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Meromorphic functions with the same preimages at several finite sets

MEROMORPHIC FUNCTIONS WITH THE SAME PREIMAGES AT SEVERAL FINITE SETS

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ABSTRACT. Let p and q be nonconstant meromorphic functions on $\mathbb{C}^m$. We show that if p and q have the same preimages as one another, counting multiplicities, at each of four nonempty pairwise disjoint subsets $S_1, \dots, S_4$ of $\mathbb{C}$, then p and q have the same preimages as one another at each of infinitely many subsets of $\mathbb{C}$, and moreover $g(p) = g(q)$ for some nonconstant rational function $g(x)$ whose degree is bounded in terms of the sizes of the S_i's. This result is new already when $m = 1$, and it implies many previous results about the extent to which a meromorphic function is determined by its preimages of a few points or a few small sets, in addition to yielding new consequences such as a classification of all possibilities when two of the S_i's have size 1.

KEYWORDS. Value distribution, meromorphic functions.

CONTENTS

1. Introduction

As a consequence of his theory of value distribution of meromorphic functions, Nevanlinna [35] showed that a nonconstant meromorphic function on the complex plane is uniquely determined by its inverse images at any five points of the Riemann sphere $\mathbb{C}_\infty$. He also showed that if nonconstant meromorphic functions p, q on the complex plane have the same preimages as one another, counting multiplicities, at each of four points in $\mathbb{C}_\infty$, then there is a Möbius transformation μ such that $p = \mu \circ q$. In this paper we develop a new theory which addresses preimages of sets rather than merely preimages of points. In case the sets have size 1, our results generalize Nevanlinna's four-values theorem and the "counting multiplicities" version of Nevanlinna's five-values theorem. We will use the following standard terminology:

Notation. We write $\mathcal{M}(\mathcal{R})$ for the set of meromorphic functions on a complex manifold $\mathcal{R}$ (which in this paper can always be assumed to be either $\mathbb{C}^m$ or a compact Riemann surface such as the Riemann sphere $\mathbb{C}_\infty$).

Definition 1.1. We say that $p, q \in \mathcal{M}(\mathcal{R})$ share CM a subset S of $\mathbb{C}_\infty$ if the p-preimages of S coincide with the q-preimages of S, counting multiplicities.

Definition 1.1 involves the multiplicity of an element of $\mathcal{R}$ under an element of $\mathcal{M}(\mathcal{R})$. We will recall the definition of this notion in Section 2. We note that this and other concepts become simpler in case $\mathcal{R}$ has dimension 1, and that all results in this paper are new in the one-dimensional case.

Our first result asserts that if nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ share CM four "essentially different" finite sets, then there is a nonconstant rational function $g(x) \in \mathbb{C}(x)$ such that $g \circ p = g \circ q$ and $\deg(g)$ is bounded in terms of the sizes of the shared sets; it follows that p and q share CM (counting multiplicities) infinitely many finite sets.

Theorem 1.2. *Pick a positive integer m and nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$. Suppose that p and q share CM each of n finite subsets $S_1, \dots, S_n$ of $\mathbb{C}_\infty$ for some $n \geq 4$, where no S_i is contained in $\cup_{j \neq i} S_j$. Then $g \circ p = g \circ q$ for some nonconstant $g \in \mathbb{C}(x)$ such that $\deg(g) \leq \frac{1}{n-3}(-2 + \sum_{i=1}^{n} |S_i|)$, where in addition if $n \geq 5$ then $\deg(g) \leq \max_i |S_i|$.*

Note that if $g \circ p = g \circ q$ for some $g \in \mathbb{C}(x) \setminus \mathbb{C}$ then $p^{-1}(g^{-1}(\alpha)) = q^{-1}(g^{-1}(\alpha))$ for each $\alpha \in \mathbb{C}_\infty$, so if α is not a critical value of $g(x)$ then $g^{-1}(\alpha)$ is a set of size $\deg(g)$ which is shared CM by p and q. This yields the following consequence of Theorem 1.2:

Corollary 1.3. *If the conditions of Theorem 1.2 hold then p and q share CM infinitely many pairwise disjoint k-element subsets of $\mathbb{C}_\infty$ for some integer k with $k \leq \frac{1}{n-2}(-2 + \sum_{i=1}^{n} |S_i|)$; moreover, if $n \geq 5$ then we may choose $k \leq \max_i |S_i|$.*

Theorem 1.2 is already new when $m = 1$, where it may be viewed as a vast generalization of Nevanlinna's "four values" result and the CM version of his "five values" result. For, if p, q share CM five points then Theorem 1.2 implies that $g \circ p = g \circ q$ with $\deg(g) = 1$, so that $p = q$. Likewise if p, q share CM four points then Theorem 1.2 implies that $g \circ p = g \circ q$ with $\deg(g) \leq 2$. If $\deg(g) = 1$ then we again obtain $p = q$. If $\deg(g) = 2$ then $g = \mu \circ x^2 \circ \nu$ for some Möbius transformations $\mu, \nu \in \mathbb{C}(x)$, so that $x^2 \circ \nu \circ p = x^2 \circ \nu \circ q$ and thus $\nu \circ p = \epsilon\nu \circ q$ for some $\epsilon \in \{1, -1\}$, whence $p = \eta \circ q$ where $\eta := \nu^{-1} \circ \epsilon\nu$ is a Möbius transformation. In Section 8 we will show that our results also imply many other results from the literature, in addition to yielding many new results when one imposes further hypotheses on the sizes of the shared sets S_i. Thus, our results provide a new perspective which connects many old and new results as being consequences of the single general Theorem 1.2.

Theorem 1.2 motivates the following definition:

Definition 1.4. We say that $p, q \in \mathcal{M}(\mathbb{C}^m)$ are *quasi-equivalent* if there exists a nonconstant $g \in \mathbb{C}(x)$ such that $g \circ p = g \circ q$.

We emphasize that quasi-equivalence is much more restrictive than algebraic dependence. For instance, any two rational functions $p, q \in \mathbb{C}(x)$ are algebraically dependent, but the vast majority of such p, q are not quasi-equivalent. Further, as explained before Corollary 1.3, quasi-equivalence is more directly related to value-sharing questions than algebraic dependence. We have seen hundreds of papers about value-sharing which include examples showing that their results would not be true with weaker hypotheses; but we checked that all such examples in these papers consist of quasi-equivalent functions, so it is conceivable that the results of the papers would remain true with weaker hypotheses, once one adds to the conclusion some pairs of quasi-equivalent functions. More generally, it seems natural to seek results showing that certain value-sharing hypotheses imply quasi-equivalence, and conversely to produce examples of non-quasi-equivalent functions with interesting value-sharing properties.

Finally, we note that for applications of Theorem 1.2 it is crucial to have a good bound on $\deg(g)$, in terms of the sizes of the shared sets. It turns out that different types of arguments are needed to prove the existence of $g(x)$ than to bound its degree.

Example 1.5. Theorem 1.2 cannot be improved to three shared sets, since for instance $p(x) := (e^x + 2)/(e^x + 1)$ does not take the values 1 or 2, so that p and $2p$ share CM $\{\infty\}$, $\{0\}$, and $\{2\}$, but there is no nonconstant $g \in \mathbb{C}(x)$ for which $g \circ p = g \circ 2p$. In this example the meromorphic functions p and $2p$ are algebraically dependent; more generally, we will show in Proposition 3.11 that if nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ are algebraically dependent and share CM three disjoint nonempty finite sets then $g(p) = \alpha g(q)$ for some $\alpha \in \mathbb{C}^*$ and some nonconstant $g \in \mathbb{C}(x)$. A different type of example is $(e^{x^2} - 1)/(e^x - 1)$ and $(e^{-x^2} - 1)/(e^{-x} - 1)$, which are algebraically independent but share CM $\{\infty\}$, $\{0\}$, and $\{1\}$.

Many authors have studied pairs of meromorphic functions which share some sets of prescribed sizes. In order to apply our theory to this type of question, and also in order to prove the bounds on $\deg(g)$ in Theorem 1.2, we describe the collection of all sets shared CM by any two quasi-equivalent meromorphic functions p and q. A routine set theory exercise shows that if p and q share two sets S and T, then p and q also share $S \cup T$, $S \cap T$, and $S \setminus T$. Thus every nonempty finite set which is shared CM by p and q can be written as the union of minimal shared sets, where we define a minimal shared set to be a nonempty shared set which does not properly contain any other nonempty shared set. Moreover, distinct minimal shared sets are disjoint, and any union of minimal shared sets is again a shared set. If p and q are quasi-equivalent then let $g(x)$ be a nonconstant rational function of the smallest possible degree such that $g(p) = g(q)$. Let Λ_g be the set of points α in $\mathbb{C}_\infty$ such that g has the same multiplicity at each g-preimage of α; thus, Λ_g includes all points which are not critical values of g (and possibly some critical values as well), so that in particular Λ_g includes all but finitely many points of $\mathbb{C}_\infty$. For each $\alpha \in \mathbb{C}_\infty$, we write $g^{-1}(\alpha)_{\text{set}}$ for the set of distinct g-preimages of α. As explained before Corollary 1.3, the set $g^{-1}(\alpha)_{\text{set}}$ is shared CM by p and q whenever $\alpha \in \Lambda_g$. Conversely, in most situations the collection of such sets $g^{-1}(\alpha)_{\text{set}}$ comprises all minimal shared sets for p and q:

Theorem 1.6. *For quasi-equivalent $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$, let $g(x) \in \mathbb{C}(x)$ be a minimal-degree nonconstant rational function for which $g \circ p = g \circ q$, and define Λ_g as above. Then one of the following occurs:*

(1.6.1) *The collection of all sets $g^{-1}(\alpha)_{set}$ with $\alpha \in \Lambda_g$ equals the collection of all minimal shared sets for p and q.*

(1.6.2) *For some $\beta \in \Lambda_g$, $g^{-1}(\beta)_{set}$ is the union of two distinct minimal shared sets S_1, S_2, and the collection of all minimal shared sets for p and q consists of S_1, S_2, and all sets $g^{-1}(\alpha)_{set}$ with $\alpha \in \Lambda_g \setminus \{\beta\}$.*

In light of Theorems 1.2 and 1.6, in order to describe the possibilities for $p, q, S_1, \ldots, S_4$ where the S_i's are disjoint nonempty finite subsets of $\mathbb{C}_\infty$ which are shared CM by $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$, there are two remaining problems:

(1.7.1) Determine all solutions to $g \circ p = g \circ q$ in nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ and $g \in \mathbb{C}(x) \setminus \mathbb{C}$.

(1.7.2) For each solution (g, p, q) to (1.4.1) in which g has minimal degree among all solutions to (1.4.1) for the relevant p and q, determine whether (1.6.2) holds.

There are dozens of papers solving (1.4.1) when g, p, q satisfy additional restrictive properties, for instance [2, 3, 4, 5, 6, 10, 15, 17, 18, 19, 23, 24, 25, 26, 29, 37, 41, 51, 52, 53]. The recent papers [9, 13, 31, 32] went beyond the cases treated previously, by solving (1.4.1) when any of the following hold:

- the numerator of $(g(x) - g(y))/(x - y)$ is irreducible

- $g(x) = f(x)^n$ for some positive integer n, where there is a primitive n-th root of unity ζ such that $f(p(x)) = \zeta f(q(x))$ and the numerator of $f(x) - \zeta f(y)$ is irreducible
- some $\alpha \in \mathbb{C}_\infty$ has at most two distinct g-preimages.

By using a more detailed version of Theorem 1.6 (namely Theorem 6.7), we will determine all situations when (1.6.2) holds in each of the above three cases, yielding Proposition 8.14, Proposition 8.16, and Theorem 8.19, respectively. An informal conclusion is that (1.6.2) rarely holds, except when β has very few g-preimages. We note that the proof in case some element has two preimages ultimately relies on the classification of finite simple groups, which has not been applied previously to this type of question. We then use these results to exhibit all nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ which share CM two points and two distinct nonempty finite sets which do not contain either point. The full result is Theorem 8.27; a concise consequence is as follows.

Theorem 1.8. *Suppose distinct nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ share CM four disjoint nonempty finite subsets of $\mathbb{C}_\infty$, of which at least two have size 1. Let $g(x)$ be a minimal-degree nonconstant rational function such that $g \circ p = g \circ q$. Then there exist Möbius transformations μ and ν such that $\mu \circ g \circ \nu$ is one of the following:*

(1) $x^m(x-1)^n$ *with* $m, n > 0$ *and* $m + n \geq 3$
(2) x^n *with* $n \geq 2$
(3) $T_n(x)$ *with* $n \geq 5$, *where* $T_n(\cos\theta) = \cos n\theta$
(4) $\big((\zeta-1)(x^2+1) + 2(\zeta+1)x\big)^n / x^n$ *with* $n \geq 3$ *and* ζ *a primitive* n*-th root of unity*
(5) $(x+\alpha-2)\big(2x^3 - 2x^2 + (\alpha+1)(x+1)\big)^3 / x^5$ *with* $\alpha^2 = 5$
(6) $(2x+11-5\alpha)^2(x^2+x-1)^4/x^5$ *with* $\alpha^2 = 5$
(7) $(x^3+6x^2+3x-1)^3/x^3$
(8) $(4x^4+16x^3+8x-1)^2/x^4$
(9) $(x^2-6x+1)^4/x^2$
(10) $(x+1)^4(x+6\alpha+10)^2/x^2$ *with* $\alpha^2 = 3$
(11) $(x^2+10x+5)^3/x^5$
(12) $(3x^2+6x-1)^2/x$.

Conversely, for each rational function $g(x)$ in the above list, and any positive integer m, there exist distinct nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ which share four disjoint nonempty finite subsets of $\mathbb{C}_\infty$, of which at least two have size 1, where in addition $g(x)$ is a minimal-degree nonconstant rational function such that $g \circ p = g \circ q$.

We now briefly explain how our results relate to previous results in the literature. Suppose nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ share CM the disjoint nonempty finite subsets $S_1, \ldots, S_n$ of $\mathbb{C}_\infty$, where $n \geq 4$. When $m = 1$, the possibilities when three S_i's have size 1 were determined in [42], generalizing previous results of [21, 50] when a fourth has size 2. We will give a very short self-contained proof of this special case of Theorem 1.8 (for all m) in Proposition 8.1, in order to illustrate how to apply our results to specific situations. Likewise, in Proposition 8.8 we give a short proof of the possibilities when four S_i's have size at most 2; when $m = 1$, this implies the combination of Nevanlinna's four-values result with the results of the papers [45, 46, 48, 50], which address the cases that the number of one-element S_i's is 1, 0, 2, 3, respectively. The paper [40] determined the possibilities when $m = 1$ and the S_i's have sizes $1, 1, 3, 3$, which again follows easily from our methods, and is a special case of the much more difficult Theorem 1.8. The previous result closest to ours is [21, Thm. 3], which asserts that if $m = 1$ and $n \geq 4$ and some S_i has size 1 then p and q must be algebraically dependent. Finally, since $p := e^x$ and $q := -e^x$ share CM each set $S = \{\alpha, -\alpha\}$ with $\alpha \in \mathbb{C} \setminus \{0\}$, it is not true that if n is big enough and the S_i's have the same size then $p = q$, contradicting the assertion in [22, XXIII].

It would be interesting to seek analogues of our results for shared sets ignoring multiplicities (IM). Some first steps in this direction are taken in [44, 47], but the following questions remain open:

Question 1.9. Is there an absolute constant N so that if nonconstant $p, q \in \mathcal{M}(\mathbb{C})$ share IM N disjoint nonempty finite subsets of $\mathbb{C}_\infty$ then $g \circ p = g \circ q$ for some $g \in \mathbb{C}(x) \setminus \mathbb{C}$?

Question 1.10. If nonconstant $p, q \in \mathcal{M}(\mathbb{C})$ share IM infinitely many finite subsets of $\mathbb{C}_\infty$ then must there be some $g \in \mathbb{C}(x) \setminus \mathbb{C}$ for which $g \circ p = g \circ q$?

Remark 1.11. Question 1.9 is open even in the simplest case when p and q are polynomials. Question 1.10 has an affirmative answer in that case, since p and q only have finitely many critical values, so that by repeatedly taking intersections and set differences of the given shared sets we obtain infinitely many IM-shared sets which contain no critical values and hence are shared CM, whence the conclusion follows from our results (or in this case from the easier Lemma 3.9). Finally, the multivariable analogues of these questions are also open.

This paper is organized as follows. In the next section we list the notation and terminology we will use. In Section 3 we show that if $p, q \in \mathcal{M}(\mathbb{C}^m)$ share four disjoint finite sets then $g \circ p = g \circ q$ for some nonconstant $g \in \mathbb{C}(x)$. Our proof combines several new ideas with ingredients from Nevanlinna's proof of his "four values" theorem, which in turn was based an earlier argument due to Pólya [38]. Our proof yields no bound on $\deg(g)$ in terms of the sizes of the shared sets, and the next four sections are required to prove such a bound. In Section 4 we describe the collection of all rational functions $g(x)$ which satisfy $g \circ p = g \circ q$ for prescribed meromorphic functions p, q on an arbitrary complex manifold $\mathcal{R}$. In Section 5 we prove some useful properties about multiplicities of preimages of points under a minimal-degree nonconstant $g(x) \in \mathbb{C}(x)$ satisfying $g \circ p = g \circ q$. The results in Section 4 and especially Section 5 are of independent interest; certainly the combination of Galois-theoretic and topological methods used in these sections is quite different from previous work in the subject. In Section 6 we describe the collection of all sets shared CM by any prescribed $p, q \in \mathcal{M}(\mathbb{C}^m)$ for which $g \circ p = g \circ q$ for some $g \in \mathbb{C}(x) \setminus \mathbb{C}$, and prove a refinement of Theorem 1.6. In Section 7 we combine the results of the previous sections in order to prove a generalization of Theorem 1.2. Finally, in Section 8 we prove several results that facilitate applying our results to specific situations, and use these to classify all possibilities for $p, q, S_1, \ldots, S_4$ when certain conditions hold. This yields simple proofs of many previous results, in addition to several new results such as a refinement of Theorem 1.8

2. Notation and terminology

In this section we list the notation and terminology used in this paper. These are also defined when first used, but we list them here for ease of reference.

We first recall the standard definition of multiplicity of points under a meromorphic function.

Definition 2.1. Let $\mathcal{R}$ be a complex manifold, let $p\colon \mathcal{R} \to \mathbb{C}$ be a holomorphic function which is not identically zero, and let α be a point in $\mathcal{Z}_p := \{\beta \in \mathcal{R} : p(\beta) = 0\}$. Further, let $\mathcal{O}_\alpha$ be the local ring consisting of the germs at α of holomorphic functions defined on a neighborhood of α, and let $\mathcal{I}$ be the ideal of $\mathcal{O}_\alpha$ consisting of all elements which vanish on $\mathcal{Z}_p$. Letting k be the maximal integer for which $p \in \mathcal{I}^k$, we say that p has a zero of multiplicity k at α, and write $m_p(\alpha) := k$. If $\alpha \in \mathcal{R} \setminus \mathcal{Z}_p$ then we define $m_p(\alpha) := 0$.

For any meromorphic function $p \in \mathcal{M}(\mathcal{R})$ and a point $\alpha \in \mathcal{R}$ for which $p(\alpha) \in \mathbb{C}$, we may write $p - p(\alpha)$ in a neighborhood of α as the quotient of two holomorphic functions q/r, and the multiplicity of p at α is $\nu_p(\alpha) := m_q(\alpha) - m_r(\alpha)$. Finally, if $p(\alpha) = \infty$ then the multiplicity of p at α is $\nu_p(\alpha) := \nu_{1/p}(\alpha)$.

- $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$
- $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere
- $\mathcal{M}(\mathcal{R})$ is the set of all meromorphic functions on the complex manifold $\mathcal{R}$
- for $p \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ we write $\mathcal{E}(p) := \mathbb{C}_\infty \setminus p(\mathbb{C}^m)$ for what is sometimes called the set of Picard exceptional values of p; Picard's little theorem says $|\mathcal{E}(p)| \leq 2$
- a *multiset* (or "set with multiplicities") is a collection of elements which need not be distinct
- $p^{-1}(\alpha)$ is the multiset of all preimages of $\alpha \in \mathbb{C}_\infty$ under some nonconstant $p \in \mathcal{M}(\mathcal{R})$, counted with multiplicities
- S_{set} is the set of distinct elements in the multiset S
- if S is a nonempty finite multiset then $\operatorname{gcdmult}(S)$ denotes the greatest common divisor of the multiplicities of all elements of S

- if S is a multiset and k is a positive integer then S^k denotes the union of k copies of S
- $\{a^{*m}, b\}$ is the multiset having m copies of a and one copy of b
- $\mathcal{G}_1(p,q)$ is defined in Definition 4.1
- minimal shared multisets are defined in Definition 6.1
- the multisets T_α are defined in Definition 6.3
- $T_n(x)$ is the degree-n Chebyshev polynomial, namely the unique polynomial such that $T_n(\cos\theta) = \cos n\theta$.

3. Four shared sets implies infinitely many

Theorem 3.1. *If nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ share CM each of four finite multisets $S_1, \ldots, S_4$ of elements of $\mathbb{C}_\infty$, where no S_i is contained in the union of the other S_j's, then $g \circ p = g \circ q$ for some nonconstant $g \in \mathbb{C}(x)$. Conversely, if $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ and $g \in \mathbb{C}(x) \setminus \mathbb{C}$ satisfy $g \circ p = g \circ q$ then p and q share CM each of infinitely many pairwise disjoint k-element subsets of $\mathbb{C}_\infty$, where $k := \deg(g)$.*

The proof of Theorem 3.1 relies on the following several-variable generalization (see [16, Thm. 3.5] or [20, p. 54]) of a classical result of Borel [8]:

Lemma 3.2. *For any $n > 0$, if $r_1, \ldots, r_n$ are entire functions on $\mathbb{C}^m$ which have no zeroes, and $r_1 + \cdots + r_n = 0$, then $r_i = \alpha r_j$ for some $i \neq j$ and some $\alpha \in \mathbb{C}^*$.*

We begin by adapting this result to our setting. It is convenient to use the language of divisors.

Definition 3.3. For any complex manifold $\mathcal{R}$, the *divisor* of a nonconstant $p \in \mathcal{M}(\mathcal{R})$ is the formal $\mathbb{Z}$-linear combination of points of $\mathcal{R}$ defined as the sum of the zeroes of p minus the sum of the poles of p, where the zeroes and poles are counted with multiplicities. If p is introduced as an element of $\mathbb{C}(x)$ then we view p as an element of $\mathcal{M}(\mathbb{C}_\infty)$ when defining its divisor – thus, in this situation we allow ∞ as a possible zero or pole of p, although we would not allow this if the same function p were instead introduced as an element of $\mathcal{M}(\mathbb{C})$.

Lemma 3.4. *Pick $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ and $f_i, g_i \in \mathbb{C}(x) \setminus \mathbb{C}$ (for $i = 1, 2, 3$), and suppose that for each i the divisor of $f_i(p)$ equals the divisor of $g_i(q)$. Then there exist integers n_1, n_2, n_3 which are not all zero and for which $F(p)/G(q)$ is in $\mathbb{C}^*$, where $F := \prod_{i=1}^3 f_i^{n_i}$ and $G := \prod_{i=1}^3 g_i^{n_i}$. If in addition each f_i has at least one zero or pole which is not a zero or pole of any other f_j, then F and G are nonconstant.*

Proof. Write $h_i(x,y) := f_i(x)/g_i(y)$ for $i = 1, 2, 3$. Since the field extension $\mathbb{C}(x,y)/\mathbb{C}$ has transcendence degree 2, the three elements $h_i \in \mathbb{C}(x,y)$ must be algebraically dependent. Thus there is a nonzero polynomial $P(u,v,w) \in \mathbb{C}[u,v,w]$ such that $P(h_1,h_2,h_3) = 0$. Writing $P(u,v,w) := \sum c_{r,s,t} u^r v^s w^t$

where the sum is over a nonempty finite set Δ of triples (r,s,t) of nonnegative integers, and each $c_{r,s,t}$ is in $\mathbb{C}^*$, it follows that $\sum c_{r,s,t}h_1^r h_2^s h_3^t = 0$. Recall that the h_i's are in $\mathbb{C}(x,y)$, and substitute p for x and q for y to obtain $\sum c_{r,s,t}H_1^r H_2^s H_3^t = 0$ where $H_i := h_i(p,q) = f_i(p)/g_i(q)$. Since $f_i(p)$ and $g_i(q)$ have the same divisor, their ratio H_i has no zeroes or poles. Thus, for each triple $(r,s,t) \in \Delta$, the function $c_{r,s,t}H_1^r H_2^s H_3^t$ is entire and has no zeroes, so by Lemma 3.2 there are two distinct triples (r,s,t) and (r',s',t') in Δ for which $H_1^r H_2^s H_3^t = \alpha H_1^{r'} H_2^{s'} H_3^{t'}$ with $\alpha \in \mathbb{C}^*$. Writing $n_1 := r - r'$, $n_2 := s - s'$, and $n_3 := t - t'$, it follows that $\prod_{i=1}^3 H_i^{n_i} = \alpha$, or equivalently $F(p)/G(q) = \alpha$ where $F := \prod_{i=1}^3 f_i^{n_i}$ and $G := \prod_{i=1}^3 g_i^{n_i}$. Here n_1, n_2, n_3 are integers which are not all zero.

Now suppose that each f_i has at least one zero or pole δ_i which is not a zero or pole of any other f_j. Since at least one n_i is nonzero, it follows that the corresponding δ_i is a zero or pole of F, so that F is nonconstant. $G(q) = F(p)/\alpha$ is also nonconstant, so that G is nonconstant as well. □

In order to apply Lemma 3.4 to specific $p, q \in \mathcal{M}(\mathcal{R})$, we need to exhibit $f_i, g_i \in \mathbb{C}(x)$ for which $f_i(p)$ and $g_i(q)$ have the same divisor. In our situation, f_i will be a product of integer powers of the characteristic polynomials of some shared multisets. By a slight abuse of notation, if S is a finite multiset of elements of a complex manifold $\mathcal{R}$ then we also write S for the divisor on $\mathcal{R}$ defined as the formal sum of the elements of the multiset S.

Lemma 3.5. *Let p and q be nonconstant meromorphic functions on a complex manifold $\mathcal{R}$, and let S_1 and S_2 be disjoint nonempty finite multisets of elements of $\mathbb{C}_\infty$ such that p and q share each S_i CM. Then there are integers $n_1, n_2 > 0$ and a nonconstant $h \in \mathbb{C}(x)$ such that the divisor of $h(x)$ is $n_1S_1 - n_2S_2$ and the divisors of $h(p)$ and $h(q)$ are equal.*

Proof. First assume that neither S_i contains ∞. Let $f_i(x) := \prod_{\alpha \in S_i}(x - \alpha)$ be the characteristic polynomial of S_i. By hypothesis, the f_i's are nonconstant coprime polynomials such that, for each i, $f_i \circ p$ and $f_i \circ q$ have the same zeroes CM. Then $h(x) := f_1(x)^{\deg f_2}/f_2(x)^{\deg f_1}$ is a nonconstant rational function whose numerator and denominator are monic polynomials of the same degree, so that $h(\infty) = 1$. Thus the zeroes of $h(p)$ coincide CM with the zeroes of $f_1(p)^{\deg f_2}$, which coincide CM with the zeroes of $f_1(q)^{\deg f_2}$, and hence with the zeroes of $h(q)$. Likewise, the poles of $h(p)$ agree CM with the poles of $h(q)$. Since the zeroes of h consist of $|S_2|$ copies of S_1, and the poles of h consist of $|S_1|$ copies of S_2, this proves the result in case neither S_i contains ∞.

If some S_i contains ∞ then let $T := S_1 \cup S_2$ and let $\mu(x)$ be a Möbius transformation such that $\mu(T)$ does not contain ∞. Then $\widehat{p} := \mu \circ p$ and $\widehat{q} := \mu \circ q$ share CM each multiset $\widehat{S}_i := \mu(S_i)$, where the $\widehat{S}_i$'s are nonempty and disjoint but do not contain ∞. Thus there is a nonconstant $\widehat{h} \in \mathbb{C}(x)$ such that $\widehat{h}(\widehat{p})$ and $\widehat{h}(\widehat{q})$ have the same divisor, where in addition the divisor of $\widehat{h}$ is $n_1\widehat{S_1} - n_2\widehat{S_2}$ for some positive integers n_1, n_2. Then $h := \widehat{h} \circ \mu$ has divisor $n_1 S_1 - n_2 S_2$, and the divisors of $h(p)$ and $h(q)$ are identical. $\square$

With these ingredients in hand, we now prove that if p and q share four multisets then we obtain a weaker version of our desired functional equation.

Proposition 3.6. *For any nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$, and any pairwise disjoint nonempty finite multisets $S_1, \dots, S_4$ of elements of $\mathbb{C}_\infty$ such that p, q share CM each S_i, there exist $h \in \mathbb{C}(x) \setminus \mathbb{C}$ and $\gamma \in \mathbb{C}^*$ such that $h \circ p = \gamma h \circ q$. Moreover, h can be chosen so that its divisor is a $\mathbb{Z}$-linear combination of S_1, S_2, S_3, S_4.*

Proof. By Lemma 3.5, for each $i = 1, 2, 3$ there exist a nonconstant $h_i \in \mathbb{C}(x)$ and positive integers u_i, v_i such that $h_i(x)$ has divisor $u_i S_i - v_i S_4$ and the divisors of $h_i(p)$ and $h_i(q)$ equal one another. By Lemma 3.4, there are integers n_1, n_2, n_3 which are not all zero and for which $h := \prod_{i=1}^3 h_i^{n_i}$ is nonconstant and $h(p) = \gamma \cdot h(q)$ for some $\gamma \in \mathbb{C}^*$. Since the divisor of h is $\sum_{i=1}^3 (n_i u_i S_i - n_i v_i S_4)$, this yields the result. $\square$

Our proof of Theorem 3.1 also uses the following result of Coman and Poletsky [11, Thm. 5.2]:

Lemma 3.7. *If nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ are algebraically dependent then there exist a compact Riemann surface $\mathcal{R}$ of genus 0 or 1, a holomorphic map $r \colon \mathbb{C}^m \to \mathcal{R}$, and $p_0, q_0 \in \mathcal{M}(\mathcal{R})$ such that $p = p_0 \circ r$ and $q = q_0 \circ r$.*

Remark 3.8. The special case $m = 1$ of Lemma 3.7 was proved in [7, Thm. 1] independently and simultaneously to [11].

In order to apply Lemma 3.7 to questions about shared multisets, we first address shared multisets on a compact Riemann surface.

Lemma 3.9. *Let $\mathcal{R}$ be a compact Riemann surface, and pick $p_0, q_0 \in \mathcal{M}(\mathcal{R}) \setminus \mathbb{C}$. If S_1, S_2, S_3 are disjoint nonempty finite multisets of elements of $\mathbb{C}_\infty$ such that p_0, q_0 share CM S_1 and S_2, and $p_0^{-1}(S_3)_{set} \subseteq q_0^{-1}(S_3)_{set}$, then $g \circ p_0 = g \circ q_0$ for some nonconstant $g \in \mathbb{C}(x)$.*

Proof. Write $v_i := \prod_{\alpha \in S_i}(x - \alpha)$ and $w := v_1^{\deg v_2}/v_2^{\deg v_1}$. Since $v_i \circ p_0$ and $v_i \circ q_0$ have the same zeroes CM, and $w(\infty) = 1$, the functions $w \circ p_0$ and $w \circ q_0$ have the same divisor. Thus $\gamma := w(p_0)/w(q_0)$ is a holomorphic map $\mathcal{R} \to \mathbb{C}_\infty$ which has no zeroes or poles, so compactness of $\mathcal{R}$ implies $\gamma \in \mathbb{C}^*$. Compactness also implies that each element of S_3 has the form $s = p_0(\theta)$ with $\theta \in \mathcal{R}$, and then $w(p_0(\theta)) = \gamma w(q_0(\theta))$ is in $\gamma w(S_3)$. Thus $w(S_3)_{\mathrm{set}} \subseteq \gamma w(S_3)_{\mathrm{set}}$. Since all zeroes and poles of w are in $S_1 \cup S_2$, the set $w(S_3)_{\mathrm{set}}$ is contained in $\mathbb{C}^*$. Since this set is finite and nonempty, and is preserved by multiplication by γ, it follows that $\gamma^n = 1$ for some positive integer n, so that $w^n \circ p_0 = w^n \circ q_0$. $\square$

We also use the following generalization of Picard's little theorem:

Lemma 3.10. *If $\mathcal{R}$ is a compact Riemann surface and $h\colon \mathbb{C}^m \to \mathcal{R}$ is a nonconstant holomorphic map which is not surjective, then there exists a biholomorphic map $\mathcal{R} \to \mathbb{C}_\infty$, and $\mathcal{R} \setminus h(\mathbb{C}^m)$ has size at most* 2.

Proof. For any nonempty finite subset $\mathcal{E}$ of $\mathcal{R} \setminus h(\mathbb{C}^m)$, write $\mathcal{R}_0 := \mathcal{R} \setminus \mathcal{E}$. Then h induces a nonconstant holomorphic map $\mathbb{C}^m \to \mathcal{R}_0$, so $\mathcal{R}_0$ cannot be hyperbolic (e.g. by [34, Lemma 2.3]). Thus $\mathcal{R}$ has genus zero (so $\mathcal{R} \cong \mathbb{C}_\infty$) and $\mathcal{E}$ has size at most 2. $\square$

Proof of Theorem 3.1. If $g \circ p = g \circ q$ then p and q share CM the multiset $S_\alpha := g^{-1}(\alpha)$ for any $\alpha \in \mathbb{C}_\infty$. Plainly $|S_\alpha| = \deg(g)$ and $S_\alpha \cap S_\beta = \emptyset$ when $\alpha \neq \beta$, and moreover S_α is a set whenever α is not one of the finitely many critical values of g. Thus p and q share CM infinitely many pairwise disjoint sets, each of which has size $\deg(g)$.

Conversely, we now assume that p and q share CM each of four pairwise disjoint finite multisets $S_1, \dots, S_4$ of elements of $\mathbb{C}_\infty$, where in addition no ${}_i$ is contained in the union of the other S_j's. Proposition 3.6 yields $h \in \mathbb{C}(x) \setminus \mathbb{C}$ and $\gamma \in \mathbb{C}^*$ such that $h \circ p = \gamma h \circ q$, and thus p and q are algebraically dependent. By Lemma 3.7, there exist a compact Riemann surface $\mathcal{R}$, a holomorphic map $r\colon \mathbb{C}^m \to \mathcal{R}$, and $p_0, q_0 \in \mathcal{M}(\mathcal{R})$ such that $p = p_0 \circ r$ and $q = q_0 \circ r$. Since p and q are nonconstant, also p_0, q_0, r are nonconstant. The identity $h \circ p = \gamma h \circ q$ now becomes $h \circ p_0 \circ r = \gamma h \circ q_0 \circ r$, so that $h \circ p_0 = \gamma h \circ q_0$. Since $\mathcal{R}$ is compact, we can speak of the degrees of p_0 and q_0 (i.e., the numbers of preimages of any point, counted with multiplicities), and the above identity implies $\deg(h) \cdot \deg(p_0) = \deg(h) \cdot \deg(q_0)$, whence $\deg(p_0) = \deg(q_0)$.

For any finite multiset S of elements of $\mathbb{C}_\infty$, the multiset $p^{-1}(S)$ is the union of all $r^{-1}(\alpha)$ with $\alpha \in p_0^{-1}(S)$. Thus S is shared CM by p and q if and only if the multiset differences $p_0^{-1}(S) \setminus q_0^{-1}(S)$ and $q_0^{-1}(S) \setminus p_0^{-1}(S)$ each consist of elements of $\mathcal{E} := \mathcal{R} \setminus r(\mathbb{C}^m)$. Since $p_0^{-1}(S)$ and $q_0^{-1}(S)$ have the same size, and they also have the same size after removing all copies of elements of $\mathcal{E}$ from both of them, it follows that $p_0^{-1}(S)$ and $q_0^{-1}(S)$ contain the same number of elements of $\mathcal{E}$ (when counted with multiplicities).

We may assume that at most two of the S_i's are shared CM by p_0 and q_0, since otherwise Lemma 3.9 produces $g \in \mathbb{C}(x) \setminus \mathbb{C}$ with $g \circ p_0 = g \circ q_0$, whence also $g \circ p = g \circ q$. By relabeling the S_i's if needed, we may assume that for $i \in \{1, 2\}$ we have $p_0^{-1}(S_i) \neq q_0^{-1}(S_i)$, so that $p_0^{-1}(S_i) \setminus q_0^{-1}(S_i)$ and $q_0^{-1}(S_i) \setminus p_0^{-1}(S_i)$ are disjoint nonempty multisets of the same size which each consist of elements of $\mathcal{E}$. We have $|\mathcal{E}| \leq 2$ by Lemma 3.10, and also the four multisets $p_0^{-1}(S_i)$ are pairwise disjoint, as are the four multisets $q_0^{-1}(S_i)$. Thus there are distinct $\alpha_1, \alpha_2 \in \mathcal{E}$, and positive integers e_1, e_2, such that for each $i \in \{1, 2\}$

- $p_0^{-1}(S_i) \setminus q_0^{-1}(S_i)$ consists of e_i copies of α_i, and
- $q_0^{-1}(S_i) \setminus p_0^{-1}(S_i)$ consists of e_i copies of α_{3-i}.

Since $\mathcal{E} = \{\alpha_1, \alpha_2\}$ is contained in $p_0^{-1}(S_1 \cup S_2)$ and $q_0^{-1}(S_1 \cup S_2)$, it follows that $p_0^{-1}(S_j) = q_0^{-1}(S_j)$ for $j \in \{3, 4\}$. Next, for $T := S_1 \cup S_2$, the multiset $p_0^{-1}(T)$ is the union of $\cup_{i=1}^{2}(p_0^{-1}(S_i) \cap q_0^{-1}(S_i))$ with $e_1 + e_2$ copies of each α_i, and this union also equals $q_0^{-1}(T)$. Hence p_0 and q_0 share CM the disjoint multisets T, S_3, and S_4, so by Lemma 3.9 there exists $g \in \mathbb{C}(x) \setminus \mathbb{C}$ such that $g \circ p_0 = g \circ q_0$, whence also $g \circ p = g \circ q$. □

We conclude this section with a variant of Theorem 3.1 addressing algebraically dependent meromorphic functions which share three multisets. This result will not be used elsewhere in this paper.

Proposition 3.11. *Suppose algebraically dependent $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ share CM three disjoint nonempty finite multisets S_1, S_2, S_3 of elements of $\mathbb{C}_\infty$. Then $g(p) = \alpha g(q)$ for some nonconstant $g(x) \in \mathbb{C}(x)$ and some $\alpha \in \mathbb{C}^*$.*

Proof. By Lemma 3.7, we can write $p = p_0 \circ r$ and $q = q_0 \circ r$ for some compact Riemann surface $\mathcal{R}$, some holomorphic map $r: \mathbb{C}^m \to \mathcal{R}$, and some $p_0, q_0 \in \mathcal{M}(\mathcal{R})$. Writing $\mathcal{E} := \mathcal{R} \setminus r(\mathbb{C}^m)$, put $A_i := \mathcal{E} \cap p_0^{-1}(S_i)$ and $B_i := \mathcal{E} \cap q_0^{-1}(S_i)$. For each $i \in \{1, 2, 3\}$, one of the following holds:

(1) $A_i = B_i = \emptyset$
(2) $A_i \neq \emptyset = B_i$
(3) $A_i = \emptyset \neq B_i$
(4) $A_i \neq \emptyset$ and $B_i \neq \emptyset$.

Since the multisets $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ agree except for copies of elements of A_i in $p_0^{-1}(S_i)$ and elements of B_i in $q_0^{-1}(S_i)$, we see that

- if (1) holds then $p_0^{-1}(S_i) = q_0^{-1}(S_i)$
- if (2) holds then $|p_0^{-1}(S_i)| > |q_0^{-1}(S_i)|$
- if (3) holds then $|p_0^{-1}(S_i)| < |q_0^{-1}(S_i)|$.

Since $|p_0^{-1}(S_i)| = \deg(p_0) \cdot |S_i|$, it follows that

- if (1) holds then $\deg(p_0) = \deg(q_0)$
- if (2) holds then $\deg(p_0) > \deg(q_0)$
- if (3) holds then $\deg(p_0) < \deg(q_0)$.

Thus there cannot be i, j for which two different cases among (1),(2),(3) hold. Since $|\mathcal{E}| \leq 2$ by Lemma 3.10, there is at least one i for which A_i is empty, so that (1) or (3) holds for that i; and likewise there is at least one j for which B_j is empty, so that (1) or (2) holds for that j. Thus (1) holds for at least one i, and every j satisfies either (1) or (4). Write $f_i(x) := \prod_{\alpha \in S_i}(x - \alpha)$, and put $n_i := |S_i|$. If $p_0^{-1}(S_i) = q_0^{-1}(S_i)$ for at least two i's, say $i = 1$ and $i = 2$, then for $g := (f_2)^{n_1}/(f_1)^{n_2}$ we see that $g \circ p_0$ and $g \circ q_0$ have the same divisor, so their ratio is constant by compactness of $\mathcal{R}$, yielding the desired conclusion. Henceforth assume that there is exactly one i for which $p_0^{-1}(S_i) = q_0^{-1}(S_i)$. We may assume that (1) holds for $i = 3$ but (4) holds for $i = 1$ and $i = 2$. Then A_1, B_1, A_2, B_2 each have size 1, and $A_1 \cup A_2$ and $B_1 \cup B_2$ are the same two-element set. Here $\deg(p_0) = \deg(q_0)$, so that $|p_0^{-1}(S_i)| = |q_0^{-1}(S_i)|$ for each i, whence since the multisets of elements of $\mathbb{C}_\infty \setminus \mathcal{E}$ in $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ coincide, it follows that the multisets of elements of $\mathcal{E}$ in $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ have the same size. Since $A_1 \cap B_1 = \emptyset$, we have $A_1 = B_2 = \{\alpha_1\}$ and $A_2 = B_1 = \{\alpha_2\}$, where, for $i \in \{1, 2\}$ and some positive integer e_i, the multisets $p_0^{-1}(S_i) \setminus q_0^{-1}(S_i)$ and $q_0^{-1}(S_i) \setminus p_0^{-1}(S_i)$ consist of e_i copies of α_i and e_i copies of α_{3-i}, respectively. Putting $h := (f_1)^{e_2}(f_2)^{e_1}$, it follows that $h(p_0)$ and $h(q_0)$ have the same zeroes CM, so for $g := h^{n_3}/(f_3)^{\deg(h)}$ the functions $g(p_0)$ and $g(q_0)$ have the same divisor and hence have constant ratio. Finally, $g(x)$ is nonconstant since each element of S_3 is a pole of $g(x)$. □

4. Minimal relations between meromorphic functions

Theorem 3.1 yields nonconstant rational functions $g(x)$ such that $g(p) = g(q)$, for prescribed $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ satisfying certain shared-multiset hypotheses. In this section we describe the collection of all rational functions $g(x)$ satisfying $g(p) = g(q)$. We also solve the analogous problem for the equation $g(p)/g(q) \in \mathbb{C}^*$.

Definition 4.1. For any complex manifold $\mathcal{R}$ and any nonconstant $p, q \in \mathcal{M}(\mathcal{R})$, let $\mathcal{G}_1(p, q)$ be the set of all $g \in \mathbb{C}(x) \setminus \mathbb{C}$ such that $g \circ p = g \circ q$. When the choices of p and q are clear, we write $\mathcal{G}_1$ for $\mathcal{G}_1(p, q)$.

Proposition 4.2. *Let $\mathcal{R}$ be a complex manifold, and pick $p, q \in \mathcal{M}(\mathcal{R}) \setminus \mathbb{C}$. If $\mathcal{G}_1$ is nonempty and $g_1(x)$ is a minimal-degree element of $\mathcal{G}_1$ then $\mathcal{G}_1 = \{d \circ g_1 : d \in \mathbb{C}(x) \setminus \mathbb{C}\}$.*

Proof. Let L be the set of all $g(x) \in \mathbb{C}(x)$ for which $g \circ p = g \circ q$. Then L contains $\mathbb{C}$ and is preserved by addition, multiplication, and division by nonzero elements, so L is a field between $\mathbb{C}$ and $\mathbb{C}(x)$. Since $L \neq \mathbb{C}$ by hypothesis, Lüroth's theorem [39, Thm. 2] implies $L = \mathbb{C}(h(x))$ for some nonconstant $h(x) \in L$. For any minimal-degree $g_1 \in \mathcal{G}_1$, since $g_1 \in L$ we have $g_1 = \mu \circ h$ for some nonconstant $\mu \in \mathbb{C}(x)$. Minimality of $\deg(g_1)$ implies $\mu(x)$ is a Möbius transformation, so that $L = \mathbb{C}(g_1(x))$, which implies the conclusion. □

In the applications of our main results in Section 8, we will use an analogue of Proposition 4.2 for the set Ω of $g \in \mathbb{C}(x)$ such that $g(p)/g(q) \in \mathbb{C}^*$. Our proof of Proposition 4.2 does not carry over to this situation, since Ω is not closed under addition. We will circumvent this issue by showing that if $g, h \in \Omega$ then the field $\mathbb{C}(g(x), h(x))$ equals $\mathbb{C}(f(x))$ for some $f \in \Omega$. This is the key to the proof of the next result.

Definition 4.3. For any complex manifold $\mathcal{R}$, and any nonconstant $p, q \in \mathcal{M}(\mathcal{R})$, let $\mathcal{G}_2(p, q)$ be the set of all $g(x) \in \mathbb{C}(x) \setminus \mathbb{C}$ for which $g(p)/g(q) \in \mathbb{C}^*$. When the choices of p and q are clear, we write $\mathcal{G}_2$ for $\mathcal{G}_2(p, q)$.

Proposition 4.4. *Let $\mathcal{R}$ be a complex manifold and pick $p, q \in \mathcal{M}(\mathcal{R}) \setminus \mathbb{C}$. Suppose $\mathcal{G}_2$ is nonempty, and let $g_2(x)$ be any element of $\mathcal{G}_2$ having the* *smallest possible degree. Writing $\alpha := g_2(p)/g_2(q)$, define*

$$s := \begin{cases} n & \text{if } \alpha \text{ is a primitive } n\text{-th root of unity} \\ 0 & \text{if } \alpha \text{ is not a root of unity.} \end{cases}$$

Then $\mathcal{G}_2$ is the set of nonconstant rational functions of the form $(x^k u(x^s)) \circ g_2$ with $k \in \mathbb{Z}$ and $u \in \mathbb{C}(x)$. Moreover, if $h \in \mathcal{G}_2$ and $\beta := h(p)/h(q)$ then the numerator of $h(x) - \beta h(y)$ is divisible by the numerator of $g_2(x) - \alpha g_2(y)$.

Proof. For any $h \in \mathcal{G}_2$, put $\beta := h(p)/h(q)$, and note that $\alpha, \beta \in \mathbb{C}^*$. Since $\mathbb{C}(g_2(x), h(x))$ is a subfield of $\mathbb{C}(x)$ which properly contains $\mathbb{C}$, Lüroth's theorem implies $\mathbb{C}(g_2(x), h(x)) = \mathbb{C}(f(x))$ for some nonconstant $f(x) \in \mathbb{C}(x)$. Since $\mathbb{C}(f(x))$ contains both $g_2(x)$ and $h(x)$, we have $g_2 = \widehat{g_2} \circ f$ and $h = \widehat{h} \circ f$ for some nonconstant $\widehat{g_2}, \widehat{h} \in \mathbb{C}(x)$. Since $f(x) \in \mathbb{C}(g_2(x), h(x))$, there is a bivariate rational function $H(x, y) \in \mathbb{C}(x, y)$ such that $H(g_2(x), h(x)) = f(x)$. Rewriting this as $H(\widehat{g_2}(f(x)), \widehat{h}(f(x))) = f(x)$ shows that $H(\widehat{g_2}(x), \widehat{h}(x)) = x$. Thus

$$f \circ p = H(g_2 \circ p, h \circ p) = H(\alpha g_2 \circ q, \beta h \circ q) = H(\alpha \widehat{g_2} \circ f \circ q, \beta \widehat{h} \circ f \circ q).$$

Let $\mu(x) := H(\alpha \widehat{g_2}(x), \beta \widehat{h}(x))$, so that $\mu \in \mathbb{C}(x)$ and $\mu \circ f \circ q = f \circ p$. Then

$$\widehat{g_2} \circ \mu \circ f \circ q = \widehat{g_2} \circ f \circ p = g_2 \circ p = \alpha g_2 \circ q = \alpha \widehat{g_2} \circ f \circ q,$$

so that $\widehat{g_2} \circ \mu = \alpha \widehat{g_2}$. Therefore $\mu(x)$ is a Möbius transformation which permutes each of the multisets $\widehat{g_2}^{-1}(0)$ and $\widehat{g_2}^{-1}(\infty)$. These two multisets are nonempty, finite, and disjoint. Every Möbius transformation has the form $\nu^{-1} \circ \theta \circ \nu$ for Möbius transformations $\nu(x)$ and $\theta(x)$ where $\theta(x)$ is either γx (with $\gamma \in \mathbb{C}^*$) or $x + 1$. Since $x + 1$ only preserves one nonempty finite subset of $\mathbb{C}_\infty$, namely $\{\infty\}$, also $\nu^{-1} \circ (x + 1) \circ \nu$ only preserves one nonempty finite subset of $\mathbb{C}_\infty$. Thus $\mu = \nu^{-1} \circ \gamma x \circ \nu$ for some $\gamma \in \mathbb{C}^*$. Since $\mu \circ f \circ q = f \circ p$, it follows that

$$\gamma \nu \circ f \circ q = \nu \circ \mu \circ f \circ q = \nu \circ f \circ p,$$

so that $\nu \circ f \in \mathcal{G}_2$. Since $g_2 = \widehat{g_2} \circ f$, minimality of $\deg(g_2)$ implies $\deg(\widehat{g_2}) = 1$, so that $f = \widehat{g_2}^{-1} \circ g_2$ and thus

$$h = \widehat{h} \circ f = \widehat{h} \circ \widehat{g_2}^{-1} \circ g_2.$$

Therefore $h = d \circ g_2$ with $d \in \mathbb{C}(x)$.

Conversely, for any nonconstant $d \in \mathbb{C}(x)$ and any $\beta \in \mathbb{C}^*$, put $h := d \circ g_2$. Then $h(p) = \beta h(q)$ if and only if

$$\beta d \circ g_2 \circ q = \beta h(q) = h(p) = d \circ g_2 \circ p = d \circ \alpha g_2 \circ q,$$

or equivalently $\beta d = d \circ \alpha x$. All solutions to the latter equation were described in [1] and [33], yielding the desired description of $\mathcal{G}_2$. For any such $h(x)$, since $x - y$ divides the numerator of $d(x) - d(y)$, substituting $g_2(x)$ for x and $\alpha g_2(y)$ for y shows that the numerator of $g_2(x) - \alpha g_2(y)$ divides the numerator of $d(g_2(x)) - d(\alpha g_2(y)) = h(x) - \beta h(y)$. $\square$

5. Complete multiple values of the minimal-degree rational function relating p and q

In this section we prove a result about the multiplicities of points under a minimal-degree $g \in \mathcal{G}_1$; this will be used in our proof of Theorem 1.2. Recall that if S is a multiset then S_{set} denotes the underlying set, and $\operatorname{gcdmult}(S)$ denotes the greatest common divisor of the multiplicities of all the elements of S.

Proposition 5.1. *For a complex manifold $\mathcal{R}$, and nonconstant $p, q \in \mathcal{M}(\mathcal{R})$, suppose that $g(p) = g(q)$ for some $g \in \mathbb{C}(x) \setminus \mathbb{C}$, and choose one such $g(x)$ of minimal degree. Then there are at most two points $\alpha \in \mathbb{C}_\infty$ for which* $\operatorname{gcdmult}(g^{-1}(\alpha)) > 1$.

We will deduce Proposition 5.1 from the following result, which is of independent interest.

Proposition 5.2. *Pick a nonconstant $g \in \mathbb{C}(x)$ and distinct $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}_\infty$. Suppose that $e_i := \operatorname{gcdmult}(g^{-1}(\alpha_i))$ is at least 2 for each $i = 1, 2, 3$. Then the triple (e_1, e_2, e_3) is a permutation of an element of*

$$\mathcal{N} := \{(2, 2, r) \colon r > 1\} \cup \{(2, 3, s) \colon 3 \le s \le 5\}.$$

Let π be a permutation of $\{1, 2, 3\}$ such that the triple $N := (e_{\pi(1)}, e_{\pi(2)}, e_{\pi(3)})$ is in $\mathcal{N}$, and let $\mu(x)$ be the unique Möbius transformation which maps the points $\alpha_{\pi(1)}$, $\alpha_{\pi(2)}$, $\alpha_{\pi(3)}$ to 1, 0, ∞, respectively. Then $\mu \circ g = f_N \circ h$ for some $h \in \mathbb{C}(x)$, where

$$\begin{aligned}
f_{(2,2,r)} &= \frac{(x^r + 1)^2}{4x^r} \\
f_{(2,3,3)} &= \frac{(x^4 + 8x)^3}{64(x^3 - 1)^3} \\
f_{(2,3,4)} &= \frac{(x^8 + 14x^4 + 1)^3}{108(x^5 - x)^4} \\
f_{(2,3,5)} &= \frac{(x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)^3}{-1728(x^{11} + 11x^6 - x)^5}.
\end{aligned}$$

Conversely, for each $N \in \mathcal{N}$ we have

$$\begin{aligned}
\operatorname{gcdmult}(f_N^{-1}(1)) &= N(1) \\
\operatorname{gcdmult}(f_N^{-1}(0)) &= N(2) \\
\operatorname{gcdmult}(f_N^{-1}(\infty)) &= N(3),
\end{aligned}$$

and there is a finite set T_N of Möbius transformations such that

$$f_N(x) - f_N(y) = \frac{\prod_{\nu \in T_N} (x - \nu(y))}{D_N(x)},$$

where $D_N(x)$ is the denominator exhibited in the definition of $f_N(x)$. Finally, for each $\nu \in T_N$ there is a positive integer k with $k < \deg(f_N)$ such that the composition $\nu \circ \nu \circ \cdots \circ \nu$ of k copies of ν equals x.

Remark 5.3. The rational functions $f_N(x)$ in Proposition 5.2 date back at least to the 19-th century book of Klein [27]. These rational functions generate the fields of rational functions invariant under the non-cyclic finite rotation groups of the sphere, namely the groups of rotational symmetries of the regular dihedron, tetrahedron, octahedron, or icosahedron. Thus the field extension $\mathbb{C}(x)/\mathbb{C}(f_N(x))$ is Galois with Galois group D_r, A_4, S_4 or A_5 according as N is $(2,2,r)$, $(2,3,3)$, $(2,3,4)$, or $(2,3,5)$; moreover, the elements of the Galois group are the maps $x \mapsto \nu(x)$ with $\nu \in T_N$. For a beautiful exposition of this material, see [49].

Proof that Proposition 5.2 implies Proposition 5.1. Let $\alpha_1, \alpha_2, \alpha_3$ be distinct points in $\mathbb{C}_\infty$, and suppose that each value $e_i := \operatorname{gcdmult}(g^{-1}(\alpha_i))$ is greater than 1. Proposition 5.2 implies that $\mu \circ g = f_N \circ h$ for some Möbius transformation $\mu(x)$, some $N \in \mathcal{N}$, and some $h \in \mathbb{C}(x)$. Since $g(p) = g(q)$, we have $f_N(h(p)) = f_N(h(q))$, so that $\prod_{\nu \in T_N}(h(p) - \nu(h(q))) = 0$, and thus $h(p) = \nu(h(q))$ for some $\nu \in T_N$. By Proposition 5.2, the order of $\nu(x)$ under composition is an integer k which is less than $\deg(f_N)$. We will give two different proofs that this information yields a contradiction, one using Galois theory and one from first principles.

We first give the algebraic proof. The function $\sigma\colon \mathbb{C}(x) \to \mathbb{C}(x)$ defined by $\sigma(u(x)) := u(\nu(x))$ is an order-k automorphism of the field $\mathbb{C}(x)$. Writing L for the set of elements of $\mathbb{C}(x)$ fixed by σ, Artin's theorem from Galois theory [28, Thm. VI.1.8] implies that L is a subfield of $\mathbb{C}(x)$ such that $[\mathbb{C}(x) : L] = k$. Since L properly contains $\mathbb{C}$, by Lüroth's theorem we have $L = \mathbb{C}(u(x))$ for some nonconstant $u(x) \in \mathbb{C}(x)$, and it is known that $[\mathbb{C}(x) : \mathbb{C}(u(x))] = \deg(u)$. But then $u(h(p)) = u(\nu(h(q)) = \sigma(u)(h(q)) = u(h(q))$, which contradicts minimality of $\deg(g)$ since $\deg(u \circ h) = k \cdot \deg(h) < \deg(f_N) \cdot \deg(h) = \deg(g)$.

We now give the self-contained proof. If $\nu(\infty) \neq \infty$ then the numerator of the rational function $\nu(x) - x$ has degree 2 and hence has a zero in $\mathbb{C}$. Thus in any case the set S of fixed points of $\nu(x)$ is nonempty. Let $\rho(x)$ be a Möbius transformation such that $\rho(\infty) \in S$ and if $|S| > 1$ then also $\rho(0) \in S$. Then $\theta := \rho^{-1} \circ \nu \circ \rho$ is a Möbius transformation having $|S|$ fixed points and having the same order under composition as does $\nu(x)$, which by Proposition 5.2 is an integer k less than $\deg(f_N)$. If $|S| = 1$ then ∞ is the unique fixed point of $\theta(x)$, so that $\theta(x)$ is a degree-one polynomial and $\theta(x) - x$ is a nonzero constant β, whence $\theta(x) = x + \beta$ has infinite order under composition, contradiction. Thus $|S| > 1$, so $\theta(x)$ fixes 0 and ∞, and hence $\theta(x) = \zeta x$ for some $\zeta \in \mathbb{C}^*$. Plainly the order of $\theta(x)$ under composition is the order of ζ under multiplication, so that ζ is a primitive k-th root of unity. Since $\rho^{-1}(h(p)) = \zeta\rho^{-1}(h(q))$, it follows that $\mathcal{G}_1$ contains $x^k \circ \rho^{-1} \circ h$, contradicting minimality of $\deg(g)$. $\square$

We have now reduced the proof of Proposition 5.1 to the proof of Proposition 5.2. Our proof of the latter result uses the following version of the Hurwitz genus formula for holomorphic maps $\mathbb{C}_\infty \to \mathbb{C}_\infty$:

Lemma 5.4. *Any $g \in \mathbb{C}(x)$ of degree $k > 0$ satisfies*

$$2k - 2 = \sum_{\alpha \in \mathbb{C}_\infty} \big(k - |g^{-1}(\alpha)_{set}|\big).$$

The Hurwitz formula immediately implies that for any nonconstant rational function $g(x)$, there cannot be four distinct points $\alpha \in \mathbb{C}_\infty$ for which $|g^{-1}(\alpha)_{\text{set}}| \le \deg(g)/2$, and hence there cannot be four distinct $\alpha \in \mathbb{C}_\infty$ for which $\operatorname{gcdmult}(g^{-1}(\alpha)) > 1$. However, there do exist nonconstant $g \in \mathbb{C}(x)$ for which $\operatorname{gcdmult}(g^{-1}(\alpha)) > 1$ for three distinct $\alpha \in \mathbb{C}_\infty$, and the goal of Proposition 5.2 is to describe them all. Although the existence of such functions was known long ago, the classification of them is new, and our proof of this classification is rather indirect and unexpected.

Proof of Proposition 5.2. Writing $k := \deg(g)$, we have $|g^{-1}(\alpha_i)_{\text{set}}| \le k/e_i$, so Lemma 5.4 implies that

$$2k - 2 \ge \sum_{i=1}^{3} \big(k - |g^{-1}(\alpha)_{\text{set}}|\big) \ge \sum_{i=1}^{3} \Big(k - \frac{k}{e_i}\Big),$$

whence $\sum_{i=1}^{3} 1/e_i > 1$. Since the e_i's are integers greater than 1, and since $1 = 1/3 + 1/3 + 1/3 = 1/2 + 1/4 + 1/4 = 1/2 + 1/3 + 1/6$, it follows that (e_1, e_2, e_3) is a permutation of an element of $\mathcal{N}$. Now let π, N, μ be as in the statement of the result. It is easy to check directly that for $\gamma \in \mathbb{C}_\infty$ the multiplicity of f_N at γ is $N(1)$, $N(2)$, $N(3)$, or 1, according as $f_N(\gamma)$ is 1, 0, ∞, or another value. Now view f_N and $\widehat{g} := \mu \circ g$ as branched coverings $S^2 \to S^2$, and let B be the set of branch points of $\widehat{g}$, which includes the branch points of f_N. Then the branched coverings f_N and $\widehat{g}$ become topological covering maps when we restrict the domain to avoid preimages of B, yielding finite topological covering maps $\psi : S^2 \setminus f_N^{-1}(B) \to S^2 \setminus B$ and $\phi : S^2 \setminus \widehat{g}^{-1}(B) \to S^2 \setminus B$. Form the pullback of ϕ along ψ as usual, yielding the diagram

$$\begin{tikzcd} X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S^2 \setminus f_N^{-1}(B) \arrow[d, "\psi"] \\ S^2 \setminus \widehat{g}^{-1}(B) \arrow[r, "\phi"] & S^2 \setminus B \end{tikzcd}$$

where $X := \{(a, b) \in (S^2 \setminus \widehat{g}^{-1}(B)) \times (S^2 \setminus f_N^{-1}(B))\colon \phi(a) = \psi(b)\}$ and π_1 and π_2 are projections on the first and second coordinates, respectively. We may compactify the topological covering map $\phi \circ \pi_1 : X \to S^2 \setminus B$ (see e.g. [14, §2]) in order to obtain a branched covering $\eta : \widehat{X} \to S^2$ which factors as $\eta = g \circ \widehat{\pi_1} = f_N \circ \widehat{\pi_2}$ where $\widehat{\pi_i}$ is the induced extension of π_i. For each $\beta \in S^2$, the multiplicity under $\widehat{g}$ of every point in $\widehat{g}^{-1}(\beta)$ is divisible by the multiplicity under f_N of every point in $f_N^{-1}(\beta)$, so by elementary covering space theory it follows that $\widehat{\pi_1}$ is an unbranched covering. Since S^2

is simply connected, this implies that the restriction of $\widehat{\pi_1}$ to any connected component Y of $\widehat{X}$ will be a homeomorphism $\theta_1\colon Y \to S^2$, so if θ_2 is the restriction of $\widehat{\pi_2}$ to Y then $\widehat{g} = f_N \circ \theta_2 \circ \theta_1^{-1}$. Here $\theta_2 \circ \theta_1^{-1}$ is a finite-degree branched covering $S^2 \to S^2$. Of course, any such branched covering induces a holomorphic function $\mathbb{C}_\infty \to \mathbb{C}_\infty$, which in turn is a rational function $h(x)$ such that $\mu \circ g = f_N \circ h$. Finally, the remaining assertions about the factorization of $f_N(x) - f_N(y)$ and the orders of elements of T_N are easy to verify directly, given that T_N is the group (under the operation of functional composition) generated by the set U_N defined as follows:

$$U_{(2,2,r)} := \{\zeta x^e \colon \zeta^r = 1,\, e \in \{1, -1\}\}$$
$$U_{(2,3,3)} := \left\{e^{2\pi i/3}x,\, \frac{x+2}{x-1}\right\}$$
$$U_{(2,3,4)} := \left\{ix,\, \frac{x+1}{x-1}\right\}$$
$$U_{(2,3,5)} := \left\{\zeta x,\, \frac{(\zeta^3+1)x+1}{x-\zeta^2-1}\right\} \quad \text{where } \zeta := e^{2\pi i/5}. \qquad \square$$

Remark 5.5. The topological argument in the above proof can be written in the language of algebraic geometry, by considering the normalizations of components of the fibered product of the morphisms $\mathbb{P}^1 \to \mathbb{P}^1$ induced by f_N and $\mu \circ g$. We chose topological language since we thought this would be more familiar to some complex analysts in our audience.

6. Minimal shared multisets

In this section we prove a generalization of Theorem 1.6, by describing the collection of all shared multisets for some nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$, under the assumption that $g(p) = g(q)$ for some nonconstant $g \in \mathbb{C}(x)$. We begin by addressing the analogous question for meromorphic functions on an arbitrary complex manifold.

6.1. Arbitrary complex manifolds.

Definition 6.1. For any complex manifold $\mathcal{R}$ and any nonconstant $p, q \in \mathcal{M}(\mathcal{R})$, a *minimal shared multiset* for p and q is a nonempty finite multiset S of elements of $\mathbb{C}_\infty$ such that S is shared CM by p and q, but no nonempty proper sub-multiset of S is shared CM by p and q.

Lemma 6.2. *If S is a finite multiset of elements of $\mathbb{C}_\infty$, then S is shared CM by p and q if and only if S is the union of finitely many minimal shared multisets for p and q.*

Proof. If S is shared, and T is a minimal shared multiset contained in S, then $S \setminus T$ is a shared multiset which is smaller than S, so by induction on $|S|$ we see that S is a union of minimal shared multisets. Conversely, any union of shared multisets is itself shared. $\square$

In light of the above result, in order to describe all shared multisets for p and q, it suffices to describe the minimal shared multisets. We now introduce a large collection of shared multisets T_α in case $g(p) = g(q)$ for some nonconstant $g \in \mathbb{C}(x)$. It will turn out that, in many situations, these T_α comprise the collection of all minimal shared multisets.

Definition 6.3. Let p and q be nonconstant meromorphic functions on a complex manifold $\mathcal{R}$, and suppose that the set $\mathcal{G}_1$ from Definition 4.1 is nonempty. Let $g \in \mathcal{G}_1$ have the smallest possible degree. For any $\alpha \in \mathbb{C}_\infty$, let R_α be the multiset $g^{-1}(\alpha)$, let $\operatorname{gcdmult}(R_\alpha)$ denote the greatest common divisor of the multiplicities of the elements of R_α, and let T_α be the multiset having the same underlying set as R_α, but in which the multiplicity of each element is $1/\operatorname{gcdmult}(R_\alpha)$ times the multiplicity of the element in R_α.

Example 6.4. If $p = e^z$ and $q = -e^z$ for $\mathcal{R} = \mathbb{C}$ then we may choose $g(x)$ to be x^2, so that $R_0 = \{0, 0\}$ has $\operatorname{gcdmult}(R_0) = 2$ and thus $T_0 = \{0\}$; likewise $T_\infty = \{\infty\}$, but for any $\alpha \notin \{0, \infty\}$ we have $R_\alpha = \{\beta, -\beta\}$ with $\beta^2 = \alpha$, so that $\operatorname{gcdmult}(R_\alpha) = 1$ and $T_\alpha = R_\alpha$.

Lemma 6.5. *Let p, q be nonconstant meromorphic functions on a complex manifold $\mathcal{R}$ such that $\mathcal{G}_1$ is nonempty. Then each T_α with $\alpha \in \mathbb{C}_\infty$ is a nonempty finite multiset which is shared CM by p and q, and every minimal shared multiset is contained in one of the multisets T_α. The collection of all T_α's depends only on p and q, and not on the choice of a minimal-degree function in $\mathcal{G}_1$.*

Proof. By taking preimages of α on both sides of the equation $g \circ p = g \circ q$, we see that p, q share CM R_α, and hence also T_α. Plainly T_α is nonempty and finite. By Proposition 4.2, any other choice of g has the form $\widehat{g} := \mu \circ g$ for some Möbius transformation μ; denoting the corresponding multisets by $\widehat{T}_\alpha$, it follows that $T_\alpha = \widehat{T}_{\mu(\alpha)}$, so that the collection of all T_α's equals the collection of all $\widehat{T}_\alpha$'s. Finally, the union of the T_α's is $\mathbb{C}_\infty$, so for any minimal shared multiset S there is some α for which $S \cap T_\alpha$ is nonempty; but then $S \cap T_\alpha$ is a shared multiset, so minimality of S implies $S \cap T_\alpha = S$, whence $S \subseteq T_\alpha$. □

We now show that if $\mathcal{R}$ is a compact Riemann surface and $\mathcal{G}_1$ is nonempty then the T_α comprise all minimal shared multisets for p and q.

Proposition 6.6. *If p and q are nonconstant meromorphic functions on a compact Riemann surface $\mathcal{R}$, and $\mathcal{G}_1$ is nonempty, then the minimal shared multisets for p and q are precisely the multisets T_α with $\alpha \in \mathbb{C}_\infty$.*

Proof. Pick a minimal-degree $g \in \mathcal{G}_1$, and suppose that some T_α is not a minimal shared multiset. Since T_α is shared CM by p and q, it is the union of two or more (not necessarily distinct) minimal shared multisets. Since $\operatorname{gcdmult}(T_\alpha) = 1$, these minimal shared multisets in T_α cannot all be equal, so T_α contains two disjoint minimal shared multisets S_1 and S_2. By

Lemma 3.5, there are integers $n_1, n_2 > 0$ and a nonconstant $h \in \mathbb{C}(x)$ such that the divisor of $h(x)$ is $n_1 S_1 - n_2 S_2$ and the functions $h(p)$ and $h(q)$ have the same divisor. Then $\gamma := h(p)/h(q)$ is in $\mathbb{C}^*$ since $\mathcal{R}$ is compact. For any $\beta \in \mathbb{C}_\infty$ with $\beta \neq \alpha$, the set $h(T_\beta)_{\text{set}}$ is a nonempty finite subset of $\mathbb{C}^*$, and for any $\delta \in T_\beta$ there is some $\epsilon \in \mathcal{R}$ such that $\delta = q(\epsilon)$, whence $\delta' := p(\epsilon)$ is an element of T_β satisfying

$$h(\delta') = h(p(\epsilon)) = \gamma \cdot h(q(\epsilon)) = \gamma \cdot h(\delta).$$

Thus $h(T_\beta)_{\text{set}}$ is preserved by multiplication by γ, so γ is a root of unity and hence $h^n(p) = h^n(q)$ for some positive integer n. By Proposition 4.2 we have $h^n = d \circ g$ for some $d \in \mathbb{C}(x)$, so the divisor of h^n is a $\mathbb{Z}$-linear combination of $g^{-1}(\alpha)$'s. But this is impossible because the divisor of h^n has positive coefficients at the elements of S_1 and negative coefficients at the elements of S_2. This contradiction shows that in fact every T_α must be a minimal shared multiset. □

6.2. **Complex m-space.** We now prove the following generalization of Theorem 1.6, which involves both the shared multisets T_α from Definition 6.3 and the set $\mathcal{G}_1$ from Definition 4.1.

Theorem 6.7. *Pick nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ for which $\mathcal{G}_1$ is nonempty, and let $g(x)$ be a minimal-degree element of $\mathcal{G}_1$. Then one of the following occurs:*

(6.7.1) *The collection of all multisets T_α with $\alpha \in \mathbb{C}_\infty$ equals the collection of all minimal shared multisets for p and q.*

(6.7.2) *For some $\beta \in \mathbb{C}_\infty$, the multiset T_β is the union of positive numbers of copies of each of two distinct minimal shared multisets S_1, S_2, and the collection of all minimal shared multisets consists of S_1, S_2, and all T_α with $\alpha \neq \beta$. In this case we can write $p = p_0 \circ r$ and $q = q_0 \circ r$ for some $r \in \mathcal{M}(\mathbb{C}^m)$ and some $p_0, q_0 \in \mathbb{C}(x)$ such that $g(p_0) = g(q_0)$, and for any such p_0, q_0, r there will be two Picard exceptional values γ, δ of r, with $\gamma \in p_0^{-1}(S_1) \cap q_0^{-1}(S_2)$ and $\delta \in p_0^{-1}(S_2) \cap q_0^{-1}(S_1)$, where in addition for each $i = 1, 2$ the multisets $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ coincide except for copies of γ and δ.*

Proof. Since $g(p) = g(q)$, the functions p and q are algebraically independent. By Lemma 3.7, there is a compact Riemann surface $\mathcal{R}$ for which $p = p_0(r)$ and $q = q_0(r)$ for some $p_0, q_0 \in \mathcal{M}(\mathcal{R})$ and some holomorphic map $r \colon \mathbb{C}^m \to \mathcal{R}$. Thus for any multiset S of elements of $\mathbb{C}_\infty$, the multiset $p^{-1}(S)$ is the union of $r^{-1}(\alpha)$ for $\alpha \in p_0^{-1}(S)$. It follows that p and q share S CM if and only if the multiset differences $A(S) := p_0^{-1}(S) \setminus q_0^{-1}(S)$ and $B(S) := q_0^{-1}(S) \setminus p_0^{-1}(S)$ both consist of elements of the set $\mathcal{E} := \mathcal{R} \setminus r(\mathbb{C}^m)$, which has size at most 2 by Lemma 3.10.

Suppose (6.7.1) does not hold, so, by Lemma 6.5, some T_α is not a minimal shared multiset. Since $\operatorname{gcdmult}(T_\alpha) = 1$, it follows that T_α contains two disjoint minimal shared multisets S_1 and S_2. The identity $g(p_0(r)) =$

$g(p) = g(q) = g(q_0(r))$ implies that $g(p_0) = g(q_0)$, so in particular $\deg(p_0) = \deg(q_0)$. Thus $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ have the same size, so also $A_i := A(S_i)$ and $B_i := B(S_i)$ have the same size n_i. Proposition 6.6 implies that S_i is not shared by p_0 and q_0, so $n_i > 0$. Thus A_i contains an element γ_i. Since A_i consists of elements of $\mathcal{E}$, and $|\mathcal{E}| \le 2$, disjointness of the S_i's implies that A_i consists of n_i copies of γ_i. Likewise, since A_i and B_i are disjoint, B_i must consist of n_i copies of γ_{3-i}, so (6.7.2) holds. □

Example 6.8. The second possibility in Theorem 6.7 can actually occur. For instance, let k, n be integers with $0 < k < n$, put $\zeta := e^{2\pi i/n}$, and let $p := (e^x + \zeta^k)/(e^x + 1)$ and $q := \zeta p$. Then we may choose $g := x^n$, so that $g^{-1}(1) = \{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\}$ is the union of $S_1 := \{\zeta, \zeta^2, \dots, \zeta^k\}$ and $S_2 := \{\zeta^{k+1}, \zeta^{k+2}, \dots, \zeta^n\}$. Here p has no preimages of 1 or ζ^k, so q has no preimages of ζ or ζ^{k+1}, whence

$$p^{-1}(S_1) = p^{-1}(\{\zeta, \zeta^2, \dots, \zeta^{k-1}\}) = q^{-1}(\{\zeta^2, \zeta^3, \dots, \zeta^k\}) = q^{-1}(S_1),$$

and likewise $p^{-1}(S_2) = q^{-1}(S_2)$.

7. Bounding the degree of a rational function relating p and q

In this section we use the results of the previous two sections in order to bound the degree of a minimal-degree element of $\mathcal{G}_1$ in terms of the sizes of shared multisets. The combination of these bounds with Theorem 3.1 yields Theorem 1.2.

Theorem 7.1. *Pick nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$, and let $S_1, \dots, S_n$ be finite multisets of elements of $\mathbb{C}_\infty$ such that p, q share CM each S_i, where $n \ge 4$ and no S_i is contained in the union of the other S_j's. Then $g(p) = g(q)$ for some nonconstant $g \in \mathbb{C}(x)$ such that $\deg(g) \le \frac{1}{n-3}(-2 + \sum_{i=1}^n |(S_i)_{set}|)$. If $n \ge 5$ then we can choose $g(x)$ to have degree at most $\max_i |S_i|$. Moreover, if $n \ge 5$ and the S_i's are minimal shared multisets then $\max_i |S_i|$ is the smallest degree of any nonconstant $g \in \mathbb{C}(x)$ for which $g(p) = g(q)$.*

Remark 7.2. In order to obtain the best bound on $\deg(g)$ from Theorem 7.1, it is sometimes advantageous to ignore some of the S_i's when applying the bounds in this result. For instance, if $n > 5$ then we can choose $g(x)$ to have degree at most the size of the fifth-smallest S_i. Likewise, if $n = 5$ and one S_i is much larger than the others then the best bound will come from applying the first bound in Theorem 7.1 to the other four S_j's. This shows that the first bound is sometimes better than the second bound when they both apply; conversely, if $n \ge 5$ and the S_i's are sets of the same size then the second bound is better than the first.

Proof. By Theorem 3.1 there is a nonconstant $g \in \mathbb{C}(x)$ such that $g(p) = g(q)$. Choose one such $g(x)$ for which $k := \deg(g)$ is as small as possible. For each i, let R_i be a minimal shared multiset contained in $S_i \setminus \cup_{j \ne i} S_j$, so that the R_i's are pairwise disjoint. Let I be the set of values i for which R_i has the form T_{α_i} with $\alpha_i \in \mathbb{C}_\infty$. Theorem 6.7 implies that $|I| \ge n - 2$, and in addition if $|I| = n-2$ then there is some $\alpha \in \mathbb{C}_\infty$ for which T_α is the union of copies of the two multisets R_i with $i \notin I$. Thus if $|I| = n-2$ then $V := g(\cup_{i=1}^n R_i)$ has size $n-1$, and $g^{-1}(V)$ is the union of copies of the R_i's, so Lemma 5.4 yields

$$\begin{aligned} 2k-2 &\geq \sum_{\alpha \in V} \bigl(k - |g^{-1}(\alpha)_{\mathrm{set}}|\bigr) \\ &= (n-1)k - \sum_{i=1}^n |(R_i)_{\mathrm{set}}| \\ &\geq (n-1)k - \sum_{i=1}^n |(S_i)_{\mathrm{set}}|, \end{aligned}$$

whence $k \leq \frac{1}{n-3}\bigl(-2 + \sum_{i=1}^n |(S_i)_{\mathrm{set}}|\bigr)$. If $|I| \geq n-1$ then $V := g(\cup_{i \in I} R_i)$ has the same size as I, so Lemma 5.4 yields

$$\begin{aligned} 2k-2 &\geq \sum_{\alpha \in V} \bigl(k - |g^{-1}(\alpha)_{\mathrm{set}}|\bigr) \\ &= k|I| - \sum_{i \in I} |(R_i)_{\mathrm{set}}| \\ &\geq (n-1)k - \sum_{i=1}^n |(S_i)_{\mathrm{set}}|, \end{aligned}$$

so that again $k \leq \frac{1}{n-3}\bigl(-2 + \sum_{i=1}^n |(S_i)_{\mathrm{set}}|\bigr)$.

By Proposition 5.1 there are at most two elements $i \in I$ for which $g^{-1}(\alpha_i)$ consists of more than one copy of R_i. Thus if $n \geq 5$ then, since $|I| \geq n-2 \geq 3$, there is some $i \in I$ for which $g^{-1}(\alpha_i) = R_i$, so that

$$k = \deg(g) = |g^{-1}(\alpha_i)| = |R_i| \leq \max(\{|S_j| \colon 1 \leq j \leq n\}).$$

Finally, if $n \geq 5$ then $k = |R_i|$ for some i, and every R_j is contained in some $g^{-1}(\alpha_j)$ and hence has size at most k, so $k = \max_{j=1}^n |R_j|$. □

8. Classification results under additional hypotheses

In this section we apply our results in order to describe all $p, q \in \mathcal{M}(\mathbb{C}^m)$ which share a collection of multisets that satisfy certain types of additional constraints.

8.1. **Three points.**

We first give a quick proof of the classification of p, q sharing four multisets of which at least three have size 1. Later in this section we will prove the analogous result in which "three" is replaced by "two". Since the proof of the latter result relies on the results of some long and difficult papers, it seems worthwhile to illustrate the techniques by proving the present special case.

Proposition 8.1. *If distinct $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ share CM three distinct $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}_\infty$ in addition to a finite nonempty multiset of elements of*

$\mathbb{C}_\infty \setminus \{\alpha_1, \alpha_2, \alpha_3\}$, then p, q satisfy either case 1 of Table 1 (on page 28) or case 1 or 6 of Table 3, after replacing p and q by $\mu \circ p$ and $\mu \circ q$ for some Möbius transformation $\mu(x)$, and possibly interchanging p and q. The minimal shared multisets for p and q are listed in case 1 of Table 2 or case 1 or 6 of Table 4.

Remark 8.2. In case the S_i are sets and $m = 1$, Proposition 8.1 was proved in [42], generalizing special cases proved previously in [21, 50].

Proof. By Theorem 3.1, we have $g \circ p = g \circ q$ for some nonconstant $g \in \mathbb{C}(x)$. Choose $g(x)$ to have minimal degree. Since $p \neq q$, we have $\deg(g) > 1$. By Lemma 5.4, at most two points have a unique g-preimage, and if there are two then all other points have $\deg(g)$ distinct g-preimages. Thus, by Theorem 6.7, there are distinct $\alpha, \beta, \gamma \in \mathbb{C}^*$ satisfying one of the following:

(1) α, β each have a unique g-preimage, and (6.7.2) applies to $g^{-1}(\gamma)$
(2) α has a unique g-preimage, β has two g-preimages, and (6.7.2) applies to $g^{-1}(\beta)$.

Replace g, p, q by $\mu \circ g \circ \nu^{-1}$, $\nu \circ p$, $\nu \circ q$ for suitable Möbius transformations μ and ν in order to assume in (1) that $g = x^k$ and in (2) that $g = x^j (x-1)^k$, where j, k are positive integers. In (1) we have $p = \zeta q$ where ζ is a primitive k-th root of unity, yielding case 6 of Table 3. In (2), if $\gcd(j, k) = 1$ then $(g(x) - g(y))/(x - y)$ is irreducible by [4, Prop. 2.4 and Lemma 4.5], yielding case 1 of Table 1. In (2), if $\ell := \gcd(j, k) > 1$ then $p(x) = \zeta p(y)$ for some primitive ℓ-th root of unity ζ, and $g = x^\ell \circ f$ where $f(x) := x^{j/\ell}(x-1)^{k/\ell}$. Then $f(x) - \zeta f(y)$ is irreducible by [4, Lemmas 3.1 and 4.5], yielding case 1 of Table 3. $\square$

8.2. **Points in components of fibered products.** One of the main questions we must address is: for prescribed nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ such that $f \circ p = f \circ q$ for some $f \in \mathbb{C}(x) \setminus \mathbb{C}$, which of the possibilities in Theorem 6.7 describes the minimal shared multisets for p and q? In particular, when does the unusual case (6.7.2) hold? We now recall some general lemmas that are useful for resolving this.

Definition 8.3. For any $f(x) \in \mathbb{C}(x) \setminus \mathbb{C}$ and any $\alpha \in \mathbb{C}_\infty$, we write $e_f(\alpha)$ for the multiplicity of f at α.

We will use the following three well-known results:

Lemma 8.4. *Let $g, h, p_0, q_0 \in \mathbb{C}(x) \setminus \mathbb{C}$ satisfy $g \circ p_0 = h \circ q_0$ and $\deg(g) = \deg(q_0)$, and suppose that the numerator of $g(x) - h(y)$ is irreducible. For any $\alpha, \beta \in \mathbb{C}_\infty$ such that $g(\alpha) = h(\beta)$, the intersection $p_0^{-1}(\alpha) \cap q_0^{-1}(\beta)$ contains precisely $m := \gcd(e_g(\alpha), e_h(\beta))$ distinct elements of $\mathbb{C}_\infty$, and the multiplicities of p_0 and q_0 at each such element are $e_h(\beta)/m$ and $e_g(\alpha)/m$, respectively.*

Lemma 8.5. *Let $g, p_0, q_0 \in \mathbb{C}(x) \setminus \mathbb{C}$ satisfy $g \circ p_0 = g \circ q_0$ and $\deg(p_0) = \deg(g) - 1$ but $p_0 \neq q_0$, and suppose that the numerator of $(g(x) - g(y))/(x - y)$*

is irreducible. For any distinct $\alpha, \beta \in \mathbb{C}_\infty$ such that $g(\alpha) = g(\beta)$, the intersection $p_0^{-1}(\alpha) \cap q_0^{-1}(\beta)$ contains precisely $m := \gcd(e_g(\alpha), e_g(\beta))$ distinct elements of $\mathbb{C}_\infty$, and the multiplicities of p_0 and q_0 at each such element are $e_g(\beta)/m$ and $e_g(\alpha)/m$, respectively.

Lemma 8.6. *Let $g, p_0, q_0 \in \mathbb{C}(x) \setminus \mathbb{C}$ satisfy $H(p_0, q_0) = 0$ where $H(x, y)$ is an irreducible factor of the numerator of $(g(x) - g(y))/(x - y)$ and the x-degree of $H(x, y)$ is $\deg(p_0)$. For any $\alpha, \gamma \in \mathbb{C}_\infty$ such that $e_g(\alpha) = e_g(\gamma) = 1$, there is at most one point δ in $p_0^{-1}(\alpha) \cap q_0^{-1}(\gamma)$, and if such a point δ exists then $g(\alpha) = g(\gamma)$ and $\alpha \neq \gamma$ and $e_{p_0}(\delta) = e_{q_0}(\delta) = 1$.*

Remark 8.7. Although variants of these results have been used for over 100 years, the only reference we know which proves a result of sufficient generality to imply Lemmas 8.4 and 8.5 is [12, Lemma 7.1], which actually addresses the more general setting of fibered products of tamely ramified branched covers of a nonsingular projective curve over an arbitrary field. The above three results follow from [12, Lemma 7.1] upon noting that the map $z \mapsto (p_0(z), q_0(z))$ is the normalization map for the curve $g(x) = h(y)$, $(g(x) - g(y))/(x - y) = 0$, or $H(x, y) = 0$, respectively.

8.3. **Four sets of size at most 2.** We now use our results to prove a several-variable generalization of the known classification of $p, q \in \mathcal{M}(\mathbb{C})$ sharing four sets of size at most 2:

Proposition 8.8. *Let p, q be nonconstant meromorphic functions on $\mathbb{C}^m$. If p, q share CM four pairwise disjoint nonempty subsets $S_1, \dots, S_4$ of $\mathbb{C}_\infty$ with $|S_i| \leq 2$ then $p = \mu \circ q$ for some Möbius transformation $\mu(x)$.*

Remark 8.9. When $m = 1$, the above result is the combination of Nevanlinna's four-values theorem with the results of the papers [45, 46, 48, 50], which address the cases that the number of S_i's of size 1 is 1, 0, 2, 3, respectively. Further, the proof in [46] relies on [30], and the result in [48] has extra hypotheses besides the sizes of the S_i's, but according to [46] the proof in [48] does not require these hypotheses. Proposition 8.8 was conjectured in [44], after special cases had been proven in [21, 36, 43, 48, 50].

Proof. Pick a minimal-degree $g(x) \in \mathcal{G}_1$, say with $n := \deg(g)$. Note that if $n = 1$ then $p = q$, and if $n = 2$ then $p = \mu(q)$ for some Möbius transformation $\mu(x)$ (as we explained after Theorem 1.2). Henceforth assume $n \geq 3$. By Theorem 6.7, at least two S_i's are unions of T_β's, say S_1 and S_2. If S_1 is the union of more than one T_β then, since $|S_1| \leq 2$, Lemma 5.4 implies that each element of $\mathbb{C}_\infty \setminus g(S_1)$ has n distinct g-preimages. Thus $|S_2| = n|g(S_2)|$, so $n \leq 2$, contradiction. Hence $S_1 = T_\alpha$ for some $\alpha \in \mathbb{C}_\infty$, and likewise $S_2 = T_\beta$. Upon replacing g by $\nu \circ g$ for a suitable Möbius transformation $\nu(x)$, we may assume that $\alpha = 0$ and $\beta = \infty$. Since S_1 and S_2 are sets, it follows from the definition of T_γ that the divisor of g is $uS_1 - vS_2$ for some positive integers u and v with $u|S_1| = n = v|S_2|$. Writing $d := \gcd(u, v)$, since $|S_i| \leq 2$ we see that $d|S_i| = n$ for at least one $i \in \{1, 2\}$, say for $i = 1$. Since all zeroes and poles of the rational function $g(x)$ have multiplicity divisible by d, we have $g = h^d$ for some $h \in \mathbb{C}(x)$, where $\deg(h) = n/d = |S_1| \le 2$ and $S_1 = h^{-1}(0)$ and $S_2 = h^{-1}(\infty)_{\text{set}}$. The identity $h^d \circ p = h^d \circ q$ implies that $h \circ p = \zeta h \circ q$ for some ζ with $\zeta^d = 1$, where minimality of $\deg(g)$ implies ζ is a primitive d-th root of unity. This yields the desired Möbius transformation if $\deg(h) = 1$, so we assume $\deg(h) = 2$. Since $n > 2$ and $n = d \cdot \deg(h) = 2d$, we must have $d > 1$. If the numerator of $h(x) - \zeta h(y)$ is reducible then each factor $H(x, y)$ of this numerator has x-degree 1 and y-degree 1, and some such factor satisfies $H(p, q) = 0$, so $p = \mu(q)$ for some Möbius transformation $\mu(x)$. Henceforth assume the numerator of $h(x) - \zeta h(y)$ is irreducible. Each $\gamma \in \mathbb{C}^*$ has $d \ge 2$ distinct preimages under x^d, and hence has at least d distinct preimages under g, with equality holding if and only if each d-th root of γ has a unique h-preimage. By Lemma 5.4, at most two values have a unique h-preimage, so all but at most one $\gamma \in \mathbb{C}^*$ have more than two g-preimages. Thus (6.7.2) holds, so $p = p_0 \circ r$ and $q = q_0 \circ r$ for some $r \in \mathcal{M}(\mathbb{C}^m)$ and some $p_0, q_0 \in \mathbb{C}(x)$, whence $h(p_0) = \zeta h(q_0)$. We may choose p_0, q_0, r so that p_0 has the smallest possible degree, which in this case is 2 since $h(x) - \zeta h(y)$ is irreducible and $\deg(h) = 2$. By Lemma 5.4, exactly two elements of $\mathbb{C}_\infty$ have a unique p_0-preimage, which by Lemma 8.5 implies that exactly one element of $\mathbb{C}_\infty$ has a unique h-preimage and two ζh-preimages. Thus if $\zeta = -1$ then the two values with a unique h-preimage are not negatives of one another, so that for any ζ each multiset T_β with $\beta \in \mathbb{C}^*$ contains at least three distinct elements. From (6.7.2), it follows that there is some $\gamma \in \mathbb{C}^*$ for which T_γ is the union of copies of S_3 and S_4. Upon replacing $g(x)$ by $g(x)/\gamma$, we may assume $\gamma = 1$, so that $S_3 \cup S_4 = \cup_{\theta^d = 1} h^{-1}(\theta)_{\text{set}}$. If $d > 2$ then, since at most two of these θ's have a unique h-preimage, it follows that $4 \ge |S_3 \cup S_4| \ge 2d - 2$, so $d = 3$ and two cube roots of unity have a unique h-preimage. Upon multiplying h by a suitable cube root of unity, we may assume that ζ and ζ^2 have unique h-preimages, so by Lemma 8.5 each of $p_0^{-1}(h^{-1}(\zeta))$ and $q_0^{-1}(h^{-1}(\zeta^2))$ consists of two points which each have unequal multiplicities under p_0 and q_0; thus all four of these points must be Picard exceptional values of r, contradiction. Hence $d = 2$, so the two values with a unique h-preimage are not negatives of one another. If an element of $\{1, -1\}$ has a unique h-preimage then we get a contradiction in the same way as above, so assume that $h^{-1}(\{1, -1\})$ consists of four distinct elements. By Lemma 8.5, for each $a \in A := h^{-1}(1)$ and each $b \in B := h^{-1}(-1)$, both $p_0^{-1}(a) \cap q_0^{-1}(b)$ and $p_0^{-1}(b) \cap q_0^{-1}(a)$ consist of a single point. Since $|A| = |B| = 2$ and $A \cup B = S_3 \cup S_4$, this yields eight points of $p_0^{-1}(S_3 \cup S_4)$ having different images under p_0 and q_0, where at most two of these points are in $p_0^{-1}(S_3) \cap q_0^{-1}(S_3)$ and at most two are in $p_0^{-1}(S_4) \cap q_0^{-1}(S_4)$. This leaves at least four points which must be Picard exceptional values of r, contradiction. $\square$

8.4. **Restrictions on β in** (6.7.2)**.** We now present three results restricting the elements β that can occur in (6.7.2). These use the set $\mathcal{G}_1(p, q)$ from

Definition 4.1. The goal of these results is to show that, for various choices of g, p, q, there are only finitely many possibilities for an element β in (6.7.2), so that testing these possibilities becomes a finite problem.

Lemma 8.10. *Pick $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ which can be written as $p = p_0 \circ r$ and $q = q_0 \circ r$ with $p_0, q_0 \in \mathbb{C}(x)$ and $r \in \mathcal{M}(\mathbb{C}^m)$, and choose p_0, q_0, r which minimize $\deg(p_0)$. Suppose that $\mathcal{G}_1 := \mathcal{G}_1(p, q)$ is nonempty, and let $g \in \mathcal{G}_1$ have minimal degree. If $\deg(p_0) > \max(1, \deg(g)/2 - 1)$ and $\beta \in \mathbb{C}_\infty$ has $\deg(g)$ distinct g-preimages then $g^{-1}(\beta)$ is a minimal shared multiset for p and q.*

Proof. Suppose otherwise. By Theorem 6.7, $g^{-1}(\beta)$ contains two minimal shared multisets S_1 and S_2, where in addition $p_0^{-1}(S_i) \cap q_0^{-1}(S_{3-i})$ consists of copies of a single point for each $i \in \{1, 2\}$. Without loss we may assume that $|S_1| \le k/2$, where $k := \deg(g)$. Since β has $\deg(g)$ distinct g-preimages, we have $e_g(\alpha) = 1$ for each $\alpha \in g^{-1}(\beta)$. By Lemma 8.6, for each $\alpha \in S_1$ we see that both $p_0^{-1}(\alpha)$ and $q_0(p_0^{-1}(\alpha))$ are sets of size $\deg(p_0)$, and the latter set does not contain α. Since $\deg(p_0) > k/2 - 1 \ge |S_1| - 1$, it follows that $q_0(p_0^{-1}(\alpha))$ contains at least $\deg(p_0) - |S_1| + 1 > 0$ elements of S_2 for each $\alpha \in S_1$. Thus $|S_1| = 1$ and $\deg(p_0) - |S_1| + 1 = 1$, so $\deg(p_0) = 1$, contradiction. $\square$

Lemma 8.11. *Pick $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ which can be written as $p = p_0 \circ r$ and $q = q_0 \circ r$ with $p_0, q_0 \in \mathbb{C}(x)$ and $r \in \mathcal{M}(\mathbb{C}^m)$, and choose p_0, q_0, r which minimize $\deg(p_0)$. Suppose that $\deg(p_0) > \max(1, (\deg(g) - 2)/4)$ and $\mathcal{G}_1 := \mathcal{G}_1(p, q)$ is nonempty, and let $g \in \mathcal{G}_1$ have minimal degree. If $\beta \in \mathbb{C}_\infty$ has $\deg(g)$ distinct g-preimages but $g^{-1}(\beta)$ is not a minimal shared multiset for p and q then there exist distinct $\alpha, \gamma \in p_0^{-1}(g^{-1}(\beta))$ such that $p_0(\alpha) = q_0(\gamma)$ and $p_0(\gamma) = q_0(\alpha)$.*

Proof. Suppose otherwise. By Theorem 6.7, $g^{-1}(\beta)$ is the union of two nonempty shared sets S_1 and S_2, where in addition $p_0^{-1}(S_i) \cap q_0^{-1}(S_{3-i})$ consists of copies of a single point for each $i \in \{1, 2\}$. Without loss we may assume that $|S_1| \le k/2$, where $k := \deg(g)$. Since β has $\deg(g)$ distinct g-preimages, we have $e_g(\alpha) = 1$ for each $\alpha \in g^{-1}(\beta)$. By Lemma 8.6, for each $\alpha \in S_1$ we see that both $p_0^{-1}(\alpha)$ and $q_0(p_0^{-1}(\alpha))$ are sets of size $\deg(p_0)$, and the latter set does not contain α. We show first that $|S_1| > 2$: for, since $T := q_0(p_0^{-1}(\alpha))$ has size $\deg(p_0) > 1$ and contains at most one element of S_2, it must contain at least one element α' of S_1; since T does not contain α, we have $\alpha' \ne \alpha$, and $U := q_0(p_0^{-1}(\alpha'))$ does not contain either α or α', but U has size $\deg(p_0) > 1$ and U contains at most one element of S_2, so U contains an element of S_1 distinct from α and α', whence $|S_1| > 2$. Since $q_0(p_0^{-1}(\alpha)) \cap S_2$ is empty for all but one $\alpha \in S_1$, and has size 1 for the excluded α, it follows that $q_0(p_0^{-1}(\alpha)) \cap S_1$ has size $\deg(p_0)$ for all but one $\alpha \in S_1$, and size $\deg(p_0) - 1$ for the excluded α. Likewise $p_0(q_0^{-1}(\alpha')) \cap S_1$ has size $\deg(p_0)$ for all but one $\alpha' \in S_1$, and size $\deg(p_0) - 1$ for the excluded α'. Since $|S_1| > 2$, there is some $\alpha \in S_1$ for which both $q_0(p_0^{-1}(\alpha)) \cap S_1$ and $p_0(q_0^{-1}(\alpha)) \cap S_1$

have size $\deg(p_0)$. By hypothesis, these sets are disjoint from each other and from $\{\alpha\}$, so $|S_1| \geq 1 + 2\deg(p_0)$, whence $\deg(g) \geq 2|S_1| \geq 2 + 4\deg(p_0)$, contradiction. □

Remark 8.12. It is straightforward to prove analogues of the previous lemma with weaker bounds on $\deg(p_0)$ but more possibilities in the conclusion.

Although the hypotheses of the following result are quite special, they cover situations arising in the proof of Theorem 8.19.

Lemma 8.13. *Pick $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ which can be written as $p = p_0 \circ r$ and $q = q_0 \circ r$ with $p_0, q_0 \in \mathbb{C}(x)$ and $r \in \mathcal{M}(\mathbb{C}^m)$, and choose p_0, q_0, r which minimize $\deg(p_0)$. Suppose that $\deg(p_0) = 3$ and $q_0 = p_0 \circ \mu$ for some Möbius transformation $\mu(x)$ such that $\mu \circ \mu = x$, and suppose in addition that $\mathcal{G}_1 := \mathcal{G}_1(p, q)$ is nonempty. Let $g \in \mathcal{G}_1$ have minimal degree, and pick $\beta \in \mathbb{C}_\infty$ which has $\deg(g)$ distinct g-preimages. If $g^{-1}(\beta)$ is not a minimal shared multiset for p and q then $\deg(g) \geq 10$, and if $\deg(g) = 10$ then there are distinct $\alpha, \gamma \in g^{-1}(\beta)$ such that $q_0(p_0^{-1}(\alpha)) = q_0(p_0^{-1}(\gamma))$.*

Proof. Since β has $k := \deg(g)$ distinct g-preimages, each $\alpha \in g^{-1}(\beta)$ satisfies $e_g(\alpha) = 1$, and $g^{-1}(\beta)$ is a k-element set. By Lemma 8.6, if $\alpha \in g^{-1}(\beta)$ then each of $p_0^{-1}(\alpha)$, $q_0^{-1}(\alpha)$, $q_0(p_0^{-1}(\alpha))$, and $p_0(q_0^{-1}(\alpha))$ is a 3-element set, where in addition the latter two sets do not contain α. Moreover, the properties of $\mu(x)$ imply that for $\gamma, \delta \in g^{-1}(\beta)$ we have $p_0^{-1}(\gamma) \cap q_0^{-1}(\delta) = \emptyset$ if and only if $p_0^{-1}(\delta) \cap q_0^{-1}(\gamma) = \emptyset$. By Proposition 6.7.2, if $g^{-1}(\beta)$ is not a minimal shared multiset for p and q then $g^{-1}(\beta)$ is the union of two disjoint minimal shared multisets S_1 and S_2, each of which is in fact a set. We may assume $|S_1| \leq |S_2|$. By Proposition 6.7.2, there is a unique $\delta \in S_1$ for which $p_0(q_0^{-1}(\delta)) \cap S_2$ is nonempty, and then $p_0(q_0^{-1}(\delta))$ contains two distinct elements α, γ of $S_1 \setminus \{\delta\}$. Assume $k \leq 10$, so that $|S_1| \leq 5$. Since $p_0(q_0^{-1}(\alpha))$ is a three-element subset of $S_1 \setminus \{\alpha\}$, it contains at least one element ϵ of $S_1 \setminus \{\alpha, \gamma, \delta\}$. Then $p_0(q_0^{-1}(\epsilon))$ is a three-element subset of $S_1 \setminus \{\delta, \epsilon\}$, so it contains at least one element π of $S_1 \setminus \{\alpha, \gamma, \delta, \epsilon\}$. Since $|S_1| \leq 5$, it follows that $S_1 = \{\alpha, \gamma, \delta, \epsilon, \pi\}$ and $k \geq 2|S_1| = 10$. Hence $p_0(q_0^{-1}(\epsilon)) = \{\alpha, \gamma, \pi\}$ and $p_0(q_0^{-1}(\pi)) = \{\alpha, \gamma, \epsilon\}$, so $q_0(p_0^{-1}(\alpha)) = \{\delta, \epsilon, \pi\} = q_0(p_0^{-1}(\gamma))$, as desired. □

8.5. **Classifications under irreducibility hypotheses.** We now determine all possibilities for $p, q \in \mathcal{M}(\mathbb{C}^m)$ and all of their shared multisets in two of the main classes of such p, q which satisfy the conditions of Theorem 1.2.

Proposition 8.14. *Let $g, p_0, q_0 \in \mathbb{C}(x) \setminus \mathbb{C}$ satisfy $g \circ p_0 = g \circ q_0$ and $\deg(p_0) = \deg(g) - 1$ but $p_0 \neq q_0$, and suppose that the numerator of $(g(x) - g(y))/(x - y)$ is irreducible. Put $p := p_0 \circ r$ and $q := q_0 \circ r$ for some $r \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$. Then $g(x)$ is a minimal-degree element of $\mathcal{G}_1$ for p and q. Pick $\beta \in \mathbb{C}_\infty$ and suppose that T_β is not a minimal shared multiset for p*

and q. Then T_β contains two minimal shared multisets, each of which has size 1. Moreover, g, p_0, q_0, β, and the set $\mathcal{E}$ of Picard exceptional values of r appear in Table 1, after we compose with Möbius transformations μ, ν, ρ to replace g, p_0, q_0, r, β by $\mu \circ g \circ \nu^{-1}$, $\nu \circ p_0 \circ \rho$, $\nu \circ q_0 \circ \rho$, $\rho^{-1} \circ r$, $\mu(\beta)$.

Table 1

Case	g	p_0	q_0	β	$\mathcal{E}$
1	$x^j(x-1)^k$	$\dfrac{x^j - 1}{x^{j+k} - 1}$	$p_0 \circ \dfrac{1}{x}$	0	$\{\infty, 0\}$
2	$\dfrac{(x^2 + 2x - \frac{1}{3})^2}{x}$	$\dfrac{-3x^3}{x^3+1}$	$p_0 \circ \dfrac{x+1}{2x-1}$	∞	$\{0, -1\}$
3	$\dfrac{(x^2 + 10x + 5)^3}{x^5}$	$p_0 = \dfrac{1}{25}x^5 + \dfrac{1}{5}x^4 + \dfrac{3}{5}x^3 + x^2 + x$ $q_0 = p_0 \circ \dfrac{5}{x},\ \beta = \infty,\ \mathcal{E} = \{\infty, 0\}$			

Note: j and k are coprime positive integers.

Proof. If $g = u \circ v$ with $u, v \in \mathbb{C}(x)$ then the numerator of $g(x) - g(y)$ is divisible by the numerator of $v(x) - v(y)$, which in turn is divisible by $x - y$. The irreducibility hypothesis therefore implies that $g(x)$ cannot be written as the composition of lower-degree rational functions, so Proposition 4.2 implies $g(x)$ is a minimal-degree element of $\mathcal{G}_1$. For $\beta \in \mathbb{C}_\infty$, define T_β with respect to $g(x)$, and suppose that T_β is not a minimal shared multiset for p and q. By Proposition 6.7, T_β is the union of more or more copies of each of two distinct minimal shared multisets S_1 and S_2, where $p_0^{-1}(S_i) \cap q_0^{-1}(S_{3-i})$ consists of one or more copies of a Picard exceptional value α_i for r, and $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ coincide except for copies of α_1 and α_2. For any $a \in S_1$ and $b \in S_2$, Lemma 8.5 implies that $p_0^{-1}(a) \cap q_0^{-1}(b)$ and $p_0^{-1}(b) \cap q_0^{-1}(a)$ each contain $\gcd(e_g(a), e_g(b))$ distinct points. Thus $\gcd(e_g(a), e_g(b)) = 1$ and each S_i consists of copies of a single point, which by minimality of S_i implies $|S_i| = 1$. Now the result follows by inspecting the list of possibilities in [13] (or alternately [31]). The statement of the result takes into account the possibility of equivalences between different examples caused by composing with suitable Möbius transformations. □

Remark 8.15. In light of Theorem 6.7, the minimal shared multisets for p and q in the three cases in Table 1 are as follows:

Proposition 8.16. *Let $f, p_0, q_0 \in \mathbb{C}(x) \setminus \mathbb{C}$ satisfy $\deg(p_0) = \deg(f)$ and $f \circ p_0 = \zeta f \circ q_0$ where ζ is a primitive n-th root of unity with $n > 1$, and*

TABLE 2. Minimal shared multisets for p, q in Table 1

Case	Minimal shared multisets not of the form $g^{-1}(\alpha)$
1	$\{\infty\}$, $\{0\}$, $\{1\}$, and if $j = k = 1$ then $\{1/2\}$
2	$\{\infty\}$, $\{0\}$, {roots of $x^2 + 2x - 1/3$}
3	$\{\infty\}$, $\{0\}$, {roots of $x^2 + 10x + 5$}

suppose that the numerator of $f(x) - \zeta f(y)$ is irreducible. Put $p := p_0 \circ r$ and $q := q_0 \circ r$ for some $r \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$. Then $g(x) := f(x)^n$ is a minimal-degree element of $\mathcal{G}_1$ for p and q. Pick $\beta \in \mathbb{C}_\infty$ and suppose that T_β is not a minimal shared multiset for p and q. Then T_β contains precisely two distinct minimal shared multisets S_1 and S_2, at least one of which has size 1. *Moreover, f, p_0, q_0, β and the set $\mathcal{E}$ of Picard exceptional values of r appear in Table 3, after we compose with Möbius transformations ν, ρ, and $\mu = \alpha x^\epsilon$ with $\epsilon \in \{1, -1\}$ to replace $f, p_0, q_0, r, \beta, \zeta$ by $\mu \circ f \circ \nu^{-1}$, $\nu \circ p_0 \circ \rho$, $\nu \circ q_0 \circ \rho$, $\rho^{-1} \circ r$, $\alpha^n \beta^\epsilon$, ζ^ϵ, and also after possibly replacing p_0, q_0, ζ by $q_0, p_0, 1/\zeta$.*

TABLE 3. Solutions where the numerator of $f(x) - \zeta f(y)$ is irreducible

Case	f	p_0	q_0	ζ	β	$\mathcal{E}$	Conditions
1	$x^j(x-1)^k$	$\dfrac{\delta x^j - 1}{\delta x^{j+k} - 1}$	$x^k p_0(x)$	δ^k	0	$\{0, \infty\}$	$j, k > 0$ coprime; $\delta^k \neq 1$
2	$\dfrac{x^2 + 2\frac{\zeta+1}{\zeta-1}x + 1}{x}$	$\dfrac{x^2 + x}{x - \zeta}$	$\dfrac{\frac{1}{\zeta}x^2 - x}{x+1}$	ζ	∞	$\{-1, \zeta\}$	$\zeta \notin \{1, -1\}$
3	$\dfrac{(x+1)^2(x+6s+10)}{x}$	$\dfrac{-(s+1)x^2(x+1)}{x^2 - sx + 1}$	$\dfrac{p_0(x)}{x^3}$	-1	∞	$\{0, \infty\}$	$s^2 = 3$
4	$\dfrac{x^3 + 6x^2 + 3x - 1}{x}$	$\dfrac{\omega x^2(x+1)}{x^2 - (2\omega+1)x - 1}$	$\dfrac{-\omega p_0(x)}{x^3}$	ω	∞	$\{0, \infty\}$	$\omega^3 = 1 \neq \omega$
5	$\dfrac{(x^2 - 6x + 1)^2}{x}$	$\dfrac{x - 1}{x^4 + 2x^3 + 2x^2 + x}$	$-x^4 p_0(x)$	-1	∞	$\{0, \infty\}$	
6	x	x	ζx	ζ	1	$\{1, \zeta^k\}$	$k \in \mathbb{Z},\ \zeta^k \neq 1$
7	$2x^2 - 1$	$\dfrac{2x}{x^2 + 1}$	$\dfrac{x^2 - 1}{x^2 + 1}$	-1	1	$\{1, \infty\}$	

Proof. Since $f(p) = \zeta f(q)$ and the numerator of $f(x) - \zeta f(y)$ is irreducible, the last assertion in Proposition 4.4 implies that $f(x)$ is a minimal-degree element of $\mathcal{G}_2$. In light of this, Proposition 4.4 implies that $g(x) := f(x)^n$ is a minimal-degree element of $\mathcal{G}_1$. For $\beta \in \mathbb{C}_\infty$, define T_β as usual (with respect to $g(x)$), and suppose that T_β is not a minimal shared multiset for p and q. By Proposition 6.7, T_β is the union of one or more copies of each of two distinct minimal shared multisets S_1 and S_2, where $p_0^{-1}(S_i) \cap q_0^{-1}(S_{3-i})$ consists of one or more copies of a Picard exceptional value α_i for r, and $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ coincide except for copies of α_1 and α_2.

First suppose $\beta \in \{0, \infty\}$. Then for any $a \in S_1$ and $b \in S_2$ we have $f(a) = \beta = \zeta f(b)$, so by Lemma 8.4 there are $\gcd(e_f(a), e_f(b))$ distinct points in $p_0^{-1}(a) \cap q_0^{-1}(b)$. Since this multiset consists of copies of a single point, it follows that $\gcd(e_f(a), e_f(b)) = 1$ and that each of S_1 and S_2 consists of copies of a single point, which by minimality of S_i implies $S_1 = \{a\}$ and $S_2 = \{b\}$. By inspecting the list in [31] of all $f(x) \in \mathbb{C}(x)$ with $|f^{-1}(\infty)_{\text{set}}| = 2$ for which $f(x) = \zeta f(y)$ defines an irreducible curve of genus 0, we find that cases (1)–(5) in Table 3 include all possibilities.

Henceforth suppose $\beta \in \mathbb{C}^*$. Then $f(S_1) \cup f(S_2)$ consists of all n-th roots of β, so there exist $a \in S_1$ and $b \in S_2$ with $f(a) = \zeta f(b)$. For any such a, b, by Lemma 8.4 there are $\gcd(e_f(a), e_f(b))$ points in $p_0^{-1}(a) \cap q_0^{-1}(b)$; thus $\gcd(e_f(a), e_f(b)) = 1$ and $f(S_1) \cap \zeta f(S_2)$ consists of a single point w, where in addition a is the unique point in $f^{-1}(w) \cap S_1$ and b is the unique point in $f^{-1}(w/\zeta) \cap S_2$. Writing $k := \deg(f)$, we may assume $k > 1$, since if $k = 1$ then we obtain case (6) in Table 3.

Now suppose $f^{-1}(w)_{\text{set}} \neq \{a\}$. Then $f^{-1}(w)$ contains an element of S_2, so that $\zeta f(S_2)$ contains ζw, whence $f^{-1}(\zeta w)$ is contained in S_2. Continuing in this way shows that S_2 contains $f^{-1}(\zeta^i w)$ for $i = 1, 2, \dots, n-1$. Thus $(S_1)_{\text{set}} = \{a\}$, so minimality of S_1 implies $S_1 = \{a\}$; also $f^{-1}(w/\zeta)_{\text{set}} = \{b\}$. Moreover, for any $c \in f^{-1}(\zeta w)$ we have $c \in S_2$, and $p_0^{-1}(c) \cap q_0^{-1}(a) \neq \emptyset$ by Lemma 8.4; thus $f^{-1}(\zeta w)_{\text{set}} = \{c\}$. By Lemma 8.4, a has a unique p_0-preimage, and each point in $f^{-1}(w)$ has at most $k/2$ distinct p_0-preimages. It follows from Lemma 5.4 that $|f^{-1}(w)_{\text{set}}| \leq 2$. Since $e_f(a)$ is coprime to $k = \deg(f)$, and $k > 1$, we have $e_f(a) \neq k$, so that $f^{-1}(w)$ contains a unique element d distinct from a. Thus $e_f(d) = k - e_f(a)$ is coprime to k, so $|p_0^{-1}(d)|_{\text{set}} = 1$, which by Lemma 5.4 implies that every element of $\mathbb{C}_\infty \setminus \{a, d\}$ has k distinct p_0-preimages. Moreover, the ζf-preimages of ζw are a and d, and the f-preimage of ζw is c, so since $e_f(a)$ and $e_f(d)$ are coprime to $k = e_f(c)$ we see that c has precisely two p_0-preimages, whence $k = 2$. By Lemma 5.4, there is a unique $\gamma \in \mathbb{C}_\infty \setminus \{w/\zeta\}$ such that $|f^{-1}(\gamma)_{\text{set}}| = 1$, and Lemma 8.4 implies that that $|f^{-1}(\zeta\gamma)_{\text{set}}| = 1$, since otherwise each element of $f^{-1}(\zeta\gamma)$ would have a unique p_0-preimage. Thus $\gamma \in \{0, \infty\}$, so since also ζw has a unique f-preimage, we must have $\zeta w = w/\zeta$ and hence $n = 2$, yielding case 7 in Table 3.

Finally, suppose that $f^{-1}(w)_{\text{set}} = \{a\}$. Since $\gcd(e_f(a), e_f(b)) = 1$ and $\deg(f) > 1$, we must have $f^{-1}(w/\zeta)_{\text{set}} \neq \{b\}$. Thus upon replacing p_0, q_0, ζ by $q_0, p_0, 1/\zeta$, we reduce to the previous case. $\square$

Remark 8.17. The minimal shared multisets for p and q in the seven cases in Table 3 are as follows, where we recall that $g = f^n$ where $n > 1$ and ζ is a primitive n-th root of unity:

Table 4. Minimal shared multisets for p, q in Table 3

Case	Minimal shared multisets not of the form $g^{-1}(\alpha)$
1	$\{\infty\}$, $\{0\}$, $\{1\}$
2	$\{\infty\}$, $\{0\}$, $f^{-1}(0)$
3	$\{\infty\}$, $\{0\}$, $f^{-1}(0)$
4	$\{\infty\}$, $\{0\}$, $f^{-1}(0)$
5	$\{\infty\}$, $\{0\}$, {roots of $x^2 - 6x + 1$}
6	$\{\infty\}$, $\{0\}$, $\{\zeta, \zeta^2, \dots, \zeta^k\}$, $\{\zeta^{k+1}, \zeta^{k+2}, \dots, \zeta^n\}$
7	$\{\infty\}$, $\{1\}$, $f^{-1}(0)$, {roots of $f(x)^n - 1$ other than 1}

Remark 8.18. For $f \in \mathbb{C}(x) \setminus \mathbb{C}$ and $\zeta \in \mathbb{C}^*$, write

$$F_{f,\zeta}(x, y) := \begin{cases} f(x) - \zeta f(y) & \text{if } \zeta \neq 1 \\ \frac{f(x)-f(y)}{x-y}, & \text{if } \zeta = 1 \end{cases}$$

and let $G_{f,\zeta}(x, y)$ be the numerator of $F_{f,\zeta}(x, y)$. The papers [13] (for $\zeta = 1$) and [32] (for $\zeta \neq 1$) determine all pairs (f, ζ) for which $G_{f,\zeta}(x, y)$ is irreducible and satisfies $G_{f,\zeta}(p, q) = 0$ for some $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$. Conversely, for all such p and q, the minimal shared multisets for p and q are determined by the combination of the results of [13, 32] with Propositions 8.14, 8.16, and 6.7. We do not state this list here, since it involves many cases; but it can be read off at once from the results in [13, 32]. Finally, we note that for any pair (f, ζ) as above, it is a routine exercise to describe all corresponding p, q.

8.6. **Laurent polynomials.** We now treat the case that some point has at most two g-preimages. This is the most general class of rational functions $g(x)$ for which the solutions to $g \circ p = g \circ q$ with $p, q \in \mathcal{M}(\mathbb{C}^m) \setminus \mathbb{C}$ have been determined at present, other than the class in Remark 8.18 whose definition involves an irreducibility hypothesis. We note that, although the previous results in this section were elementary, the proof of the next result ultimately relies on the classification of finite simple groups.

Theorem 8.19. *Pick nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$ such that $h \circ p = h \circ q$ for some $h \in \mathbb{C}(x) \setminus \mathbb{C}$ such that some $\alpha \in \mathbb{C}_\infty$ has at most two distinct h-preimages. Let $g(x)$ be a minimal-degree nonconstant rational function for which $g \circ p = g \circ q$, and suppose there is some $\beta \in \mathbb{C}_\infty$ for which T_β is not a minimal shared multiset for p and q. Then, after replacing the triple (g, p, q) by $(\mu \circ g \circ \nu^{-1}, \nu \circ p, \nu \circ q)$ for suitable Möbius transformations μ and ν, we can write $p = p_0 \circ r$ and $q = q_0 \circ r$ for some $p_0, q_0 \in \mathbb{C}(x)$ and some $r \in \mathcal{M}(\mathbb{C}^m)$ with set of Picard exceptional values $\mathcal{E}$ such that one of the following occurs:*

(1) *one of the possibilities in Proposition 8.14 or Proposition 8.16*
(2) *one of the possibilities in Table 5, where all minimal shared multisets not of the form $g^{-1}(\alpha)$ are presented in Table 6.*

Table 5. Solutions where some element of $\mathbb{C}_\infty$ has at most two g-preimages

Case	g	p_0	q_0	β	$\mathcal{E}$
1	$\dfrac{\left(x^4+4x^3+2x-\frac{1}{4}\right)^2}{x^4}$	$\dfrac{x^2-2x+2}{x^3+2x^2+2x}$	$p_0\circ\dfrac{2}{x}$	∞	$\{\infty,0\}$
2				27	$\{t-1,t+1\}$
3	$\dfrac{\left(x+\frac{11-5s}{2}\right)^2(x^2+x-1)^4}{x^5}$	$\dfrac{x^2+x+\frac{3-s}{2}}{\frac{-3-s}{2}x^3+x^2-x}$	$p_0\circ\dfrac{3-s}{2x}$	∞	$\{\infty,0\}$
4				0	$\left\{-s+2,\dfrac{-s-1}{2}\right\}$
5	$\dfrac{(x+s-2)\left(x^3-x^2+\frac{s+1}{2}(x+1)\right)^3}{x^5}$	$\dfrac{\frac{s+1}{2}\left(x^3+x^2+\frac{3-s}{2}x\right)}{x^2+\frac{3-s}{2}(x+1)}$	$p_0\circ\dfrac{3-s}{2x}$	∞	$\{\infty,0\}$
6				0	$\left\{-1,\dfrac{s-3}{2}\right\}$
7	$T_3\circ\dfrac{x^2-6x+1}{8x}$	$A(x)$	$p_0\circ\dfrac{2\gamma}{x}$	1	E
8	$T_n(x)\circ\dfrac{x}{2}$	$x+\dfrac{1}{x}$	$\zeta x+\dfrac{1}{\zeta x}$	1	$\{\zeta^k,\zeta^{-k-1}\}$
9				-1	$\{\eta^{2k+1},\eta^{-2k-3}\}$

Here $t^2=3,\ s^2=5,\ A(x):=\dfrac{(2-\gamma)x(x+\gamma)(x+2\gamma)}{(x+1)(x+2)(x+\gamma+1)(3x+2\gamma+2)}$ where $\gamma=\theta^2$ and $\theta^4-\theta^2+1=0$, $E:=\{\theta^3-1,-\theta^3-\theta^2+\theta\}$, and $\zeta=\eta^2$ where η is a primitive $2n$-th root of unity with $n\geq 5$, and if n is odd then $0\leq k\leq\frac{n-3}{2}$ while if n is even then $0\leq k\leq\frac{n-2}{4}$.

Table 6. Minimal shared multisets for p, q from Table 5

Case	Minimal shared multisets not of the form $g^{-1}(\alpha)$
1	$\{\infty\},\ \{0\},\ \Big\{\text{roots of } x^4+4x^3+2x-\frac{1}{4}\Big\}$
2	$\Big\{\frac{3t-5}{2}\Big\},\ \Big\{\Big(\frac{t-1}{2}\Big)^{*3}, \Big(\frac{-t-1}{2}\Big)^{*3}, \frac{-3t-5}{2}\Big\},$ $\{\infty, 0\},\ \Big\{\text{roots of } x^4+4x^3+2x-\frac{1}{4}\Big\}$
3	$\{\infty\},\ \{0\},\ \Big\{\frac{5s-11}{2}, \Big(\frac{s-1}{2}\Big)^{*2}, \Big(\frac{-s-1}{2}\Big)^{*2}\Big\}$
4	$\Big\{\frac{5s-11}{2}\Big\},\ \Big\{\frac{s-1}{2}, \frac{-s-1}{2}\Big\},\ \{\infty, 0\}$
5	$\{\infty\},\ \{0\}$
6	$\{-s+2\},\ \Big\{\frac{-s+1}{2}, \frac{(ti+1)(s+1)}{4}, \frac{(-ti+1)(s+1)}{4}\Big\},\ \{\infty, 0\}$
7	$\{\infty, 0\}$, $\{\text{roots of } x^3+1-9(x^2+x)\}$, $\{7+\delta\}$, $\{1^{*4}, 7-\delta\}$
8	$\{\infty\}$, $\{2\} \cup \{(\zeta^j+\zeta^{-j})^{*2} \colon 1 \le j \le k\}$, and either S (if n odd) or $\{-2\} \cup S^2$ (if n even), where $S := \{\zeta^j+\zeta^{-j} \colon k+1 \le j \le \frac{n-1}{2}\}$
9	$\{\infty\}$, $\{\eta^{2j+1}+\eta^{-2j-1} \colon 0 \le j \le k\}$, $\{\eta^{2j+1}+\eta^{-2j-1} \colon k < j < n\}$

Note: γ^{*k} indicates k copies of γ; also $\delta := 4\theta^3 - 8\theta$

Proof. By Theorem 6.7, we have $p = p_0 \circ r$ and $q = q_0 \circ r$ for some $p_0, q_0 \in \mathbb{C}(x)$ and some $r \in \mathcal{M}(\mathbb{C}^m)$, and then $g \circ p_0 = g \circ q_0$. We may assume p_0 has the smallest possible degree. Also T_β contains precisely two distinct minimal shared multisets S_1 and S_2, where $p_0^{-1}(S_i)$ and $q_0^{-1}(S_i)$ coincide except for elements of the set $\mathcal{E}$ of Picard exceptional values of r, with each $p_0^{-1}(S_i) \cap q_0^{-1}(S_{3-i})$ containing an element of $\mathcal{E}$.

Case 1: First suppose the numerator of $(g(x)-g(y))/(x-y)$ is irreducible. Then the possibilities are described in Proposition 8.14.

Case 2: Next suppose $\mu(g(x)) = f(x)^n$ for some $f(x) \in \mathbb{C}(x)$, some Möbius transformation $\mu(x)$, and some integer $n > 1$, where in addition there is a primitive n-th root of unity ζ such that $f \circ p_0 = \zeta f \circ q_0$ and the numerator of $f(x) - \zeta f(y)$ is irreducible. Then the possibilities are described in Proposition 8.16.

Case 3: Now suppose that some point has a unique g-preimage, but neither of the previous cases applies. After composing with Möbius transformations as in the conclusion, by [4] we may assume that $g(x) = T_n(x/2)$ with $n \geq 5$, where $p_0(x) = x + 1/x$ and $q_0(x) = p_0(\zeta x)$ for some primitive n-th root of unity ζ. Thus $p_0^{-1}(S_i) = \zeta q_0^{-1}(S_i)$ for each i.

We must have $\beta \neq \infty$, since $T_\infty = \{\infty\}$ is a minimal shared multiset. Since $2g(p_0(x)) = x^n + x^{-n}$, it follows that $p_0^{-1}(S_1) \setminus q_0^{-1}(S_1)$ consists of one or more copies of a single element $\alpha \in \mathbb{C}^*$, where $\beta = (\alpha^n + \alpha^{-n})/2$.

If $\beta^2 \neq 1$ then $M := \{\alpha\gamma \colon \gamma^n = 1\}$ is disjoint from $N := \{1/\gamma \colon \gamma \in M\}$, and $p_0^{-1}(T_\beta) = M \cup N$. Since $p_0(N) = p_0(M) = T_\beta$, each S_i must contain an element of $p_0(N)$, so that $R := N \cap p_0^{-1}(S_1)$ is a nonempty proper subset of N. Since $\alpha \notin R$, we must have $R \subseteq q_0^{-1}(S_i)$, which equals $\zeta^{-1}R$. But $|\zeta^{-1}R| = |R|$, so that $R = \zeta^{-1}R$, contradicting the fact that R is a nonempty proper subset of N.

Thus we may assume $\beta^2 = 1$. Here $p_0^{-1}(T_\beta)_{\mathrm{set}} = \{\alpha\gamma \colon \gamma^n = 1\}$, so that $\mathcal{E} \setminus \{\alpha\} = \{\alpha\zeta^{-\ell}\}$ for some integer ℓ with $0 < \ell < n$. Writing $A := p_0^{-1}(S_1)_{\mathrm{set}}$, we have $q_0^{-1}(S_1)_{\mathrm{set}} = \zeta^{-1}A$, so that $A \setminus \zeta^{-1}A = \{\alpha\}$ and $\zeta^{-1}A \setminus A = \{\alpha\zeta^{-\ell}\}$. It follows that $A = \{\alpha\zeta^{-j} \colon 0 \leq j < \ell\}$, so $|A| = \ell$. Since $p_0 \circ x^{-1} = p_0$, the map $x \mapsto x^{-1}$ preserves the set A. Since $|A_{\mathrm{set}}| < n$, it follows that $\alpha^{-1} = \alpha\zeta^{1-\ell}$, so that $\zeta^\ell = \zeta\alpha^2$. If n is odd then, since g, p_0, q_0 are all preserved by conjugating by $-x$, we may assume $\beta = 1$, so that $\alpha = \zeta^k$ with $0 \leq k < n$. After interchanging the S_i's if needed, we may assume $1 \in S_1$, so that $\ell > k$, which together with the bounds on k and ℓ and the congruence $\ell \equiv 2k + 1 \pmod{n}$ yields $\ell = 2k + 1$. Thus $|A| = \ell = 2k + 1$, so since $|A| < n$ we get $k < (n-1)/2$. If n is even then let η be a fixed square root of ζ such that η is a primitive $2n$-th root of unity. By interchanging the S_i's if needed, we may assume $|A| \leq n/2$. Since the triple $(g(x), p_0(x), q_0(x))$ equals $(g(-x), -p_0(-x), -q_0(-x))$, we may replace the S_i's by their negatives if needed, in order to assume that $\alpha = \eta^j$ with $0 \leq j < n$. If j is even then $j = 2k$ where as above we have $\ell = 2k + 1$, so since $\ell = |A| \leq n/2$ we get $k \leq (n-2)/4$. Finally, if j is odd then $j = 2k+1$ where $\ell \equiv 2k+2 \pmod{n}$, so that $\ell = 2k + 2$ and thus $k \leq (n-4)/4$.

Case 4: Henceforth suppose that none of the previous three cases applies. By [31], after replacing g, p_0, q_0 by $\mu \circ g \circ \nu^{-1}$, $\nu \circ p_0$, $\nu \circ q_0$ for suitable Möbius transformations $\mu(x)$ and $\nu(x)$, one of the following occurs:

$$g(x) = \frac{(x^2+1)^3}{x^2} \tag{8.20}$$

$$g(x) = x^2 \circ \frac{x^4+4x^3+2x-\frac{1}{4}}{x^2} \tag{8.21}$$

$$g(x) = \frac{(2x+11-5s)^2(x^2+x-1)^4}{x^5} \quad \text{where } s^2=5 \tag{8.22}$$

$$g(x) = \frac{(x+s-2)\big(2(x^3-x^2)+(s+1)(x+1)\big)^3}{x^5} \quad \text{where } s^2=5 \tag{8.23}$$

$$\begin{gathered} g(x) = T_n \circ g_0 \ \text{ where } g_0(x) := \frac{(\zeta-1)^2(x^2+1)+2x(\zeta+1)^2}{8\zeta x} \\ \text{with } \zeta \text{ a primitive } (2n)\text{-th root of unity and } n \geq 3. \end{gathered} \tag{8.24}$$

If any of (8.20)–(8.23) holds then we determine all irreducible factors $H(x,y)$ of the numerator of $g(x)-g(y)$ such that the normalization of the curve $H(x,y)=0$ has genus 0 (and hence admits a parametrization by rational functions) and also $H(x,y)$ does not divide the numerator of $f(x)-f(y)$ for any $f \in \mathbb{C}(x) \setminus \mathbb{C}$ with $\deg(f) < \deg(g)$. For each such $H(x,y)$, we compute $p_0, q_0 \in \mathbb{C}(x)$ such that $H(p_0,q_0)=0$ and $\deg(p_0)$ equals the x-degree of $H(x,y)$. In each case we find that $q_0 = p_0 \circ \mu$ for some Möbius transformation $\mu(x)$ satisfying $\mu \circ \mu = x$. Also in each case, either Lemma 8.10 or Lemma 8.13 implies that β is a critical value of $g(x)$. We then exhaustively check all nonempty proper submultisets S of T_β for each critical value β of $g(x)$, in order to determine all such S which are shared by p and q. We find that the only possibilities are the first six cases in Table 6.

Finally, assume that (8.24) holds. By [31] we may assume that $p_0(x) = N(x)/D(x)$ where

$$N(x) := -(\zeta-1)^2 x\big(x^2+x(\zeta+1)^2+2\zeta^3+2\zeta\big)$$

and

$$\begin{aligned} D(x) := &\big((\zeta+1)(x^2+2\zeta^2+2)+(3\zeta^2+2\zeta+3)x\big) \\ &\cdot \big(\zeta x^2+(\zeta+1)^2 x+2(\zeta^2+1)\big) \end{aligned}$$

and $q_0 = p_0 \circ \mu$ where $\mu(x) := 2(\zeta^2+1)/x$. Here $\rho(x) := -(\zeta^2+1)(x+2\zeta)/(x+\zeta^2+1)$ satisfies $p_0 \circ \rho = p_0$ and $\rho \circ \rho = x = \mu \circ \mu$, and also the fourth iterate $(\mu \circ \rho)^{\circ 4}$ is x. Let α be the unique element of $\mathbb{C}_\infty$ contained in $p_0^{-1}(S_1) \setminus q_0^{-1}(S_1)$. Define $\alpha_2 := \rho(\alpha)$, $\alpha_3 := \rho(\mu(\alpha_2))$, and $\alpha_4 := \rho(\mu(\alpha_3))$.

First suppose $\alpha \neq \alpha_i$ for each $i \in \{2,3,4\}$. Since $p_0(\alpha_2) = p_0(\alpha)$, we have $\alpha_2 \in p_0^{-1}(S_1)$, so since $\alpha_2 \neq \alpha$ we get $\alpha_2 \in q_0^{-1}(S_1)$. Likewise $p_0(\alpha_3) = q_0(\alpha_2)$, so $\alpha_3 \in p_0^{-1}(S_1)$, whence also $\alpha_3 \in q_0^{-1}(S_1)$. Next, $p_0(\alpha_4) = q_0(\alpha_3)$, so $\alpha_4 \in p_0^{-1}(S_1)$, whence also $\alpha_4 \in q_0^{-1}(S_1)$. Then

$$\begin{aligned} q_0(\alpha_4) &= p_0(\mu(\alpha_4)) = p_0(\mu(\rho(\mu(\alpha_3)))) = p_0(\mu(\rho(\mu(\rho(\mu(\alpha_2)))))) \\ &= p_0(\mu(\rho(\mu(\rho(\mu(\rho(\alpha))))))) = p_0(\rho \circ (\mu \circ \rho)^{\circ 3}(\alpha)) = p_0(\mu(\alpha)) = q_0(\alpha), \end{aligned}$$

so $\alpha \in q_0^{-1}(S_1)$, contradiction.

If $\alpha = \alpha_i$ for some $i \in \{2,3,4\}$ then we will exhibit $\gamma, \delta, \epsilon \in \mathbb{C}_\infty$ such that

$$p_0(\alpha) = p_0(\gamma),\ q_0(\gamma) = p_0(\delta),\ q_0(\delta) = p_0(\epsilon), \text{ and } q_0(\epsilon) = q_0(\alpha). \tag{8.25}$$

In case

$$\alpha \notin \{\gamma, \delta, \epsilon\}, \tag{8.26}$$

it follows that $p_0^{-1}(S_1) \cap q_0^{-1}(S_1)$ contains each of γ, δ, ϵ, which yields the contradiction $\alpha \in q_0^{-1}(S_1)$.

If $\alpha_2 = \alpha$ then $\alpha^2 + 2(\zeta^2+1)(\alpha+\zeta) = 0$. Let γ be a root of

$$\zeta^3 x^2/2 + (\alpha + \zeta^4 + 2\zeta^2 - \zeta + 2)x + (\zeta^2+1)\alpha + 2\zeta^4 - \zeta^3 + 4\zeta^2 - \zeta + 2,$$

put $\epsilon := -\alpha - 2\zeta^2 - 2$, and let $\delta \in \mathbb{C}$ satisfy

$$\begin{aligned}\delta(1-\zeta) :=& ((\zeta^2+\zeta)\alpha + 2\zeta^4 + \zeta^3 + \zeta^2)\gamma + (2\zeta^3 + 2\zeta)\alpha \\ &+ 4\zeta^5 - 2\zeta^4 + 8\zeta^3 - 4\zeta^2 + 4\zeta - 2.\end{aligned}$$

These satisfy (8.25). If $n > 3$ then (8.26) holds, which we already know is impossible. If $n = 3$ then a routine computation shows that in each example there is a square root θ of ζ such that the S_i's are $\{4\theta^3 - 8\theta + 7\}$ and $\{1^{*4}, -4\theta^3 + 8\theta + 7\}$ with $\mathcal{E} = \{\theta^3 - 1, -\theta^3 - \theta^2 + \theta\}$.

If $\alpha_3 = \alpha$ then $\alpha^2 + 2(\zeta^2+1)(1 + 2\alpha/(\zeta+1)) = 0$. Let γ be a root of

$$\begin{aligned}&x^2(\zeta^3 - \zeta)/2 + ((\zeta+1)\alpha + \zeta^4 + 2\zeta^2 - 2\zeta + 3)x + (\zeta^3 + \zeta^2 + \zeta + 1)\alpha \\ &\quad + 3\zeta^4 - 2\zeta^3 + 6\zeta^2 - 2\zeta + 3,\end{aligned}$$

put $\epsilon := ((\zeta+1)\alpha + 2\zeta^2 + 2)/(\zeta - 1)$, and let $\delta \in \mathbb{C}$ satisfy

$$\begin{aligned}\delta(\zeta+1)(\zeta-1)^2 =& \gamma(\zeta+1)\big((\zeta^2+\zeta)\alpha + 3\zeta^3 + \zeta\big) \\ &+ 2(\zeta^3+\zeta)(\zeta+1)\alpha + 2(3\zeta^5 - \zeta^4 + 6\zeta^3 - 2\zeta^2 + 3\zeta - 1).\end{aligned}$$

These satisfy (8.25) and (8.26), which we know is impossible.

Finally, assume that $\alpha_4 = \alpha$. Then $\alpha^2 + 2(\zeta + \zeta^{-1})(\alpha + 1) = 0$. Let δ be a root of

$$x^2\zeta^3/2 + (\zeta\alpha + \zeta^4 + 2\zeta^2 - \zeta + 2)x + (\zeta^3 + \zeta)\alpha + 2\zeta^4 - \zeta^3 + 4\zeta^2 - \zeta + 2$$

put $\gamma := -\alpha - 2(\zeta + \zeta^{-1})$, and let ϵ satisfy

$$\begin{aligned}\epsilon(1-\zeta) =& \delta\big((\zeta^3+\zeta^2)\alpha + 2\zeta^4 + \zeta^3 + \zeta^2\big) \\ &+ 2(\zeta^4+\zeta^2)\alpha + 2(2\zeta^5 - \zeta^4 + 4\zeta^3 - 2\zeta^2 + 2\zeta - 1).\end{aligned}$$

Once again, these satisfy (8.25) and (8.26), which is impossible. $\square$

8.7. **Two points.** We now classify the meromorphic p, q which share two points in addition to two other sets.

Theorem 8.27. *Pick distinct nonconstant $p, q \in \mathcal{M}(\mathbb{C}^m)$, and suppose that p and q share CM four disjoint nonempty finite multisets $S_1, \ldots, S_4$ of elements of $\mathbb{C}_\infty$, where at least two of the S_i's have size 1. Then, after replacing p and q by $\mu \circ p$ and $\mu \circ q$ for a suitable Möbius transformation $\mu(x)$, there exist $g, p_0, q_0 \in \mathbb{C}(x) \setminus \mathbb{C}$ and $r \in \mathcal{M}(\mathbb{C}^m)$ such that $p = p_0 \circ r$, $q = q_0 \circ r$, and $g \circ p_0 = g \circ q_0$, where g, p_0, q_0 and the set $\mathcal{E}$ of Picard exceptional values of r are one of the following:*

(8.27.1) *One of the three cases in Table 1*
(8.27.2) *One of the seven cases in Table 3, where $g = f^n$ with n being the multiplicative order of ζ*
(8.27.3) *Cases 1, 3, or 5 in Table 5*
(8.27.4) *$g(x) = x^n$ with $n > 1$, $p_0(x) = x$, and $q_0(x) = \zeta x$ where ζ is a primitive n-th root of unity, and either $|\mathcal{E}| < 2$ or $\mathcal{E} = \{\beta, \gamma\}$ with $\beta^n \neq \gamma^n$*
(8.27.5) *$g(x) = T_n(x/2)$ with $n \geq 5$, $p_0(x) = x + 1/x$, $q_0(x) = \zeta x + 1/(\zeta x)$ where ζ is a primitive n-th root of unity, and $\mathcal{E}$ is either $\{1, 1/\zeta\}$ or $\{\delta, 1/\delta^3\}$ with $\delta^2 = \zeta$ and $\delta^n = -1$.*

In cases (8.27.1), (8.27.2) and (8.27.3) the minimal shared multisets for p and q which do not have the form $g^{-1}(\alpha)$ with $\alpha \in \mathbb{C}_\infty$ are listed in Table 2, 4 or 6, respectively. In case (8.27.4) the only such minimal shared multisets are $\{\infty\}$ and $\{0\}$. In case (8.27.5) the only such minimal shared multisets are listed in Table 7.

Table 7. Minimal shared multisets for case (8.27.5)

$\mathcal{E}$	Condition	Minimal shared multisets not of the form $g^{-1}(\alpha)$
$\left\{1, \frac{1}{\zeta}\right\}$	n odd	$\{\infty\}$, $\{2\}$, $\left\{\zeta^k + \frac{1}{\zeta^k} : 1 \leq k \leq \frac{n-1}{2}\right\}$
	n even	$\{\infty\}$, $\{2\}$, $\{-2\} \cup \left\{\left(\zeta^k + \frac{1}{\zeta^k}\right)^{*2} : 1 \leq k \leq \frac{n-2}{2}\right\}$
$\left\{\delta, \frac{1}{\delta^3}\right\}$	n odd	$\{\infty\}$, $\left\{\delta + \frac{1}{\delta}\right\}$, $\{-2\} \cup \left\{\left(\delta^k + \frac{1}{\delta^k}\right)^{*2} : 3 \leq k \leq n-2,\ k \text{ odd}\right\}$
	n even	$\{\infty\}$, $\left\{\delta + \frac{1}{\delta}\right\}$, $\left\{\delta^k + \frac{1}{\delta^k} : 3 \leq k \leq n-1,\ k \text{ odd}\right\}$

Proof. By Theorem 3.1, there is some $g \in \mathbb{C}(x) \setminus \mathbb{C}$ for which $g \circ p = g \circ q$. Choose one such $g(x)$ having the smallest possible degree. If two points in $\mathbb{C}_\infty$ each have a unique g-preimage then, after replacing g, p, q by $\mu \circ p \circ \nu^{-1}$, $\nu \circ p$, and $\nu \circ q$ for suitable Möbius transformations $\mu(x)$ and $\nu(x)$, we may assume that $g(x) = x^n$ for some positive integer n, which yields (8.27.4). Henceforth assume that at most one point in $\mathbb{C}_\infty$ has a unique g-preimage. Then Theorem 6.7.2 implies there is some $\beta \in \mathbb{C}_\infty$ for which T_β is not a minimal shared multiset for p and q. Also Theorem 6.7.2 implies that some $\alpha \in \mathbb{C}_\infty$ has at most two distinct g-preimages, so the hypotheses of Theorem 8.19 are satisfied, and the present result follows from Theorem 8.19. $\square$

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