) = \int_0^1 p_2^*(v_1, v_2) g_1(v_2) dv_2 = \int_0^{v_2} g_1(v_1) dv_1 = G_1(v_2) = (v_2)^2.$$

The equilibrium bidding strategies are given by (25):

$$\tilde{b}_1^*(v_1) = v_1 \tilde{P}_1^*(v_1) - \int_0^{v_1} \tilde{P}_1^*(t) dt = (v_1)^2 - \int_0^{v_1} t dt = \frac{(v_1)^2}{2},$$

and

$$\tilde{b}_2^*(v_2) = v_2 \tilde{P}_2^*(v_2) - \int_0^{v_2} \tilde{P}_2^*(t) dt = (v_2)^3 - \int_0^{v_2} t^2 dt = \frac{2(v_2)^3}{3}.$$

Therefore, we have

$$(\tilde{B}^*)^{-1}(b_2) = \tilde{b}_1^* \circ \tilde{b}_2^{*-1}(b_2) = (\frac{1}{2})^{5/3}(3b_2)^{2/3}, \text{ and } \tilde{B}^*(b_1) = \tilde{b}_2^* \circ \tilde{b}_1^{*-1}(b_1) = \frac{2(2b_1)^{3/2}}{3}.$$

Note

$$\frac{\tilde{B}^*(b_1)}{b_1} = \frac{2^{5/2}(b_1)^{1/2}}{3}, b_1 \in [0, \frac{1}{2}].$$

Let $\hat{b}_1 = \frac{9}{32}$. Then we have

$$\frac{\tilde{B}^*(b_1)}{b_1} \begin{cases} < 1, & \text{if } b_1 \in [0, \hat{b}_1), \\ > 1, & \text{if } b_1 \in (\hat{b}_1, \frac{1}{2}]. \end{cases}$$

Remark 8. *Therefore, to achieve efficient allocation in this example setting, the contest organizer must favor the weaker player (i.e. player 2) if the stronger player's bid is lower than* $\hat{b}_1$*; and the* *contest organizer must favor the stronger player (i.e. player* 1*) if the stronger player's bid is higher than* $\hat{b}_1$*.*

The winning chance of the weaker player (player 2) under the efficient favoritism rule is

$$\tilde{P}_2^* = \int_0^1 G_1(v_2)dv_2 = \int_0^1 (v_2)^2 dv_2 = \frac{1}{3},$$

which is smaller than that (i.e. $P_2^* = \frac{11}{27}$) under the effort-maximizing favoritism rule. This means the effort-maximizing favoritism rule indeed enhances the weaker player's winning chance if we use the efficient allocation as reference point.

4 Discussions: when players' values are public information

The literature on favoritism in asymmetric competitions also studies the environment with complete information. In this section, we adopt such a setting and study the effort-maximizing favoritism rule. We will reveal that the optimal rule would favor the weak player to the extreme. However, at the optimum, the weak player would (almost) always lose the competition.

We adopt an analytical framework of all pay auctions with complete information. There are two players $i = 1, 2$. Player i's value of winning the auction is v_i, $i = 1, 2$, which are public information.

The players make their bids/effort simultaneously. The player with higher bid wins and pays his bid/effort cost, which equals his/her bid. The ties are broken randomly, unless it will be specified alternatively. The contest organizer and players' payoffs are specified in the same way as in Section 2: A player's expected payoff is his/her value multiplied by his/her winning probability minus his/her effort cost. The contest organizer maximizes players' expected total bids/effort.

4.1 Analysis with complete information

Without loss of generality, we assume $v_1 > v_2$, i.e. bidder 1 is a stronger bidder. Bidder i's bid/effort is denoted by $b_i \geq 0$, $i = 1, 2$. The higher bidder wins and both bidders incur the cost of their bid, which is b_i.[7]

[7] The bidder with higher value wins when there is a tie in their bids.

Definition 2. *The favoritism rule is specified by bidder 2's winning threshold* $B(b_1) \in [0, v_2]$*, which is an increasing function defined on* $[0, v_1]$*. This rule means that bidder 1 placing bid* b_1 *wins if and only if bidder 2's bid* b_2 *is no greater than* $B(b_1)$*. We use* $B^{-1}(b_2)$ *denote the inverse function.*

We first provide a general result of equilibrium construction under an arbitrary favoritism rule, which is defined as above.

Proposition 3. *Consider favoritism rule* $B_2(b_1)$ *such that* $B(0) = 0$ *and* $B(\bar{b}_1) = v_2$ *where* $\bar{b}_1 \in [v_2, v_1]$*. Then the following is a mixed strategy equilibrium:*

$$F_1(b_1) = \frac{B(b_1)}{v_2}, b_1 \in [0, \bar{b}_1], \text{ and } F_2(b_2) = \frac{v_1 - \bar{b}_1}{v_1} + \frac{B^{-1}(b_2)}{v_1}, b_2 \in [0, v_2]. \tag{27}$$

At equilibrium, bidder 1 enjoys an expected payoff of $v_1 - \bar{b}_1$*; and bidder 1 has an expected payoff of zero.*

Proof: Given bidder 2 plays $F_2(b_2)$ and the favoritism rule $B(b_1)$, bidder 1's expected payoff is as follows if he bids $b_1 \in [0, \bar{b}_1]$:

$$\pi_1(b_1) = F_2(B(b_1))v_1 - b_1 = \left[\frac{v_1 - \bar{b}_1}{v_1} + \frac{B^{-1}(B(b_1))}{v_1}\right] v_1 - b_1 = v_1 - \bar{b}_1.$$

Given bidder 1 plays $F_1(b_1)$ and the favoritism rule $B^{-1}(b_2)$, bidder 2's expected payoff is as follows if he bids $b_2 \in [0, v_2]$:

$$\pi_2(b_2) = F_1(B^{-1}(b_2))v_2 - b_2 = \frac{B(B^{-1}(b_2))}{v_2} v_2 - b_2 = 0.$$

□

We next construct the following favoritism rule denoted by bidder 2's winning threshold $B(b_1)$ and derive the associated equilibrium $F_i(b_i), i = 1, 2$ by Proposition 3. Recall that this rule $B(b_1)$ means that bidder 1 placing bid b_1 wins if and only if bidder 2's bid b_2 is no greater than $B(b_1)$.

The restriction we impose on $B^{-1}(b_2)$ is that it is increasing and $B^{-1}(0) = 0$. In particular, one can write $B^{-1}(b_2) = b_2 \cdot \kappa(b_2)$ where $\kappa(b_2) \in [0, +\infty)$.

Take small positive numbers ε and δ. Let

$$\frac{B(b_1)}{v_2} = \begin{cases} \frac{b_1}{v_1-\delta}\varepsilon, & \text{if } 0 \le b_1 \le v_1-\delta, \\ \varepsilon + \frac{1-\varepsilon}{\delta}[b_1-(v_1-\delta)], & v_1-\delta \le b_1 \le v_1. \end{cases} \tag{28}$$

Note $B(b_1) \in [0, v_2]$.

The inverse function of $B(b_1)$ is denoted by

$$\frac{B^{-1}(b_2)}{v_1} = \begin{cases} \frac{b_2/v_2}{\varepsilon}(1-\delta/v_1), & \text{if } 0 \le b_2 \le v_2\varepsilon, \\ (1-\delta/v_1) + \frac{\delta/v_1}{1-\varepsilon}[b_2/v_2-\varepsilon], & \text{if } v_2\varepsilon \le b_2 \le v_2. \end{cases} \tag{29}$$

Define bidding strategies

$$F_1(b_1) = \frac{B(b_1)}{v_2}, \text{ and } F_2(b_2) = \frac{B^{-1}(b_2)}{v_1}. \tag{30}$$

Based on Proposition 3, we have the following result.

Corollary 1. *$F_1(b_1)$ and $F_2(b_2)$ constitute a mixed strategy bidding equilibrium in the all pay auction under favoritism rule denoted by bidder 2's winning threshold $B(b_1)$, which is described in (28).*

Each bidder i's expected effort equals $v_i \Pr(i \text{ wins})$ minus his equilibrium payoff. Therefore the total expected effort from any equilibrium must be smaller than $\sum_i v_i \Pr(i \text{ wins}) \le v_1$. We thus have the following result.

Proposition 4. *The total expected effort induced in an all pay auction under any favoritism rule must be no greater than v_1.*

Note at the above identified equilibrium that bidders' equilibrium payoff is zero. Therefore, each bidder i's expected effort equals $v_i \Pr(i \text{ wins})$. We have the total expected effort is

$$\begin{aligned} TE &= v_1 [1-\Pr(\text{bidder 2 wins})] + v_2 \Pr(\text{bidder 2 wins}) \\ &= v_1 + [v_2 - v_1] \Pr(\text{bidder 2 wins}). \end{aligned}$$

We next show that for the constructed equilibrium and favoritism rule, the total expected effort converges to the upper bound v_1 when ε and δ converge to zero.

Proposition 5. *For the constructed equilibrium and favoritism rule, the total expected effort converges to the upper bound v_1 when ε and δ converge to zero.*

Proof: Recall $TE = v_1 + [v_2 - v_1]\Pr(\text{bidder 2 wins})$ since both players' equilibrium payoffs are zero. For our purpose, we only need to show that Pr(bidder 2 wins) converges to zero.

$$\Pr(\text{bidder 2 wins}) = \int_0^{v_2}\int_0^{B^{-1}(b_2)} dF_1(b_1)dF_2(b_2) = \int_0^{v_2} F_1(B^{-1}(b_2))dF_2(b_2)$$

$$\begin{aligned} &= \int_0^{v_2} \frac{B(B^{-1}(b_2))}{v_2}dF_2(b_2) = \int_0^{v_2}\frac{b_2}{v_2}dF_2(b_2) = \frac{b_2}{v_2}F_2(b_2)|_0^{v_2} - \int_0^{v_2}\frac{1}{v_2}F_2(b_2)db_2 \\ &= 1 - \frac{1}{v_2}\left[\int_0^{v_2\varepsilon}\frac{b_2/v_2}{\varepsilon}(1-\delta/v_1)db_2 + \int_{v_2\varepsilon}^{v_2}\left\{(1-\delta/v_1) + \frac{\delta/v_1}{1-\varepsilon}[b_2/v_2 - \varepsilon]\right\}db_2\right].\end{aligned}$$

Note that $0 \le \int_0^{v_2\varepsilon}\frac{b_2/v_2}{\varepsilon}(1-\delta/v_1)db_2 < \frac{\varepsilon/v_2}{\varepsilon}(1-\delta/v_1)\cdot[v_2\varepsilon - 0] = (1-\delta/v_1)\cdot\varepsilon$. Therefore $\int_0^{v_2\varepsilon}\frac{b_2/v_2}{\varepsilon}(1-\delta/v_1)db_2$ converges to zero when $\varepsilon \to 0^+$.

Note $\int_{v_2\varepsilon}^{v_2}\left\{(1-\delta/v_1) + \frac{\delta/v_1}{1-\varepsilon}[b_2/v_2-\varepsilon]\right\}db_2$ converges to $\int_0^{v_2} db_2 = v_2$ when $\varepsilon \to 0^+$ and $\delta \to 0^+$.

Therefore, Pr(bidder 2 wins) converges to 0 when $\varepsilon \to 0^+$ and $\delta \to 0^+$. □

Remark 9. *Such nonlinear favoritism rule favors bidder 2 extremely to incentivize bidder 1, i.e. the stronger bidder, to bidder close to his value; at the same time, at the equilibrium, bidder 2 wins with probability 0 even he is extremely favored. Note that in a standard all pay auction, the weaker player wins with probability $\frac{v_2}{2v_1}(>0)$ at equilibrium.*[8].

Remark 10. *For the above identified optimal favoritism rule, we have that at the limit, the stronger player wins with probability 1, and both players have zero expected payoff. We claim these properties must be satisfied by any optimal favoritism rule, which induces the upper bound effort v_1. which is identified in Proposition 4. Let $TE_i, i = 1, 2$ denote player i's expected effort. Note player i's*

[8] See Hillman and Riley (1989) or Baye et al. (1996) for the equilibrium bidding strategies in a standard all pay auction. This bidding equilibrium can also be obtained by Proposition 3 and setting $B_1(b_2)$ and $B_2(b_1)$ as identity functions.

expected payoff is

$$\pi_i = v_i \Pr(\textit{bidder } i \textit{ wins}) - TE_i.$$

Therefore, the total expected effort

$$\begin{aligned} TE_1 + TE_2 &= v_1 [1 - \Pr(\textit{bidder } 2 \textit{ wins})] + \ v_2 \Pr(\textit{bidder } 2 \textit{ wins}) - [\pi_1 + \pi_2] \\ &= v_1 + \ [v_2 - v_1] \Pr(\textit{bidder } 2 \textit{ wins}) - [\pi_1 + \pi_2]. \end{aligned}$$

Note that players' participation constraints mean that we must have $\pi_i \geq 0$. *To have* $TE_1 + TE_2 = v_1$, *we must have* $\Pr(\textit{bidder } 2 \textit{ wins}) = 0$ *and* $\pi_i = 0$.

Geometric interpretation

Next, we present a graphical interpretation of the logic behind the above identified optimal design. Consider the bidding equilibriums of Proposition 3. Player 1's expected effort is

$$\begin{aligned} TE_1 &= \int_0^{\bar{b}_1} b_1 dF_1(b_1) = \int_0^{\bar{b}_1} b_1 d\frac{B(b_1)}{v_2} \\ &= [b_1 \frac{B(b_1)}{v_2}]|_0^{\bar{b}_1} - \frac{1}{v_2} \int_0^{\bar{b}_1} B(b_1) db_1 \\ &= \bar{b}_1 - \frac{1}{v_2} \int_0^{\bar{b}_1} B(b_1) db_1, \end{aligned}$$

and player 2's expected effort is

$$\begin{aligned} TE_1 &= \int_0^{v_2} b_2 dF_2(b_2) = \int_0^{v_2} b_2 d[\frac{v_1 - \bar{b}_1}{v_1} + \frac{B^{-1}(b_2)}{v_1}] \\ &= b_2[\frac{v_1 - \bar{b}_1}{v_1} + \frac{B^{-1}(b_2)}{v_1}]|_0^{v_2} - \int_0^{v_2} [\frac{v_1 - \bar{b}_1}{v_1} + \frac{B^{-1}(b_2)}{v_1}] db_2 \\ &= v_2 - \frac{v_1 - \bar{b}_1}{v_1} v_2 - \frac{1}{v_1} \int_0^{v_2} B^{-1}(b_2) db_2 \\ &= \frac{v_2}{v_1} \bar{b}_1 - \frac{1}{v_1} \int_0^{v_2} B^{-1}(b_2) db_2. \end{aligned}$$

Therefore, we have

$$TE_1 + TE_2 = \bar{b}_1 + \frac{v_2}{v_1}\bar{b}_1 - \left[\frac{1}{v_2}\int_0^{\bar{b}_1} B(b_1)db_1 + \frac{1}{v_1}\int_0^{v_2} B^{-1}(b_2)db_2\right].$$

Note that $\int_0^{\bar{b}_1} B(b_1)db_1 + \int_0^{v_2} B^{-1}(b_2)db_2$ is simply the area of $[0, \bar{b}_1] \times [0, v_2]$. Thus,

$$\underbrace{\int_0^{\bar{b}_1} B(b_1)db_1}_{X} + \underbrace{\int_0^{v_2} B^{-1}(b_2)db_2}_{Y} = \bar{b}_1 v_2,\ \forall B(\cdot),\ \text{s.t.: } B(0) = 0,\ B(\bar{b}_1) = v_2.$$

Since $\frac{1}{v_2} > \frac{1}{v_1}$, to maximize $TE_1 + TE_2$, we want to minimize $\int_0^{\bar{b}_1} B(b_1)db_1$. This means that $X = \int_0^{\bar{b}_1} B(b_1)db_1 = 0$ and $Y = \int_0^{v_2} B^{-1}(b_2)db_2 = \bar{b}_1 v_2$ at the limit. This requires that $B(b_1)$ takes very low value when b_1 is lower than $\bar{b}_1$.

As a result, in the limit, we have $\frac{1}{v_2}X + \frac{1}{v_1}Y = \frac{\bar{b}_1 v_2}{v_1}$ at the optimum.

Thus,

$$TE_1 + TE_2 = \bar{b}_1 + \frac{v_2}{v_1}\bar{b}_1 - \frac{\bar{b}_1 v_2}{v_1} = \bar{b}_1,$$

which means that to maximize $TE_1 + TE_2$, we should set $\bar{b}_1^* = v_1$ at optimum.

For this optimal $\bar{b}_1^* = v_1$, the identified favoritism rule $B(b_1)$ in (28) indeed makes $\int_0^{\bar{b}_1^*} B(b_1)db_1 = \int_0^{v_1} B(b_1)db_1$ close to zero as $B(b_1)$ takes very low value when b_1 is lower than $v_1 - \delta$.

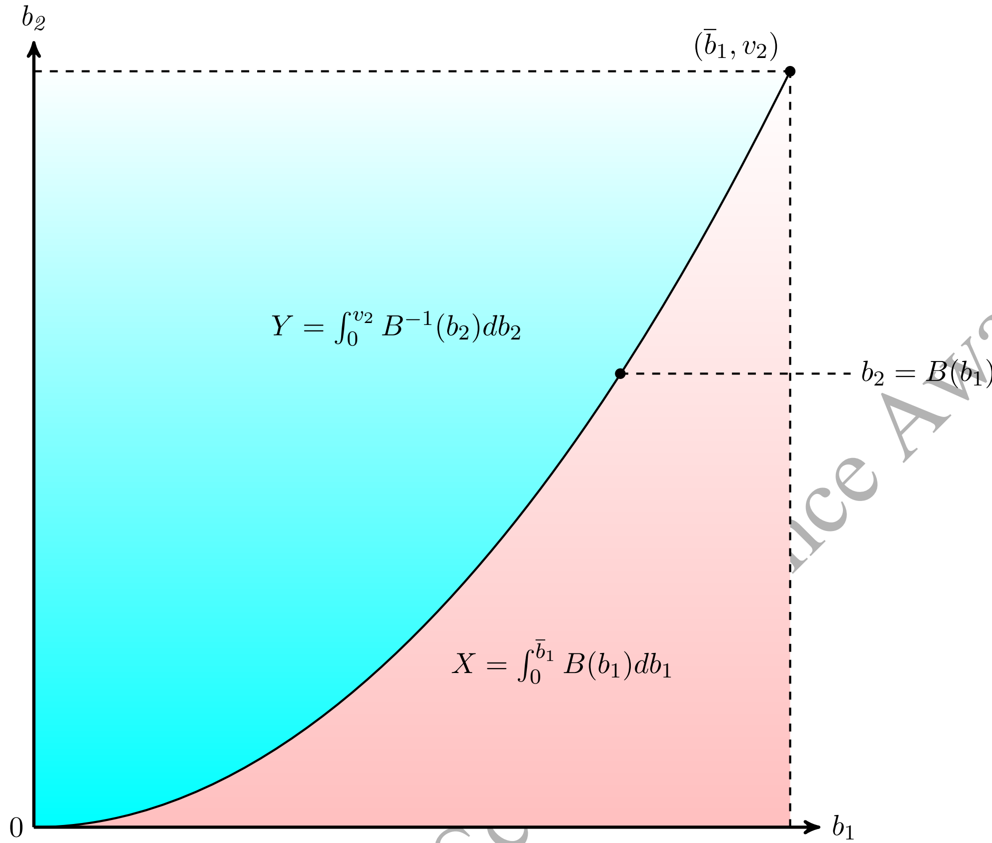

Figure 1: Illustration of Optimal Choice of Favoritism Rule

5 Concluding remarks

Asymmetric competitions are everywhere. Introducing favoritism in these competitions is an effective way to mitigate the discouragement effect due to the heterogeneity of players. The literature on favoritism in asymmetric competitions has so far been focusing on linear instruments such as headstart and handicap. In this paper, we generalize the analysis to accommodate fully nonlinear favoritism rules, and explicitly characterize the effort-maximizing favoritism rules and the induced equilibria in an analytical framework of all pay auctions with incomplete and/or complete information. Our optimal designs provide further guidance to better incentivize many asymmetric competitions in economic, political, social, and athletic contexts.

Besides effort elicitation, winner diversity is also one important consideration as evidenced by the rationale for the introduction of affirmative actions in school admissions, etc. Our findings reveal that favoritism focusing on pure effort-maximization can rather hurt the diversity compared to a standard all pay auction without favoritism. This result indicates that policy makers need to exercise caution when designing desirable favoritism rule. One possible extension of our study is to consider an objective of weight average of effort supply and winner diversity. We leave this to future work.

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