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Disjoint Cycles in Ordinary Multipartite Tournaments and Round-Robin Tournaments

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Abstract In 2010, N. Lichiardopol conjectured for $q \geq 3$ and $k \geq 1$ that any tournament with minimum out-degree at least $(q-1)k-1$ contains k disjoint cycles of length q , which has been established for tournaments. In this paper, we demonstrate that the conjecture holds for ordinary multipartite tournaments when $q = 3$, and for round-robin tournaments when $q \geq 3$. Moreover, we point out several flaws found in the proof for tournaments when $q = 4$.

Keywords: Lichiardopol conjecture; Minimum out-degree; Disjoint cycles; Ordinary multipartite tournaments; Round-robin tournaments

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1 Introduction

This paper mainly presents a study of digraphs, and we refer the readers to the book [2] for standard definitions related to digraphs. For a digraph $D = (V, A)$, if there is an arc from x to y , we write $x \rightarrow y$, and call y an *out-neighbor* of x , x an *in-neighbor* of y . For a vertex x of D , we let $d_D^+(x)$ denote the *out-degree* of x , which is the number of out-neighbors of x . Similarly, we let $d_D^-(x)$ denote the *in-degree* of x , let $\delta^+(D)$ denote the minimum out-degree of V . For $U \subseteq V$, we let $d_U^+(x)$ denote the number of out-neighbors of x in U , similarly $d_U^-(x)$ denote the number of in-neighbors of x in U . For two vertex sets A and B , we let $d^+(A, B)$ denote the number of arcs from A to B . We say there exists a *k-matching* from A to B if there exist k arcs from A to B which have no common endpoints. A *tournament* is a digraph such that for each pair of vertices x and y , there exists exactly one arc between x and y . There are many interesting results about tournaments, we refer interested readers to Chapter 2 of [2]. A digraph $D = (V, A)$ is called a *multipartite tournament* if V can be partitioned into sets V_1, V_2, \dots, V_m , such that for any $i < j$ and any $v_1 \in V_i, v_2 \in V_j$, there exists exactly one arc between v_1 and v_2 , and for any $v_1, v_2 \in V_i$, there exists no arc between v_1 and v_2 . Moreover, if for any $i < j$, the arcs from V_i to V_j have a common orientation, then the multipartite tournament is called an *ordinary multipartite tournament*. Some properties of ordinary multipartite tournaments are presented by the work of Bang-Jensen et al [3]. An ordinary multipartite tournament is also called a *uniform multipartite tournament* [11]. Note that a tournament is always an ordinary multipartite tournament, if we partition its vertex set into sets each of which contains exactly one vertex.

In this work, we also consider the problem of multi-digraph. The concept of a multi-digraph is a generalization of the concept of a digraph. In a multi-digraph there might be multi-arcs from vertex x to y . But multi-digraphs with loops are not addressed in this work. In this paper we define a *round-robin tournament* to be a multi-digraph such that there are exactly two arcs between each pair of vertices. To our best knowledge, round-robin tournaments were first studied in [8]. In this master's thesis round-robin tournaments were called double-arc tournaments. We call them round-robin tournaments because they can model round-robin tournaments in real world.

Problems related to disjoint cycles in digraphs have always been an area of

focus. J. C. Bermond and C. Thomassen gave the following conjecture in 1981:

Conjecture 1.1. [4] *For any digraph D , if $\delta^+(D) \geq 2k - 1$, then D contains k disjoint cycles.*

The conjecture is trivial for $k = 1$ and it was proved for $k = 2$ in [10] and for $k = 3$ in [7].

J. Bang-Jensen, S. Bessy and S. Thomassé had a great contribution on this conjecture. In [1] they proved the following theorem:

Theorem 1.2. *For $k \geq 1$, every tournament T with $\delta^+(T) \geq 2k - 1$ has k disjoint cycles, each of which has length 3.*

For convenience, we call a cycle with length q an q -cycle. Note that if a tournament has an q -cycle, then we can find a 3-cycle whose vertex set is a subset of the vertex set of the q -cycle. This can be easily proved by induction on q . Thus whenever a tournament T contains k disjoint cycles, it contains k disjoint 3-cycles.

Tending to generalize Theorem 1.2 in another dimension, N. Lichiardopol raised another conjecture in 2010:

Theorem 1.3. [6] *For $k \geq 1$ and $q \geq 3$, every tournament T with $\delta^+(T) \geq (q - 1)k - 1$ has k disjoint q -cycles.*

When $q = 3$ this conjecture is exactly Theorem 1.2. The case $q = 4$ was proved in the master's thesis of S. Zhu [13]. F. Ma, D. B. West and J. Yan proved this conjecture for $q \geq 5$ in [9].

In this paper we firstly generalize Theorem 1.2 to ordinary multipartite tournament case: •

Theorem 1.4. *For $k \geq 1$, every ordinary multipartite tournament T with $\delta^+(T) \geq 2k - 1$ has k disjoint 3-cycles.*

Theorem 1.4 will be proved in Section 2.

In fact, at first we wanted to generalize Theorem 1.3 to ordinary multipartite tournament case, but we found a counterexample when $q = 4$. Hence, we extend Theorem 1.3 to another case, namely the round-robin tournament:

Theorem 1.5. *For $k \geq 1$ and $q \geq 3$, every round-robin tournament T with $\delta^+(T) \geq 2(q - 1)k - 2$ has k disjoint q -cycles.*

Theorem 1.5 will be proved in Section 3.

Furthermore, we find that although the final result in [13] is correct but with flaws in the proof. So we list all the flaws in [13] and then present a proof with better completeness in Section 4.

2 Disjoint 3-cycles in Ordinary Multipartite Tournaments

2.1 Preparation

In order to prove Theorem 1.4, we prove a slightly stronger theorem:

Theorem 2.1. *Let k be a positive integer with $k \geq 1$. Suppose T is an ordinary multipartite tournament with $\delta^+(T) \geq 2k - 1$. For any $k - 1$ disjoint 3-cycles $\mathcal{F} = \{C_1, C_2, \dots, C_{k-1}\}$ in T , let $W = V(C_1) \cup V(C_2) \cup \dots \cup V(C_{k-1})$, $U = V(T) \setminus W$, there exist k disjoint 3-cycles whose vertex set intersects U on at most 4 vertices.*

It deserves to be noted that Theorem 1.4 can be directly deduced from Theorem 2.1 by induction on k .

We still denote $V(C_i)$ by C_i when it causes no confusion.

First of all, some lemmas frequently used in the proofs are listed below.

Lemma 2.2. *In an ordinary multipartite tournament, if $x \rightarrow y, y \rightarrow z$, then there is an arc between x and z .*

Proof. Since $x \rightarrow y, y \rightarrow z$, x and z belong to different parts, thus there is an arc between x and z . \square

Lemma 2.3. *For every acyclic multipartite tournament with n vertices, there is an ordering of the vertices v_1, v_2, \dots, v_n , such that for any $i < j$, there is no arc from v_i to v_j .*

We can prove this lemma by induction on n .

Note that “no arc from v_i to v_j ” means that either there exists no arc between v_i and v_j , or there exists an arc $v_j \rightarrow v_i$.

Lemma 2.4. *If a multipartite tournament has a cycle, then it has a 3-cycle.*

Proof. Suppose the cycle has length k . We prove it by induction on k .

When $k = 3$, this is obvious.

For $k \geq 4$, let the cycle be $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \dots \rightarrow v_k \rightarrow v_1$. By Lemma 2.2, there is an arc between v_1 and v_3 . If $v_3 \rightarrow v_1$, we get a 3-cycle $v_1 \rightarrow v_2 \rightarrow v_3$. If $v_1 \rightarrow v_3$, we get a $(k - 1)$ -cycle $v_1 \rightarrow v_3 \rightarrow \dots \rightarrow v_k \rightarrow v_1$. By induction hypothesis we can get the result. \square

Lemma 2.5. [5] (See also Theorem 3.1.16 of [12]) Suppose A and B are two disjoint set of vertices in a tournament, and there is no k -matching from A to B . Then there exists a subset C of $A \cup B$ containing at most $k - 1$ vertices, such that the endpoints of all arcs from A to B belong to C .

We will prove Theorem 2.1 by induction on k .

When $k = 1$, obviously T has a cycle. By Lemma 2.4 it has a 3-cycle. Thus Theorem 2.1 holds.

For $k \geq 2$, we argue by contradiction. Suppose there exist $k - 1$ disjoint 3-cycles $\mathcal{F} = \{C_1, C_2, \dots, C_{k-1}\}$ in T , but do not exist k disjoint 3-cycles which fit the theorem. We name it “the ultimate assumption”.

Recall that $W = V(C_1) \cup V(C_2) \cup \dots \cup V(C_{k-1})$, $U = V(T) \setminus W$. Note that the sub-multipartite tournament of T induced by U is an acyclic ordinary multipartite tournament. Otherwise this sub-multipartite tournament contains a 3-cycle, which contradicts the ultimate assumption.

Moreover, here are two important definitions we need to present:

Definition 2.6. For i 3-cycles in \mathcal{F} , $i \in \{1, 2\}$, we say that they can be extended if and only if there exist $i + 1$ disjoint 3-cycles whose vertices belong to the initial 3-cycles and U , and intersect U on at most four vertices.

Note that once there exists 1 or 2 cycles in \mathcal{F} that can be extended, the ultimate assumption would be violated.

Definition 2.7. For arc xy , $x, y \in W$, and vertices $z, z \in U$, z is a breaker of xy if and only if $x \rightarrow y \rightarrow z \rightarrow x$ forms a 3-cycle.

Notation: Below we denote the 3-cycle $x \rightarrow y \rightarrow z \rightarrow x$ by $(xyzx)$.

2.2 Several Prepositive Claims

Claim 1. For every $C_i \in \mathcal{F}$, C_i has at most two arcs with breakers, and every arc has at most three breakers. Thus C_i has at most six breakers.

Proof. Let $C_i = (xyzx)$.

(1) Suppose for contradiction that every arc of C_i has a breaker. Let v_{xy} , v_{yz} , v_{zx} be breakers of xy , yz and zx respectively. Since $v_{zx} \rightarrow z \rightarrow v_{yz}$, there exists an arc between v_{zx} and v_{yz} . Furthermore, we have $v_{zx} \rightarrow v_{yz}$. Otherwise, 3-cycles xyv_{xy} , $zv_{yz}v_{zx}$ can extend C_i , which violates the ultimate assumption. Symmetrically, we have $v_{yz} \rightarrow v_{xy}$ and $v_{xy} \rightarrow v_{zx}$, which forms a 3-cycle in U . This contradicts the fact that U is acyclic;

(2) For the sake of contradiction, we assume that the arc xy has four breakers, which were named v_1, v_2, v_3, v_4 respectively.

Consider the ordinary multipartite tournament $T' = T - \{x, y\}$. We have $\delta^+(T') \geq 2(k-1) - 1$, and T' has $k-2$ disjoint 3-cycles, i.e. $\mathcal{F} \setminus \{C_i\}$. Applying induction on k , we know that there is a collection \mathcal{F}' in T' , which contains $k-1$ disjoint 3-cycles, and \mathcal{F}' intersects $U \cup \{z\}$ on at most four vertices. Since $v_1, v_2, v_3, v_4, z \in U \cup \{z\}$, at least one of those five vertices is not included in \mathcal{F}' . In the following content, the selection of vertex is presented.

(2.1) Suppose that z is not included in \mathcal{F}' . Thus, the collection $\mathcal{F}' \cup C_i$ contains k disjoint 3-cycles, and its vertices intersect U on at most four vertices. This contradicts the ultimate assumption.

(2.2) Suppose that z is included in \mathcal{F}' . Without loss of generality, v_1 is not included in \mathcal{F}' . Therefore $\mathcal{F}' \cup \{(xv_1yx)\}$ contains k disjoint 3-cycles, and its vertex set intersects U on at most four vertices. This contradicts the ultimate assumption. \square

Thanks to Lemma 2.3, U can be divided into two subsets U_1 and U_2 , such that there is no arc from U_1 to U_2 . Also the size of U_1 can be an arbitrary integer between 0 and $|U|$.

Claim 2. *For arbitrarily chosen partition U_1, U_2 of U such that there is no arc from U_1 to U_2 , and for all $i \in [1, k-1]$, the following three properties are true:*

- (1) *If $d^+(U_1, C_i) \geq 4$, then there exists a 2-matching from U_1 to C_i .*
- (2) *If $d^+(U_1, C_i) \geq 8$, then there exist a 3-matching from U_1 to C_i , otherwise C_i can be represented as $(xyzx)$ such that yz has three breakers in U_1 , xy has two breakers in U_1 , $d_{U_1}^-(x) \geq 5$, $d_{U_2}^+(x) = 0$, $d_{U_2}^+(y) \leq 1$, $d_{U_2}^+(z) \leq 1$.*
- (3) *If $d^+(U_1, C_i) \geq 8$, then there does not exist a 2-matching from C_i to U_2 , and $d^+(C_i, U_2) \leq 3$.*

By reversing each arc in T and using Claim 2, we can easily obtain the following results:

The reversing claim. (1) If $d^+(C_i, U_2) \geq 4$, then there exists a 2-matching from C_i to U_2 ;

(2) If $d^+(C_i, U_2) \geq 8$, then there exist a 3-matching from C_i to U_2 , otherwise C_i can be represented as $(x'y'z'x')$ such that $x'y'$ has three breakers in U_2 , $y'z'$ has two breakers in U_2 , $d_{U_2}^+(z') \geq 5$, $d_{U_1}^-(z') = 0$, $d_{U_1}^-(x') \leq 1$, $d_{U_1}^-(y') \leq 1$;

(3) If $d^+(C_i, U_2) \geq 8$, then there does not exist a 2-matching from U_1 to C_i , and $d^+(U_1, C_i) \leq 3$.

Now we start to prove Claim 2.

Proof. (1) We prove it by contradiction. Assume that there exists no 2-matching from U_1 to C_i . According to Lemma 2.5, and $E(U_1, C_i) \geq 4$, there exists $x \in C_i$ such that x belongs to all arcs from U_1 to C_i . Assume in-neighbors of x on U_1 are u_1, u_2, u_3, u_4 . Suppose $x \rightarrow y$ in C_i , we find that for any $j \in [1, 4]$, $u_j \rightarrow x \rightarrow y$. u_j and y must therefore be adjacent, and furthermore $y \rightarrow u_j$. Hence u_j is a breaker of xy , which contradicts Claim 1.

(2) Suppose that there exists no 3-matching from U_1 to C_i . By Lemma 2.5, there exist $x, y \in U_1 \cup C_i$ such that all arcs from U_1 to C_i contain x or y . Since $E(U_1, C_i) \geq 8$, x and y can not both be in U_1 . Without loss of generality, assume that $x \in U_1$ and $y \in C_i$. Now suppose $y \rightarrow z$ in C_i . Since $E(U_1, C_i) \geq 8$, there exist four vertices (different from x) $u_1, u_2, u_3, u_4 \in U_1$ such that for any $j \in [1, 4]$, $u_j \rightarrow y \rightarrow z$. Hence u_j and z are adjacent. Moreover, $z \rightarrow u_j$. Therefore, u_1, u_2, u_3, u_4 are yz breakers, which contradicts Claim 1. Thus $x, y \in C_i$.

Without loss of generality, let $C_i = (xyzx)$. Assume $d_{U_1}^-(y) \leq 2$. Since $E(U_1, C_i) \geq 8$, and all arcs from U_1 to C_i contain x or y , we have $d_{U_1}^-(x) \geq 6$. Thus, there exist four vertices in U_1 that have arcs to x but have no arc to y . Proceeding with an argument similar to the above, we know that these four vertices are breakers of yz , which poses a contradiction. Thus $d_{U_1}^-(y) \geq 3$.

Assume $d_{U_1}^-(y) \geq 4$. There exist four vertices in U_1 that have arcs to y . Proceeding with an argument similar to the above, we know these four vertices are breakers of yz , which poses a contradiction. Thus $d_{U_1}^-(y) \leq 3$.

Now we have $d_{U_1}^-(y) = 3$. Since $E(U_1, C_i) \geq 8$, and all arcs from U_1 to C_i contain x or y , we have $d_{U_1}^-(x) \geq 5$. Hence there exist at least two vertices in

U_1 which have arcs to x but have no arc to y . This means xy has at least two breakers in U_1 . Similarly, we can prove yz has three breakers in U_1 . Next, let one xy breaker be x_1 , one yz breaker be y_1 .

Assume there exists $x' \in U_2$ such that $x \rightarrow x'$. We have $x_1 \rightarrow x \rightarrow x'$, which means $x' \rightarrow x_1$. Therefore, $(xx'x_1x)$ and (yzy_1y) extend C_i , which contradicts the ultimate assumption. Thus $d_{U_2}^+(x) = 0$.

Assume there exist two vertices $y_a, y_b \in U_2$ which are out-neighbors of y . Then we can find that $x \rightarrow y \rightarrow y_a, y_b$, hence y_a, y_b are both adjacent to x . Since x has no out-neighbor in U_2 , we have $y_a, y_b \rightarrow x$. xy has two more breakers, which contradicts Claim 1. Thus $d_{U_2}^+(y) \leq 1$.

Assume there exist two vertices $z_a, z_b \in U_2$ which are out-neighbors of z . For the same reason, these two vertices are adjacent to y . Because we already have $d_{U_2}^+(y) \leq 1$, then either z_a or z_b is an in-neighbor of y . Then yz has one more breaker, which contradicts Claim 1. Thus $d_{U_2}^+(z) \leq 1$.

(3) We prove it by contradiction. Suppose there exists a 2-matching from C_i to U_2 .

(3.1) If there exists a 3-matching from U_1 to C_i , then C_i can be extended by two 3-cycles which have the form $U_1 \rightarrow C_i \rightarrow U_2 \rightarrow U_1$;

(3.2) If there does not exist a 3-matching from U_1 to C_i , then suppose that $C_i = (xyzx)$, $d_{U_2}^+(x) = 0$, and the 2-matching from C_i to U_2 is $\{yy', zz'\}$. At this time, $y \rightarrow z'$. Otherwise z' will be the fourth breaker of yz . Thus, we have $x \rightarrow y \rightarrow z', x \rightarrow y \rightarrow y'$. Furthermore, since $d_{U_2}^+(x) = 0$, we have $y', z' \rightarrow x$. Hence y', z' are two more breakers of the xy , which poses a contradiction.

Consequently, there is no 2-matching from C_i to U_2 . And by the reversing claim (1) we immediately have $d^+(C_i, U_2) \leq 3$. \square

Claim 3. For any $C_i, C_j \in \mathcal{F}$, it is impossible that there exist 3-matchings from U_1 to C_i , C_i to C_j , and C_j to U_2 .

Proof. Assume that there exist 3-matchings from U_1 to C_i , C_i to C_j , and C_j to U_2 .

Suppose that $C_i = (xyzx)$ and $C_j = (x'y'z'x')$. Three 3-matchings are $\{x_1x, y_1y, z_1z\}$, $\{xx', yy', zz'\}$, and $\{x'x_2, y'y_2, z'z_2\}$ respectively. Since $x \rightarrow x' \rightarrow x_2$, x and x_2 are adjacent. For the same reason, y and y_2 , z and z_2 are adjacent. Next, we say w is ‘‘positively adjacent’’ if $w \rightarrow w_2$, ‘‘negatively adjacent’’ if $w_2 \rightarrow w$, where $k \in \{x, y, z\}$.

(1) All x, y, z are negatively adjacent. At this time, C_i, C_j can be extended by $(xx'x_2x), (yy'y_2y), (zz'z_2z)$;

(2) Exactly one of x, y, z is positively adjacent. Without loss of generality, suppose this vertex is x . At this time $x_1 \rightarrow x \rightarrow x_2$, hence $x_2 \rightarrow x_1$. C_i, C_j can be extended by $(xx_2x_1x), (yy'y_2y), (zz'z_2z)$;

(3) At least two of x, y, z are positively adjacent. Without loss of generality, suppose two of them are x and y . Now C_i can be extended by $(xx_2x_1x), (yy_2y_1y)$. \square

Claim 4. For any $C_i, C_j \in \mathcal{F}$, it is impossible that $d^+(U_1, C_i) \geq 8$, $d^+(C_i, C_j) \geq 7$, and $d^+(C_j, U_2) \geq 8$.

Proof. Assume C_i and C_j , whose vertices are $\{x, y, z\}$ and $\{x', y', z'\}$ respectively, satisfy $d^+(U_1, C_i) \geq 8$, $d^+(C_i, C_j) \geq 7$, and $d^+(C_j, U_2) \geq 8$. Since $d^+(C_i, C_j) \geq 7$, we see that there is a 3-matching from C_i to C_j .

(1) Assume there exists no 3-matching from both U_1 to C_i , and C_j to U_2 . Suppose that $C_i = (xyzx)$ and $C_j = (x'y'z'x')$. According to Claim 2 and the reversing claim, without loss of generality, we assume that yz has three breakers in U_1 , xy has at least two breakers in U_1 , $x'y'$ has three breakers in U_2 , and $y'z'$ has at least two breakers in U_2 . Also $d_{U_2}^+(x) = d_{U_1}^-(z') = 0$, $d_{U_2}^+(z) \leq 1$, $d_{U_1}^-(x') \leq 1$. Now suppose $x_1 \in U_1$ is an xy breaker but not an in-neighbor of x' , $z_2 \in U_2$ is a $y'z'$ breaker but not an out-neighbor of z . And we denote y_1, y_2 as breakers of yz and $x'y'$ respectively, which are different from x_1 and z_2 .

Firstly, there is no arc from x to z' . Otherwise C_i, C_j can be extended by $(z_2xz'z_2), (yz_1y_1), (y'y_2x'y')$. Secondly, there is no arc from x to x' . Otherwise, as we already know $x_1 \rightarrow x \rightarrow x'$, we have $x' \rightarrow x_1$. Thus C_i, C_j can be extended by $(x'x_1x'), (z_2y'z'z_2), (yz_1y_1)$. Thirdly, there is no arc from z to z' . Otherwise, we have $z \rightarrow z' \rightarrow z_2$, thus $z_2 \rightarrow z$. C_i, C_j can be extended by $(z'z_2z_2'), (x'y'y_2x'), (x_1xyz_1)$.

However, $d^+(C_i, C_j) \geq 7$, there must exist an arc from x to C_j , we get a contradiction.

(2) Without loss of generality, assume that there exists no 3-matching from U_1 to C_i , but one from C_j to U_2 . We let $C_i = (xyzx)$, and denote 3-matching of C_i to C_j , C_j to U_2 by $\{xx', yy', zz'\}, \{x'x_2, y'y_2, z'z_2\}$ respectively. According to Claim 2, yz has three breakers in U_1 , xy has two breakers in U_1 . Specifically, we denote one of the yz breakers as y_1 . Besides, we know that $d_{U_2}^+(x) = 0$,

$d_{U_2}^+(y) \leq 1$, and $d_{U_2}^+(z) \leq 1$. Next, we will construct the contradiction step by step:

Firstly, we claim that $z \rightarrow z_2$. Otherwise, because $z \rightarrow z' \rightarrow z_2$, we have $z_2 \rightarrow z$. Now there are two disjoint 3-cycles, i.e. $(z_2 z z' z_2), (x x' x_2)$, because $y_1 \rightarrow y \rightarrow y', y_1$ and y' are adjacent.

(a) If $y' \rightarrow y_1$, then there exists a 3-cycle $(y y' y_1 y)$, which can extend C_i, C_j with $(z_2 z z' z_2), (x x' x_2)$;

(b) If $y_1 \rightarrow y'$, then we have $y_1 \rightarrow y' \rightarrow y_2$, which means y_1 and y_2 are adjacent. Furthermore, $y_2 \rightarrow y_1$. Thus, there exists a 3-cycle $(y_1 y' y_2 y_1)$, which can extend C_i, C_j with $(z_2 z z' z_2), (x x' x_2)$.

Secondly, we claim that $y \rightarrow z_2$. Otherwise, we have $z_2 \rightarrow y$, and z_2 becomes the fourth breaker of yz ;

Thirdly, we claim that $\{x_2, y_2\} \rightarrow \{y, z\}$. Note that z_2 is the only out-neighbor of y, z in U_2 . Since $y \rightarrow y' \rightarrow y_2$, we have $y_2 \rightarrow y$. Since $y_2 \rightarrow y \rightarrow z$, we have $y_2 \rightarrow z$. Since $x_2 \rightarrow x \rightarrow y$, we have $x_2 \rightarrow y$. Since $x_2 \rightarrow y \rightarrow z$, we have $x_2 \rightarrow z$;

Fourthly we claim that the sub-multipartite tournament of T induced by $\{y', y_2, z, z'\}$ is acyclic. Otherwise, the cycle inside can extend C_i, C_j with $(y_1 y z_2), (x x' x_2)$. Moreover, we have $C_j = (x' y' z' x')$. Otherwise, we have $z' \rightarrow y'$, implying that $(y' y_2 z z')$ is a cycle, a contradiction. Also we have $y' \rightarrow z$. Otherwise since $y' \rightarrow y_2 \rightarrow z, z \rightarrow y'$, $(y' y_2 z y')$ is a cycle, which poses a contradiction;

Fifthly, we claim that $x \rightarrow y'$. Otherwise, since $x \rightarrow x' \rightarrow y$, we have $y' \rightarrow x$. Besides, we have $y' \rightarrow z$ and $d^+(C_i, C_j) \geq 7$. Hence $x \rightarrow z'$, and $z \rightarrow x'$. Therefore, the 3-cycles $(x z' z_2 x), (x' x_2 z x'), (y y' y_2 y)$ extend C_i, C_j ;

Sixthly, we claim that $z' \rightarrow x_2$. Otherwise, since $z' \rightarrow x' \rightarrow x_2, x_2 \rightarrow z'$. So we find that the 3-cycles $(x_2 z' x' x_2), (x y' y_2), (z y_1 y z)$ extend C_i, C_j .

From the above analysis, we know that $(z z' x_2 z), (y' y_2 x y'), (z_2 y_1 y z_2)$ extend C_i, C_j , which contradicts the ultimate assumption. \square

2.3 Analysis on range of k

Here, we want to prove that $k \leq 6$. Let $|T| = n$. Then $\frac{n(n-1)}{2} \geq (2k-1)n$, so $|T| \geq 4k-1$. Thus $|U| = |T| - |W| \geq (4k-1) - (3k-3) = k+2$. When $k \geq 3$, we let $|U_1| = 5$. Among all 3-cycles in \mathcal{F} , we define I as the set of cycles receiving at least eight arcs from U_1 , O as the set of cycles sending at least eight

arcs to U_2 . At last we define P as $P = \mathcal{F} \setminus (O \cup I)$. Note that i, o, p below are defined as the cardinality of I, O, P respectively.

Firstly, we obtain lower and upper bounds of arcs leaving U_1 and entering $T \setminus U_1$ respectively with the following:

$$5(2k - 1) - 10 \leq 15i + 7p + 3o. \quad (2.1)$$

The $5(2k - 1)$ on the left-hand side is a lower bound of arcs with heads in U_1 , and 10 represents the number of arcs with heads and tails both in U_1 . On the right-hand side, since $|U_1| = 5$ and one cycle has 3 vertices, the number of arcs from U_1 to I is at most $15i$. By the definition of I and P , the number of arcs from U_1 to P is at most $7p$. By the definition of O and the reversing claim (3), the number of arcs from U_1 to O is at most $3o$. We need to point out that there is no arc from U_1 to U_2 . Thus we get the right side of (2.1). As we have $p = k - 1 - i - o$, we can simplify (2.1) to the following form:

$$3k + 4o - 8 \leq 8i. \quad (2.2)$$

Secondly, we estimate the upper and lower bound of arcs which leave $P \cup I$ and enter $T \setminus (P \cup I)$:

$$3(p+i)(2k-1) - \frac{1}{2} \cdot 3(p+i)[3(p+i)-1] \leq 9po + 6io + 7p + 3i + [15(p+i) - (10k - 15 - 3o)]. \quad (2.3)$$

The left-hand side bounds the number of arcs which have heads in $P \cup I$ but no tails there from below. On the right-hand side, $9po, 6io, 7p, 3i$ bound the number of arcs which are from P to O , from I to O (by Claim 4), from P to U_2 (by the definition of O and P), and from I to U_2 (by Claim 2(3)), respectively. $15(p+i)$ is an upper bound of number of arcs between $P \cup I$ and U_1 (regardless of direction). And $10k - 15 - 3o = 5(2k - 1) - 10 - 3o$ is a lower bound of arcs leaving U_1 and entering $P \cup I$. Thus, $15(p+i) - (10k - 15 - 3o)$ is an upper bound of arcs from $P \cup I$ to U_1 . Substitute $p = k - 1 - i - o$ and (2.2) into (2.3), we have

$$16o^2 + (52 - 13k)o + 4k^2 - 24k \leq 0. \quad (2.4)$$

Since o is a real number, the discriminant of (2.4)

$$-87k^2 + 184k + 2704 \quad (2.5)$$

is at least 0. Hence we can get $k \leq 6$.

2.4 Small Cases of k

Claim 5. For every $C \in \mathcal{F}$, if $d^+(U_1, C) \geq 10$, then there exists at least one 3-matching from U_1 to C .

Proof. We prove this by contradiction. Assume that there exists a 3-cycle C such that $d^+(U_1, C) \geq 10$, and there is no 3-matching from U_1 to C .

By Lemma 2.5, there exist $x, y \in U_1 \cup C$ such that all arcs from U_1 to C are adjacent to at least one of them.

- (1) If $x, y \in U_1$, we have at most six arcs from U_1 to C , which is impossible;
- (2) Without loss of generality, we assume $x \in U_1, y \in C$. Since $E(U_1, C) \geq 10$, y has at least seven in-neighbors in $U_1 \setminus \{x\}$. These vertices dominate y , and y dominates z . Thus, these seven vertices must be adjacent to z , and all of them are z 's out-neighbors. Hence yz has at least seven breakers;

(3) If $x, y \in C$, then we let $C = (xyzx)$. Now, it is true that $d_{U_1}^-(y) \leq 3$, otherwise yz has four breakers. It is also true that $d_{U_1}^-(x) \leq 6$, otherwise xy has four breakers. Therefore, $d^+(U_1, C) \leq 9$, which contradicts the assumption. Consequently, there exists at least one 3-matching from U_1 to C . \square

Definition 2.8. We say that 3-cycle C has a 3-cover if and only if there exists a 3-matching from U_1 to C , or two 2-matchings from U_1 to C such that they cover all the three vertices of C .

Claim 6. If there exists a 3-cover from U_1 to $C \in \mathcal{F}$, then there is no 2-matching from C to U_2 . Moreover, we have $d^+(C, U_2) \leq 3$.

Proof. We argue by contradiction. Assume that a 3-cycle $C = (xyzx)$ has a 3-cover from U_1 , and a 2-matching to U_2 ($\{xx', zz'\}$).

(1) Suppose the 3-cover is formed by a 3-matching. We can then extend C in an obvious manner;

(2) Suppose the 3-cover is formed by 2-matchings, and one of them has tails in $\{x, z\}$. We name this 2-matching $\{x_1x, z_1z\}$. This time we find that C can be extended by $(x_1xx'x_1), (z_1zz'z_1)$;

(3) Suppose the 3-cover is formed by 2-matchings $\{ax, by\}$ and $\{cy, dz\}$. Then there exists a 3-cycle $(axx'a)$.

(3.1) If z' and b are adjacent, then the 4-cycle $(byzz'b)$ contains a 3-cycle, which can extend C together with $(axx'a)$.

(3.2) If z' and b are not adjacent, we have $z' \rightarrow y$ because $b \rightarrow y$. Then 3-cycles $(yzz'y)$ and $(axx'a)$ extend C . \square

We should point out that Claim 5 and Claim 6 are correct if we reverse U_1 and U_2 , “receiving” and “sending” in the statement.

We name a 3-cycle $C_i \in \mathcal{F}$ as 2- m , 3- c , and 3- m if and only if there exists a 2-matching, 3-cover, and 3-matching from U_1 to C_i respectively.

Claim 7. *Suppose a, b, c are arbitrarily chosen vertices in U_1 and $Y = V(\mathcal{F}')$, where \mathcal{F}' is a subset of \mathcal{F} containing p 3-cycles. If $d_Y^+(a) \geq 2p$, $d_Y^+(b) \geq 2p - 1$, $d_Y^+(c) \geq 2p - 2$, then \mathcal{F}' has a 3- c cycle, or all the cycles in \mathcal{F}' are 2- m .*

Proof. We prove this claim by induction on p .

When $p = 1$, the claim is true.

When $p \geq 2$, let $k = d^+(\{a, b, c\}, Y)$. We have $k \geq 6p - 3 > 3p$. Hence there exists a cycle $C_i \in \mathcal{F}'$ such that $d^+(\{a, b, c\}, C_i) \geq 4$. So C_i is 2- m .

(1) If C_i is 3- c , the induction is proved;

(2) If C_i is not 3- c , then for any $x \in \{a, b, c\}$, $d^+(x, C_i) \leq 2$. By applying the induction hypothesis on $\mathcal{F}' \setminus \{C_i\}$, we know the statement is true as well. \square

Next we will only consider $k \in \{2, 3, 4, 5, 6\}$.

(1) $k = 2$

When $k = 2$, we have $\delta^+(T) \geq 3$. According to Thomassen's work in [10], there exist two disjoint 3-cycles C'_1 and C'_2 in T . Now \mathcal{F} has exactly one 3-cycle C_1 .

(1.1) If there exists a cycle $C \in \{C'_1, C'_2\}$ such that $V(C) \cap V(C_1) = \emptyset$, then C, C_1 extend C_1 .

(1.2) If both C'_1 and C'_2 have common vertices with C_1 , then C'_1, C'_2 extend C_1 .

(2) $k = 3$

When $k = 3$, we have $\delta^+(T) \geq 5$. Suppose that $\{C_i, C_j\}$ are two 3-cycles in \mathcal{F} . Let $U_1 = \{v_1, v_2\}$. Then there are at least $2 \cdot 5 - 1 = 9$ arcs from U_1 to W , at most $2 \cdot 6 - 9 = 3$ arcs from W to U_1 . There are at least $6 \cdot 5 - \frac{1}{2} \cdot 6 \cdot 5 = 15$ arcs from W to U , and at least $15 - 3 = 12$ arcs from W to U_2 . Therefore, we know that there exists a cycle $C_i \in \mathcal{F}$ such that $d^+(U_1, C_i) \geq 5$. Meanwhile $d^+(U_1, C_j) \geq 3$.

(2.1) $d^+(U_1, C_j) \geq 4$: In this case C_j is a 2- m cycle, and $d^+(C_j, U_2) \leq 7$. As a result, $d^+(C_i, U_2) \geq 5$.

Since $d^+(U_1, C_i) \geq 5$, C_i is 3-c. By Claim 6, there does not exist a 2-matching from C_i to U_2 . But $d^+(C_i, U_2) \geq 5$, by the reversing claim, hence there exists a 2-matching from C_i to U_2 , which poses a contradiction.

(2.2) $d^+(U_1, C_j) = 3$: If C_j is a 2-m cycle, we can get a contradiction in the same way as (2.1). Thus C_j is not 2-m. Then either $v_1 \rightarrow C_j \rightarrow v_2$, or $v_2 \rightarrow C_j \rightarrow v_1$. If $d^+(U_1, C_i) = 5$, then $d^+(U_1, C_j) \geq 4$, C_j is 2-m, which poses a contradiction. Now we only need to consider the case $d^+(U_1, C_i) = 6$, $U_1 \rightarrow C_i$. If there exists an arc from C_j to C_i , then $d^+(C_i, U_2) \geq 5 \times 3 - 3 \times 3 - 3 = 3$. When $E(C_i, U_2) \geq 4$, we can get a contradiction in the same way as (2.1). Now we can suppose $C_i \rightarrow C_j$, and $d^+(C_i, U_2) = 3$.

Let $C_i = (xyzx)$ and $C_j = (x'y'z'x')$. From the above we know each vertex in C_j has at least 3 out-neighbors in U_2 . Without loss of generality, suppose x has the largest number of out-neighbors in U_2 . If $d^+(x, U_2) \leq 2$, there exists an out-neighbor of x' in U_2 which is not an out-neighbor of x (denoted by v'_1). Now $d^+(y, U_2) \leq 1$, $d^+(z, U_2) \leq 1$. There exists an out-neighbor of y' in U_2 different from v'_1 which is not an out-neighbor of y (denoted by v'_2). There exists an out-neighbor of z' in U_2 different from v'_1, v'_2 . Now we have three cycles $(xx'v'_1x)$, $(yy'v'_2y)$ and $(zz'v'_3v_iz)$ ($i = 1$ if $v_1 \rightarrow z$ and $i = 2$ if $v_2 \rightarrow z$) to extend C_i and C_j , a contradiction. If $d^+(x, U_2) = 3$, then $d^+(y, U_2) = d^+(z, U_2) = 0$. Let v'_1 be one out-neighbor of x in U_2 . There exists an out-neighbor of y' in U_2 different from v'_1 . There exists an out-neighbor of z' in U_2 different from v'_1, v'_2 . Now we have three cycles $(v_ixv'_1v_i)$ ($i = 1$ if $v_1 \rightarrow x$ and $i = 2$ if $v_2 \rightarrow x$), $(yy'v'_2y)$ and $(zz'v'_3z)$ to extend C_i and C_j , which poses a contradiction.

In the following proof we assume that $U_1 = \{v_1, v_2, v_3\}$.

(3) $k = 4$

When $k = 4$, we have $\delta^+(T) \geq 7$, and there are three 3-cycles in \mathcal{F} , we call them C_1, C_2 and C_3 . As $|U_1| = 3$, we know that there are at least $3 \cdot 7 - 3 = 18$ arcs from U_1 to W , at most $3 \cdot 9 - 18 = 9$ arcs from W to U_1 . There are at least $9 \cdot 7 - \frac{1}{2} \cdot 9 \cdot 8 = 27$ arcs from W to U , and at least $27 - 9 = 18$ arcs from W to U_2 . Therefore, 3-cycles in \mathcal{F} cannot be 2-m, 2-m, and 3-c respectively. Otherwise, there are at most $7 + 7 + 3 = 17$ arcs from W to U_2 .

Claim 8. *At least two 3-cycles in \mathcal{F} are 3-c.*

Proof. Because $d^+(U_1) \geq 7$, there are at least 7 arcs from u_1 to W . So without loss of generality, we suppose $d^+(v_1, C_1) = 3$.

(3.1) If $d^+(\{v_2, v_3\}, C_1) = 0$, then $d^+(v_2, C_2 \cup C_3) \geq 6$, and $d^+(v_3, C_2 \cup C_3) \geq 5$. Thus, C_2 and C_3 are 3-c;

(3.2) If $d^+(\{v_2, v_3\}, C_1) \geq 1$, then C_1 is a 3-c cycle. Note that there are at least 4,3,2 arcs from v_1, v_2, v_3 to $C_2 \cup C_3$ respectively. As Claim 7 goes, there are at least one 3-c cycle in $C_2 \cup C_3$. Consequently, the claim is true. \square

Without loss of generality, let C_1 and C_2 be 3-c cycles. There does not exist a 2-matching from U_1 to C_3 , which means $d^+(U_1, C_3) \leq 3$. As a result, $d^+(U_1, C_1 \cup C_2) \geq 15$, and thus one of C_1 and C_2 (let it be C_1) is 3-m.

Moreover, we can know there are at least $6 \cdot 7 - \frac{1}{2} \cdot 6 \cdot 5 = 27$ arcs from $C_1 \cup C_2$ to $U \cup C_3$. Among them at most $18 - 15 = 3$ to U_1 , and $2 \cdot 3 = 6$ to U_2 . Thus there are at least 18 remaining arcs which must go to C_3 . Therefore there exists a 3-matching from C_1 to C_3 .

As the last step, we know $d^+(C_3, U_2) \geq 18 - 3 - 3 = 12$, hence there exists a 3-matching from C_3 to U_2 . Consequently, there are 3-matchings from U_1 to C_1 , C_1 to C_3 , and C_3 to U_2 . This contradicts Claim 3.

(4) $k = 5$

When $k = 5$, we have $\delta^+(T) \geq 9$, and there are four 3-cycles in \mathcal{F} , we call them C_1, C_2, C_3 and C_4 . Similar to case (3), we get $d^+(U_1, W) \geq 24$, $d^+(W, U_1) \leq 12$, $d^+(W, U) \geq 42$, and $d^+(W, U_2) \geq 30$. Therefore, 3-cycles in \mathcal{F} cannot all be 2-m cycles. Otherwise, $d^+(W, U_2) \leq 4 \cdot 7 = 28$.

Claim 9. *At least two cycles in \mathcal{F} are 3-c cycles.*

Proof. As there are at least 9 arcs from u_1 to W , we suppose $d^+(u_1, C_1) = 3$ without loss of generality.

(4.1) If $d^+(\{v_2, v_3\}, C_1) = 0$, then $d^+(v_2, C_2 \cup C_3 \cup C_4) \geq 8$, and $d^+(v_3, C_2 \cup C_3 \cup C_4) \geq 7$. Without loss of generality, let $v_2 \rightarrow C_2 \cup C_3$. As we know $d^+(v_3, C_2 \cup C_3) \geq 4$, then u_3 is adjacent to both cycles. Thus, C_2, C_3 are 3-c cycles.

(4.2) If $d^+(v_2, v_3, C_1) \geq 1$, then C_1 is a 3-c cycle. There are at least 6,5,4 arcs from v_1, v_2, v_3 to $C_2 \cup C_3 \cup C_4$ respectively. By Claim 7, there are at least one 3-c cycle in $C_2 \cup C_3 \cup C_4$. Consequently, the claim is true. \square

Without loss of generality, let C_1 and C_2 be 3-c cycles. Then there exist at least 3,2,1 arcs from v_1, v_2, v_3 to $C_3 \cup C_4$ respectively. Therefore one of

these two cycles, let it be C_3 , is 2-m. Then C_4 is not 2-m, which means that $d^+(U_1, C_4) \leq 3$. As a result, $d^+(U_1, C_1 \cup C_2 \cup C_3) \geq 21$. We observe that for any $C_i, i = 1, 2, 3$, if C_i is not a 3-m cycle, $d^+(U_1, C_i) \leq 6$. Hence at least one of C_1, C_2, C_3 is 3-m. Let this cycle be C_1 .

Moreover, we notice that $d^+(C_1 \cup C_2 \cup C_3, U_1) \leq 3 \cdot 9 - 21 = 6$, $d^+(C_1 \cup C_2 \cup C_3, U_2) \leq 3 + 3 + 7 = 13$, and $d^+(C_1 \cup C_2 \cup C_3, T \setminus (C_1 \cup C_2 \cup C_3)) \geq 9 \cdot 9 - \frac{1}{2} \cdot 9 \cdot 8 = 45$. Thus $d^+(C_1 \cup C_2 \cup C_3, C_4) \geq 45 - 13 - 6 = 26$, which means that $d^+(C_1, C_4) \geq 8$, and there exists a 3-matching from C_1 and C_4 .

As the last step we know that $d^+(C_4, U_2) \geq 30 - 13 = 17$. Hence there exists a 3-matching from C_4 to U_2 . So far, we have found 3-matchings from U_1 to C_1 , C_1 to C_4 , and C_4 to U_2 . This contradicts Claim 3.

(5) $k = 6$

When $k = 6$, we have $\delta^+(T) \geq 11$, and there are five 3-cycles in \mathcal{F} , we call them C_1, C_2, C_3, C_4 and C_5 . Similar to case (3), we can get $d^+(U_1, W) \geq 30$, $d^+(W, U_1) \leq 15$, $d^+(W, U) \geq 60$, and $d^+(W, U_2) \geq 45$. Therefore, 3-cycles in \mathcal{F} cannot all be 2-m cycles. Otherwise, $d^+(W, U_2) \leq 5 \cdot 7 = 35$.

Claim 10. *There are either at least three 3-c cycles, or two 3-c and two 2-m cycles in \mathcal{F} .*

Proof. As there are at least 11 arcs from u_1 to W , we suppose $d^+(v_1, C_1) = 3$ without loss of generality.

(5.1) If $d^+(\{v_2, v_3\}, C_1) = 0$, then $d^+(v_2, C_2 \cup C_3 \cup C_4 \cup C_5) \geq 10$, and $d^+(v_3, C_2 \cup C_3 \cup C_4 \cup C_5) \geq 9$. Without loss of generality, let $v_2 \rightarrow C_2 \cup C_3$. We know that $d^+(v_3, C_2 \cup C_3) \geq 3$.

(5.1.1) If v_3 is connected to both C_2 and C_3 , then C_2, C_3 are both 3-c cycles.

(5.1.2) If v_3 has no arc to one of these two cycles (let it be U_2), then $v_3 \rightarrow C_3 \cup C_4 \cup C_5$, and $d^+(v_2, C_4 \cup C_5) \geq 4$. Now, C_4 and C_5 are 3-c cycles.

(5.2) If $d^+(\{v_2, v_3\}, C_1) \geq 1$, then C_1 is a 3-c cycle. Note that there are at least 8, 7, 6 arcs from v_1, v_2, v_3 to $C_2 \cup C_3 \cup C_4 \cup C_5$ respectively. As Claim 7 goes, there are at least one 3-c cycle in $C_2 \cup C_3 \cup C_4 \cup C_5$.

As a result, we know that there exist at least two 3-c cycles in \mathcal{F} (let them be C_1 and C_2). Now there are at least 5, 4, 3 arcs from v_1, v_2, v_3 to $C_3 \cup C_4 \cup C_5$ respectively. Without loss of generality, we assume that $d^+(\{v_1, v_2, v_3\}, C_3) \geq 4$, C_3 is then a 2-m cycle.

(5.2.1) If C_3 is a 3-c cycle, we get three 3-cycles C_1, C_2 and C_3 , and the claim is proven.

(5.2.2) If C_3 is not a 3-c cycle, then each vertex of $\{u_1, u_2, u_3\}$ has at least two arcs to C_3 . This means there are at least 3,2,1 arcs from v_1, v_2, v_3 to $C_4 \cup C_5$ respectively. Thus without loss of generality, C_4 is 2-m. Hence the claim is also true. \square

Now consider the case when C_1, C_2, C_3 are 3-c cycles. Now we assume C_4, C_5 are not 2-m. Otherwise, it can be dealt with as in the next case. Thus, $d^+(U_1, C_4) \leq 3$, and $d^+(U_1, C_5) \leq 3$. As a result, we have $d^+(U_1, C_1 \cup C_2 \cup C_3) \geq 30 - 3 \cdot 2 = 24$. This result indicates that without loss of generality, C_1, C_2 are both 3-m cycles, and $d^+(C_1 \cup C_2 \cup C_3, U_1) \leq 3 \cdot 9 - 24 = 3$.

According to Claim 6, $d^+(C_1 \cup C_2 \cup C_3, U_2) \leq 9$, and $d^+(C_1 \cup C_2 \cup C_3, T \setminus (C_1 \cup C_2 \cup C_3)) \geq 9 \cdot 11 - \frac{1}{2} \cdot 9 \cdot 8 = 63$. Hence $d^+(C_1 \cup C_2 \cup C_3, C_4 \cup C_5) \geq 63 - 9 - 3 = 51$. So one of C_4, C_5 (let it be C_4), has 3-matchings to both C_1 and C_2 .

As the last step, we have $d^+(C_4 \cup C_5, U_2) \geq 45 - 9 = 36$. Assume that $d^+(C_4, U_2) \geq 18$, then there exists a 3-matching from C_4 to U_2 . Now, there are 3-matchings from U_1 to C_1, C_1 to C_4 , and C_5 to U_2 . This contradicts Claim 3.

Next, we consider the case where C_1, C_2 are 3-c, and C_3, C_4 are 2-m. Obviously, C_5 is not 2-m and thus $d^+(U_1, C_5) \leq 3$. Therefore, $d^+(U_1, C_1 \cup C_2 \cup C_3 \cup C_4) \geq 27$, $d^+(C_1 \cup C_2 \cup C_3 \cup C_4, U_1) \leq 9$, $d^+(C_1 \cup C_2 \cup C_3 \cup C_4, U_2) \leq 3 \cdot 2 + 7 \cdot 2 = 20$, and $d^+(C_1 \cup C_2 \cup C_3 \cup C_4, T \setminus (C_1 \cup C_2 \cup C_3 \cup C_4)) \geq 12 \cdot 11 - \frac{1}{2} \cdot 12 \cdot 11 = 66$. Consequently, we have $d^+(C_1 \cup C_2 \cup C_3 \cup C_4, C_5) \geq 37$, which is obviously impossible.

Now, when $k = 2, 3, 4, 5, 6$, we can always reach a contradiction. Thus the proof of Theorem 2.1 is completed.

3 Disjoint Cycles in Round-Robin Tournaments

In this section we suppose T is a round-robin tournament.

Definition 3.1. If wv and vu are both arcs in T , we call vu an opposite arc of wv .

To prove Theorem 1.5, we prove the following theorem in order to “extract” an tournament from an round-robin tournament:

Theorem 3.2. For any positive integer $d \geq 1$, any round-robin tournament T satisfying $\delta^+(T) \geq 2d$ contains a tournament T' satisfying $\delta^+(T') \geq d$.

Proof. We delete the arcs of T step by step until it becomes an empty graph, and reconstruct T' .

- For any pair of vertices u and v , if there are two arcs from u to v , we delete exactly one of them and reach a digraph T'_1 , we delete both of them and reach a digraph T_1 .
- If there exists a cycle C_1 in T_1 , we delete arcs in C_1 and opposite arcs of C_1 . If there still exists a cycle C_2 , we delete arcs in C_2 and opposite arcs of C_2 . Continue this process until there are no cycles. We get a series of edge-disjoint cycles C_1, C_2, \dots, C_p and finally we reach an acyclic digraph T_2 .
- If there exists some paths in T_2 , we find a longest path P_1 , and delete arcs in P_1 and opposite arcs of P_1 . If there still exists some paths, we find a longest path P_2 , and delete arcs in P_2 and opposite arcs of P_2 . Continue this process until we reach an empty graph. We get a series of path P_1, P_2, \dots, P_q .

Let T' be $T'_1 \cup C_1 \cup C_2 \cup \dots \cup C_p \cup P_1 \cup P_2 \cup \dots \cup P_q$. Obviously T' is a tournament. Next we only need to prove $\delta^+(T') \geq d$.

For any vertex v , if v is not an endpoint of any path P_i , then the out-degree of v in T must be an even number $2k$, and its out-degree in T' must be k .

If v is an endpoint of some paths in $\{P_1, \dots, P_q\}$, then suppose P_i is the path which has the smallest subscription. Before deleting arcs in P_i , the digraph is acyclic, and P_i is a longest path. Thus after deleting arcs in P_i and opposite arcs of P_i , v becomes an isolated vertex. Hence v is not an endpoint of $P_{i+1}, P_{i+2}, \dots, P_q$. This means that v is the endpoint of exactly one path in $\{P_1, \dots, P_q\}$. Consequently, the out-degree of v in T must be an odd number $2k + 1$, and its out degree in T' must be k or $k + 1$. As $2k + 1 \geq 2d$, we have $k \geq d$. Then we complete the proof. \square

4 List of Flaws

In [13], the author proved Theorem 1.3 when $q = 4$. We restate it as the following theorem:

Theorem 4.1. *For any positive integer $k \geq 1$, any tournament with minimum out-degree at least $3k - 1$ contains k disjoint cycles of length 4.*

Actually the author proved a slightly stronger argument than Theorem 4.1.

Theorem 4.2. *For any positive integer $k \geq 1$, if T is a tournament which has $\delta^+(T) \geq 3k - 1$, then for any $k - 1$ disjoint 4-cycles $\mathcal{F} = \{C_1, C_2, \dots, C_{k-1}\}$, let $W = V(C_1) \cup V(C_2) \cup \dots \cup V(C_{k-1})$ and $U = V(T) \setminus W$, there exist k disjoint 4-cycles whose vertex set intersects U on at most seven vertices.*

We carefully read [13], and found the proofs in [13] are mostly correct. However, there are some typos in [13]. And some proofs are not rigorous. Here we correct those typos and offer rigorous proofs of some claims, so that other readers can understand the result easier.

The author proved Theorem 4.2 by induction on k . For $k \geq 2$, the author assumed that there exist $k - 1$ disjoint 4-cycles $\mathcal{F} = \{C_1, C_2, \dots, C_{k-1}\}$, let $W = V(C_1) \cup V(C_2) \cup \dots \cup V(C_{k-1})$ and $U = V(T) \setminus W$, there do not exist k disjoint 4-cycles whose vertex set intersects U on at most seven vertices.

According to Rédei's Theorem (see Theorem 2.2.4 of [1]), any tournament has a Hamiltonian dipath. We can order the vertices of U as v_1, v_2, \dots, v_n , such that for every i , $v_{i+1} \rightarrow v_i$. Obviously the sub-digraph of T induced by U has no 4-cycles, thus there is no arc from v_i to v_j when $j - i \geq 3$.

From Claim 1 to Claim 6 the author supposed that $U_1 = \{v_1, v_2, \dots, v_6\}$, $S = \{v_7\}$, and $U_2 = U \setminus (U_1 \cup S)$.

Claim 1. *For any 4-cycle $C \in \mathcal{F}$, every 3-path of C has at most six breakers.*

Claim 2. *Let $C \in \mathcal{F}$. If $d^+(U_1, C) \geq 13$, then there exists a 3-matching from U_1 to C .*

Claim 3. *Let $C \in \mathcal{F}$. If $d^+(C, U_2) \geq 7$, then there exists a 2-matching from C to U_2 . If $d^+(C, U_2) \geq 13$, then there exists a 3-matching from C to U_2 .*

Claim 4. *Suppose C_i and C_j are two 4-cycles in \mathcal{F} . If $d^+(U_1, C_i) \geq 13$ and $d^+(C_j, U_2) \geq 13$, then $d^+(C_i, C_j) \leq 12$.*

Claim 5. *Let S_1 and S_2 be two disjoint vertex sets satisfying $|S_1| \leq 4$ and $|S_2| = 4$. If $d^+(S_1, S_2) \geq 5$, then there exists a 2-matching from S_1 to S_2 . If $d^+(S_1, S_2) \geq 9$, then there exists a 3-matching from S_1 to S_2 .*

Claim 6. Let $C \in \mathcal{F}$. If there exists a 3-matching from U_1 to C , then there is no 3-matching from C to U_2 . Conversely, if there exists a 3-matching from C to U_2 , then there is no 3-matching from U_1 to C .

After proving Claim 1 to Claim 6, the author proved $k \leq 7$. After that the author proved that $k = 2, 3, 4, 5, 6$ are all impossible. During the course of the proofs the author supposed that $U_1 = \{v_1, v_2, v_3, v_4\}$, $S = \{v_5, v_6\}$, and $U_2 = U \setminus (U_1 \cup S)$.

We point that the author mainly has four flaws in the proof:

Flaw 1. The proof of Claim 4 is not rigorous.

In order to offer a rigorous proof of Claim 4, firstly we prove another claim:

Claim 4.1. Suppose C_i and C_j are two 4-cycles in \mathcal{F} . If $\{v_{i_1}u_1, v_{i_2}u_2, v_{i_3}u_3\}$ is a 3-matching from U_1 to C_i and $\{u'_1v_{j_1}, u'_2v_{j_2}, u'_3v_{j_3}\}$ is a 3-matching from C_j to U_2 , then there does not exist a 3-matching from $\{u_1, u_2, u_3\}$ to $\{u'_1, u'_2, u'_3\}$.

Proof. Without loss of generality suppose that $i_3 > i_2 > i_1$ and $j_1 > j_2 > j_3$. Assume that there exists a 3-matching from $\{u_1, u_2, u_3\}$ to $\{u'_1, u'_2, u'_3\}$.

- (1) If $u_3u'_3$ is in the matching, we extend C_i and C_j by $(v_{i_3}u_3u'_3v_{j_3} \dots v_{i_3})$, $(v_{i_2}u_2u'_2v_{j_2}v_{i_2})$ and $(v_{i_1}u_1u'_1v_{j_1}v_{i_1})$ (if $u_1u'_1$ is in the matching), or by $(v_{i_3}u_3u'_3v_{j_3} \dots v_{i_3})$, $(v_{i_2}u_2u'_1v_{j_1}v_{i_2})$ and $(v_{i_1}u_1u'_2v_{j_2}v_{i_1})$ (if $u_1u'_2$ is in the matching);
- (2) If $u_3u'_2$ is in the matching, we extend C_i and C_j by $(v_{i_3}u_3u'_2v_{j_2}v_{i_3})$, $(v_{i_2}u_2u'_3v_{j_3}v_{i_2})$, and $(v_{i_1}u_1u'_1v_{j_1}v_{i_1})$ (if $u_1u'_1$ is in the matching), or by $(v_{i_3}u_3u'_2v_{j_2}v_{i_3})$, $(v_{i_2}u_2u'_1v_{j_1}v_{i_2})$, and $(v_{i_1}u_1u'_3v_{j_3}v_{i_1})$ (if $u_1u'_3$ is in the matching);
- (3) If $u_3u'_1$ is in the matching, we extend C_i and C_j by $(v_{i_3}u_3u'_1v_{j_1}v_{i_3})$, $(v_{i_2}u_2u'_3v_{j_3}v_{i_2})$, and $(v_{i_1}u_1u'_2v_{j_2}v_{i_1})$ (if $u_1u'_2$ is in the matching), or by $(v_{i_3}u_3u'_1v_{j_1}v_{i_3})$, $(v_{i_2}u_2u'_2v_{j_2}v_{i_2})$, and $(v_{i_1}u_1u'_3v_{j_3}v_{i_1})$ (if $u_1u'_3$ is in the matching).

□

Now we are ready to offer the proof of Claim 4.

Proof. Let C_i be $(xyzt_x)$ and C_j be $(x'y'z't'x')$ respectively. Because $d^+(U_1, C_i) \geq 13$ and $d^+(C_j, U_2) \geq 13$, by Claim 2 and Claim 3 there exists one 3-matching from U_1 to C_i , and one from C_j to U_2 . Suppose they are $\{v_{i_1}x, v_{i_2}y, v_{i_3}z\}$ and $\{x'v_{j_1}, y'v_{j_2}, z'v_{j_3}\}$ without loss of generality.

Assume that $d^+(C_i, C_j) \geq 13$. As $d^+(\{t\}, C_j) \leq 4$, we have $d^+(\{x, y, z\}, C_j) \geq 9$. We consider three sub-cases:

(1) At least two of x, y, z have 4 arcs to C_j . Without loss of generality, we suppose that $d^+(x, C_j) = 4, d^+(y, C_j) = 4$. Since $d^+(\{x, y, z\}, C_j) \geq 9$, we have $d^+(z, C_j) \geq 1$. If z dominates at least one vertex in $\{x', y', z'\}$, then there exists a 3-matching from $\{x, y, z\}$ to $\{x', y', z'\}$. This contradicts Claim 4.1. If z does not dominate anyone of $\{x', y', z'\}$, then $z \rightarrow t'$. In this case we can construct an “almost” 3-matching $\{xy', yz', z't'x'\}$ from $\{x, y, z\}$ to $\{x', y', z'\}$. This can cause a contradiction in the same way as the proof of Claim 4.1;

(2) Exactly one of x, y, z has 4 arcs to C_j . Without loss of generality we assume that $d^+(x, C_j) = 4, d^+(y, C_j) = 3$, and $d^+(z, C_j) \geq 2$. As a result, z dominates at least one vertex in $\{x', y', z'\}$. Suppose $z \rightarrow z'$. y dominates at least one vertex in $\{x', y'\}$. Suppose $y \rightarrow y'$. At last, we have $x \rightarrow x'$. Hence we have a 3-matching from C_i to C_j , which contradicts Claim 4.1;

(3) All of x, y, z have at most 3 arcs to C_j . In this case we have $d^+(x, C_j) = d^+(y, C_j) = d^+(z, C_j) = 3$, and thus $d^+(t, C_j) = 4$. Hence, x dominates at least one vertex in $\{x', y', z'\}$. Suppose $x \rightarrow x'$. y dominates at least one vertex in $\{y', z'\}$. Suppose $y \rightarrow y'$. At last, we have $t \rightarrow z'$, which means that there exists an “almost” 3-matching $\{xx', yy', ztz'\}$ from $\{x, y, z\}$ to $\{x', y', z'\}$. This can cause a contradiction in the same way as the proof of Claim 4.1. \square

Flaw 2. *There are four typos in the proof of Claim 6.*

(1) In sub-case (1), third paragraph, sixth row, “ $C = (v_3u_{j_1} \dots u_{j_2}u_{k_2}v_3)$ ” should be “ $(v_3u_{j_1} \dots u_{j_2}u_{k_2}v_3)$.”

(2) In sub-case (2), third paragraph, sixth row, “ $B = (v_3u_{j_2} \dots u_{j_1}u_{k_1})$ ” should be “ $(v_3u_{j_2} \dots u_{j_1}u_{k_1}v_3)$.”

(3) In sub-case (3), third paragraph, third row, “ $(v_1u_{j_3} \dots u_7u_{k_3}v_1)$ ” should be “ $(v_1u_{j_3} \dots u_7u_{k_1}v_1)$.”

(4) In sub-case (3), third paragraph, fifth row, “ $(v_1u_{j_3} \dots u_7u_{k_3}v_1)$ ” should be “ $(v_1u_{j_3} \dots u_7u_{k_1}v_1)$.”

Flaw 3. *The proof of “ $k = 4$ is impossible” is not rigorous.*

Proof. When $k = 4$, we have $\delta^+(T) \geq 11$, and there are three cycles in \mathcal{F} . On account of $d^+(u_1, \mathcal{F}) \geq 10$ and $d^+(u_2, \mathcal{F}) \geq 9$, there exists a cycle C in \mathcal{F} such that $d^+(U_1, C) \geq 7$. Let C be $(xyzt)$. Without loss of generality, we suppose $x \rightarrow y, y \rightarrow z, z \rightarrow t, x$ and $t \rightarrow x$.

(1) $d^+(v_1, C) = 4, d^+(v_2, C) \geq 3$. Obviously at most one vertex in C is not dominated by v_2 . There are four sub-cases:

Firstly, $v_2 \rightarrow \{x, y, z\}$. We will have $d^+(x, U_2) \geq 2$, $d^+(t, U_2) \geq 1$. There exist arcs from x to U_2 xv_i, xv_j , ($i < j$) and an arc from t to U_2 tv_l . Therefore, C can be extended by: (a) $(v_2ytv_l v_2)$ and $(v_1zxv_j v_1)$ ($j \neq l$), (b) $(v_2zxv_i v_2)$ and $(v_1y tv_l v_1)$ ($j = l$).

Secondly, $v_2 \rightarrow \{x, y, t\}$. We will have $d^+(x, U_2) \geq 2$, $d^+(t, U_2) \geq 2$. There exist arcs from x to U_2 xv_i, xv_j , ($i < j$) and arcs from t to U_2 tv_l, tv_m , ($l < m$). Therefore, C can be extended by: (a) $(v_2ytv_l v_2)$ and $(v_1zxv_j v_1)$ ($j \neq l$), (b) $(v_2xv_j \dots v_2)$ and $(v_1y tv_m v_1)$ ($j = l$).

Thirdly, $v_2 \rightarrow \{y, z, t\}$. We will have $d^+(x, U_2) \geq 1$, $d^+(t, U_2) \geq 2$. There exists an arc x to U_2 xv_j and arcs from t to U_2 tv_l, t, v_m , ($l < m$). Therefore, C can be extended by: (a) $(v_2ytv_l v_2)$ and $(v_1zxv_j v_1)$ ($j \neq l$), (b) $(v_2zxv_j v_2)$ and $(v_1y tv_m v_1)$ ($j = l$).

Fourthly, $v_2 \rightarrow \{x, z, t\}$. We will have $d^+(x, U_2) \geq 2$, $d^+(t, U_2) \geq 2$. There exist arcs from x to U_2 xv_i, xv_j , ($i < j$) and arcs from t to U_2 tv_l, tv_m , ($l < m$). Therefore, C can be extended by: (a) $(v_2zxv_i v_2)$ and $(v_1y tv_m v_1)$ ($m \neq i$), (b) $(v_2zxv_j v_2)$ and $(v_1y tv_m v_1)$ ($m = i$).

(2) $d^+(v_1, C) \geq 3$, $d^+(v_2, C) = 4$. We can exchange the role of v_1 and v_2 above and get a proof.

Consequently, we can extend C in all cases, which poses a contradiction. \square

Flaw 4. *The proof of “ $k = 3$ is impossible” is not rigorous.*

Proof. When $k = 3$, we have $\delta^+(T) \geq 8$, and there are two cycles in \mathcal{F} . On account of $d^+(u_1, \mathcal{F}) \geq 7$ and $d^+(u_2, \mathcal{F}) \geq 6$, there exists a cycle C in \mathcal{F} such that $d^+(U_1, C) \geq 7$. Hence, at most one vertex in C is not dominated by one vertex in U_1 . Let C be $(xyzt)$. Without loss of generality, we suppose $x \rightarrow y$, $y \rightarrow z$, $z \rightarrow t$, x and $t \rightarrow x$.

It is easy to see that $d^+(x, U_2) \geq 2$ and $d^+(t, U_2) \geq 2$. Let one out-neighbor of x in U_2 be v_i . Let one out-neighbor of t in U_2 different from v_i be v_j . Without loss of generality, we suppose $v_1 \rightarrow y$ and $v_2 \rightarrow z$. Then two 4-cycles $(v_1y tv_j v_1)$ and $(v_2zxv_i v_2)$ can extend C , which poses a contradiction. \square

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