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Properties to Determine Inscribed Ellipses of Polygons

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Abstract

In this paper, we extend the result of [1] by calculating some examples in detail, including the inscribed ellipses in triangles, quadrilaterals, and pentagons. We also refine the original proof and reduce the requirements through projective geometry methods in the quadrilateral and pentagon cases. Furthermore, we see the inscribed ellipse problems from the perspective of two projective planes simultaneously, which offers a new way to determine the inscribed ellipses in triangles.

Keywords: Projective Geometry; Inscribed Ellipse.

Contents

1 Introduction

The ellipses is an important component of high school geometry and we often use Geogebra, a drawing tool, to visualize ellipses. However, when we try to draw an inscribed ellipse of a polygon, we can only draw the ellipse first by assigning five distinct points and then draw the subscribe polygon. We can not assign the polygon first. This paper aim to investigate the problem of inscribed ellipse from the view of tangent line by using projective geometry methods[3] [4] and technics [2].

1.1 Homogenous Coordinate of Points and Lines in Projective Plane

Projective Plane is the extension of Euclidean Plane. If we add infinite points and infinite line to the Euclidean Plane, we will get a Projective Plane. Each group of parallel lines in the Projective Plane is defined to meet at a unique infinite point. All the infinite points will compose the infinite line.

Let R be the field of real number and R^2 will stands for Euclidean plane. Let P^2 be the real projective plane. For each point (x, y) in R^2, we associate it with its homogenous point $[x : y : 1]$ in the P^2. For each line $ax + by + c = 0$ in R^2, we associate it with its homogeneous line $ax + by + ch = 0$ in P^2. On line $ax + by + ch = 0$ lies point $[-b : a : 0]$, which is the infinite point of this line. All the infinite points lie on line $h = 0$ at infinity. Since $\forall k \in Z$ and $k \neq 0$, $kax + kby + kch = 0$ represents the same line as $ax + by + ch = 0$. We can represent a line using its homogenous coordinate $[a : b : 1]$.

1.2 Duality

We can set up a unique dual relationship between the point Q on $xyh-$plane and the line L_Q on $\alpha\beta\gamma-$plane.

$$Q = [x : y : h] \iff L_Q = Q \cdot (\alpha, \beta, \gamma)$$

Similarly, there is a unique dual relationship between the line L_P on $xyh-$plane and the point P on $\alpha\beta\gamma-$plane.

$$L_P = P \cdot (x, y, h) \iff P = [\alpha : \beta : \gamma]$$

Notice that we can get the coordinate of a point by finding the partial derivative of the line: $\nabla L_P = P$ and $\nabla L_Q = Q$. The dual of a homogenous line can be seen as its gradient.

Therefore, for a homogenous curve $\varphi(\alpha, \beta, \gamma)$, we can define its homogenous dual curve: $\hat{\varphi}(x, y, z)$ as

$$\hat{\varphi} = \{[x : y : z] | \exists(\alpha, \beta, \gamma) \in \varphi, \text{such that } (x, y, z) \cdot (\alpha, \beta, \gamma) = 0\}$$

So the homogenous coordinate of $\hat{\varphi}$ is same as $\nabla\varphi$. Therefore, we can get $\hat{\varphi}$ by calculating $\nabla\varphi$.

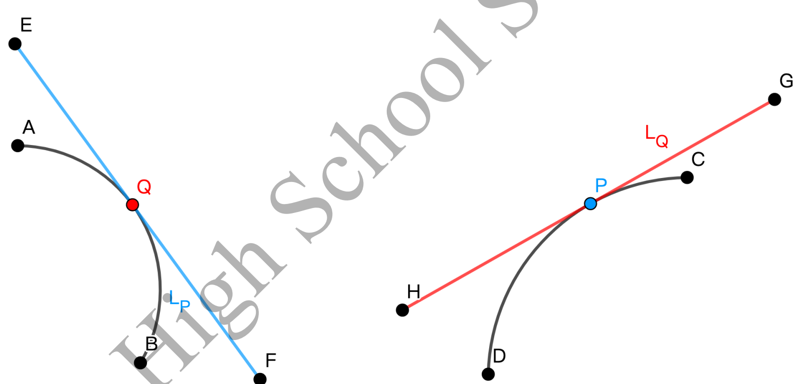

Figure 1: Explanation of Duality

1.3 Conic Curve

Since in this paper we mainly focus on conic curves, this section will introduce some basic knowledge of a conic curve.

Definition 1 *The collections of points* (x_1, x_2, x_3) *that satisfy* $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 = 0$ *are known as conic curves. Here* $a_{ij}(1 \leq i < j \leq 3)$ *are real numbers.*

The conic curves can also be represented as

$$F(x_1, x_2, x_3) = (x_1, x_2, x_3) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We often write $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$ as A.

1.3.1 Tangent Line of Conic Curve

Let point $P(p_1, p_2, p_3)$ be a point on conic curve

$$S : (x_1, x_2, x_3)A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Then the equation for the tangent line at P is

$$(p_1, p_2, p_3)A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

2 Duality and Inscribed Ellipses

2.1 Notations and Linfield's Function

We will use the homogenous coordinate of points and lines defined in Chapter 1 for later calculation. Here we define homogenous curve $\varphi = \varphi(\alpha, \beta, \gamma) \in P^2$ and its homogenous dual curve $\hat{\varphi} = \hat{\varphi}(x, y, h) \in P^2$.

We want to find the inscribed ellipse in polygon $Q_1Q_2Q_3 \cdots Q_n$. The main idea of the method is to find the ellipse φ that pass the dual points of the sides of the polygon $Q_1Q_2Q_3 \cdots Q_n$: P_{ij}, where $1 \leq i < j \leq n$. Then, according to definition of dual curve, the dual curve $\hat{\varphi}$ must be tangent to the sides of the polygon.

In this paper, we will use Linfield's function [1] as a way to represent φ:

$$\varphi = \sum_{i=1}^{n} m_i L_1 L_2 \cdots L_{i-1} L_{i+1} \cdots L_n$$

Here L_i is the dual of Q_i, and m_i is the positive real constant coefficients that lies in $(0,1)$. Since P_{ij} is the intersection of L_i, L_j and every term in φ contains at least one of L_i and L_j, then φ passes all P_{ij}, where $1 \le i < j \le n$. According to the property of duality, we can know that $\hat{\varphi}$ is tangent to polygon $Q_1Q_2Q_3 \cdots Q_n$, and the tangent points of $\hat{\varphi}$ depend on the tangent lines of φ at P_{ij}. This means the tangent points of $\hat{\varphi}$ can be determined using m_i and $Q_i (1 \le i \le n)$. Let the tangent point on Q_iQ_j be Q_{ij}. To get the homogenous coordinate of the tangent point of $\hat{\varphi}$, we just need to find the homogenous coordinate of the tangent line of φ. We can write $\varphi = (m_iL_j + m_jL_i)X + L_iL_jY$, where X, Y are products of polynomials.

$$\nabla\varphi(P_{ij}) = (m_iQ_j + m_jQ_i)X(P_{ij})$$

Then, we normalize the equation, and get $\nabla\varphi(P_{ij}) = \frac{m_i}{m_i+m_j}Q_j + \frac{m_j}{m_i+m_j}Q_i$.

Therefore, with the information of Q_i and m_i, we can get the inscribed ellipses $\hat{\varphi}$ using the Linfield Function. (Shown in Figure 2)

In this paper we will use the Linfield's function and methods in projective geometry to extend the following theorem in [1]:

Theorem 1 *Ellipses inscribed in convex non -degenerated n-gons:*
(1) In triangles, there exists a two-parameter family of inscribed ellipses.
(2) In quadrilaterals, there exists a one-parameter family of inscribed ellipses.
(3) In pentagons, there exists a zero-parameter family of inscribed ellipse.
(4) For $n \ge 6$, if there exists inscribed ellipse, it is unique.

Also we will refine the proofs provided in [1].

2.2 In triangles, there exists a unique two-parameter family of inscribed ellipses.

Let T denote the triangle with vertices Q_1, Q_2, Q_3. Using the Linfield's function, we can get a formula for φ

$$\varphi = m_1L_2L_3 + m_2L_1L_3 + m_3L_1L_2$$

To set the three unknown coefficients m_1, m_2, m_3, we need to fix two parameters $0 < r, s < 0$. So that

$$\frac{m_1}{m_2} = \frac{r}{1-r}, \frac{m_2}{m_3} = \frac{s}{1-s}$$

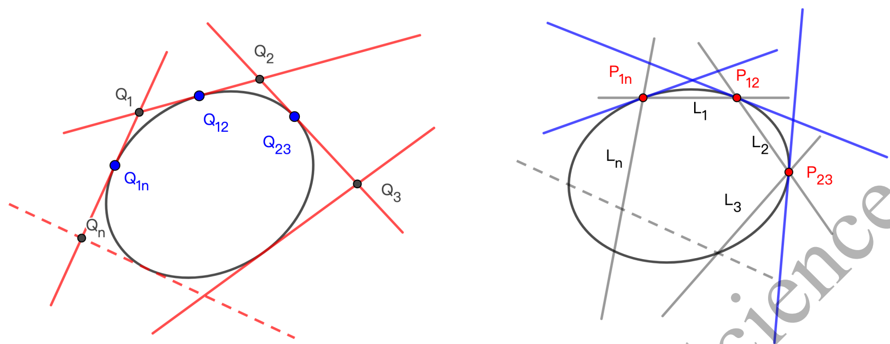

Figure 2: Explanation of Two Dual Plane

Since we just concern about the ratio , we can set $m_2 = 1$. Then, $m_1 = \frac{r}{1-r}, m_3 = \frac{1-s}{s}$. So

$$\varphi = \frac{r}{1-r}L_2L_3 + L_1L_3 + \frac{1-s}{s}L_1L_2$$

We fix r, s by fixing the tangent point on T. Let the points at which $\hat{\varphi}$ tangent to T be Q_{12} (on side Q_1Q_2) and Q_{23} (on side Q_2Q_3). $Q_{12} = (1-r)Q_1 + rQ_2, Q_{23} = (1-s)Q_2 + sQ_3$. Since φ is quadratic, $\hat{\varphi}$ is also quadratic, and it tangent to T at all three sides. As r, s are all changeable parameters, the inscribed ellipses form a two-parameter family.(Shown in Figure 3)

Because the inscribed ellipses depend on two parameters, then we can set up the relationship between these two parameters by letting the ellipses pass a certain point. Then we will get a unique family of one-parameter ellipse that is inscribed in the triangle T. This part will show explicitly in 2.2.2.

2.2.1 An Example of Triangle Case

We can set $Q_1 = [-1 : 0 : 1], Q_2 = [1 : 0 : 1], Q_3 = [0 : 1 : 1]$, and Linfield's function is $\varphi(\alpha, \beta, \gamma) = m_3(-\alpha+\gamma)(\alpha+\gamma) + m_2(-\alpha+\gamma)(\beta+\gamma) + m_1(\alpha+\gamma)(\beta+\gamma)$. Denote (x, y, h) as a point on $\hat{\varphi}$, so

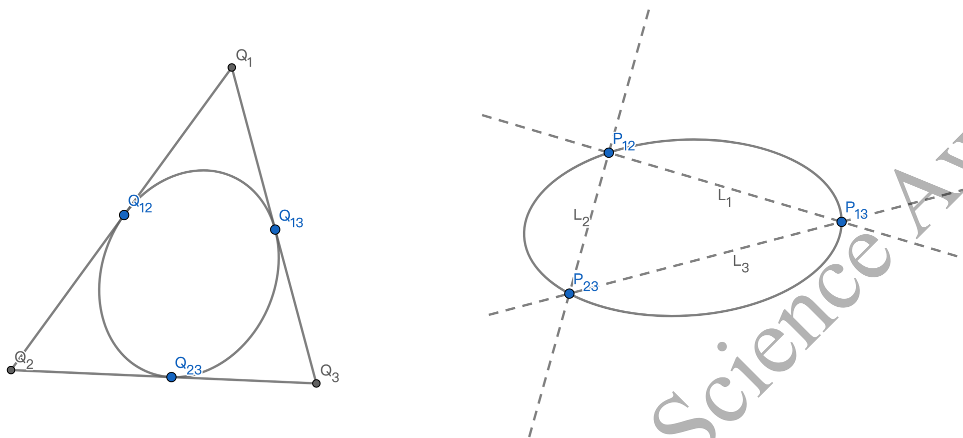

Figure 3: Triangle Case

$(x, y, h) = \nabla\varphi(\alpha, \beta, \gamma)$. Therefore, we can get

$$\begin{aligned} x =& -2m_3\alpha + (m_1 - m_2)\beta + (m_1 - m_2)\gamma \\ y =& (m_1 - m_2)\alpha + (m_1 + m_2)\gamma \\ h =& (m_1 - m_2)\alpha + (m_1 + m_2)\beta + 2(m_1 + m_2 + m_3)\gamma \end{aligned}$$

Solving the equations above, we can get

$$\begin{aligned} 8m_1m_2m_3\alpha =& -(m_1 + m_2)^2x - (m_1^2 - m_2^2 + 2m_3(m_1 - m_2))y + (m_1^2 - m_2^2)h \\ 8m_1m_2m_3\beta =& (m_1^2 - m_2^2 + 2m_1m_3 - 2m_2m_3)x - (m_1^2 + m_2^2 + 4m_3^2 - 2m_1m_2 + \\ & 4m_1m_3 + 4m_2m_3)y + (m_1^2 + m_2^2 - 2m_1m_2 + 2m_1m_3 + 2m_2m_3)h \\ 8m_1m_2m_3\gamma =& (m_1^2 - m_2^2)x + (m_1^2 + m_2^2 - 2m_1m_2 + 2m_1m_3 + 2m_2m_3)y - \\ & (m_1^2 + m_2^2 - 2m_1m_2)h \end{aligned}$$

Because $m_1m_2m_3 \neq 0$, the equations can be simplified to be

$$\begin{aligned} \alpha =& -(m_1 + m_2)^2x + (m_1^2 - m_2^2 + 2m_1m_3 - 2m_2m_3))y - (m_1^2 - m_2^2)h \\ \beta =& (m_1^2 - m_2^2 + 2m_1m_3 - 2m_2m_3)x - (m_1^2 + m_2^2 + +4m_3^2 - 2m_1m_2 + \\ & 4m_1m_3 + 4m_2m_3)y + (m_1^2 + m_2^2 - 2m_1m_2 + 2m_1m_3 + 2m_2m_3)h \\ \gamma =& (m_1^2 - m_2^2)x + (m_1^2 + m_2^2 - 2m_1m_2 + 2m_1m_3 + 2m_2m_3)y - (m_1^2 + m_2^2 - 2m_1m_2)h \end{aligned}$$

Substitute these values into φ, we can get $\hat{\varphi}$

$$
\begin{aligned}
\hat{\varphi} = & -4m_1m_2m_3(h^2(m_1-m_2)^2+m_1^2(x+y)^2+2m_1(x+y)(m_2(x-y)+2m_3y) \\
& +(2m_3y+m_2(-x+y))^2-2h(2m_1(-m_2+m_3)y+m_1^2(x+y)+m_2(-m_2x \\
& +m_2y+2m_3y)))
\end{aligned}
$$

De-homogenize the formula we can get

$$
\begin{aligned}
\hat{\varphi} = & -4m_1m_2m_3((m_1-m_2)^2+m_1^2(x+y)^2+2m_1(x+y)(m_2(x-y)+2m_3y) \\
& +(2m_3y+m_2(-x+y))^2-2(2m_1(-m_2+m_3)y+m_1^2(x+y)+m_2(-m_2x \\
& +m_2y+2m_3y)))
\end{aligned}
$$

Substitute m_1 for $\frac{r}{1-r}$, m_2 for 1, m_3 for $\frac{1-s}{s}$.

$$
\begin{aligned}
\hat{\varphi} = & -\frac{1}{(-1+r)^3s^3}4r(-1+s)(4(-1+r)^2y^2-4(-1+r)sy(-1+(-1+2r)x+(-1+2r)y) \\
& +s^2((1+x+y)^2-4r(1+x-y+2xy+2y^2)+r^2(4+8(-1+x)y+8y^2)))
\end{aligned}
$$

Setting $(r,s)=(\frac{2}{7},\frac{1}{4}),(\frac{1}{2},\frac{1}{3}),(\frac{2}{3},\frac{3}{5})$, we can get the flowing picture, which we can see that the ellipse $\hat{\varphi}$ are tangent to the triangle. (Shown in Figure 4: Ellipse1:$(r,s)=(\frac{2}{7},\frac{1}{4})$, Ellipse2:$(\frac{1}{2},\frac{1}{3})$, Ellipse3:$(\frac{2}{3},\frac{3}{5})$)

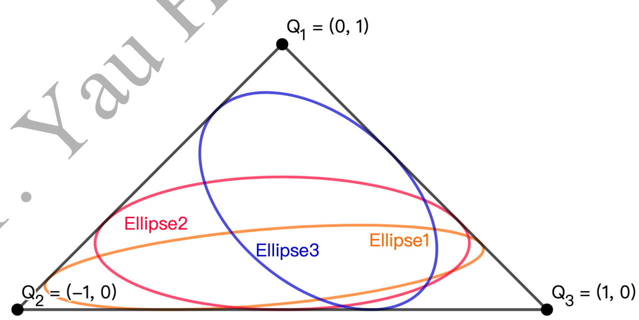

Figure 4: The Example of Triangle Case

2.2.2 The Extension of Triangle Case Example

There are still two possible family of ellipse in the case of triangle. Since the constraints we have put on the ellipse is about the tangent points, we now try to add constraints about fix points that the ellipse passes through. This will involve considering two projective planes simultaneously.

Let the ellipse $\hat{\varphi}$ in the example pass $[0:\frac{1}{2}:1]$. Then we can get

$$\hat{\varphi}_{[0:\frac{1}{2}:1]} = -\frac{1}{(-1+r)^3s^3}4r(-1+s)(4(-1+r)^2(\frac{1}{2})^2 - 4(-1+r)s(\frac{1}{2})(-1+(-1+2r)(\frac{1}{2})) + s^2((1+\frac{1}{2})^2 - 4r(1-\frac{1}{2}+2(\frac{1}{2})^2) + r^2(4-8(\frac{1}{2})+8(\frac{1}{2})^2))) = 0$$

Simplify it, we will get

$$-\frac{1}{(-1+r)^3s^3}r(-1+s)(4-8r+4r^2-12s+20rs-8r^2s+9s^2-16rs^2+8r^2s^2) = 0$$

From this equation, we can get a relationship between r and s.

$$s = \frac{2\left(2r^2-5r+3-2\sqrt{-r^4+3r^3-3r^2+r}\right)}{8r^2-16r+9}$$

or

$$s = \frac{2\left(2r^2-5r+3+2\sqrt{-r^4+3r^3-3r^2+r}\right)}{8r^2-16r+9}$$

Then we can reduce the original expression of $\hat{\varphi}$ into an expression that only relies on one parameter r.

When $s = \frac{2\left(2r^2-5r+3-2\sqrt{-r^4+3r^3-3r^2+r}\right)}{8r^2-16r+9}$, plug in $r = \frac{1}{3}$, and we can get the the ellipse:

$$\frac{1}{8}(-9(2\sqrt{2}+3)x^2 + 2x(2(9\sqrt{2}+13)y - 6\sqrt{2} - 9) - (2y-1)(6(12\sqrt{2}+17)y - 2\sqrt{2} - 3)) = 0$$

This is shown in Figure 5 (Ellipse 1).

Similarly, when $s = \frac{2\left(2r^2-5r+3+2\sqrt{-r^4+3r^3-3r^2+r}\right)}{8r^2-16r+9}$, plug in $r = \frac{1}{3}$, and we can get the the ellipse:

$$\frac{1}{8}(9(2\sqrt{2}-3)x^2 - 2x(2(9\sqrt{2}-13)y - 6\sqrt{2} + 9) + (2y-1)(6(12\sqrt{2}-17)y - 2\sqrt{2} + 3)) = 0$$

This is shown in Figure 5 (Ellipse 2).

From the figure we can see that the two ellipses share one same tangent point. The reason behind this phenomenon is a left question to be discussed.

Actually, we can make $\hat{\varphi}$ to pass another point to determine the value of r, but we can not ensure that there is always a real solution to the equation. However, there will always be an ellipse that is tangent to three non-parallel lines and pass two distinct points in the complex plane.

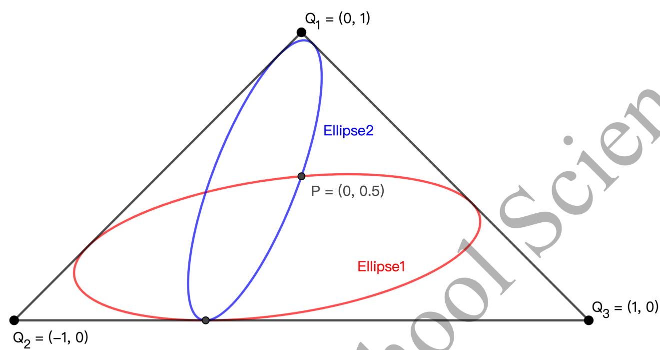

Figure 5: The Extension of Triangle Case Example

2.3 In quadrilaterals, there exists a unique one-parameter family of inscribed ellipses.

Let Q denote the quadrilateral with vertices Q_1, Q_2, Q_3, Q_4. Using the Linfield's function, we can get a formula for φ

$$\varphi = m_1L_2L_3L_4 + m_2L_1L_3L_4 + m_3L_1L_2L_4 + m_4L_1L_2L_3$$

WLOG, we can assume that the intersection of diagonals Q_2Q_4, Q_1Q_3 is the origin.(Figure 6) Therefore, we will have two constraint

$$(1-\theta)Q_1 + \theta Q_3 = [0:0:1], (1-\phi)Q_2 + \phi Q_4 = [0:0:1] \quad (1)$$

where $0 < \theta, \phi < 1$.

So we just need to fix one parameter $0 < r < 0$, and

$$\frac{m_1}{m_2} = \frac{r}{1-r}, \frac{m_2}{m_4} = \frac{\phi}{1-\phi}, \frac{m_1}{m_3} = \frac{\theta}{1-\theta}$$

Let $m_2 = 1$, we can get $m_1 = \frac{r}{1-r}, m_3 = \frac{r(1-\theta)}{\theta(1-r)}, m_4 = \frac{1-\phi}{\phi}$.

We can write out the dual of the constraints in equation 1

$$(1-\theta)L_1 + \theta L_3 = \gamma, (1-\phi)L_2 + \phi L_4 = \gamma$$

Then we can represent L_2, L_3 using L_4, L_1

$$L_3 = \frac{\gamma - (1-\theta)L_1}{\theta}, L_2 = \frac{\gamma - \phi L_4}{1-\phi}$$

Therefore,

$$\varphi = (m_2L_4 + m_4L_2)L_1L_3 + (m_1L_3 + m_3L_1)L_4L_2 = \frac{\gamma}{\phi}L_1L_3 + \frac{r\gamma}{\theta(1-r)}L_4L_2$$

Then, $\varphi = \frac{\gamma}{(1-r)\theta\phi}((1-r)\theta L_1L_3 + r\phi L_2L_4)$. The dual of the first part $\frac{\gamma}{(1-r)\theta\phi}$ is the origin and the dual of the second part $(1-r)\theta L_1L_3 + r\phi L_2L_4$ is an ellipse that tangent to the four sides of Q from the interior.

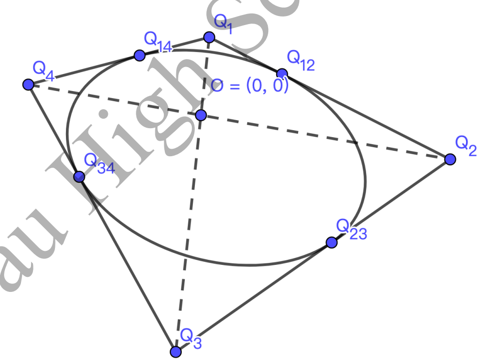

Figure 6: Quadrilateral Case

2.3.1 An Example of Quadrilateral Case

Let the original quadrilateral be $A_1A_2A_3A_4$, with $A_1 = [2:2:1], A_2 = [3:1:1], A_3 = [0:0:1], A_4 = [0:1:1]$. In order to put the intersection of the diagonals to the origin, we translate the quadrilateral into $Q_1Q_2Q_3Q_4$, with $Q_1 = [1:1:1], Q_2 = [2:0:$

$1], Q_3 = [-1 : -1 : 1], Q_4 = [-1 : 0 : 1]$.Then $\theta = \frac{1}{2}, \phi = \frac{2}{3}$. Suppose $m_2 = 1$, we can get $m_1 = \frac{r}{1-r}, m_2 = 1, m_3 = \frac{r}{1-r} m_4 = \frac{1}{2}$.

$$\begin{aligned}\varphi =&(m_2 L_4 + m_4 L_2) L_1 L_3 + (m_1 L_3 + m_3 L_1) L_4 L_2 = \frac{3}{2}\gamma L_1 L_3 + \frac{2r}{1-r}\gamma L_4 L_2 \\ =&\frac{3\gamma}{1-r}(\frac{1}{2}(1-r) L_1 L_3 + \frac{2}{3} r L_4 L_2)\end{aligned}$$

So the quadratic part of is

$$\varphi = \frac{1}{2}(1-r)(\alpha+\beta+\gamma)(-\alpha-\beta+\gamma) + \frac{2}{3} r(-\alpha+\gamma)(2\alpha+\gamma)$$

Then,

$$\varphi = -\frac{\alpha^2}{2} - \alpha\beta - \frac{\beta^2}{2} + \frac{\gamma^2}{2} - \frac{5\alpha^2 r}{6} + \alpha\beta r + \frac{2\alpha\gamma r}{3} + \frac{\beta^2 r}{2} + \frac{\gamma^2 r}{6}$$

Denote (x, y, h) as a point on $\hat{\varphi}$。Because $\hat{\varphi} = \nabla\varphi$, then $(x, y, h) = \nabla\varphi(\alpha, \beta, \gamma)$. As a result, we can get

$$\begin{aligned} x =& -\frac{1}{3}\alpha(5r+3) + \beta(r-1) + \frac{2\gamma r}{3} \\ y =& (r-1)(\alpha+\beta) \\ h =& \frac{2r}{3}\alpha + \frac{3+r}{3}\gamma \end{aligned}$$

As a result,

$$\begin{pmatrix} x \\ y \\ h \end{pmatrix} = \begin{pmatrix} -\frac{3+5r}{3} & ,-1+r & ,\frac{2r}{3} \\ -1+r & ,-1+r & ,0 \\ \frac{2r}{3} & ,0 & ,\frac{3+r}{3} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Solving this matrix ,we can get

$$\begin{aligned} (-\frac{4r^3}{3} - \frac{4r^2}{3} + \frac{8r}{3})\alpha =& (\frac{r^2}{3} + \frac{2r}{3} - 1)x + (-\frac{r^2}{3} - \frac{2r}{3} + 1)y + (-\frac{2r^2}{3} + \frac{2r}{3})h \\ (-\frac{4r^3}{3} - \frac{4r^2}{3} + \frac{8r}{3})\beta =& (-\frac{r^2}{3} - \frac{2r}{3} + 1)x + (-r^2 - 2r - 1)y + (\frac{2r^2}{3} - \frac{2r}{3})h \\ (-\frac{4r^3}{3} - \frac{4r^2}{3} + \frac{8r}{3})\gamma =& (-\frac{2r^2}{3} + \frac{2r}{3})x + (\frac{2r^2}{3} - \frac{2r}{3})y + (-\frac{8r^2}{3} + \frac{8r}{3})h \end{aligned}$$

Because $0 < r < 1$, $-\frac{4r^3}{3} - \frac{4r^2}{3} + \frac{8r}{3} \neq 0$.

Simplify the equations

$$\begin{aligned} \alpha =& (r^2 + 2r - 3)x + (-r^2 - 2r + 3)y + (-2r^2 + 2r)h \\ \beta =& (-r^2 - 2r + 3)x + (-3r^2 - 6r - 3)y + (2r^2 - 2r)h \\ \gamma =& (-2r^2 + 2r)x + (2r^2 - 2r)y + (-8r^2 + 8r)h \end{aligned}$$

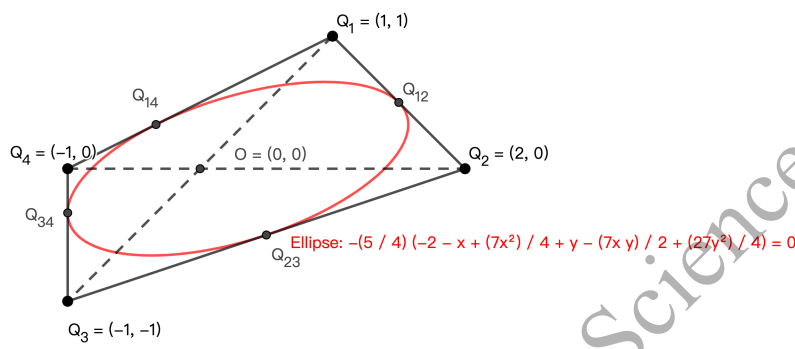

Figure 7: The Example of Quadrilateral Case

Substituting these value into φ, we will get the dual of φ

$$\begin{aligned}\hat{\varphi} =& 2r(r^2 + r - 2)(8h^2(r - 1)r + 4h(r - 1)r(x - y) - (r^2 + 2r - 3)x^2 \\ &+ 2(r^2 + 2r - 3)xy + 3(r + 1)^2y^2)\end{aligned}$$

De-homogenizing the formula, we can get

$$\begin{aligned}\hat{\varphi} =& 2r(r^2 + r - 2)(8(r - 1)r + 4(r - 1)r(x - y) - (r^2 + 2r - 3)x^2 \\ &+ 2(r^2 + 2r - 3)xy + 3(r + 1)^2y^2)\end{aligned}$$

Let $r = \frac{1}{2}$, we can get Figure 7.

2.4 In pentagons, there exists a unique zero-parameter family of inscribed ellipses.

Let P denote the pentagon with vertices Q_1, Q_2, Q_3, Q_4, Q_5. Using the Linfield's function, we can get a formula for φ

$$\varphi = m_1L_2L_3L_4L_5 + m_2L_1L_3L_4L_5 + m_3L_1L_2L_4L_5 + m_4L_1L_2L_3L_5 + m_5L_1L_2L_3L_4$$

First, for every pentagon $Q_1Q_2Q_3Q_4Q_5$, we can extend side Q_1Q_5 and Q_3Q_4 that will interact at Q_0. Then the big ellipse can be seen as inscribed in quadrilateral $Q_0Q_1Q_2Q_3$. On the other hand, we can generate $Q_1Q_2Q_3Q_4Q_5$ by adding an edge Q_4Q_5 that is tangent to the ellipse, and Q_4 is on Q_0Q_3 and Q_5 is on Q_0Q_1 Therefore, we just need to show that the ellipse we get from the quadrilateral case can be set to tangent to line Q_4Q_5. (Shown in Figure 8)

From the quadrilateral case, we know that we can write the big ellipse as $\varphi_1 = (1-r)\theta L_1L_3 + r\phi L_0L_2$. Because of the property of duality, $L_1L_3(P_{45}) = Q_4Q_5(Q_1)Q_4Q_5(Q_3)$. Since Q_1, Q_3 are on the same side of line Q_4Q_5, $Q_4Q_5(Q_1)$ and $Q_4Q_5(Q_3)$ are both positive or negative. Then $L_1L_3(P_{45}) = Q_4Q_5(Q_1)Q_4Q_5(Q_3) > 0$. Similarly, because of the property of duality, $L_0L_2(P_{45}) = Q_4Q_5(Q_0)Q_4Q_5(Q_2)$. Since Q_0, Q_2 are on the different sides of line Q_4Q_5, $Q_4Q_5(Q_0)$ and $Q_4Q_5(Q_2)$ have one positive and one negative number. Then $L_0L_2(P_{45}) = Q_4Q_5(Q_0)Q_4Q_5(Q_2) < 0$. As a result, we can always find and $r \in (0,1)$ such that $\varphi_1(P_{45}) = (1-r)\theta L_1L_3(P_{45}) + r\phi L_0L_2(P_{45}) = 0$, which means that φ_1 pass the dual of line Q_4Q_5. So $\hat{\varphi}_1$ tangent to five sides of $Q_1Q_2Q_3Q_4Q_5$.

Since φ and φ_1 are both tangent to the pentagon P, we know that

$$\nabla\varphi(P_{12}) \propto \nabla\varphi_1(P_{12}); \nabla\varphi(P_{23}) \propto \nabla\varphi_1(P_{23});$$
$$\nabla\varphi(P_{015}) \propto \nabla\varphi_1(P_{015}); \nabla\varphi(P_{034}) \propto \nabla\varphi_1(P_{034})$$

As φ is a curve of fourth power, φ_1 is proportional to φ with these constraints. Therefore, φ has a quadratic branch that is tangent to the pentagon $Q_1Q_2Q_3Q_4Q_5$.

2.4.1 An Example of Pentagon Case

Consider the pentagon $Q_1Q_2Q_3Q_4Q_5$, where $Q_1 = [0:2:1], Q_2 = [1:0:1], Q_3 = [0:-2:1], Q_4 = [-1:-1:1], Q_5 = [-1:1:1]$. Then

$$\begin{aligned}\varphi =& m_1(\alpha+\gamma)(-2\beta+\gamma)(-\alpha-\beta+\gamma)(-\alpha+\beta+\gamma)\\ &+ m_2(2\beta+\gamma)(-2\beta+\gamma)(-\alpha-\beta+\gamma)(-\alpha+\beta+\gamma)\\ &+ m_3(2\beta+\gamma)(\alpha+\gamma)(-\alpha-\beta+\gamma)(-\alpha+\beta+\gamma)\\ &+ m_4(2\beta+\gamma)(\alpha+\gamma)(-2\beta+\gamma)(-\alpha+\beta+\gamma)\\ &+ m_5(2\beta+\gamma)(\alpha+\gamma)(-2\beta+\gamma)(-\alpha-\beta+\gamma)\end{aligned}$$

Extend Q_1Q_5, Q_3Q_4 and meet at Q_0, so $Q_0 = [-2:0:1]$.

From the quadrilateral case, we can know φ_1 is inscribed in $Q_0Q_1Q_2Q_3$. In this case, $\theta = \frac{1}{2}, \phi = \frac{1}{3}$. So $\varphi_1 = \frac{1-r}{2}L_1L_3 + \frac{r}{3}L_0L_2 = \frac{1-r}{2}(2\beta+\gamma)(-2\beta+\gamma) + \frac{r}{3}(-2\alpha+\gamma)(\alpha+\gamma)$. Now, I just need to prove that φ_1 pass the dual of Q_4Q_5, which is P_{45}. Because the expression for Q_4Q_5 is $\alpha+\gamma = 0$, then $P_{45} = [1:0:1]$. Therefore $\varphi_1(P_{45}) =$

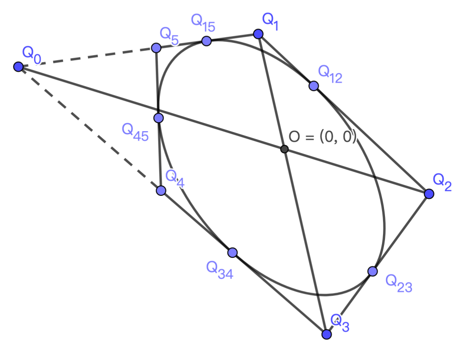

Figure 8: Pentagon Case

$\frac{1-r}{2}(1)(1) + \frac{r}{3}(-1)(2) = \frac{1-r}{2} - \frac{2}{3}r$. As a result, $r = \frac{3}{7}$ satisfy the requirement that φ_1 pass P_{45}.

$$\varphi_1 = \frac{1}{7}(-2\alpha^2 - 8\beta^2 - \alpha\gamma + 3\gamma^2)$$

Denote (x, y, h) as a point on $\hat{\varphi}_1$. Because $\hat{\varphi}_1 = \nabla\varphi_1$, then $(x, y, h) = \nabla\varphi_1(\alpha, \beta, \gamma)$. As a result, we can get

$$x = -\frac{4}{7}\alpha - \frac{1}{7}\gamma$$
$$y = -\frac{16}{7}\beta$$
$$h = -\frac{1}{7}\alpha + \frac{6}{7}\gamma$$

Solve these equations for α, β, γ:

$$\frac{400}{343}\alpha = -\frac{96}{49}x - \frac{16}{49}h$$
$$\frac{400}{343}\beta = -\frac{25}{49}y$$
$$\frac{400}{343}\gamma = -\frac{16}{49}x + \frac{64}{49}h$$

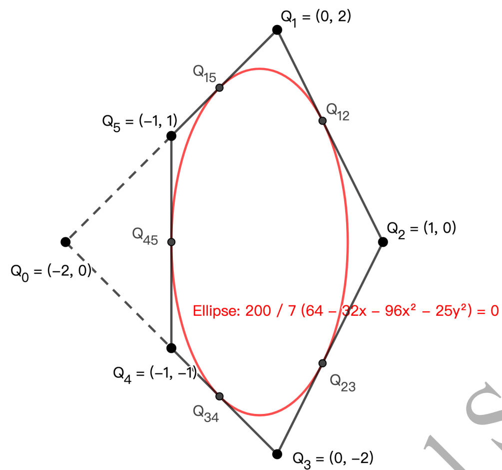

Figure 9: The Example of Pentagon Case

Since we just consider the ratio , we can simplify the equations into

$$\begin{aligned}\alpha &= -96x - 16h \\ \beta &= -25y \\ \gamma &= -16x + 64h\end{aligned}$$

Therefore,

$$\hat{\varphi}_1 = -\frac{200}{7}(96x^2 + 25y^2 + 32xh - 64h^2)$$

De-homogenizing the formula, we can get

$$\hat{\varphi}_1 = -\frac{200}{7}(96x^2 + 25y^2 + 32x - 64)$$

It is the inscribed ellipse we want. (Figure 9)

2.4.2 Some Other Things to Discuss

Actually, we can deduce quadrilateral case based on triangle case using the similar method that we use when deducing pentagon case. However, in this case we will have a completely different form of φ_1 and

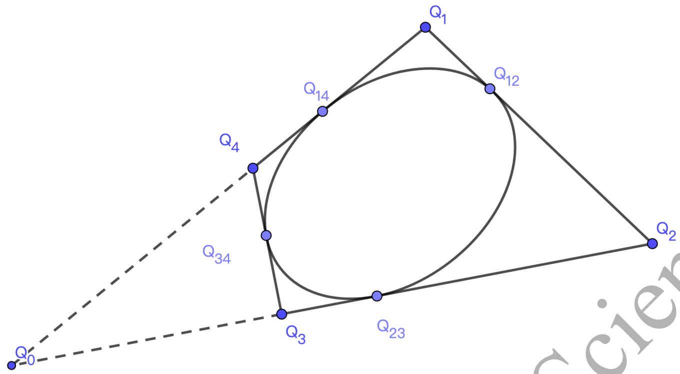

Figure 10: Explanation of The Deduction Quadrilateral Case

we can not continue the deduction to pentagon. This is a left question to answer.

The deduction from triangle to quadrilateral is produced as follow. Let Q denote the quadrilateral with vertices Q_1, Q_2, Q_3, Q_4. (Shown in Figure 10) Using the Linfield's function, we can get a formula for φ

$$\varphi = m_1L_2L_3L_4 + m_2L_1L_3L_4 + m_3L_1L_2L_4 + m_4L_1L_2L_3$$

We can extend Q_1Q_4 and Q_2Q_3 to meet at Q_0.

From the triangular case, there is a two-parameter family of inscribed ellipse $\varphi_1 = \frac{r}{1-r}L_0L_2 + L_0L_1 + \frac{1-s}{s}L_1L_2$. I will prove that we can choose proper s to make φ_1 pass the dual of Q_3Q_4, which is P_{34}.

Since Q_1, Q_2 are on the same side of Q_3Q_4 and are on the different side with Q_0, we can get $Q_3Q_4(Q_1)Q_3Q_4(Q_2) > 0$, $Q_3Q_4(Q_0)Q_3Q_4(Q_1) < 0$,and $Q_3Q_4(Q_0)Q_3Q_4(Q_2) < 0$, which means $L_1L_2(P_{34}) > 0$, $L_0L_1(P_{34}) < 0$, and $L_0L_2(P_{34}) < 0$. Because $0 < r, s < 1$, then we can get all positive real number by choosing proper r and s for $\frac{r}{1-r}, \frac{1-s}{s}$. Therefore, for every r, we can always find a corresponding s such that $\frac{r}{1-r}L_0L_2 + L_0L_1 + \frac{1-s}{s}L_1L_2 = 0$

According to the formula of the tangent point, we can know that

$$\nabla\varphi(P_{12}) \propto \nabla\varphi_1(P_{12}), \nabla\varphi(P_{23}) \propto \nabla\varphi_1(P_{23}), \nabla\varphi_1(P_{14}) \propto \nabla\varphi_1(P_{14})$$

Because φ is a cubic curve, φ is proportional to φ_1

Therefore, there is a one-parameter ellipse that tangent to the four sides of Q from the interior.

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