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Title: On sharp upper estimate of lattice points: Yau geometric conjecture

ON SHARP UPPER ESTIMATE OF LATTICE POINTS: YAU GEOMETRIC CONJECTURE

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Abstract. The simple problem of counting the number of lattice points in n-dimensional simplexes, in fact has a much greater significance in singularity theory and number theory. The number of lattice points is equal to the geometric genus of an isolated singularity of a weighted homogeneous polynomial. This paper estimates the number of lattice points in a seven-dimensional simplex, and proves the Yau Geometric Conjecture in seven dimensions, which gives an upper bound to the number. We do so by dividing the simplex to several layers of cross section sixth-dimensional simplexes and sums up the upper bound of lattice points in each layer. This proof provides potential insight to extend the upper bound estimate to the general n-dimensional case.

Keywords: Yau Geometric Conjecture, lattice points, sharp upper bound.
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1. Introduction

For fixed real numbers $a_1 \geq a_2 \geq \cdots \geq a_n \geq 1$, an n-dimensional real right-angled simplex Δ_n is defined by the inequality

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1, \text{ where } x_1, \cdots, x_n \geq 0.$$

We define P_n to be the number of positive integral points in Δ_n, i.e.,

$$P_n = \# \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{Z}_+^n \,\middle|\, \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1 \right\}.$$

We use Q_n to denote the number of non-negative integral points in Δ_n, i.e.,

$$Q_n = \# \left\{ (x_1, x_2, \cdots, x_n) \in (\mathbb{Z}_+ \cup \{0\})^n \,\middle|\, \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1 \right\}.$$

Note that the numbers P_n and Q_n are intimately linked through the equation (see [5])

$$P_n(a_1, a_2, \cdots, a_n) = Q_n(a_1(1-a), a_2(1-a), \cdots, a_n(1-a)),$$

where a is defined as $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$. Hence, computing P_n is equivalent to computing Q_n.

Finding a sharp estimate of P_n (Q_n) for real right-angled simplex is related to many other mathematical problems. In number theory, recall that a function known as the Dickman–de Bruijn function $\psi(x, y)$ is defined as the number of positive integers n such that $n \leq x$, and all of the prime factors of n are at most y, where x and y are positive integers.

The numbers P_n and Q_n are intimately related to the *Dickman–de Bruijn function* in [7]. The connection, described by Luo-Yau-Zuo [19], is most readily observed by noting that, assume that $p_1 < p_2 < \cdots < p_k$ are the primes less than or equal to y, it follows from

$$p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \leq x,$$

that

$$e_1 \log p_1 + e_2 \log p_2 + \cdots + e_k \log p_k \leq \log x.$$

It can be rewritten as,

$$\frac{e_1}{\frac{\log x}{\log p_1}} + \frac{e_2}{\frac{\log x}{\log p_2}} + \cdots + \frac{e_k}{\frac{\log x}{\log p_k}} \leq 1,$$

that is exactly an expression in the format of the condition in the definition of Q_n. Hence, enumerating the Dickman–de Bruijn function is equivalent to calculating Q_n.

The connections between P_n and Q_n and other areas of number theory, including primality testing, determining large gaps in the sequence of the primes, and discovering new algorithms for prime factorization are described by Granville [7]. Furthermore, Lin, et. al. [17] describes the connection between P_n, Q_n and singularity theory. Therefore, it is a very interesting important problem to find and estimate P_n and Q_n.

The quest to compute P_n and Q_n dates back to 1899, when Pick [23] discovered the famous *Pick's theorem,* or a formula for Q_2. This formula tells us that the number of lattice points inside Δ_2 is determined by the area of the Δ_2 and the number of integer points on the boundary.

$$Q_2 = area(\Delta_2) + \frac{|\partial \Delta_2 \cap \mathbb{Z}^2|}{2} + 1,$$

where $\partial \Delta_2$ represents the boundary of the triangle, and $|\partial \Delta \cap \mathbb{Z}^2|$ represents the number of integral points on the boundary. Mordell [22] continued by discovering a formula for Q_3 using Dedekind sums. Erhart [6] followed with the discovery of *Ehrhart polynomials*, which facilitate the calculation of Q_n. The theory of *Ehrhart polynomials* can be seen as a higher-dimensional generalization of *Pick's theorem.* More formally, consider a lattice $\mathcal{L}$ in $\mathbb{R}^n$ and a d-dimensional polytope P in $\mathbb{R}^n$ with the property that all vertices of the polytope are points of the lattice. For any positive integer t, let tP be the t-fold dilation of P, i.e., the polytope formed by multiplying each vertex coordinate, in a basis for the lattice, by a factor of t, and let

$$L(P, t) = \#(tP \cap \mathcal{L})$$

be the number of lattice points contained in the polytope tP. Ehrhart [6] showed that L is a rational polynomial of degree d in t, i.e. there exist rational numbers $L_0(P), \ldots, L_d(P)$ such that:

$$L(P, t) = L_d(P)t^d + L_{d-1}(P)t^{d-1} + \cdots + L_0(P)$$

for all positive integers t. The Ehrhart polynomial of the interior of a closed convex polytope P can be computed as:

$$L(\mathrm{int}(P), t) = (-1)^d L(P, -t)$$

where d is the dimension of P. This result is known as Ehrhart-Macdonald reciprocity.

However, these *Ehrhart polynomials* are only useful if every coefficient is known, a condition that is extremely difficult to meet in the general case, i.e., $a_1, a_2, \cdots, a_n$ are not integers.

The difficulty of this problem eventually led mathematicians to start trying to bound P_n and Q_n instead of finding precise formulas. Lehmer [13] found that if $a = a_1 = a_2 = \cdots = a_n$, then

$$Q_n = \binom{\lfloor a \rfloor + n}{n},$$

where $\lfloor x \rfloor$(round down) denotes the integral part of a real number x. This formula naturally yields a nice definition of sharpness of an estimate T_n of Q_n. We consider the estimate sharp if and only if

$$T_n|_{a_1=a_2=\cdots=a_n=a\in\mathbb{Z}} = \binom{a+n}{n}.$$

In other words, any upper or lower bound is sharp if and only if its estimate is exact when $a_1 = a_2 = \cdots = a_n \in \mathbb{Z}$.

Many other mathematicians (Hardy and Littlewood [9, 10, 11]; Lehmer [8]; Spencer [25, 26]; Brion and Vergneetc [4]; Beck [1, 2, 3]; etc) had also studied this problem from other point of view, for more details one can see the introduction section in [29]. Here we want to introduce another estimate which is the two-parts GLY Conjecture, an upper bound for P_n formulated by Lin-Yau [16]. Before we state the GLY Conjecture, we need to first introduce the *signed Stirling numbers of the first kind* and a notation A^n_k for elementary symmetric polynomials as follows [34].

Definition 1.1. *The (signed) Stirling numbers of the first kind $s(n,k)$ are defined by the following property:*

$$\prod_{i=0}^{n-1}(x-i) = \sum_{k=0}^{n} s(n,k)x^k.$$

Define $s(0,0) = 1$ and $s(n,0) = s(0,n) = 0$.

Definition 1.2. *Let $a_1, a_2, \cdots, a_n$ be positive real numbers. We denote*

$$A^n_{n-k} = \left(\prod_{i=1}^{n} a_i\right) \sum_{1\le i_1<i_2<\cdots<i_k\le n} \frac{1}{a_{i_1}a_{i_2}\cdots a_{i_k}}.$$

Thus, A^n_{n-k} is the elementary symmetric polynomial of $a_1, a_2, \cdots, a_n$ with degree $n-k$.

Conjecture 1.3 (GLY (Granville-Lin-Yau) Conjecture)**.** *Let*

$$P_n = \#\left\{(x_1, x_2, \cdots, x_n) \in \mathbb{Z}^n_+ : \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \le 1\right\},$$

where $a_1 \ge a_2 \ge \cdots \ge a_n \ge 1$ are real numbers. If $n \ge 3$, then:

(1) Rough (general) upper estimate: For all $a_n > 1$,

$$n!\, P_n < q_n := \prod_{i=1}^{n}(a_i - 1).$$

(2) Sharp upper estimate: For $n \ge 3$, if $a_1 \ge a_2 \ge \cdots \ge a_n \ge n-1$, then

$$n!\, P_n \le\ A^n_0 + \frac{s(n,n-1)}{n}A^n_1 + \sum_{l=1}^{n-2}\frac{s(n,n-1-l)}{\binom{n-1}{l}}A^{n-1}_l,$$

and equality holds if and only if $a_1 = a_2 = \cdots = a_n \in \mathbb{Z}_+$.

The sharp GLY Conjecture has been proven to be true for $3 \leq n \leq 6$ [31, 12, 29, 15], and the rough GLY upper estimate holds for all n, as proven by Yau and Zhang [32]. Unfortunately, the sharp GLY Conjecture is not true for $n = 7$, and there is a counter-example which was given in [29].

Conjecture 1.4 (Modified GLY Conjecture)**.** *There exists an integer $\alpha(n)$ which depends only on n such that the sharp estimate GLY-Conjecture (2) holds when $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant \alpha(n)$.*

The above Modified GLY Conjecture is studied in Wang Xuejun thesis, he proved that $\alpha(7) = 7$ [28].

In this paper, we will use the following theorem (the modified sharp GLY conjecture for $n = 7$):

Theorem 1.5 (modified GLY Conjecture for $n = 7$ [28])**.** *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 \geq 7$ be real numbers and P_7 be the number of positive integral solutions of*

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} + \frac{x_7}{a_7} \leq 1.$$

Then,

$$\begin{aligned} 7!\,P_7 \leq{} & a_1a_2a_3a_4a_5a_6a_7 - 3(a_1a_2a_3a_4a_5a_6 + a_1a_2a_3a_4a_5a_7 + a_1a_2a_3a_4a_6a_7 + a_1a_2a_3a_5a_6a_7 \\ & + a_1a_2a_4a_5a_6a_7 + a_1a_3a_4a_5a_6a_7 + a_2a_3a_4a_5a_6a_7) + \frac{175}{6}(a_1a_2a_3a_4a_5 + a_1a_2a_3a_4a_6 \\ & + a_1a_2a_3a_5a_6 + a_1a_2a_4a_5a_6 + a_1a_3a_4a_5a_6 + a_2a_3a_4a_5a_6) - 49(a_1a_2a_3a_4 + a_1a_2a_3a_5 \\ & + a_1a_2a_3a_6 + a_1a_2a_4a_5 + a_1a_2a_4a_6 + a_1a_2a_5a_6 + a_1a_3a_4a_5 + a_1a_3a_4a_6 + a_1a_3a_5a_6 \\ & + a_1a_4a_5a_6 + a_2a_3a_4a_5 + a_2a_3a_4a_6 + a_2a_3a_5a_6 + a_2a_4a_5a_6 + a_3a_4a_5a_6) + \frac{406}{5}(a_1a_2a_3 \\ & + a_1a_2a_4 + a_1a_2a_5 + a_1a_2a_6 + a_1a_3a_4 + a_1a_3a_5 + a_1a_3a_6 + a_1a_4a_5 + a_1a_4a_6 + a_1a_5a_6 \\ & + a_2a_3a_4 + a_2a_3a_5 + a_2a_3a_6 + a_2a_4a_5 + a_2a_4a_6 + a_2a_5a_6 + a_3a_4a_5 + a_3a_4a_6 + a_3a_5a_6 \\ & + a_4a_5a_6) - \frac{588}{5}(a_1a_2 + a_1a_3 + a_1a_4 + a_1a_5 + a_1a_6 + a_2a_3 + a_2a_4 + a_2a_5 + a_2a_6 \\ & + a_3a_4 + a_3a_5 + a_3a_6 + a_4. \end{aligned}$$

Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 \in \mathbb{Z}_+$.

Research over the GLY Conjecture have caused the proposal of another upper bound estimate, the *Yau Number-Theoretic Conjecture* (Conjecture 1.6) made by Yau in 1995.

Conjecture 1.6 (Yau Number-Theoretic Conjecture)**.** *Let $n \geq 3$ be a positive integer, and let $a_1 \geq a_2 \cdots \geq a_n > 1$ be real numbers. If $P_n > 0$, then*

$$n!\,P_n \leq (a_1 - 1)\cdots(a_n - 1) - (a_n - 1)^n + a_n(a_n - 1)\cdots(a_n - (n-1)),$$

and equality holds if and only if $a_1 = \cdots = a_n \in \mathbb{Z}$.

In this paper, we will use the $n = 6$ case of Conjecture 1.6, proven by Liang-Yau-Zuo [14], extensively. We reproduce it as a theorem below for easy reference.

Theorem 1.7 (Yau Number-Theoretic Conjecture for $n = 6$)**.** *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 > 1$ be real numbers. If $P_6 > 0$, then*

$$720P_n \leq \mu - (a_6 - 1)^6 + a_6(a_6 - 1)(a_6 - 2)(a_6 - 3)(a_6 - 4)(a_6 - 5),$$

where $\mu = (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)$. Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 \in \mathbb{Z}$.

For recent progress of Yau Number-Theoretic Conjecture, one can see [14, 37]. In order to state the Yau Geometric Conjecture, we need to first define a *weighted homogeneous polynomial* as follows:

Definition 1.8. *A polynomial $f(x_1, x_2, \cdots, x_n)$ is a weighted homogeneous polynomial if it is a sum of monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ such that, for some fixed positive rational numbers $w_1, w_2, \cdots, w_n$,*

$$\frac{i_1}{w_1} + \frac{i_2}{w_2} + \cdots + \frac{i_n}{w_n} = 1,$$

for every monomial of f. The numbers $w_1, w_2, \cdots, w_n$ are known as the weights of the polynomial.

Furthermore, for any weighted homogeneous polynomial $f(x_1, \cdots, x_n)$ defines an isolated singularity at the origin, i.e., $f = \partial f/\partial x_1 = \cdots = \partial f/\partial x_n = 0$ has only a zero solution, then there are two important invariants associated to f. These two invariants are the Milnor number μ and the geometric genus p_g which can be calculated from the weights $w_1, w_2, \cdots, w_n$ as follows.

$$\mu = (w_1 - 1)(w_2 - 1) \cdots (w_n - 1),$$

see [21] and

$$P_g = \# \left\{ (x_1, x_2, \cdots, x_n) \in \mathbb{Z}_+^n : \frac{x_1}{w_1} + \frac{x_2}{w_2} + \cdots + \frac{x_n}{w_n} \leq 1 \right\},$$

see [20].

The Durfee conjecture, althought the inequality is not sharp, but remains signficant as it relates the above two important invariants of isolated hypersurface singularities by a famous inequality $n!\,\mu < p_g$. Although the Durfee conjecture was verified for weighted homogeneous isolated hypersurface singularity in [32], it is still open for general isolated hypersurface singularity. In 1995, Yau announced his conjecture which proposed a sharp inequality (see Conjecture 1.9). The Yau Geometric Conjecture is harder to prove comparing with the Durfee conjecture. In this paper, we will prove the Yau Geometric Conjecture for seven dimensional weighted homogeneous isolated hypersurface singularity.

Conjecture 1.9 (Yau Geometric Conjecture)**.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ, p_g, and v be the Milnor number, geometric genus, and multiplicity of the singularity $V = \{z : f(z) = 0\}$. Then,*

$$\mu - p(v) \geq n!\, p_g,$$

where $p(v) = (v-1)^n - v(v-1)\cdots(v-n+1)$. Equality holds if and only if f is a homogeneous polynomial.

Remark. *Note that p_g counts the number of positive lattice points in the simplex*

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1,$$

where the a_i are the weights of the weighted *homogeneous polynomial f and $a_1 \geq a_2 \geq \cdots \geq a_n > 1$ (cf. [20]). Thus, the equality case of Conjecture 1.9 is $a_1 = a_2 = \cdots = a_n \in \mathbb{Z}$. Furthermore, Saeki [24] tells us that $v = \lceil a_n \rceil$ (i.e. round up) and it is known that $\mu = (a_1 - 1)(a_2 - 1) \cdots (a_n - 1)$ (cf. [21]). Chen-Yau-Zuo [5] also proved that the* *fractional part of* a_n *has to be one of* $\frac{a_n}{a_1}, \frac{a_n}{a_2}, \cdots,$ *or* $\frac{a_n}{a_{n-1}}$*. Finally, we can also define the polynomial* $p_n(v) = (v-1)^n - v(v-1)\cdots(v-n+1)$*. Thus,*

$$p_7(v) = -1 - 713 + 1743v^2 - 1589v^3 + 700v^4 - 154v^5 + 14v^6, \tag{1}$$

$$p_6(v) = 1 + 114v - 259v^2 + 205v^3 - 70v^4 + 9v^5. \tag{2}$$

Conjecture 1.9 is proven to hold for $3 \le n \le 6$ [5, 30, 16, 18]. In this paper, we prove Conjecture 1.9 for $n = 7$. It extends the Yau Geometric Conjecture to yet another dimension. This is difficult because the number of cases has increased from 6 in the 6-dimensional case to 7 in the 7-dimensional one, adding the number of layers to consider and increasing its complexity. We applied some new analysis techniques are used to prove our main theorem. This has the potential to be able to generalize the Yau Geometric Conjecture to any positive integer n.

Since most of the subcases have specified numeric values and the algebraic expressions are great in size, all computations in this paper were done using Maple 2018. Our main theorem is as follows.

Theorem 1.10 (**Main Theorem**)**.** *Let* $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5 \ge a_6 \ge a_7 > 1$ *be real numbers and let* P_7 *be the number of positive integral solutions of* $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} + \frac{x_7}{a_7} \le 1$*. Define* $\mu = (a_1-1)(a_2-1)\cdots(a_7-1)$*. If* $P_7 > 0$*, then*

$$7!\, P_7 \le \mu - (-1 - 713v + 1743v^2 - 1589v^3 + 700v^4 - 154v^5 + 14v^6),$$

where v *is calculated as* $v = \lceil a_7 \rceil$*. Note that the fractional part* β *of* a_7 *is one of* $\frac{a_7}{a_1}, \frac{a_7}{a_2}, \frac{a_7}{a_3}, \frac{a_7}{a_4}, \frac{a_7}{a_5},$ *or* $\frac{a_7}{a_6}$*. Equality holds if and only if* $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 \in \mathbb{Z}$*.*

Remark. *In fact, Chen-Lin-Yau-Zuo (cf.* [5] *Theorem 2.5), proved that the fractional part of* a_7 *has to be one of* $\frac{a_7}{a_1}, \frac{a_7}{a_2}, \cdots,$ *or* $\frac{a_7}{a_6}$*. In* [36]*, Yau-Zuo verified Conjecture 1.9 when* $p_g = 0$*, In fact the condition* $P_7 > 0$ *and "Note that the fractional part* β *of* a_7 *is one of* $\frac{a_7}{a_1}, \frac{a_7}{a_2}, \frac{a_7}{a_3}, \frac{a_7}{a_4}, \frac{a_7}{a_5},$ *or* $\frac{a_7}{a_6}$*." can be removed from the above main theorem, please also see section 4.*

2. Some Lemmas

The following two lemmas will frequently use to decide the positivity of polynomials in some restricted domains.

Lemma 2.1 ([29] Lemma 3.1)**.** *Let* $f(\beta)$ *be a polynomial defined by*

$$f(\beta) = \sum_{i=0}^{n} c_i \beta^i$$

where $\beta \in (0,1)$*. If for any* $k = 0, 1, \cdots, n$

$$\sum_{i=0}^{k} c_i \ge 0$$

then $f(\beta) \ge 0$ *for* $\beta \in (0,1)$*.*

Though the Lemma 2.1 is easy to use, the condition of Lemma 2.1 may not be satisfied in some situation. In that case, we can use the following lemma.

Lemma 2.2 (Sturm's Theorem)**.** *Starting from a given polynomial* $X_0 = f(x)$*, let* $X_1 = f'(x)$ *and the polynomials* X_2*,* X_3*,* $\cdots$*,* X_r *be determined by Euclidean algorithm as follows:*

$$\begin{array}{rcl} X_0 & = & Q_1X_1 - X_2, \\ X_1 & = & Q_2X_2 - X_3, \\ \cdots & \cdots & \cdots\cdots \\ X_{r-2} & = & Q_{r-1}X_{r-1} - X_r, \\ X_{r-1} & = & Q_rX_r \end{array}$$

where $\deg X_k > \deg X_{k+1}$ *for* $k = 1, \cdots, r-1$*. For every real number* a *which is not a root of* $f(x)$ *let* $w(a)$ *be the number of variations in sign in the number sequence*

$$X_0(a), X_1(a), \cdots, X_r(a)$$

in which all zeros are omitted. If b and c are any numbers (b < c) for which f(x) does not vanish, then the number of the various roots in the interval $b \leq x \leq c$*(multiple roots to be counted only once) is equal to*

$$w(b) - w(c).$$

Proof. See [27]. □

Note that the computation in Lemma 2.2 is more complicated than that in Lemma 2.1. Therefore, we prefer Lemma 2.1 when it works. However, the condition of Lemma 2.2 is necessary and sufficient, so it can be applied to judge the positivity of any such polynomials in some intervals for general case.

3. Proof of Main Theorem 1.10

Note that all computations in this paper were done using Maple 2018.

To prove the Main Theorem, we first fix the value for $\lceil a_7 \rceil$,leave the other variables free, and calculate the upper bound. We then sum these values to estimate the upper bound for $7!\,P_7$. It then remains to show that the obtained upper bound is less than or equal to the RHS of the main theorem. We shall seperate the proof into the following cases, based on the value of $\lceil a_7 \rceil$:

Case I: $1 < a_7 \leq 2$. Thus, $\lceil a_7 \rceil = 2$;
Case II: $2 < a_7 \leq 3$. Thus, $\lceil a_7 \rceil = 3$;
Case III: $3 < a_7 \leq 4$. Thus, $\lceil a_7 \rceil = 4$;
Case IV: $4 < a_7 \leq 5$. Thus, $\lceil a_7 \rceil = 5$;
Case V: $5 < a_7 \leq 6$. Thus, $\lceil a_7 \rceil = 6$;
Case VI: $6 < a_7 \leq 7$. Thus, $\lceil a_7 \rceil = 7$;
Case VII: $7 < a_7$.

To prove that the RHS is greater, we find the difference between that and the estimated upper bound. Using the partial differentiation test, we show that the partial derivative of the difference is positive for each order, so that the difference is increasing or constant over its entire domain. It then simply remains to show that the minimal value of the original difference is greater than or equal to 0.

3.1. Case I.

In this case, $\lceil a_7 \rceil = 2$. Plugging that into the Main Theorem, we obtain the following:

Theorem 3.1. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 > 1$ be real numbers and let P_7 be the number of positive integral solutions of* $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} + \frac{x_7}{a_7} \leq 1$. *If $P_7 > 0$ and $1 < a_7 \leq 2$, then*

$$7!\, P_7 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)(a_7 - 1) - 1.$$

Proof. Since $a_7 \in (1, 2]$, we only have to consider the level $x_7 = 1$. In this case, given $P_7 > 0$, we know $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 1)$ is a solution to the inequality in the theorem above. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{1}{a_7} = \alpha,$$

Then $\alpha \in (0, \frac{1}{2}]$. Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewritting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Using the Yau Number Theoretic Conjecture for n=6, we have:

$$\begin{aligned} 7!\, P_7 = 7!\, P_6(x_7 = 1) \leq 7[&(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 \\ &+ A_6(A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] = B. \end{aligned}$$

Evaluate the difference between the RHS of the inequality in Theorem 3.1 and the above inequality, where we substitute $a_i = \dfrac{A_i}{\alpha}$. We get rid of the denominator, and set Δ_1 as the following:

$$\Delta_1 := \alpha^5(1 - \alpha)((a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)(a_7 - 1) - 1 - B).$$

Then we apply the partial differentiation test. We first must determine the domain of Δ_1. Note that $\frac{1}{A_6} < 1$, $\frac{2}{A_5} \leq \frac{1}{A_5} + \frac{1}{A_6}$. This is true involving other A_i. Thus we know

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2 \text{ and } A_6 > 1.$$

Now we begin partial differentiation test

$$\frac{\partial^6 \Delta_1}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 7\alpha^6 - 7\alpha^5 + 1 > 0.$$

For all $\alpha \in (0, 1]$ Thus, the partial derivative of Δ_1 with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = 1$.

$$\frac{\partial^5 \Delta_1}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=1} = 1 - \alpha > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq 1$, $\alpha \in (0, 1)$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value:

$$\frac{\partial^4 \Delta_1}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=1, A_6=1} = (1 - \alpha)^2 > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$:

$$\frac{\partial^3 \Delta_1}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=1, A_5=1, A_6=1} = (1 - \alpha)^3 > 0,$$

$$\frac{\partial^2 \Delta_1}{\partial A_1 \partial A_2}\Big|_{A_3=1, A_4=1, A_5=1, A_6=1} = (1 - \alpha)^4 > 0,$$

$$\frac{\partial \Delta_1}{\partial A_1}\Big|_{A_2=1,A_3=1,A_4=1,A_5=1,A_6=1} = (1-\alpha)^5 > 0.$$

Over $\alpha \in (0,1]$. By symmetry of Δ_1 in A_1, A_2, A_3, A_4, and A_5, all $\frac{\partial \Delta_1}{\partial A_2}, \frac{\partial \Delta_1}{\partial A_3}, \frac{\partial \Delta_1}{\partial A_4}, \frac{\partial \Delta_1}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_2 := \Delta_1\Big|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2}.$$

We must show that Δ_2 is positive for all $\alpha \in (0,1]$ and $A_6 \geq 1$. By directly applying the partial differentiation test, we realize that not all derivatives are positive over the domain. Therefore, we evaluate the polynomial by dividing it into 2 cases.

Subcase(a): $A_5 \geq 2.10$,
Subcase(b): $A_5 < 2.10$.

We determine the number 2.10 through testing, where all derivatives are positive when $A_5 \geq 2.10$.

3.1.1. *Subcase I(a).* In this subcase, we can apply the partial differentiation test normally

$$\frac{\partial^5 \Delta_2}{\partial A_6^5} = 7560\alpha^5(1-\alpha) > 0,$$

$$\frac{\partial^4 \Delta_2}{\partial A_6^4}\Big|_{A_6=2.10} = 4116.00\alpha^5(1-\alpha) > 0,$$

$$\frac{\partial^3 \Delta_2}{\partial A_6^3}\Big|_{A_6=2.10} = 583.80\alpha^5(1-\alpha) > 0,$$

$$\frac{\partial^2 \Delta_2}{\partial A_6^2}\Big|_{A_6=2.10} = 193.06\alpha^5(1-\alpha) > 0,$$

$$\frac{\partial \Delta_2}{\partial A_6}\Big|_{A_6=2.10} = 720.0 + 697.0\alpha^6 - 698.0\alpha^5 + 20.0\alpha^4 - 155.0\alpha^3 + 580.0\alpha^2 - 1044.0\alpha > 0.$$

Over the set domain. Thus we know Δ_2 is positive over the entire domain in the subcase. Finally, we evaluate Δ_2 at its minimum

$$\Delta_2\Big|_{A_6=2.10} = 1512.0 + 905.6\alpha^6 - 926.7\alpha^5 + 197.0\alpha^4 - 905.5\alpha^3 + 2262.0\alpha^2 - 2912.4\alpha > 0.$$

3.1.2. *Subcase I(b).* In this subcase, since both A_6 and α have fixed, finite domains, we can plot them in Maple to verify that it is non-negative over the entire region. Hence, Subcase I(b) is complete, thus completing the proof for Theorem 3.1. □

3.2. **Case II.** In this case, $\lceil a_7 \rceil = 3$. Plugging that into the Main Theorem, we obtain the following

Theorem 3.2. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 > 1$ be real numbers and let P_7 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} + \frac{x_7}{a_7} \leq 1$. If $P_7 > 0$ and $2 < a_7 \leq 3$, then*

$$7!\, P_7 \leq (a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 128.$$

Proof. Since $a_7 \in (2,3]$, we have two levels to consider: $x_7 = 1$ and $x_7 = 2$. There must be solutions on the $x_7 = 1$ level. Hence, we have the two following subcases

Subcase II(a): $P_6(x_7 = 2) = 0$,
Subcase II(b): $P_6(x_7 = 2) > 0$.

3.2.1. *Subcase II(a).* We know that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.2. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{1}{a_7} := \alpha.$$

Then $\alpha \in (\frac{1}{2}, \frac{2}{3}]$.Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Number Theoretic Conjecture for n=6, we have

$$7!\,P_7 = 7!\,P_6(x_7 = 2) \leq 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 + A_6(A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] = B.$$

Evaluate the difference between the RHS of the inequality in Theorem 3.2 and the above inequality. We denote this difference as Δ_3. We note that

$$\Delta_3 := \Delta_2 - 127\alpha^5(1 - \alpha).$$

The partial derivatives to Δ_3 is the same to that of Δ_1, which has been proven in the last case. Therefore, we only need evaluate Δ_3 with each variable taking its minimum.

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2 \text{ and } A_6 > 1.$$

We must only check

$$\Delta_3\Big|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1} = 129\alpha^6 - 149\alpha^5 + 175\alpha^4 - 735\alpha^3 + 1624\alpha^2 - 1764\alpha + 720 > 0.$$

Therefore, Δ_3 is always positive, and the subcase is complete.

3.2.2. *Subcase II(b).* We know that $P_6(x_7 = 2) > 0$, so

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 2)$$

is a solution to the inequality in Theorem 3.2. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{2}{a_7} := \alpha,$$

then $\alpha \in (0, \frac{1}{3}]$ because $a_7 \in (2, 3]$. Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Number Theoretic Conjecture for n=6, we have

$$7!\,P_7 = 7!\,P_6(x_7 = 2) \leq 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 + A_6(A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] = B,$$

and for the $x_7 = 1$ layer

$$7!\,P_6(x_7 = 1) \leq 7[(A_1 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_2 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_3 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_4 \cdot \frac{1+\alpha}{2\alpha} - 1)$$
$$(A_5 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_6 \cdot \frac{1+\alpha}{2\alpha} - 1) - (A_6 \cdot \frac{1+\alpha}{2\alpha} - 1)^6 + (A_6 \cdot \frac{1+\alpha}{2\alpha})(A_6 \cdot \frac{1+\alpha}{2\alpha} - 1)$$
$$(A_6 \cdot \frac{1+\alpha}{2\alpha} - 2)(A_6 \cdot \frac{1+\alpha}{2\alpha} - 3)(A_6 \cdot \frac{1+\alpha}{2\alpha} - 4)(A_6 \cdot \frac{1+\alpha}{2\alpha} - 5)] := B_2.$$

$7!\,P_7 = 7!\,(P_6(x_7 = 1) + P_6(x_7) = 2)$, so we can subtract the RHS of Theorem 3.2 by the sum of the RHS of the above equalities to get Δ_4, and we can rid the denominator without changing the sign. Then, we merely need to apply the partial differentiation test to Δ_4.

$\Delta_4 := 64\alpha^5(1-\alpha)[(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1)-128-(B_1+B_2)]$.

We are trying to show that it is positive for

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2 \text{ and } A_6 > 1.$$

For convenience in later cases, we show that this is positive over $(0, \frac{1}{2}]$ despite only having to show that it is positive over $(0, \frac{1}{3}]$. First, we determine

$$\frac{\partial^6 \Delta_4}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 455\alpha^7 - 413\alpha^6 + 63\alpha^5 + 35\alpha^4 - 35\alpha^3 - 63\alpha^2 + 29\alpha + 57 > 0.$$

For all $\alpha \in (0, \frac{1}{2}]$ Thus, the partial derivative of Δ_4 with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = 1$.

$$\frac{\partial^5 \Delta_4}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=1} = -7\alpha^7 - 21\alpha^6 - 7\alpha^5 + 35\alpha^4 + 35\alpha^3 - 71\alpha^2 - 21\alpha + 57 > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq 1$, $\alpha \in (0, 1)$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value:

$$\frac{\partial^4 \Delta_4}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=1, A_6=1} = (1-\alpha)^2 > 0.$$

Over the domain of α. For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$:

$$\frac{\partial^3 \Delta_4}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=1, A_5=1, A_6=1} = 7(-1+\alpha)^2(\alpha^4 + 2\alpha^3 - 2\alpha + 57/7)(1+\alpha) > 0,$$

$$\frac{\partial^2 \Delta_4}{\partial A_1 \partial A_2}\Big|_{A_3=1, A_4=1, A_5=1, A_6=1} = -7(-1+\alpha)^3(\alpha^3 + \alpha^2 - \alpha + 57/7)(1+\alpha) > 0,$$

$$\frac{\partial \Delta_4}{\partial A_1}\Big|_{A_2=1, A_3=1, A_4=1, A_5=1, A_6=1} = (-1+\alpha)^4(1+\alpha)(7\alpha^2 + 57) > 0.$$

Over the interval $\alpha \in (0, \frac{1}{2}]$. By symmetry of Δ_4 in A_1, A_2, A_3, A_4, and A_5, all

$$\frac{\partial \Delta_4}{\partial A_2}, \frac{\partial \Delta_4}{\partial A_3}, \frac{\partial \Delta_4}{\partial A_4}, \frac{\partial \Delta_4}{\partial A_5}$$

are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_5 := \Delta_4\Big|_{A_1=6, A_2=5, A_3=4, A_4=3, A_5=2}.$$

We must show that Δ_5 is positive for all $\alpha \in (0, \frac{1}{2}]$ and $A_6 \geq 1$. If we apply the partial differentiation test normally, we will obtain negative values for certain partial derivatives. Through testing, we realize that if $A_6 \geq 2$, then all partial derivatives are positive. Therefore, we first test the case where $A_6 \geq 2$.

$$\frac{\partial^5 \Delta_5}{\partial A_6^5} = -15120\alpha(33\alpha^6 - 28\alpha^5 + 5\alpha^4 - 5\alpha^2 - 4\alpha - 1) > 0,$$

$$\frac{\partial^4 \Delta_5}{\partial A_6^4}\Big|_{A_6=2} = -3360\alpha(-1+\alpha)(59\alpha^5 - 11\alpha^4 + 6\alpha^3 + 34\alpha^2 + 31\alpha + 9) > 0,$$

$$\frac{\partial^3\Delta_5}{\partial A_6^3}\Big|_{A_6=2} = -1680\alpha(-1+\alpha)(11\alpha^5 - 11\alpha^4 - 33\alpha^3 - 3\alpha^2 + 34\alpha + 18) > 0,$$

$$\frac{\partial^2\Delta_5}{\partial A_6^2}\Big|_{A_6=2} = -224\alpha(-1+\alpha)(70\alpha^5 + 97\alpha^4 - 34\alpha^3 - 165\alpha^2 + 30\alpha + 90) > 0,$$

$$\frac{\partial\Delta_5}{\partial A_6}\Big|_{A_6=2} = 51072\alpha^7 - 57072\alpha^6 - 11496\alpha^5 + 33248\alpha^4 + 35040\alpha^3 - 55232\alpha^2 - 21240\alpha + 41040 > 0.$$

Over the set domain. Therefore, Δ_5 is increasing when $A_6 \geq 2$. We confirm that it is positive by evaluating it at its minimum

$$\Delta_2\Big|_{A_6=2} = 61568\alpha^7 - 52096\alpha^6 - 10656\alpha^5 - 34736\alpha^4 + 112880\alpha^3 - 49072\alpha^2 - 94608\alpha + 82080 > 0,$$

for $\alpha \in (0, \frac{1}{2}]$. As for when $A_6 \in [1, 2]$, similar to Subcase I(b), since both A_6 and α have fixed, finite domains, we can plot them in Maple to verify that it is non-negative over the entire region. Hence, Subcase II(b) is complete, thus completing the proof for Theorem 2.2. □

3.3. **Case III.** In this case, $\lceil a_7 \rceil = 4$. Plugging that into the Main Theorem, we obtain the following:

Theorem 3.3. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 > 1$ be real numbers and let P_7 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} + \frac{x_7}{a_7} \leq 1$. If $P_7 > 0$ and $3 < a_7 \leq 4$, then*

$$7!\, P_7 \leq (a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 2187.$$

Proof. Since $a_7 \in (3, 4]$, we have three levels to consider: $x_7 = 1, x_7 = 2$ and $x_7 = 3$. There must be solutions on the $x_7 = 1$ level. Hence, we have the three following subcases:

Subcase III(a): $P_6(x_7 = 3) = P_6(x_7 = 2) = 0$,
Subcase III(b): $P_6(x_7 = 3) = 0, P_6(x_7 = 2) > 0$,
Subcase III(c): $P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$.

3.3.1. *Subcase III(a).* We know that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.3. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{1}{a_7} := \alpha,$$

then $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Geometric Conjecture for n=6, we have

$$7!\, P_7 = 7!\, P_6(x_7 = 1) \leq 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - (A_6-1)^6 + A_6(A_6-1)(A_6-2)(A_6-3)(A_6-4)(A_6-5)] := B_1.$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.3 and rid the denominator. Let this difference be Δ_6

$$\Delta_6 := [(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 2187 - B_1] * \alpha^5(1-\alpha).$$

Here we need to show Δ_6 is positive. First, we must determine the domain of Δ_6. Using the same logic as Case I and Case II, we have

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq A_6 \geq \frac{\alpha}{1-\alpha}.$$

because $A_5 \geq A_6 = \dfrac{\alpha}{1-\alpha}$, and $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_6}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 7\alpha^6 - 7\alpha^5 + 1 > 0.$$

For all $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ Thus, the partial derivative of Δ_6 with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = \dfrac{\alpha}{1-\alpha}$.

$$\frac{\partial^5 \Delta_6}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=\frac{\alpha}{1-\alpha}} = \frac{-14\alpha^7 + 21\alpha^6 - 7\alpha^5 - \alpha^2}{-1+\alpha} > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \dfrac{\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{3}{4}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_6}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{28\alpha^8 - 56\alpha^7 + 35\alpha^6 - 7\alpha^5 + \alpha^4}{(-1+\alpha)^2} > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$,

$$\frac{\partial^3 \Delta_6}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=3, A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{(56\alpha^4 - 112\alpha^3 + 70\alpha^2 - 15\alpha + 3)\alpha^4)}{(-1+\alpha)^2} > 0,$$

$$\frac{\partial^2 \Delta_6}{\partial A_1 \partial A_2}\Big|_{A_3=4, A_4=3, A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{(168\alpha^4 - 336\alpha^3 + 211\alpha^2 - 49\alpha + 12)\alpha^4}{-1+\alpha)^2} > 0,$$

$$\frac{\partial \Delta_6}{\partial A_1}\Big|_{A_2=5, A_3=4, A_4=3, A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{(672\alpha^4 - 1345\alpha^3 + 852\alpha^2 - 215\alpha + 60)\alpha^4}{(-1+\alpha)^2} > 0.$$

Over $\alpha \in (\frac{2}{3}, \frac{3}{4}]$. By symmetry of Δ_6 in A_1, A_2, A_3, A_4, and A_5, all $\dfrac{\partial \Delta_6}{\partial A_2}, \dfrac{\partial \Delta_6}{\partial A_3}, \dfrac{\partial \Delta_6}{\partial A_4}, \dfrac{\partial \Delta_6}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_7 := \Delta_6\Big|_{A_1=6, A_2=5, A_3=4, A_4=3, A_5=2}.$$

We must show Δ_7 is positive over its domain. To do so, we normally apply the partial differentiation test

$$\frac{\partial^5 \Delta_7}{\partial A_6^5} = 7560\alpha^5(1-\alpha) > 0,$$

$$\frac{\partial^4 \Delta_7}{\partial A_6^4}\Big|_{A_6=\frac{\alpha}{1-\alpha}} = 19320\alpha^6 - 11760\alpha^5 > 0,$$

$$\frac{\partial^3 \Delta_7}{\partial A_6^3}\Big|_{A_6=\frac{\alpha}{1-\alpha}} = \frac{-210(115\alpha^2 - 138\alpha + 41)\alpha^5}{-1+\alpha} > 0,$$

$$\frac{\partial^2\Delta_7}{\partial A_6^2}\Big|_{A_6=\dfrac{\alpha}{1-\alpha}} = \frac{14(1384\alpha^3 - 2427\alpha^2 + 1392\alpha - 259)\alpha^5}{(-1+\alpha)^2} > 0,$$

$$\frac{\partial\Delta_7}{\partial A_6}\Big|_{A_6=\dfrac{\alpha}{1-\alpha}} = \frac{1}{(-1+\alpha)^3}(-10164\alpha^9 + 21279\alpha^8 - 14908\alpha^7 + 3240\alpha^6 + 1148\alpha^5 - 3269\alpha^4 + 5747\alpha^3 - 5872\alpha^2 + 3204\alpha - 720) > 0.$$

This confirms that Δ_7 is increasing. We evaluate Δ_7 at its minimum:

$$\Delta_7\Big|_{A_6=\dfrac{\alpha}{1-\alpha}} = \frac{1}{(-1+\alpha)^4}(5100\alpha^8 - 15354\alpha^7 + 20240\alpha^6 - 16151\alpha^5 + 9927\alpha^4 - 7087\alpha^3 + 5872\alpha^2 - 3204\alpha + 720)\alpha^2) > 0.$$

This completes the subcase.

3.3.2. *Subcase III(b).* We know that $P_6(x_7 = 2) > 0$, so $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 2)$ is a solution to the inequality in Theorem 3.3. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \le 1 - \frac{2}{a_7} := \alpha,$$

then $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ because $a_7 \in (3, 4]$. Let Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \le 1.$$

Thus, by the Yau Geometric Conjecture for n=6, we have

$$7!\,P_6(x_7 = 2) \le 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 + A_6(A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] := B_1.$$

and for the $x_7 = 1$ layer,

$$7!\,P_6(x_7 = 1) \le 7[(A_1 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1)(A_2 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1)(A_3 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1)(A_4 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1)(A_5 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1)(A_6 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1) - (A_6 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1)^6 + (A_6 \cdot \frac{1+\alpha_1}{2\alpha_1})(A_6 \cdot \frac{1+\alpha_1}{2\alpha_1} - 1)(A_6 \cdot \frac{1+\alpha_1}{2\alpha_1} - 2)(A_6 \cdot \frac{1+\alpha_1}{2\alpha_1} - 3)(A_6 \cdot \frac{1+\alpha_1}{2\alpha_1} - 4)(A_6 \cdot \frac{1+\alpha_1}{2\alpha_1} - 5)] := B_2.$$

$7!\,P_7 = 7!\,(P_6(x_7 = 1) + P_6(x_7) = 2)$, so we can subtract the RHS of Theorem 3.3 by the sum of the RHS of the above equalities and multiply by the denominator to get Δ_8. Note that this difference is equal to $\Delta_4 - 2059 \cdot 64\alpha^6(1-\alpha)$.

$$\Delta_8 = \Delta_4 - 2059 \cdot 64\alpha^6(1-\alpha).$$

Recall that we have

$$A_1 \ge 6, A_2 \ge 5, A_3 \ge 4, A_4 \ge 3, A_5 \ge 2 \text{ and } A_6 > 1.$$

It has already been proven that Δ_4 is increasing when $A_6 \ge 2$, so the same should be true for Δ_8. Over this domain we only check the value of Δ_8 at the minimum.

$$\Delta_8\Big|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=2} = 193344\alpha^7 - 183872\alpha^6 - 10656\alpha^5 - 34736\alpha^4 + 112880\alpha^3$$

$$-49072\alpha^2 - 94608\alpha + 82080 > 0.$$

As for when $A_6 \in (1,2]$, we can again plot Δ_8 while fixing all values aside from A_6 and α, over the region on Maple or Mathematica and check for negative values, since the remaining domain is finite. Testing the minimum,it is shown to be all positive over the domain, and the subcase is complete.

3.3.3. *Subcase III(c).* We know that $P_6(x_7 = 3) > 0$, so $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 3)$ is a solution to the inequality in Theorem 3.3. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \le 1 - \frac{3}{a_7} := \alpha,$$

then $\alpha \in (0, \frac{1}{4}]$ because $a_7 \in (3, 4]$. Let Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \le 1.$$

Thus, by the Yau Geometric Conjecture for n=6, we have

$$\begin{aligned} 7!\, P_6(x_7 = 3) \le & 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 \\ & + A_6(A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] := B_1. \end{aligned}$$

For the $x_7 = 2$ layer,

$$\begin{aligned} 7!\, P_6(x_7 = 2) \le & 7[(A_1 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_2 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_3 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_4 \cdot \frac{1+2\alpha}{3\alpha} - 1) \\ & (A_5 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1) - (A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1)^6 \\ & + (A_6 \cdot \frac{1+2\alpha}{3\alpha})(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 2) \\ & (A_6 \cdot \frac{1+2\alpha}{3\alpha} - 3)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 4)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 5)] := B_2. \end{aligned}$$

And for the $x_7 = 1$ layer,

$$\begin{aligned} 7!\, P_6(x_7 = 1) \le & 7[(A_1 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_2 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_3 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_4 \cdot \frac{2+\alpha}{3\alpha} - 1) \\ & (A_5 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 1) - (A_6 \cdot \frac{2+\alpha}{3\alpha} - 1)^6 + (A_6 \cdot \frac{2+\alpha}{3\alpha}) \\ & (A_6 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 2)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 3)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 4) \\ & (A_6 \cdot \frac{2+\alpha}{3\alpha} - 5)] := B_3. \end{aligned}$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.3. Let this difference be Δ_9.

$$\Delta_9 := (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)(a_7 - 1) - 2187 - (B_1 + B_2 + B_3).$$

Here we need to show Δ_9 is positive. In order to apply the partial differentiation test to Δ_9, we must

$$A_1 \ge 6, A_2 \ge 5, A_3 \ge 4, A_4 \ge 3, A_5 \ge A_6 > 1.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_9}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 5558\alpha^7 - 4130\alpha^6 + 672\alpha^5 + 140\alpha^4 - 140\alpha^3 - 672\alpha^2 - 244\alpha + 1003 > 0.$$

For all $\alpha \in (0, \frac{1}{4}]$ Thus, the partial derivative of Δ_9 with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = 1$.

$$\frac{\partial^5 \Delta_9}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=1} = -238\alpha^7 - 224\alpha^6 + 42\alpha^5 + 140\alpha^4 + 490\alpha^3 - 204\alpha^2 - 1009\alpha + 1003 > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq 1$, $\alpha \in (0, \frac{1}{4}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_9}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=1, A_6=1} = (-1+\alpha)^2(140\alpha^5 + 308\alpha^4 + 392\alpha^3 + 112\alpha^2 + 232\alpha + 1003) > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$

$$\frac{\partial^3 \Delta_9}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^3(112\alpha^4 + 308\alpha^3 + 294\alpha^2 + 470\alpha + 1003) > 0,$$

$$\frac{\partial^2 \Delta_9}{\partial A_1 \partial A_2}\Big|_{A_3=1, A_4=1, A_5=1, A_6=1} = (-1+\alpha)^4(140\alpha^3 + 336\alpha^2 + 708\alpha + 1003) > 0,$$

$$\frac{\partial \Delta_9}{\partial A_1}\Big|_{A_2=1, A_3=1, A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^5(238\alpha^2 + 946\alpha + 1003) > 0.$$

Over $\alpha \in (0, \frac{1}{4}]$. By symmetry of Δ_9 in A_1, A_2, A_3, A_4, and A_5, all $\frac{\partial \Delta_9}{\partial A_2}, \frac{\partial \Delta_9}{\partial A_3}, \frac{\partial \Delta_9}{\partial A_4}, \frac{\partial \Delta_9}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_{10} := \Delta_9\Big|_{A_1=6, A_2=5, A_3=4, A_4=3, A_5=2}.$$

We must show Δ_{10} is positive over its domain. To do so, we normally apply the partial differentiation test

$$\frac{\partial^5 \Delta_{10}}{\partial A_6^5} = -6259680\alpha^7 + 4218480\alpha^6 - 680400\alpha^5 + 680400\alpha^3 + 1292760\alpha^2 + 748440\alpha > 0,$$

$$\frac{\partial^4 \Delta_{10}}{\partial A_6^4}\Big|_{A_6=1} = 7560\alpha(544\alpha^5 + 290\alpha^4 + 312\alpha^3 + 200\alpha^2 - 32\alpha - 99)(-1+\alpha) > 0,$$

$$\frac{\partial^3 \Delta_{10}}{\partial A_6^3}\Big|_{A_6=1} = -1890\alpha(596\alpha^5 + 514\alpha^4 + 246\alpha^3 - 413\alpha^2 - 412\alpha + 198)(-1+\alpha),$$

$$\frac{\partial^2 \Delta_{10}}{\partial A_6^2}\Big|_{A_6=1} = -378\alpha(302\alpha^5 + 154\alpha^4 + 1665\alpha^3 + 1135\alpha^2 - 1480\alpha + 330)(-1+\alpha) > 0,$$

$$\frac{\partial \Delta_{10}}{\partial A_6}\Big|_{A_6=1} = 857430\alpha^7 - 487026\alpha^6 - 83961\alpha^5 - 631935\alpha^4 + 845730\alpha^3 - 16803\alpha^2 - 943155\alpha + 722160 > 0.$$

This confirms that Δ_{10} is increasing. We evaluate Δ_{10} at its minimum:

$$\Delta_{10}\Big|_{A_6=1} =1450782\alpha^7 - 1435995\alpha^6 + 268254\alpha^5 - 665091\alpha^4 + 272790\alpha^3 + 906003\alpha^2 - 1518903\alpha + 722160 > 0.$$

This completes the subcase, therefore completing the proof for Case III. □

3.4. **Case IV.** In this case, $\lceil a_7 \rceil = 5$. Plugging that into the Main Theorem, we obtain the following

Theorem 3.4. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 > 1$ be real numbers and let P_7 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} + \frac{x_7}{a_7} \leq 1$. If $P_7 > 0$ and $4 < a_7 \leq 5$, then*

$$7!\,P_7 \leq (a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 16384.$$

Proof. Since $a_7 \in (4, 5]$, we have four levels to consider: $x_7 = 1, x_7 = 2, x_7 = 3$ and $x_7 = 4$. There must be solutions on the $x_7 = 1$ level. Hence, we have the four following subcases:

Subcase IV(a): $P_6(x_7 = 4) = P_6(x_7 = 3) = P_6(x_7 = 2) = 0$,
Subcase IV(b): $P_6(x_7 = 4) = P_6(x_7 = 3) = 0, P_6(x_7 = 2) > 0$,
Subcase IV(c): $P_6(x_7 = 4) = 0, P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$,
Subcase IV(d): $P_6(x_7 = 4) > 0, P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$.

3.4.1. *Case IV(a).* We know that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.4. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{1}{a_7} := \alpha.$$

Then $\alpha \in (\frac{3}{4}, \frac{4}{5}]$.Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Geometric Conjecture for n=6, we have

$$7!\,P_7 = 7!\,P_6(x_7 = 1) \leq 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - p_6(\lceil A_6 \rceil)].$$

We know the minimum value for $\lceil A_6 \rceil$ is 4, as $A_6 = a_6 \cdot \alpha > \frac{3}{4} \cdot 4$.Note that $p_6(v)$ is increasing for $v \geq 4$, so we can plug in $\lceil A_6 \rceil = 4$ to maximize the RHS of the above inequality. This gives

$$7!\,P_7 = 7!\,P_6(x_7 = 1) \leq 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - 729)] := B_1.$$

Subtracting the upper bound of this inequality from the RHS of Theorem 3.4, then getting rid of the denominator, we define Δ_{11}

$$\Delta_{11} := [7!\,P_7 \leq (a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 16384 - B_1]\alpha^5(1-\alpha).$$

Here we need to show Δ_{11} is positive. First, we must determine the domain of Δ_{11}. Using the same logic as previous cases, we have

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq A_5 \geq A_6 \geq \frac{\alpha}{1-\alpha} > 3.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{11}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 7\alpha^6 - 7\alpha^5 + 1 > 0.$$

For all $\alpha \in (\frac{3}{4}, \frac{4}{5}]$ Thus, the partial derivative of Δ_{11} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = \dfrac{\alpha}{1-\alpha}$.

$$\frac{\partial^5 \Delta_{11}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=\dfrac{\alpha}{1-\alpha}} = \frac{-14\alpha^7 + 21\alpha^6 - 7\alpha^5 - \alpha^2}{-1+\alpha} > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \dfrac{\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{4}, \frac{4}{5}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{11}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=\dfrac{\alpha}{1-\alpha}, A_6=\dfrac{\alpha}{1-\alpha}} = \frac{28\alpha^8 - 56\alpha^7 + 35\alpha^6 - 7\alpha^5 + \alpha^4}{(-1+\alpha)^2} > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$

$$\frac{\partial^3 \Delta_{11}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=\dfrac{\alpha}{1-\alpha}, A_5=\dfrac{\alpha}{1-\alpha}, A_6=\dfrac{\alpha}{1-\alpha}} = \frac{-(56\alpha^4 - 14\alpha^3 + 126\alpha^2 - 48\alpha + 7)\alpha^5}{(-1+\alpha)^3} > 0,$$

$$\frac{\partial^2 \Delta_6}{\partial A_1 \partial A_2}\Big|_{A_3=4, A_4=\dfrac{\alpha}{1-\alpha}, A_5=\dfrac{\alpha}{1-\alpha}, A_6=\dfrac{\alpha}{1-\alpha}} = \frac{-(168\alpha^4 - 420\alpha^3 + 377\alpha^2 - 143\alpha + 21)\alpha^5}{(-1+\alpha)^3} > 0,$$

$$\frac{\partial \Delta_6}{\partial A_1}\Big|_{A_2=5, A_3=4, A_4=\dfrac{\alpha}{1-\alpha}, A_5=\dfrac{\alpha}{1-\alpha}, A_6=\dfrac{\alpha}{1-\alpha}} = \frac{-(672\alpha^4 - 1345\alpha^3 + 852\alpha^2 - 215\alpha + 60)\alpha^4}{(-1+\alpha)^2} > 0.$$

Δ_{11} is symmetric for all $A_i, i = 1, 2, 3, 4, 5, 6$. Therefore, Δ_{11} is increasing over the entire given domain for α. We see that $A_6 = a_6\alpha \geq 6 \cdot \frac{3}{4} = \frac{9}{2}$. Therefore, the minimum is

$$\Delta_{11}\Big|_{A_1=A_2=A_3=A_4=A_5=A_6=\frac{9}{2}} = \frac{1}{64}(1545591\alpha^6 - 1547255\alpha^5 + 19440\alpha^4 - 116640\alpha^3 + 393660\alpha^2 - 708588\alpha + 531441) > 0,$$

over the interval. This completes the subcase.

3.4.2. *Case IV(b).* We know that $P_6(x_7 = 2) > 0$, so

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 2)$$

is a solution to the inequality in Theorem 3.4. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{2}{a_7} := \alpha,$$

then $\alpha \in (\frac{1}{2}, \frac{3}{5}]$ because $a_7 \in (4, 5]$. Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Number Theoretic Conjecture for n=6, we have

$$7!\, P_6(x_7 = 2) \leq 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6$$

$$+ A_6(A_6-1)(A_6-2)(A_6-3)(A_6-4)(A_6-5)] := B_1.$$

and for the $x_7 = 1$ layer

$$\begin{aligned}7!\,P_6(x_7=2) \leq& 7[(A_1\cdot\frac{1+\alpha}{2\alpha}-1)(A_2\cdot\frac{1+\alpha}{2\alpha}-1)(A_3\cdot\frac{1+\alpha}{2\alpha}-1)(A_4\cdot\frac{1+\alpha}{2\alpha}-1)\\ &(A_5\cdot\frac{1+\alpha}{2\alpha}-1)(A_6\cdot\frac{1+\alpha}{2\alpha}-1)-(A_6\cdot\frac{1+\alpha}{2\alpha}-1)^6+(A_6\cdot\frac{1+\alpha}{2\alpha})\\ &(A_6\cdot\frac{1+\alpha}{2\alpha}-1)(A_6\cdot\frac{1+\alpha}{2\alpha}-2)(A_6\cdot\frac{1+\alpha}{2\alpha}-3)(A_6\cdot\frac{1+\alpha}{2\alpha}-4)\\ &(A_6\cdot\frac{1+\alpha}{2\alpha}-5)] := B_2.\end{aligned}$$

$7!\,P_7 = 7!\,(P_6(x_7=1)+P_6(x_7)=2)$, so we can subtract the RHS of Theorem 3.4 by the sum of the RHS of the above equalities to get Δ_{12}, and we can rid the denominator without changing the sign. Then, we merely need to apply the partial differentiation test to Δ_{12}

$$\Delta_{12} := 64\alpha^6(1-\alpha)[(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1)-16384-(B_1+B_2)].$$

We are trying to show that it is positive for

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq A_6 > \frac{2\alpha}{1-\alpha}.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6\Delta_{12}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5\partial A_6} = 455\alpha^7-413\alpha^6+63\alpha^5+35\alpha^4-35\alpha^3-63\alpha^2+29\alpha+57>0.$$

For all $\alpha \in (\frac{1}{2},\frac{3}{5}]$ Thus, the partial derivative of Δ_{12} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = \frac{2\alpha}{1-\alpha}$.

$$\left.\frac{\partial^5\Delta_{12}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5}\right|_{A_6=\frac{2\alpha}{1-\alpha}} = \frac{-4\alpha(343\alpha^7-420\alpha^6+147\alpha^5-35\alpha^3-12\alpha^2+25\alpha+16)}{-1+\alpha} > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \frac{2\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{2},\frac{3}{5})$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value:

$$\left.\frac{\partial^4\Delta_{12}}{\partial A_1\partial A_2\partial A_3\partial A_4}\right|_{A_5=A_6=\frac{2\alpha}{1-\alpha}} = \frac{16\alpha^2}{(-1+\alpha)^2}(259\alpha^7-399\alpha^6+210\alpha^5-42\alpha^4-17\alpha^3+5\alpha^2+12\alpha+4)>0.$$

Over the domain of α. For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$:

$$\left.\frac{\partial^3\Delta_{12}}{\partial A_1\partial A_2\partial A_3}\right|_{A_4=A_5=A_6=\frac{2\alpha}{1-\alpha}} = \frac{-64\alpha^3}{(-1+\alpha)^3}(196\alpha^7-364\alpha^6+252\alpha^5-83\alpha^4+4\alpha^3+6\alpha^2+4\alpha+1)>0,$$

$$\frac{\partial^2 \Delta_{12}}{\partial A_1 \partial A_2}\Big|_{A_3=A_4=A_5=A_6=\dfrac{2\alpha}{1-\alpha}} = \frac{64\alpha^4}{(-1+\alpha)^4}(595\alpha^7 - 1295\alpha^6 + 1107\alpha^5 - 485\alpha^4 + 101\alpha^3 + 3\alpha^2 + 5\alpha + 1) > 0,$$

$$\frac{\partial \Delta_{12}}{\partial A_1}\Big|_{A_2=A_3=A_4=A_5=A_6=\dfrac{2\alpha}{1-\alpha}} = \frac{-64\alpha^5}{(-1+\alpha)^5}(1813\alpha^7 - 4535\alpha^6 + 4619\alpha^5 - 2505\alpha^4 + 755\alpha^3 - 97\alpha^2 + 13\alpha + 1) > 0.$$

Over the interval $\alpha \in (\frac{1}{2}, \frac{3}{5}]$. By symmetry of Δ_{12} in A_1, A_2, A_3, A_4, and A_5, all

$$\frac{\partial \Delta_{12}}{\partial A_2}, \frac{\partial \Delta_{12}}{\partial A_3}, \frac{\partial \Delta_{12}}{\partial A_4}, \frac{\partial \Delta_{12}}{\partial A_5}$$

are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_{13} := \Delta_{12}\Big|_{A_1=6, A_2=5, A_3=4, A_4=3, A_5=2}.$$

We must show that Δ_{13} is positive for all $\alpha \in (\frac{1}{2}, \frac{3}{5}]$ and $A_6 \geq 1$. Now we apply the partial differentiation test

$$\frac{\partial^5 \Delta_{13}}{\partial A_6^5} = -15120\alpha(33\alpha^6 - 28\alpha^5 + 5\alpha^4 - 5\alpha^2 - 4\alpha - 1) > 0,$$

$$\frac{\partial^4 \Delta_{13}}{\partial A_6^4}\Big|_{A_6=\dfrac{2\alpha}{1-\alpha}} = 3360\alpha^2(535\alpha^5 - 137\alpha^4 + 118\alpha^3 + 62\alpha^2 + 3\alpha - 5) > 0,$$

$$\frac{\partial^3 \Delta_{13}}{\partial A_6^3}\Big|_{A_6=\dfrac{2\alpha}{1-\alpha}} = \frac{-1680\alpha^3(1915\alpha^5 - 1253\alpha^4 + 538\alpha^3 - 14\alpha^2 - 37\alpha + 3}{-1+\alpha} > 0,$$

$$\frac{\partial^2 \Delta_{13}}{\partial A_6^2}\Big|_{A_6=\dfrac{2\alpha}{1-\alpha}} = \frac{448\alpha^4(8470\alpha^5 - 8801\alpha^4 + 4010\alpha^3 - 844\alpha^2 + 32\alpha + 13}{(-1+\alpha)^2} > 0,$$

$$\frac{\partial \Delta_{13}}{\partial A_6}\Big|_{A_6=\dfrac{2\alpha}{1-\alpha}} = \frac{8}{(-1+\alpha)^3}(405216\alpha^{10} - 550738\alpha^9 + 280615\alpha^8 - 57317\alpha^7 - 1539\alpha^6 + 17367\alpha^5 - 21637\alpha^4 - 183\alpha^3 + 23031\alpha^2 - 19305\alpha + 5130) > 0.$$

Over the set domain. Therefore, Δ_{13} is increasing. We confirm that it is positive by evaluating it at its minimum

$$\Delta_{13}\Big|_{A_6=\dfrac{2\alpha}{1-\alpha}} = \frac{32\alpha}{(-1+\alpha)^4}(97268\alpha^{10} - 261798\alpha^9 + 369655\alpha^8 - 321716\alpha^7 + 157955\alpha^6 - 31844\alpha^5 - 6379\alpha^4 + 6076\alpha^3 + 1565\alpha^2 - 4158\alpha + 1440) > 0.$$

This completes the subcase.

3.4.3. *Case IV(c).* We know that $P_6(x_7 = 3) > 0$, so

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 3)$$

is a solution to the inequality in Theorem 3.4. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{3}{a_7} := \alpha,$$

then $\alpha \in (\frac{1}{4}, \frac{2}{5}]$ because $a_7 \in (4, 5]$. Let Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Number Theoretic Conjecture for $n = 6$, we have

$$\begin{aligned} 7!\, P_6(x_7 = 3) \leq & 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 \\ & + A_6(A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] := B_1, \end{aligned}$$

for the $x_7 = 2$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 2) \leq & 7[(A_1 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_2 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_3 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_4 \cdot \frac{1+2\alpha}{3\alpha} - 1) \\ & (A_5 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1) - (A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1)^6 + (A_6 \cdot \frac{1+2\alpha}{3\alpha}) \\ & (A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 2)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 3)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 4) \\ & (A_6 \cdot \frac{1+2\alpha}{3\alpha} - 5)] := B_2, \end{aligned}$$

and for the $x_7 = 1$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 1) \leq & 7[(A_1 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_2 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_3 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_4 \cdot \frac{2+\alpha}{3\alpha} - 1) \\ & (A_5 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 1) - (A_6 \cdot \frac{2+\alpha}{3\alpha} - 1)^6 + (A_6 \cdot \frac{2+\alpha}{3\alpha}) \\ & (A_6 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 2)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 3)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 4) \\ & (A_6 \cdot \frac{2+\alpha}{3\alpha} - 5)] := B_3. \end{aligned}$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.4 and eliminate the denominator. Let this difference be Δ_{14}.

$$\Delta_{14} := 729\alpha^6(1-\alpha)[(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1)-16384-(B_1+B_2+B_3)].$$

Here we need to show Δ_{14} is positive. In order to apply the partial differentiation test to Δ_{14}, we must first determine its domain

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, \text{ and } A_6 > 1.$$

Now, we can apply the partial differentiation test

$$\begin{aligned} \frac{\partial^6 \Delta_{14}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = & 5558\alpha^7 - 4130\alpha^6 + 672\alpha^5 + 140\alpha^4 - 140\alpha^3 - 672\alpha^2 - 244\alpha \\ & + 1003 > 0. \end{aligned}$$

For all $\alpha \in (\frac{1}{4}, \frac{2}{5}]$ Thus, the partial derivative of Δ_{14} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = 1$.

$$\frac{\partial^5 \Delta_{14}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=1} = -238\alpha^7 - 224\alpha^6 + 42\alpha^5 + 140\alpha^4 + 490\alpha^3 - 204\alpha^2 - 1009\alpha + 1003 > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq 1$, $\alpha \in (\frac{1}{4}, \frac{2}{5}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{14}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=1, A_6=1} = (-1+\alpha)^2(140\alpha^5 + 308\alpha^4 + 392\alpha^3 + 112\alpha^2 + 232\alpha + 1003) > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$

$$\frac{\partial^3 \Delta_{14}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^3(112\alpha^4 + 308\alpha^3 + 294\alpha^2 + 470\alpha + 1003) > 0,$$

$$\frac{\partial^2 \Delta_{14}}{\partial A_1 \partial A_2}\Big|_{A_3=1, A_4=1, A_5=1, A_6=1} = (-1+\alpha)^4(140\alpha^3 + 336\alpha^2 + 708\alpha + 1003) > 0,$$

$$\frac{\partial \Delta_{14}}{\partial A_1}\Big|_{A_2=1, A_3=1, A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^5(238\alpha^2 + 946\alpha + 1003) > 0.$$

Over $\alpha \in (\frac{1}{4}, \frac{2}{5}]$. By symmetry of Δ_{14} in A_1, A_2, A_3, A_4, and A_5, all $\frac{\partial \Delta_{14}}{\partial A_2}, \frac{\partial \Delta_{14}}{\partial A_3}, \frac{\partial \Delta_{14}}{\partial A_4}, \frac{\partial \Delta_{14}}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_{15} := \Delta_{14}\Big|_{A_1=6, A_2=5, A_3=4, A_4=3, A_5=2}.$$

We must show Δ_{15} is positive over its domain. To do so, we normally apply the partial differentiation test

$$\frac{\partial^5 \Delta_{15}}{\partial A_6^5} = -6259680\alpha^7 + 4218480\alpha^6 - 680400\alpha^5 + 680400\alpha^3 + 1292760\alpha^2 + 748440\alpha > 0,$$

$$\frac{\partial^4 \Delta_{15}}{\partial A_6^4}\Big|_{A_6=1} = 7560\alpha(544\alpha^5 + 290\alpha^4 + 312\alpha^3 + 200\alpha^2 - 32\alpha - 99)(-1+\alpha) > 0,$$

$$\frac{\partial^3 \Delta_{15}}{\partial A_6^3}\Big|_{A_6=1} = -1890\alpha(596\alpha^5 + 514\alpha^4 + 246\alpha^3 - 413\alpha^2 - 412\alpha + 198)(-1+\alpha),$$

$$\frac{\partial^2 \Delta_{15}}{\partial A_6^2}\Big|_{A_6=1} = -378\alpha(302\alpha^5 + 154\alpha^4 + 1665\alpha^3 + 1135\alpha^2 - 1480\alpha + 330)(-1+\alpha) > 0,$$

$$\frac{\partial \Delta_{15}}{\partial A_6}\Big|_{A_6=1} = 857430\alpha^7 - 487026\alpha^6 - 83961\alpha^5 - 631935\alpha^4 + 845730\alpha^3 - 16803\alpha^2 - 943155\alpha + 722160 > 0.$$

This confirms that Δ_{15} is increasing. We evaluate Δ_{15} at its minimum:

$$\Delta_{15}\Big|_{A_6=1} = 1450782\alpha^7 - 1435995\alpha^6 + 268254\alpha^5 - 665091\alpha^4 + 272790\alpha^3 + 906003\alpha^2 - 1518903\alpha + 722160 > 0.$$

This completes the proof for this subcase.

3.4.4. *Case IV(d).* We know that $P_6(x_7 = 4) > 0$, so

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 4)$$

is a solution to the inequality in Theorem 3.4. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \le 1 - \frac{4}{a_7} := \alpha,$$

then $\alpha \in (0, \frac{1}{5}]$ because $a_7 \in (4, 5]$. Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \le 1.$$

Thus, by the Yau Number Theoretic Conjecture for $n = 6$, we have

$$\begin{aligned} 7!\, P_6(x_7 = 4) \le & 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 \\ & + A_6(A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] := B_1, \end{aligned}$$

for the $x_7 = 3$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 3) \le & 7[(A_1 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_2 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_3 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_4 \cdot \frac{1+3\alpha}{4\alpha} - 1) \\ & (A_5 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 1) - (A_6 \cdot \frac{1+3\alpha}{4\alpha} - 1)^6 + (A_6 \cdot \frac{1+3\alpha}{4\alpha}) \\ & (A_6 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 2)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 3)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 4) \\ & (A_6 \cdot \frac{1+3\alpha}{4\alpha} - 5)] := B_2, \end{aligned}$$

for the $x_7 = 2$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 2) \le & 7[(A_1 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_2 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_3 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_4 \cdot \frac{2+2\alpha}{4\alpha} - 1) \\ & (A_5 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 1) - (A_6 \cdot \frac{2+2\alpha}{4\alpha} - 1)^6 + (A_6 \cdot \frac{2+2\alpha}{4\alpha}) \\ & (A_6 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 2)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 3)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 4) \\ & (A_6 \cdot \frac{2+2\alpha}{4\alpha} - 5)] := B_3, \end{aligned}$$

and for the $x_7 = 1$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 1) \le & 6[(A_1 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_2 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_3 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_4 \cdot \frac{3+\alpha}{4\alpha} - 1) \\ & (A_5 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 1) - (A_6 \cdot \frac{3+\alpha}{4\alpha} - 1)^6 + (A_6 \cdot \frac{3+\alpha}{4\alpha}) \\ & (A_6 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 2)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 3) \\ & (A_6 \cdot \frac{3+\alpha}{4\alpha} - 4)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 5)] := B_4. \end{aligned}$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.4 and eliminate the denominator. Let this difference be Δ_{16}.

$$\Delta_{16} := 2048\alpha^6(1-\alpha)[(a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)(a_7 - 1) - 16384$$

$- (B_1 + B_2 + B_3 + B_4)].$

Here we need to show Δ_{16} is positive. In order to apply the partial differentiation test to Δ_{16}, we must first determine its domain

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, \text{ and } A_6 > 1.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{16}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 17115\alpha^7 - 10605\alpha^6 + 1575\alpha^5 + 175\alpha^4 - 175\alpha^3 - 1575\alpha^2 - 1683\alpha + 3365 > 0.$$

For all $\alpha \in (0, \frac{1}{5}]$ Thus, the partial derivative of Δ_{16} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = 1$.

$$\frac{\partial^5 \Delta_{16}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=1} = -1085\alpha^7 - 525\alpha^6 + 175\alpha^5 + 175\alpha^4 + 1225\alpha^3 + 633\alpha^2 - 3963\alpha + 3365 > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq 1$, $\alpha \in (0, \frac{1}{5}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{16}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=1, A_6=1} = (-1+\alpha)^2(539\alpha^5+1113\alpha^4+1582\alpha^3+1106\alpha^2+487\alpha+3365) > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$,

$$\frac{\partial^3 \Delta_{16}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^3(413\alpha^4 + 1204\alpha^3 + 1638\alpha^2 + 1572\alpha + 3365) > 0,$$

$$\frac{\partial^2 \Delta_{16}}{\partial A_1 \partial A_2}\Big|_{A_3=1, A_4=1, A_5=1, A_6=1} = (-1+\alpha)^4(539\alpha^3 + 1631\alpha^2 + 2657\alpha + 3365) > 0,$$

$$\frac{\partial \Delta_{16}}{\partial A_1}\Big|_{A_2=1, A_3=1, A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^5(1085\alpha^2 + 3742\alpha + 3365) > 0.$$

Over $\alpha \in (0, \frac{1}{5}]$. By symmetry of Δ_{16} in A_1, A_2, A_3, A_4, and A_5, all $\frac{\partial \Delta_{16}}{\partial A_2}, \frac{\partial \Delta_{16}}{\partial A_3}, \frac{\partial \Delta_{16}}{\partial A_4}, \frac{\partial \Delta_{16}}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_{17} := \Delta_{16}\Big|_{A_1=6, A_2=5, A_3=4, A_4=3, A_5=2}.$$

We must show Δ_{17} is positive over its domain. To do so, we normally apply the partial differentiation test

$$\frac{\partial^5 \Delta_{17}}{\partial A_6^5} = -19656000\alpha^7+10886400\alpha^6-1512000\alpha^5+1512000\alpha^3+4596480\alpha^2+4173120\alpha > 0,$$

$$\frac{\partial^4 \Delta_{17}}{\partial A_6^4}\Big|_{A_6=1} = 6720\alpha(2031\alpha^5 + 1271\alpha^4 + 1326\alpha^3 + 1046\alpha^2 + 67\alpha - 621)(-1+\alpha) > 0,$$

$$\frac{\partial^3 \Delta_{17}}{\partial A_6^3}\Big|_{A_6=1} = -3360\alpha(1213\alpha^5 + 1073\alpha^4 + 738\alpha^3 - 670\alpha^2 - 1439\alpha + 621)(-1+\alpha),$$

$$\frac{\partial^2 \Delta_{17}}{\partial A_6^2}\Big|_{A_6=1} = -224\alpha(1125\alpha^5+245\alpha^4+9602\alpha^3+13290\alpha^2-14055\alpha+3105)(-1+\alpha) > 0,$$

$$\frac{\partial \Delta_{17}}{\partial A_6}\Big|_{A_6=1} = 2643480\alpha^7 - 1120368\alpha^6 + 188088\alpha^5 - 3640912\alpha^4 + 3701640\alpha^3 + 206512\alpha^2 - 3418200\alpha + 2422800 > 0.$$

This confirms that Δ_{17} is increasing. We evaluate Δ_{17} at its minimum:

$$\Delta_{17}\Big|_{A_6=1} = 32964056\alpha^7 - 33012816\alpha^6 + 1348312\alpha^5 - 2592016\alpha^4 + 588168\alpha^3 + 3480400\alpha^2 - 5198904\alpha + 2422800 > 0.$$

This completes the subcase, therefore completing the proof for Case IV. ☐

3.5. **Case V.** In this case, $\lceil a_7 \rceil = 6$. Plugging that into the Main Theorem, we obtain the following

Theorem 3.5. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 > 1$ be real numbers and let P_7 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} + \frac{x_7}{a_7} \leq 1$. If $P_7 > 0$ and $5 < a_7 \leq 6$, then*

$$7!\, P_7 \leq (a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)(a_7 - 1) - 78125.$$

Proof. Since $a_7 \in (5, 6]$, we have five levels to consider: $x_7 = 1, x_7 = 2, x_7 = 3, x_7 = 4$, and $x_7 = 5$. There must be solutions on the $x_7 = 1$ level. Hence, we have the five following subcases:

Subcase V(a): $P_6(x_7 = 5) = P_6(x_7 = 4) = P_6(x_7 = 3) = P_6(x_7 = 2) = 0$,
Subcase V(b): $P_6(x_7 = 5) = P_6(x_7 = 4) = P_6(x_7 = 3) = 0, P_6(x_7 = 2) > 0$,
Subcase V(c): $P_6(x_7 = 5) = P_6(x_7 = 4) = 0, P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$,
Subcase V(d): $P_6(x_7 = 5) = 0, P_6(x_7 = 4) > 0, P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$,
Subcase V(e): $P_6(x_7 = 5) > 0, P_6(x_7 = 4) > 0, P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$.

3.5.1. *Case V(a).* We know that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.5. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{1}{a_7} := \alpha,$$

then $\alpha \in (\frac{4}{5}, \frac{5}{6}]$. Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Geometric Conjecture for n=6, we have

$$7!\, P_7 = 7!\, P_6(x_7 = 1) \leq 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - p_6(\lceil A_6 \rceil)].$$

We know the minimum value for $\lceil A_6 \rceil$ is 5, as $A_6 = a_6 \cdot \alpha > \frac{4}{5} \cdot 6$. Note that $p_6(v)$ is increasing for $v \geq 5$, so we can plug in $\lceil A_6 \rceil = 5$ to maximize the RHS of the above inequality. This gives

$$7!\, P_7 = 7!\, P_6(x_7 = 1) \leq 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - 28672)] := B_1.$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.5 and rid the denominator. Let this difference be Δ_{18}

$$\Delta_{18} := [(a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)(a_7 - 1) - 78125 - B_1]\alpha^5(1 - \alpha).$$

Here we need to show Δ_{18} is positive. First, we must determine the domain of Δ_{18}. Using the same logic as previous cases, we have

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq A_4 \geq A_5 \geq A_6 \geq \frac{\alpha}{1-\alpha}.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{18}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 7\alpha^6 - 7\alpha^5 + 1 > 0.$$

For all $\alpha \in (\frac{4}{5}, \frac{5}{6}]$ Thus, the partial derivative of Δ_{18} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = \dfrac{\alpha}{1-\alpha}$.

$$\frac{\partial^5 \Delta_{18}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=\frac{\alpha}{1-\alpha}} = \frac{-14\alpha^7 + 21\alpha^6 - 7\alpha^5 - \alpha^2}{-1+\alpha} > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \dfrac{\alpha}{1-\alpha}$, $\alpha \in (\frac{4}{5}, \frac{5}{6}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{18}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{28\alpha^8 - 56\alpha^7 + 35\alpha^6 - 7\alpha^5 + \alpha^4}{(-1+\alpha)^2} > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$.

$$\frac{\partial^3 \Delta_{18}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=\frac{\alpha}{1-\alpha}, A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{-(56\alpha^4 - 140\alpha^3 + 126\alpha^2 - 48\alpha + 7)\alpha^5}{(-1+\alpha)^3} > 0,$$

$$\frac{\partial^2 \Delta_{18}}{\partial A_1 \partial A_2}\Big|_{A_3=\frac{\alpha}{1-\alpha}, A_4=\frac{\alpha}{1-\alpha}, A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{(112\alpha^5 - 336\alpha^4 + 393\alpha^3 - 224\alpha^2 + 63\alpha - 7)\alpha^5}{(-1+\alpha)^4} > 0,$$

$$\frac{\partial \Delta_{18}}{\partial A_1}\Big|_{A_2=5, A_3=\frac{\alpha}{1-\alpha}, A_4=\frac{\alpha}{1-\alpha}, A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{(448\alpha^5 - 1345\alpha^4 + 1573\alpha^3 - 896\alpha^2 + 252\alpha - 28)\alpha^5}{(-1+\alpha)^4} > 0.$$

Δ_{18} is symmetric for all $A_i, i = 1, 2, 3, 4, 5, 6$. Therefore, Δ_{18} is increasing over the entire given domain for α. We only need to evaluate Δ_{18} at its minimum. If $A_6 \geq 5$,

$$\Delta_{18}\Big|_{A_1=6, A_2=A_3=A_4=A_5=A_6=5} = 85294\alpha^6 - 85324\alpha^5 + 400\alpha^4 - 2750\alpha^3 + 10625\alpha^2 - 21875\alpha + 18750 > 0,$$

over the interval. Now we consider the case where $4 \leq \dfrac{\alpha}{1-\alpha} A_6 < 5$. We set the value for A_6 which minimizes Δ_{18}. Thus, we have $A_2 = A_3 = A_4 = A_5 = A_6 = x$ for some

$x \in (4,5]$, and $A_1 \geq \dfrac{x}{x-5}$. We consider

$$\Delta_{18}\Big|_{A_1=\frac{x}{x-5}, A_2=A_3=A_4=A_5=A_6=x} = \frac{1}{x-5}((35x^5 - 175x^4 + 350x^3 - 350x^2 + 49629x - 247305)\alpha^6$$
$$+ (-35x^5 + 175x^4 - 350x^3 + 345x^2 - 49604x + 247300)\alpha^5 + (10x^3 - 45x^2)\alpha^4$$
$$+ (-10x^4 + 40x^3)\alpha^3 + (5x^5 - 15x^4)\alpha^2 - \alpha x^6 + x^6)) > 0,$$

for $\alpha \in (\frac{4}{5}, \frac{5}{6}]$ and $x \in (4,5]$. This completes the subcase.

3.5.2. *Case V(b).* We know that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1,1,1,1,1,1,2)$ is a solution to the inequality in Theorem 3.5. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{2}{a_7} := \alpha,$$

then $\alpha \in (\frac{3}{5}, \frac{2}{3}]$. Let $A_i = a_i \cdot \alpha$ for $i = 1,2,3,4,5,6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Geometric Conjecture for n=6, we have

$$7!\,P_7 = 7!\,P_6(x_7 = 2) \leq 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - p_6(\lceil A_6 \rceil)].$$

We know the minimum value for $\lceil A_6 \rceil$ is 4, as $A_6 = a_6 \cdot \alpha > \frac{3}{5} \cdot 6$.Note that $p_6(v)$ is increasing for $v \geq 4$, so we can plug in $\lceil A_6 \rceil = 4$ to maximize the RHS of the above inequality. This gives

$$7!\,P_7 = 7!\,P_6(x_7 = 2) \leq 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - 4096)] := B_1$$

and for the $x_7 = 1$ layer

$$7!\,P_6(x_7 = 2) \leq 7[(A_1 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_2 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_3 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_4 \cdot \frac{1+\alpha}{2\alpha} - 1)$$
$$(A_5 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_6 \cdot \frac{1+\alpha}{2\alpha} - 1) - 4096] := B_2.$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.5 and rid the denominator. Let this difference be Δ_{19}

$$\Delta_{19} := [(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 78125 - (B_1+B_2)]64\alpha^6(1-\alpha).$$

Here we need to show Δ_{19} is positive. First, we must determine the domain of Δ_{19}. Using the same logic as previous cases, we have

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq A_5 \geq A_6 \geq \frac{2\alpha}{1-\alpha}.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{19}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 455\alpha^7 - 413\alpha^6 + 63\alpha^5 + 35\alpha^4 - 35\alpha^3 - 63\alpha^2 + 29\alpha + 57 > 0,$$

for all $\alpha \in (\frac{3}{5}, \frac{2}{3}]$. Thus, the partial derivative of Δ_{19} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = \dfrac{2\alpha}{1-\alpha}$.

$$\frac{\partial^5 \Delta_{19}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=\frac{2\alpha}{1-\alpha}} = \frac{-4\alpha(343\alpha^7 - 420\alpha^6 + 147\alpha^5 - 35\alpha^3 - 12\alpha^2 + 25\alpha + 16}{-1+\alpha}$$

$$> 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \frac{2\alpha}{1-\alpha}$, $\alpha \in (\frac{3}{5}, \frac{2}{3}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{19}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=\frac{2\alpha}{1-\alpha}, A_6=\frac{2\alpha}{1-\alpha}} = \frac{16\alpha^2(259\alpha^7 - 399\alpha^6 + 210\alpha^5 - 42\alpha^4 - 17\alpha^3 + 5\alpha^2 + 12\alpha + 4)}{(-1+\alpha)^2} > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$.

$$\frac{\partial^3 \Delta_{19}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=\frac{2\alpha}{1-\alpha}, A_5=\frac{2\alpha}{1-\alpha}, A_6=\frac{2\alpha}{1-\alpha}} = \frac{64\alpha^3(196\alpha^7 - 364\alpha^6 + 252\alpha^5 - 83\alpha^4 + 4\alpha^3 + 6\alpha^2 + 4\alpha + 1)}{(-1+\alpha)^3} > 0,$$

$$\frac{\partial^2 \Delta_{19}}{\partial A_1 \partial A_2}\Big|_{A_3=4, A_4=\frac{\alpha}{1-\alpha}, A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{-64\alpha^3(581\alpha^7 - 1092\alpha^6 + 769\alpha^5 - 252\alpha^4 + 3\alpha^3 + 20\alpha^2 + 15\alpha + 4)}{(-1+\alpha)^3} > 0,$$

$$\frac{\partial \Delta_{19}}{\partial A_1}\Big|_{A_2=5, A_3=4, A_4=\frac{\alpha}{1-\alpha}, A_5=\frac{\alpha}{1-\alpha}, A_6=\frac{\alpha}{1-\alpha}} = \frac{-64\alpha^3(2310\alpha^7 - 4381\alpha^6 + 3117\alpha^5 - 1004\alpha^4 - 26\alpha^3 + 85\alpha^2 + 71\alpha + 20)}{(-1+\alpha)^3} > 0.$$

Δ_{19} is symmetric for all $A_i, i = 1, 2, 3, 4, 5, 6$. Therefore, Δ_{19} is increasing over the entire given domain for α. We only need to evaluate Δ_{19} at its minimum. If $A_6 \geq 5$,

$$\Delta_{19}\Big|_{A_1=6, A_2=A_3=A_4=A_5=A_6=5} = 6186991\alpha^7 - 6137753\alpha^6 + 183675\alpha^5 + 123875\alpha^4 + 558125\alpha^3 - 796875\alpha^2 - 484375\alpha + 890625,$$

over the interval. Now we consider the case where $3 \leq \frac{2\alpha}{1-\alpha} A_6 < 5$. We set the value for A_6 which minimizes Δ_{19}. Thus, we have $A_2 = A_3 = A_4 = A_5 = A_6 = x$ for some $x \in (3, 5]$, and $A_1 \geq \frac{x}{x-5}$. We consider

$$\Delta_{19}\Big|_{A_1=\frac{x}{x-5}, A_2=A_3=A_4=A_5=A_6=x} = \frac{1}{x-5}((-7x^6 + 2380x^5 - 12180x^4 + 25760x^3 - 28560x^2 + 4363904x - 21738880)\alpha^7 + (-21x^6 - 1820x^5 + 8820x^4 - 16800x^3 + 17040x^2 - 4356864x + 21738240)\alpha^6 + (-7x^6 + 280x^5 - 840x^4 - 480x^3 + 2960x^2 - 3840x)\alpha^5 + (35x^6 - 280x^5 + 1320x^4 - 2400x^3 + 2160x^2)\alpha^4 + (35x^6 - 100x^5 + 220x^4 + 320x^3)\alpha^3$$

$$+ (-71x^6 + 180x^5 - 540x^4)\alpha^2 - 21\alpha x^6 + 57x^6) > 0.$$

for $\alpha \in (\frac{3}{5}, \frac{2}{3}]$ and $x \in (3, 5]$. This completes the subcase.

3.5.3. *Case V(c).* We know that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 3)$ is a solution to the inequality in Theorem 3.5. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{3}{a_7} := \alpha,$$

Then $\alpha \in (\frac{2}{5}, \frac{1}{2}]$.Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Geometric Conjecture for n=6, we have

$$7!\, P_7 = 7!\, P_6(x_7 = 3) \leq 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - p_6(\lceil A_6 \rceil)].$$

We know the minimum value for $\lceil A_6 \rceil$ is 3, as $A_6 = a_6 \cdot \alpha > \frac{2}{5} \cdot 6$.Note that $p_6(v)$ is increasing for $v \geq 3$, so we can plug in $\lceil A_6 \rceil = 3$ to maximize the RHS of the above inequality. This gives

$$7!\, P_7 = 7!\, P_6(x_7 = 3) \leq 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - 64)] := B_1,$$

for the $x_7 = 2$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 2) \leq & 7[(A_1 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_2 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_3 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_4 \cdot \frac{1+2\alpha}{3\alpha} - 1) \\ & (A_5 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1) - 2^6] := B_2, \end{aligned}$$

and for the $x_7 = 1$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 1) \leq & 7[(A_1 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_2 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_3 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_4 \cdot \frac{2+\alpha}{3\alpha} - 1) \\ & (A_5 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 1) - 3^6]. \end{aligned}$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.5 and rid the denominator. Let this difference be Δ_{20}.

$$\Delta_{20} := 729\alpha^6(1-\alpha)[(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 78125 - (B_1+B_2+B_3)].$$

Here we need to show Δ_{20} is positive. First, we must determine the domain of Δ_{20}. Using the same logic as previous cases, we have

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq A_6 \geq \frac{3\alpha}{1-\alpha}.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{20}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 5558\alpha^7 - 4130\alpha^6 + 672\alpha^5 + 140\alpha^4 - 140\alpha^3 - 672\alpha^2 - 244\alpha + 1003 > 0.$$

For all $\alpha \in (\frac{2}{5}, \frac{1}{2}]$ Thus, the partial derivative of Δ_{20} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = \dfrac{3\alpha}{1-\alpha}$.

$$\frac{\partial^5 \Delta_{20}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6 = \frac{3\alpha}{1-\alpha}} =$$

$$\frac{-3\alpha(7490\alpha^7 - 7364\alpha^6 + 2184\alpha^5 - 70\alpha^4 - 350\alpha^3 - 618\alpha^2 + 167\alpha + 748)}{-1+\alpha}$$
$$> 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \dfrac{3\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{5}, \frac{1}{2}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{20}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=\frac{3\alpha}{1-\alpha}, A_6=\frac{3\alpha}{1-\alpha}} = \frac{9\alpha^2(10108\alpha^7 - 12376\alpha^6 + 5250\alpha^5 - 854\alpha^4 - 472\alpha^3 - 417\alpha^2 + 412\alpha + 536)}{(-1+\alpha)^2} > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$

$$\frac{\partial^3 \Delta_{20}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=3, A_5=\frac{3\alpha}{1-\alpha}, A_6=\frac{3\alpha}{1-\alpha}} = \frac{27\alpha^2(6552\alpha^7 - 8260\alpha^6 + 3612\alpha^5 - 468\alpha^4 - 277\alpha^3 - 489\alpha^2 + 252\alpha + 536)}{(-1+\alpha)^2} > 0,$$

$$\frac{\partial^2 \Delta_{20}}{\partial A_1 \partial A_2}\Big|_{A_3=4, A_4=3, A_5=\frac{3\alpha}{1-\alpha}, A_6=\frac{3\alpha}{1-\alpha}} = \frac{27\alpha^2(19404\alpha^7 - 24892\alpha^6 + 10968\alpha^5 - 1203\alpha^4 - 535\alpha^3 - 2040\alpha^2 + 528\alpha + 2144}{(-1+\alpha)^2} > 0,$$

$$\frac{\partial \Delta_{20}}{\partial A_1}\Big|_{A_2=5, A_3=4, A_4=3, A_5=\frac{3\alpha}{1-\alpha}, A_6=\frac{3\alpha}{1-\alpha}} = \frac{27\alpha^2(76986\alpha^7 - 99896\alpha^6 + 44115\alpha^5 - 4500\alpha^4 - 509\alpha^3 - 10140\alpha^2 + 720\alpha + 10720)}{(-1+\alpha)^2} > 0.$$

Δ_{20} is symmetric for all $A_i, i = 1, 2, 3, 4, 5, 6$. Therefore, Δ_{20} is increasing over the entire given domain for α. We only need to evaluate Δ_{20} at its minimum. If $A_6 \geq 4$,

$$\Delta_{20}\Big|_{A_1=6, A_2=5, A_3=A_4=A_5=A_6=4} = 61123110\alpha^7 - 59876259\alpha^6 + 1353159\alpha^5 - 598638\alpha^4 + 1687296\alpha^3 + 2054592\alpha^2 - 9903360\alpha + 7703040,$$

over the interval. Now we consider the case where $2 \leq \dfrac{3\alpha}{1-\alpha} A_6 < 4$. We set the value for A_6 which minimizes Δ_{20}. Thus, we have $A_2 = A_3 = A_4 = A_5 = A_6 = x$ for some

$x \in (2,4]$, and $A_1 \geq \dfrac{x}{x-5}$. We consider

$$\begin{aligned}
&\Delta_{20}\Big|_{A_1=\frac{x}{x-5},A_2=A_3=A_4=A_5=A_6=x} \\
&= \frac{1}{x-5}((-238x^6 + 30870x^5 - 160650x^4 + 351540x^3 - 408240x^2 + 52840836x \\
&- 262979460)\alpha^7 + (-224x^6 - 18270x^5 + 88830x^4 - 170100x^3 + 174960x^2 \\
&- 52698681x + 262968525)\alpha^6 + (42x^6 + 2520x^5 - 7560x^4 - 9720x^3 + 61965x^2 \\
&- 87480x)\alpha^5 + (140x^6 - 2520x^5 + 17280x^4 - 52650x^3 + 61965x^2)\alpha^4 + (490x^6 \\
&- 3600x^5 + 13230x^4 - 9720x^3)\alpha^3 + (-204x^6 + 1935x^5 - 5805x^4)\alpha^2 - 1009\alpha x^6 \\
&+ 1003x^6) > 0,
\end{aligned}$$

for $\alpha \in (\frac{2}{5}, \frac{1}{2}]$ and $x \in (2,4]$. This completes the subcase.

3.5.4. *Case V(d).* We know that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1,1,1,1,1,1,4)$ is a solution to the inequality in Theorem 3.5. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{4}{a_7} := \alpha,$$

then $\alpha \in (\frac{1}{5}, \frac{1}{3}]$. Let $A_i = a_i \cdot \alpha$ for $i = 1,2,3,4,5,6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Geometric Conjecture for n=6, we have

$$7!\,P_7 = 7!\,P_6(x_7 = 4) \leq 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - p_6(\lceil A_6 \rceil)].$$

We know the minimum value for $\lceil A_6 \rceil$ is 2, as $A_6 = a_6 \cdot \alpha > \frac{1}{5} \cdot 6$.Note that $p_6(v)$ is increasing for $v \geq 2$, so we can plug in $\lceil A_6 \rceil = 2$ to maximize the RHS of the above inequality. This gives

$$7!\,P_7 = 7!\,P_6(x_7 = 4) \leq 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - 1)] := B_1,$$

for the $x_7 = 3$ layer

$$\begin{aligned}
7!\,P_6(x_7 = 3) \leq & 7[(A_1 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_2 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_3 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_4 \cdot \frac{1+3\alpha}{4\alpha} - 1) \\
& (A_5 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 1) - 1] := B_2,
\end{aligned}$$

for the $x_7 = 2$ layer

$$\begin{aligned}
7!\,P_6(x_7 = 2) \leq & 7[(A_1 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_2 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_3 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_4 \cdot \frac{2+2\alpha}{4\alpha} - 1) \\
& (A_5 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 1) - 64] := B_3,
\end{aligned}$$

and for the $x_7 = 1$ layer

$$\begin{aligned}
7!\,P_6(x_7 = 1) \leq & 7[(A_1 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_2 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_3 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_4 \cdot \frac{3+\alpha}{4\alpha} - 1) \\
& (A_5 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 1) - 64] := B_4.
\end{aligned}$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.5 and rid the denominator. Let this difference be Δ_{21}.

$$\Delta_{21} := 2048\alpha^6(1-\alpha)[(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1)-78125-(B_1+B_2+B_3+B_4)].$$

Here we need to show Δ_{21} is positive. First, we must determine the domain of Δ_{21}. Using the same logic as previous cases, we have

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq \frac{4\alpha}{1-\alpha}.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{21}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 17115\alpha^7 - 10605\alpha^6 + 1575\alpha^5 + 175\alpha^4 - 175\alpha^3 - 1575\alpha^2 - 1683\alpha + 3365 > 0.$$

For all $\alpha \in (\frac{1}{5}, \frac{1}{3}]$ Thus, the partial derivative of Δ_{21} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = \dfrac{4\alpha}{1-\alpha}$.

$$\frac{\partial^5 \Delta_{21}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=\frac{4\alpha}{1-\alpha}} = -\frac{1}{-1+\alpha}[4\alpha(21665\alpha^7 - 17675\alpha^6 + 4445\alpha^5 - 175\alpha^4 - 525\alpha^3 - 1777\alpha^2 - 561\alpha + 2795)] > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \dfrac{4\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{5}, \frac{1}{3}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{21}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=2, A_6=\frac{4\alpha}{1-\alpha}} = -\frac{1}{-1+\alpha}[8\alpha(10087\alpha^7 - 8967\alpha^6 + 2415\alpha^5 + 105\alpha^4 + 381\alpha^3 - 1101\alpha^2 - 1619\alpha + 2795] > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$

$$\frac{\partial^3 \Delta_{21}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=3, A_5=2, A_6=\frac{4\alpha}{1-\alpha}} = -\frac{1}{-1+\alpha}[8\alpha(19817\alpha^7 - 18165\alpha^6 + 4865\alpha^5 - 101\alpha^4 + 1747\alpha^3 - 1383\alpha^2 - 6973\alpha + 8385] > 0,$$

$$\frac{\partial^2 \Delta_{21}}{\partial A_1 \partial A_2}\Big|_{A_3=4, A_4=3, A_5=2, A_6=\frac{4\alpha}{1-\alpha}} = -\frac{1}{-1+\alpha}[32\alpha(14700\alpha^7 - 13713\alpha^6 + 3744\alpha^5 - 389\alpha^4 + 1636\alpha^3 + 341\alpha^2 - 8560\alpha + 8385)] > 0,$$

$$\frac{\partial \Delta_{21}}{\partial A_1}\Big|_{A_2=5, A_3=4, A_4=3, A_5=2, A_6=\frac{4\alpha}{1-\alpha}} = -\frac{1}{-1+\alpha}[32\alpha(58380\alpha^7 - 55201\alpha^6 + 15364\alpha^5 - 2321\alpha^4 + 6124\alpha^3 + 9453\alpha^2 - 49148\alpha + 41925)] > 0.$$

Δ_{21} is symmetric for all $A_i, i = 1,2,3,4,5,6$. Therefore, Δ_{21} is increasing over the entire given domain for α. We only need to evaluate Δ_{21} at its minimum. If $A_6 \geq 4$

$$\begin{aligned}\Delta_{21}\Big|_{A_1=6,A_2=5,A_3=4,A_4=A_5=A_6=3} =&165564024\alpha^7 - 163813208\alpha^6 + 1253992\alpha^5 - 790856\alpha^4 - \\ &536184\alpha^3 + 8747352\alpha^2 - 17395560\alpha + 10902600,\end{aligned}$$

over the interval. Now we consider the case where $1 \leq \dfrac{4\alpha}{1-\alpha}A_6 < 3$. We set the value for A_6 which minimizes Δ_{21}. Thus, we have $A_2 = A_3 = A_4 = A_5 = A_6 = x$ for some $x \in (1,3]$, and $A_1 \geq \dfrac{x}{x-5}$. We consider

$$\begin{aligned}\Delta_{21}\Big|_{A_1=\frac{x}{x-5},A_2=A_3=A_4=A_5=A_6=x} =&\frac{1}{x-5}[((-1085x^6 + 99120x^5 - 521360x^4 + 1164800x^3 - \\ &1388800x^2 + 159055872x - 790978560)\alpha^7 + (-525x^6 - \\ &47600x^5 + 232400x^4 - 448000x^3 + 464640x^2 - 158482432x \\ &+ 790937600)\alpha^6 + (175x^6 + 5600x^5 - 16800x^4 - 33280x^3 + \\ &226560x^2 - 368640x)\alpha^5 + (175x^6 - 5600x^5 + 50080x^4 - \\ &197120x^3 + 288000x^2)\alpha^4 + (1225x^6 - 13840x^5 + 60720x^4 - \\ &76800x^3)\alpha^3 + (633x^6 + 3280x^5 - 9840x^4)\alpha^2 - 3963\alpha x^6 + \\ &3365x^6] > 0,\end{aligned}$$

for $\alpha \in (\frac{1}{5}, \frac{1}{3}]$ and $x \in (1,3]$. This completes the subcase.

3.5.5. *Case V(e).* We know that $P_6(x_7 = 5) > 0$, so $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1,1,1,1,1,1,5)$ is a solution to the inequality in Theorem 3.5. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{5}{a_7} := \alpha,$$

then $\alpha \in (0, \frac{1}{6}]$ because $a_7 \in (5,6]$. Let $A_i = a_i \cdot \alpha$ for $i = 1,2,3,4,5,6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Number Theoretic Conjecture for n=6, we have

$$\begin{aligned}7!\,P_6(x_7 = 5) \leq &6[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - (A_6-1)^6 + A_6 \\ &(A_6-1)(A_6-2)(A_6-3)(A_6-4)(A_6-5)] := B_1,\end{aligned}$$

for the $x_7 = 4$ layer

$$\begin{aligned}7!\,P_6(x_7 = 4) \leq &7[(A_1 \cdot \frac{1+4\alpha}{5\alpha} - 1)(A_2 \cdot \frac{1+4\alpha}{5\alpha} - 1)(A_3 \cdot \frac{1+4\alpha}{5\alpha} - 1)(A_4 \cdot \frac{1+4\alpha}{5\alpha} - 1) \\ &(A_5 \cdot \frac{1+4\alpha}{5\alpha} - 1)(A_6 \cdot \frac{1+4\alpha}{5\alpha} - 1) - (A_6 \cdot \frac{1+4\alpha}{5\alpha} - 1)^6 + (A_6 \cdot \frac{1+4\alpha}{5\alpha}) \\ &(A_6 \cdot \frac{1+4\alpha}{5\alpha} - 1)(A_6 \cdot \frac{1+4\alpha}{5\alpha} - 2)(A_6 \cdot \frac{1+4\alpha}{5\alpha} - 3)(A_6 \cdot \frac{1+4\alpha}{5\alpha} - 4)(A_6 \\ &\cdot \frac{1+4\alpha}{5\alpha} - 5)] := B_2,\end{aligned}$$

for the $x_7 = 3$ layer

$$7!\,P_6(x_7=3) \le 7[(A_1\cdot\frac{2+3\alpha}{5\alpha}-1)(A_2\cdot\frac{2+3\alpha}{5\alpha}-1)(A_3\cdot\frac{2+3\alpha}{5\alpha}-1)(A_4\cdot\frac{2+3\alpha}{5\alpha}-1)$$
$$(A_5\cdot\frac{2+3\alpha}{5\alpha}-1)(A_6\cdot\frac{2+3\alpha}{5\alpha}-1)-(A_6\cdot\frac{2+3\alpha}{5\alpha}-1)^6+(A_6\cdot\frac{2+3\alpha}{5\alpha})$$
$$(A_6\cdot\frac{2+3\alpha}{5\alpha}-1)(A_6\cdot\frac{2+3\alpha}{5\alpha}-2)(A_6\cdot\frac{2+3\alpha}{5\alpha}-3)(A_6\cdot\frac{2+3\alpha}{5\alpha}-4)(A_6$$
$$\cdot\frac{2+3\alpha}{5\alpha}-5)] := B_3,$$

for the $x_7 = 2$ layer

$$7!\,P_6(x_7=2) \le 7[(A_1\cdot\frac{3+2\alpha}{5\alpha}-1)(A_2\cdot\frac{3+2\alpha}{5\alpha}-1)(A_3\cdot\frac{3+2\alpha}{5\alpha}-1)(A_4\cdot\frac{3+2\alpha}{5\alpha}-1)$$
$$(A_5\cdot\frac{3+2\alpha}{5\alpha}-1)(A_6\cdot\frac{3+2\alpha}{5\alpha}-1)-(A_6\cdot\frac{3+2\alpha}{5\alpha}-1)^6+(A_6\cdot\frac{3+2\alpha}{5\alpha})$$
$$(A_6\cdot\frac{3+2\alpha}{5\alpha}-1)(A_6\cdot\frac{3+2\alpha}{5\alpha}-2)(A_6\cdot\frac{3+2\alpha}{5\alpha}-3)(A_6\cdot\frac{3+2\alpha}{5\alpha}-4)(A_6$$
$$\cdot\frac{3+2\alpha}{5\alpha}-5)] := B_4,$$

and for the $x_7 = 1$ layer

$$7!\,P_6(x_7=1) \le 7[(A_1\cdot\frac{4+\alpha}{5\alpha}-1)(A_2\cdot\frac{4+\alpha}{5\alpha}-1)(A_3\cdot\frac{4+\alpha}{5\alpha}-1)(A_4\cdot\frac{4+\alpha}{5\alpha}-1)$$
$$(A_5\cdot\frac{4+\alpha}{5\alpha}-1)(A_6\cdot\frac{4+\alpha}{5\alpha}-1)-(A_6\cdot\frac{4+\alpha}{5\alpha}-1)^6+(A_6\cdot\frac{4+\alpha}{5\alpha})$$
$$(A_6\cdot\frac{4+\alpha}{5\alpha}-1)(A_6\cdot\frac{4+\alpha}{5\alpha}-2)(A_6\cdot\frac{4+\alpha}{5\alpha}-3)(A_6\cdot\frac{4+\alpha}{5\alpha}-4)(A_6$$
$$\cdot\frac{4+\alpha}{5\alpha}-5)] := B_5.$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.5 and rid the denominator. Let this difference be Δ_{22}.

$$\Delta_{22} := [(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 78125 - (B_1+$$
$$B_2+B_3+B_4+B_5)]3125\alpha^6(1-\alpha).$$

Here we need to show Δ_{22} is positive. First, we must determine the domain of Δ_{22}. Using the same logic as previous cases, we have

$$A_1 \ge 6, A_2 \ge 5, A_3 \ge 4, A_4 \ge 3, A_5 \ge 2, A_6 > 1.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6\Delta_{22}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5\partial A_6} = 28721\alpha^7 - 15197\alpha^6 + 2016\alpha^5 + 140\alpha^4 - 140\alpha^3 - 2016\alpha^2 -$$
$$3553\alpha + 5654 > 0.$$

For all $\alpha \in (0, \frac{1}{6}]$ Thus, the partial derivative of Δ_{22} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = 1$.

$$\frac{\partial^5\Delta_{22}}{\partial A_1\partial A_2\partial A_3\partial A_4\partial A_5}\Big|_{A_6=1} = -2254\alpha^7 - 672\alpha^6 + 266\alpha^5 + 140\alpha^4 + 1610\alpha^3 + 2209\alpha^2 -$$
$$6953\alpha + 5654 > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq 1$, $\alpha \in (0, \frac{1}{6}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{22}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=1, A_6=1} = (-1+\alpha)^2(1036\alpha^5 + 2100\alpha^4 + 3080\alpha^3 + 2800\alpha^2 + 955\alpha + 5654) > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$

$$\frac{\partial^3 \Delta_{22}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^3(784\alpha^4 + 2324\alpha^3 + 3654\alpha^2 + 3209\alpha + 5654) > 0,$$

$$\frac{\partial^2 \Delta_{22}}{\partial A_1 \partial A_2}\Big|_{A_3=1, A_4=1, A_5=1, A_6=1} = (-1+\alpha)^4(1036\alpha^3 + 3472\alpha^2 + 5463\alpha + 5654) > 0,$$

$$\frac{\partial \Delta_{22}}{\partial A_1}\Big|_{A_2=1, A_3=1, A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^5(2254\alpha^2 + 7717\alpha + 5654) > 0.$$

Over $\alpha \in (0, \frac{1}{6}]$. By symmetry of Δ_{22} in A_1, A_2, A_3, A_4, and A_5, all $\frac{\partial \Delta_{22}}{\partial A_2}, \frac{\partial \Delta_{22}}{\partial A_3}, \frac{\partial \Delta_{22}}{\partial A_4}, \frac{\partial \Delta_{22}}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_{23} := \Delta_{22}\Big|_{A_1=6, A_2=5, A_3=4, A_4=3, A_5=2}.$$

We must show Δ_{22} is positive over its domain. To do so, we normally apply the partial differentiation test

$$\frac{\partial^5 \Delta_{22}}{\partial A_6^5} = -189000\alpha(177\alpha^6 - 83\alpha^5 + 10\alpha^4 - 10\alpha^2 - 42\alpha - 52) > 0,$$

$$\frac{\partial^4 \Delta_{22}}{\partial A_6^4}\Big|_{A_6=1} = 4200\alpha(5741\alpha^6 - 1795\alpha^5 + 110\alpha^4 - 560\alpha^3 - 2770\alpha^2 - 3066\alpha + 2340) > 0,$$

$$\frac{\partial^3 \Delta_{22}}{\partial A_6^3}\Big|_{A_6=1} = -1050\alpha(7231\alpha^6 - 725\alpha^5 - 1340\alpha^4 - 8010\alpha^3 - 8520\alpha^2 + 16044\alpha - 4680) > 0,$$

$$\frac{\partial^2 \Delta_{22}}{\partial A_6^2}\Big|_{A_6=1} = -700\alpha(422\alpha^6 - 495\alpha^5 + 5245\alpha^4 + 5730\alpha^3 - 21540\alpha^2 + 12978\alpha - 2340) > 0,$$

$$\frac{\partial \Delta_{22}}{\partial A_6}\Big|_{A_6=1} = 4440870\alpha^7 - 1456115\alpha^6 + 1123520\alpha^5 - 8722575\alpha^4 + 8231950\alpha^3 - 115270\alpha^2 - 5698260\alpha + 4070880 > 0.$$

This confirms that Δ_{22} is increasing. We evaluate Δ_{22} at its minimum:

$$\Delta_{22}\Big|_{A_6=1} = 242963480\alpha^7 - 243207940\alpha^6 + 3113495\alpha^5 - 5547675\alpha^4 + 1702750\alpha^3 + 5378870\alpha^2 - 8473860\alpha + 4070880 > 0.$$

This completes the subcase, therefore completing the proof for Case V. $\square$

3.6. **Case VI.** In this case, $\lceil a_7 \rceil = 7$. Plugging that into the Main Theorem, we obtain the following theorem.

Theorem 3.6. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 > 1$ be real numbers and let P_7 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} + \frac{x_7}{a_7} \leq 1$. If $P_7 > 0$ and $6 < a_7 \leq 7$, then*

$$7!\,P_7 \leq (a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 279936.$$

Proof. Since $a_7 \in (6, 7]$, we have six levels to consider: $x_7 = 1, x_7 = 2, x_7 = 3, x_7 = 4, x_7 = 5,$ and $x_7 = 6$. There must be solutions on the $x_7 = 1$ level. Hence, we have the six following subcases:

Subcase VI(a): $P_6(x_7 = 6) = P_6(x_7 = 5) = P_6(x_7 = 4) = P_6(x_7 = 3) = P_6(x_7 = 2) = 0$,

Subcase VI(b): $P_6(x_7 = 6) = P_6(x_7 = 5) = P_6(x_7 = 4) = P_6(x_7 = 3) = 0, P_6(x_7 = 2) > 0$,

Subcase VI(c): $P_6(x_7 = 6) = P_6(x_7 = 5) = P_6(x_7 = 4) = 0, P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$,

Subcase VI(d): $PP_6(x_7 = 6) =_6 (x_7 = 5) = 0, P_6(x_7 = 4) > 0, P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$,

Subcase VI(e): $P_6(x_7 = 6) = 0, P_6(x_7 = 5) > 0, P_6(x_7 = 4) > 0, P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$,

Subcase VI(f): $P_6(x_7 = 6) > 0, P_6(x_7 = 5) > 0, P_6(x_7 = 4) > 0, P_6(x_7 = 3) > 0, P_6(x_7 = 2) > 0$.

3.6.1. *Case VI(a).* We know that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 1)$ is a solution to the inequality in Theorem 3.6. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{1}{a_7} := \alpha.$$

Then $\alpha \in (\frac{5}{6}, \frac{6}{7}]$.Let $A_i = a_i{\cdot}\alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Geometric Conjecture for n=6, we have

$$7!\,P_7 = 7!\,P_6(x_7 = 1) \leq 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - p_6(\lceil A_6 \rceil)].$$

We know the minimum value for $\lceil A_6 \rceil$ is 6, as $A_6 = a_6 \cdot \alpha > \frac{6}{7} \cdot 6$.Note that $p_6(v)$ is increasing for $v \geq 6$, so we can plug in $\lceil A_6 \rceil = 6$ to maximize the RHS of the above inequality. This gives

$$7!\,P_7 = 7!\,P_6(x_7 = 1) \leq 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - 14905)] := B_1.$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.6 and rid the denominator. Let this difference be Δ_{23}

$$\Delta_{23} := [(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 279936 - B_1]\alpha^5(1-\alpha).$$

Here we need to show Δ_{23} is positive. First, we must determine the domain of Δ_{23}. Using the same logic as previous cases, we have

$$A_1 \geq 6, A_2 \geq A_3 \geq A_4 \geq A_5 \geq A_6 \geq \frac{\alpha}{1-\alpha}.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{23}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 7\alpha^6 - 7\alpha^5 + 1 > 0,$$

for all $\alpha \in (\frac{5}{6}, \frac{6}{7}]$ Thus, the partial derivative of Δ_{23} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = \dfrac{\alpha}{1-\alpha}$.

$$\frac{\partial^5 \Delta_{23}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6 = \frac{\alpha}{1-\alpha}} = \frac{1}{-1+\alpha}[-14\alpha^7 + 21\alpha^6 - 7\alpha^5 - \alpha^2] > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \dfrac{\alpha}{1-\alpha}$, $\alpha \in (\frac{5}{6}, \frac{6}{7}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{23}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5 = A_6 = \frac{\alpha}{1-\alpha}} = \frac{1}{(-1+\alpha)^2}[28\alpha^8 - 56\alpha^7 + 35\alpha^6 - 7\alpha^5 + \alpha^4] > 0,$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$

$$\frac{\partial^3 \Delta_{23}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4 = A_5 = A_6 = \frac{\alpha}{1-\alpha}} = -\frac{1}{(-1+\alpha)^3}[(56\alpha^4 - 140\alpha^3 + 126\alpha^2 - 48\alpha + 7)\alpha^5] > 0,$$

$$\frac{\partial^2 \Delta_{23}}{\partial A_1 \partial A_2}\Big|_{A_3 = A_4 = A_5 = A_6 = \frac{\alpha}{1-\alpha}} = \frac{1}{(-1+\alpha)^4}[(112\alpha^5 - 336\alpha^4 + 393\alpha^3 - 224\alpha^2 + 63\alpha - 7)\alpha^5] > 0,$$

$$\frac{\partial \Delta_{23}}{\partial A_1}\Big|_{A_2 = A_3 = A_4 = A_5 = A_6 = \frac{\alpha}{1-\alpha}} = -\frac{1}{(-1+\alpha)^5}[(224\alpha^6 - 783\alpha^5 + 1120\alpha^4 - 840\alpha^3 + 350\alpha^2 - 77\alpha + 7)\alpha^5] > 0.$$

Δ_{23} is symmetric for all $A_i, i = 1, 2, 3, 4, 5, 6$. Therefore, Δ_{23} is increasing over the entire given domain for α. We only need to evaluate Δ_{23} at its minimum. If $A_6 \geq 7$

$$\Delta_{23}\Big|_{A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = 7} = 502194\alpha^6 - 502235\alpha^5 + 735\alpha^4 - 6860\alpha^3 + 36015\alpha^2 - 100842\alpha + 117649 > 0,$$

over the interval. Now we consider the case where $5 \leq \dfrac{\alpha}{1-\alpha} A_6 < 7$. We set the value for A_6 which minimizes Δ_{23}. Thus, we have $A_2 = A_3 = A_4 = A_5 = A_6 = x$ for some $x \in (5, 7]$, and $A_1 \geq \dfrac{x}{x-5}$. We consider

$$\Delta_{23}\Big|_{A_1 = \frac{x}{x-5}, A_2 = A_3 = A_4 = A_5 = A_6 = x} = \frac{1}{x-5}[((35x^5 - 175x^4 + 350x^3 - 350x^2 + 175777x - 878045)\alpha^6 + (-35x^5 + 175x^4 - 350x^3 + 345x^2 - 175752x + 878040)\alpha^5 + (10x^3 - 45x^2)\alpha^4 + (-10x^4 + 40x^3)\alpha^3 + (5x^5 - 15x^4)\alpha^2 - \alpha x^6 + x^6)] > 0,$$

for $\alpha \in (\frac{5}{6}, \frac{6}{7}]$ and $x \in (5, 7]$. This completes the subcase.

3.6.2. *Case VI(b).* We know that $P_6(x_7 = 2) > 0$, so

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 2)$$

is a solution to the inequality in Theorem 3.6. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{2}{a_7} := \alpha,$$

then $\alpha \in (\frac{2}{3}, \frac{5}{7}]$ because $a_7 \in (6, 7]$. Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Number Theoretic Conjecture for n=6, we have

$$\begin{aligned} 7!\, P_6(x_7 = 2) \leq & 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 + A_6 \\ & (A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] := B_1, \end{aligned}$$

and for the $x_7 = 1$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 2) \leq & 7[(A_1 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_2 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_3 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_4 \cdot \frac{1+\alpha}{2\alpha} - 1) \\ & (A_5 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_6 \cdot \frac{1+\alpha}{2\alpha} - 1) - (A_6 \cdot \frac{1+\alpha}{2\alpha} - 1)^6 + (A_6 \cdot \frac{1+\alpha}{2\alpha}) \\ & (A_6 \cdot \frac{1+\alpha}{2\alpha} - 1)(A_6 \cdot \frac{1+\alpha}{2\alpha} - 2)(A_6 \cdot \frac{1+\alpha}{2\alpha} - 3)(A_6 \cdot \frac{1+\alpha}{2\alpha} - 4)(A_6 \\ & \cdot \frac{1+\alpha}{2\alpha} - 5)] := B_2. \end{aligned}$$

Since $7!\, P_7 = 7!\, (P_6(x_7 = 1) + P_6(x_7) = 2)$, so we can subtract the RHS of Theorem 3.6 by the sum of the RHS of the above equalities to get Δ_{24}, and we can rid the denominator without changing the sign. Then, we merely need to apply the partial differentiation test to Δ_{24}

$$\Delta_{24} := 64\alpha^6(1-\alpha)[(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 279936 - (B_1+B_2)].$$

We are trying to show that it is positive for

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq A_4 \geq A_5 \geq A_6 > \frac{2\alpha}{1-\alpha}.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{24}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 455\alpha^7 - 413\alpha^6 + 63\alpha^5 + 35\alpha^4 - 35\alpha^3 - 63\alpha^2 + 29\alpha + 57 > 0.$$

For all $\alpha \in (\frac{2}{3}, \frac{5}{7}]$ Thus, the partial derivative of Δ_{24} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = \dfrac{2\alpha}{1-\alpha}$.

$$\begin{aligned} \frac{\partial^5 \Delta_{24}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6 = \frac{2\alpha}{1-\alpha}} = & -\frac{1}{-1+\alpha}[4\alpha(343\alpha^7 - 420\alpha^6 + 147\alpha^5 - 35\alpha^3 - \\ & 12\alpha^2 + 25\alpha + 16)] > 0, \end{aligned}$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \dfrac{2\alpha}{1-\alpha}$, $\alpha \in (\frac{2}{3}, \frac{5}{7})$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value:

$$\frac{\partial^4 \Delta_{24}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=A_6=\frac{2\alpha}{1-\alpha}} = \frac{1}{(-1+\alpha)^2}[16\alpha^2(259\alpha^7 - 399\alpha^6 + 210\alpha^5 - 42\alpha^4 - 17\alpha^3 + 5\alpha^2 + 12\alpha + 4)] > 0,$$

over the domain of α. For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$:

$$\frac{\partial^3 \Delta_{24}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=A_5=A_6=\frac{2\alpha}{1-\alpha}} = -\frac{1}{(-1+\alpha)^3}[64\alpha^3(196\alpha^7 - 364\alpha^6 + 252\alpha^5 - 83\alpha^4 + 4\alpha^3 + 6\alpha^2 + 4\alpha + 1)] > 0,$$

$$\frac{\partial^2 \Delta_{24}}{\partial A_1 \partial A_2}\Big|_{A_3=A_4=A_5=A_6=\frac{2\alpha}{1-\alpha}} = \frac{1}{(-1+\alpha)^4}[64\alpha^4(595\alpha^7 - 1295\alpha^6 + 1107\alpha^5 - 485\alpha^4 + 101\alpha^3 + 3\alpha^2 + 5\alpha + 1)] > 0,$$

$$\frac{\partial \Delta_{24}}{\partial A_1}\Big|_{A_2=A_3=A_4=A_5=A_6=\frac{2\alpha}{1-\alpha}} = -\frac{1}{(-1+\alpha)^5}[64\alpha^5(1813\alpha^7 - 4535\alpha^6 + 4619\alpha^5 - 2505\alpha^4 + 755\alpha^3 - 97\alpha^2 + 13\alpha + 1)] > 0,$$

over the interval $\alpha \in (\frac{2}{3}, \frac{5}{7}]$. By symmetry of Δ_{24} in A_1, A_2, A_3, A_4, and A_5, all $\dfrac{\partial \Delta_{24}}{\partial A_2}, \dfrac{\partial \Delta_{24}}{\partial A_3}, \dfrac{\partial \Delta_{24}}{\partial A_4}, \dfrac{\partial \Delta_{24}}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_{25} := \Delta_{24}\Big|_{A_1=6, A_2=5, A_3=4, A_4=3, A_5=2}.$$

We must show that Δ_{25} is positive for all $\alpha \in (\frac{2}{3}, \frac{5}{7}]$ and $A_6 \geq 1$. Now we apply the partial differentiation test. We note that for certain values of $A_6 \geq 1$, not all derivatives are positive. We first set $A_6 \geq 5.5$

$$\frac{\partial^5 \Delta_{25}}{\partial A_6^5} = -15120\alpha(33\alpha^6 - 28\alpha^5 + 5\alpha^4 - 5\alpha^2 - 4\alpha - 1) > 0,$$

$$\frac{\partial^4 \Delta_{25}}{\partial A_6^4}\Big|_{A_6=5.5} = -1.94 \cdot 10^6\alpha^7 + 1.72 \cdot 10^6\alpha^6 - 321720.0\alpha^5 - 94080.0\alpha^4 + 274680.0\alpha^3 + 285600.0\alpha^2 + 83160.0\alpha > 0,$$

$$\frac{\partial^3 \Delta_{25}}{\partial A_6^3}\Big|_{A_6=5.5} = -0.00019 - 3.77 \cdot 10^6\alpha^7 + 3.45 \cdot 10^6\alpha^6 - 626009.99\alpha^5 - 379680.0\alpha^4 + 436170.0\alpha^3 + 656040.0\alpha^2 + 228690.0\alpha > 0,$$

$$\frac{\partial^2 \Delta_{25}}{\partial A_6^2}\Big|_{A_6=5.5} = -0.000096 - 4.86 \cdot 10^6\alpha^7 + 4.59 \cdot 10^6\alpha^6 - 731381.0\alpha^5 - 723296.0\alpha^4 + 340725.0\alpha^3 + 965580.0\alpha^2 + 419265.0\alpha > 0,$$

$$\frac{\partial \Delta_{25}}{\partial A_6}\Big|_{A_6=5.5} = 41040.0 - 4.65 \cdot 10^6\alpha^7 + 4.48 \cdot 10^6\alpha^6 - 563278.9\alpha^5 - 845028.0\alpha^4 + 46156.9\alpha^3 + 968745.5\alpha^2 + 545169.4\alpha > 0,$$

over the set domain. Therefore, Δ_{25} is increasing. We confirm that it is positive by evaluating it at its minimum

$$\Delta_{25}\Big|_{A_6=5.5} = 225720.00 + 1.45 \cdot 10^7\alpha^7 - 1.46 \cdot 10^7\alpha^6 - 295086.6\alpha^5 - 687031.0\alpha^4 - 82290.9\alpha^3 + 594278.7\alpha^2 + 425878.3\alpha > 0.$$

As for when $A_6 \in (4, 5.5]$, we can verify numerically through Maple or Mathematica to confirm that it indeed is positive. Thus, this subcase is complete. This completes the subcase.

3.6.3. *Case VI(c).* We know that $P_6(x_7 = 3) > 0$, so

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 3)$$

is a solution to the inequality in Theorem 3.6. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \le 1 - \frac{3}{a_7} := \alpha,$$

then $\alpha \in (\frac{1}{2}, \frac{4}{7}]$ because $a_7 \in (6, 7]$. Let Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \le 1.$$

Thus, by the Yau Number Theoretic Conjecture for n=6, we have

$$7!\, P_6(x_7 = 3) \le 7[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 + A_6 (A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] := B_1,$$

for the $x_7 = 2$ layer

$$7!\, P_6(x_7 = 2) \le 7[(A_1 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_2 \cdot \frac{1+2\alpha}{3\alpha} 1)(A_3 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_4 \cdot \frac{1+2\alpha}{3\alpha} - 1) (A_5 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1) - (A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1)^6 + (A_6 \cdot \frac{1+2\alpha}{3\alpha}) (A_6 \cdot \frac{1+2\alpha}{3\alpha} - 1)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 2)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 3)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 4)(A_6 \cdot \frac{1+2\alpha}{3\alpha} - 5)] := B_2,$$

and for the $x_7 = 1$ layer

$$7!\, P_6(x_7 = 1) \le 7[(A_1 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_2 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_3 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_4 \cdot \frac{2+\alpha}{3\alpha} - 1) (A_5 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 1) - (A_6 \cdot \frac{2+\alpha}{3\alpha} - 1)^6 + (A_6 \cdot \frac{2+\alpha}{3\alpha}) (A_6 \cdot \frac{2+\alpha}{3\alpha} - 1)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 2)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 3)(A_6 \cdot \frac{2+\alpha}{3\alpha} - 4)(A_6$$

$$\cdot \frac{2+\alpha}{3\alpha} - 5)] := B_3.$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.6 and eliminate the denominator. Let this difference be Δ_{26}

$$\Delta_{26} := 729\alpha^6(1-\alpha)[(a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - 279936 - (B_1+B_2+B_3)].$$

Here we need to show Δ_{26} is positive. In order to apply the partial differentiation test to Δ_{26}, we must first determine its domain

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq A_5 \geq A_6 > \frac{\alpha}{1-\alpha} > 3.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{26}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 5558\alpha^7 - 4130\alpha^6 + 672\alpha^5 + 140\alpha^4 - 140\alpha^3 - 672\alpha^2 - 244\alpha + 10030.$$

For all $\alpha \in (\frac{1}{2}, \frac{4}{7}]$ Thus, the partial derivative of Δ_{26} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = 3$.

$$\left.\frac{\partial^5 \Delta_{26}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\right|_{A_6=3} = 10878\alpha^7 - 8484\alpha^6 + 1386\alpha^5 + 420\alpha^4 + 210\alpha^3 - 1548\alpha^2 - \alpha + 3009 > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq \frac{2\alpha}{1-\alpha}$, $\alpha \in (\frac{1}{2}, \frac{4}{7})$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value:

$$\left.\frac{\partial^4 \Delta_{26}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\right|_{A_5=A_6=3} = 21420\alpha^7 - 17388\alpha^6 + 2772\alpha^5 + 756\alpha^4 + 1800\alpha^3 - 2853\alpha^2 - 6786\alpha + 9027 > 0.$$

Over the domain of α. For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$:

$$\left.\frac{\partial^3 \Delta_{26}}{\partial A_1 \partial A_2 \partial A_3}\right|_{A_4=A_5=A_6=3} = 42336\alpha^7 - 35532\alpha^6 + 5670\alpha^5 + 216\alpha^4 + 6993\alpha^3 - 2025\alpha^2 - 27243\alpha + 27081 > 0,$$

$$\left.\frac{\partial^2 \Delta_{26}}{\partial A_1 \partial A_2}\right|_{A_3=4, A_4=A_5=A_6=3} = 126252\alpha^7 - 108108\alpha^6 + 18306\alpha^5 - 5373\alpha^4 + 27729\alpha^3 + 14985\alpha^2 - 129627\alpha + 108324 > 0,$$

$$\left.\frac{\partial \Delta_{26}}{\partial A_1}\right|_{A_2=5, A_3=4, A_4=A_5=A_6=3} = 503118\alpha^7 - 437508\alpha^6 + 82539\alpha^5 - 51570\alpha^4 + 124794\alpha^3 + 177714\alpha^2 - 730755\alpha + 541620 > 0,$$

over the interval $\alpha \in (\frac{1}{2}, \frac{4}{7}]$. By symmetry of Δ_{26} in A_1, A_2, A_3, A_4, and A_5, all $\frac{\partial \Delta_{26}}{\partial A_2}$, $\frac{\partial \Delta_{26}}{\partial A_3}$, $\frac{\partial \Delta_{26}}{\partial A_4}$, $\frac{\partial \Delta_{26}}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for

A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_{27} := \Delta_{26}\Big|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=3}.$$

We must show that Δ_{27} is positive for all $\alpha \in (\frac{1}{2}, \frac{4}{7}]$ and $A_6 \geq 1$. Now we apply the partial differentiation test. We note that for certain values of $A_6 \geq 1$, not all derivatives are positive. We first set $A_6 \geq 4$,

$$\frac{\partial^5 \Delta_{27}}{\partial A_6^5} = -6259680\alpha^7 + 4218480\alpha^6 - 680400\alpha^5 + 680400\alpha^3 + 1292760\alpha^2 + 748440\alpha > 0,$$

$$\frac{\partial^4 \Delta_{27}}{\partial A_6^4}\Big|_{A_6=4} = -15120\alpha(970\alpha^5 + 260\alpha^4 + 384\alpha^3 + 440\alpha^2 + 421\alpha + 198)(-1+\alpha) > 0,$$

$$\frac{\partial^3 \Delta_{27}}{\partial A_6^3}\Big|_{A_6=4} = -1890\alpha(8972\alpha^5 + 1894\alpha^4 + 2982\alpha^3 + 3667\alpha^2 + 4832\alpha + 3168)(-1+\alpha) > 0,$$

$$\frac{\partial^2 \Delta_{27}}{\partial A_6^2}\Big|_{A_6=4} = -378\alpha(34802\alpha^5 + 6064\alpha^4 + 9675\alpha^3 + 9340\alpha^2 + 19520\alpha + 21120)(-1+\alpha) > 0,$$

$$\begin{aligned}\frac{\partial \Delta_{27}}{\partial A_6}\Big|_{A_6=4} =& -6534738\alpha^7 + 5446566\alpha^6 - 980541\alpha^5 + 1770183\alpha^4 - 1709901\alpha^3 - 5209479\alpha^2 \\ &+ 6659550\alpha + 1083240 > 0,\end{aligned}$$

over the set domain. Therefore, Δ_{27} is increasing. We confirm that it is positive by evaluating it at its minimum

$$\begin{aligned}\Delta_{27}\Big|_{A_6=4} =& 204026949\alpha^7 - 203972616\alpha^6 - 817407\alpha^5 + 1032435\alpha^4 + 3052413\alpha^3 - \\ & 6345342\alpha^2 + 265248\alpha + 4332960 > 0.\end{aligned}$$

As for when $A_6 \in (3, 4]$, we can verify numerically through Maple or Mathematica to confirm that it indeed is positive. Thus, this subcase is complete. This completes the subcase.

3.6.4. *Case VI(d).* We know that $P_6(x_7 = 4) > 0$, so

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 4)$$

is a solution to the inequality in Theorem 3.6. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{4}{a_7} := \alpha,$$

then $\alpha \in (\frac{1}{3}, \frac{3}{7}]$ because $a_7 \in (6, 7]$. Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Number Theoretic Conjecture for n=6, we have

$$\begin{aligned}7!\, P_6(x_7 = 4) \leq& 7[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1) - (A_6-1)^6 + A_6 \\ & (A_6-1)(A_6-2)(A_6-3)(A_6-4)(A_6-5)] := B_1,\end{aligned}$$

for the $x_7 = 3$ layer

$$7!\, P_6(x_7 = 3) \leq 7[(A_1 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_2 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_3 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_4 \cdot \frac{1+3\alpha}{4\alpha} - 1)$$

$$(A_5 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 1) - (A_6 \cdot \frac{1+3\alpha}{4\alpha} - 1)^6 + (A_6 \cdot \frac{1+3\alpha}{4\alpha})$$
$$(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 1)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 2)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 3)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 4)(A_6 \cdot \frac{1+3\alpha}{4\alpha} - 5)] := B_2,$$

for the $x_7 = 2$ layer

$$7!\, P_6(x_7 = 2) \leq 7[(A_1 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_2 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_3 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_4 \cdot \frac{2+2\alpha}{4\alpha} - 1)$$
$$(A_5 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 1) - (A_6 \cdot \frac{2+2\alpha}{4\alpha} - 1)^6 + (A_6 \cdot \frac{2+2\alpha}{4\alpha})$$
$$(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 1)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 2)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 3)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 4)(A_6 \cdot \frac{2+2\alpha}{4\alpha} - 5)] := B_3,$$

and for the $x_7 = 1$ layer

$$7!\, P_6(x_7 = 1) \leq 7[(A_1 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_2 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_3 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_4 \cdot \frac{3+\alpha}{4\alpha} - 1)$$
$$(A_5 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 1) - (A_6 \cdot \frac{3+\alpha}{4\alpha} - 1)^6 + (A_6 \cdot \frac{3+\alpha}{4\alpha})$$
$$(A_6 \cdot \frac{3+\alpha}{4\alpha} - 1)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 2)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 3)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 4)(A_6 \cdot \frac{3+\alpha}{4\alpha} - 5)] := B_4.$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.6 and eliminate the denominator. Let this difference be Δ_{28}

$$\Delta_{28} := [(a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)(a_7 - 1) - 279936 - (B_1 + B_2 + B_3 + B_4)]2048\alpha^6(1 - \alpha).$$

Here we need to show Δ_{28} is positive. First, we must determine the domain of Δ_{28}. Using the same logic as previous cases, we have

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq A_6 \geq \frac{4\alpha}{1-\alpha} > 2.$$

Now, we can apply the partial differentiation test

$$\frac{\partial^6 \Delta_{28}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = 17115\alpha^7 - 10605\alpha^6 + 1575\alpha^5 + 175\alpha^4 - 175\alpha^3 - 1575\alpha^2 - 1683\alpha + 3365 > 0,$$

for all $\alpha \in (\frac{1}{3}, \frac{3}{7}]$. Thus, the partial derivative of Δ_{28} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = 2$.

$$\left.\frac{\partial^5 \Delta_{28}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\right|_{A_6=2} = 16030\alpha^7 - 11130\alpha^6 + 1750\alpha^5 + 350\alpha^4 + 1050\alpha^3 - 942\alpha^2 - 5646\alpha + 6730 > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq 2$, $\alpha \in (\frac{1}{3}, \frac{3}{7}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{28}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=2,A_6=2} = 15484\alpha^7 - 11620\alpha^6 + 1820\alpha^5 - 420\alpha^4 + 2132\alpha^3 + 3188\alpha^2 - 15852\alpha + 13460 > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$,

$$\frac{\partial^3 \Delta_{28}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=3,A_5=2,A_6=2} = 30548\alpha^7 - 23660\alpha^6 + 4340\alpha^5 - 2412\alpha^4 + 2844\alpha^3 + 21020\alpha^2 - 56676\alpha + 40380 > 0,$$

$$\frac{\partial^2 \Delta_{28}}{\partial A_1 \partial A_2}\Big|_{A_3=4,A_4=3,A_5=2,A_6=2} = 91056\alpha^7 - 72240\alpha^6 + 15152\alpha^5 - 10672\alpha^4 - 2672\alpha^3 + 121072\alpha^2 - 254064\alpha + 161520 > 0,$$

$$\frac{\partial \Delta_{28}}{\partial A_1}\Big|_{A_2=5,A_3=4,A_4=3,A_5=2,A_6=2} = 362544\alpha^7 - 292912\alpha^6 + 72432\alpha^5 - 62448\alpha^4 - 72048\alpha^3 + 761200\alpha^2 - 1379760\alpha + 807600 > 0,$$

over $\alpha \in (\frac{1}{3}, \frac{3}{7}]$. By symmetry of Δ_{28} in A_1, A_2, A_3, A_4, and A_5, all $\frac{\partial \Delta_{28}}{\partial A_2}, \frac{\partial \Delta_{28}}{\partial A_3}, \frac{\partial \Delta_{28}}{\partial A_4}, \frac{\partial \Delta_{28}}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_{29} := \Delta_{28}\Big|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2}.$$

We must show Δ_{29} is positive over its domain. To do so, we normally apply the partial differentiation test

$$\frac{\partial^5 \Delta_{29}}{\partial A_6^5} = -19656000\alpha^7 + 10886400\alpha^6 - 1512000\alpha^5 + 1512000\alpha^3 + 4596480\alpha^2 + 4173120\alpha > 0,$$

$$\frac{\partial^4 \Delta_{29}}{\partial A_6^4}\Big|_{A_6=2} = -13440\alpha(447\alpha^5 + 17\alpha^4 + 102\alpha^3 + 242\alpha^2 + 619\alpha + 621)(-1+\alpha) > 0,$$

$$\frac{\partial^3 \Delta_{29}}{\partial A_6^3}\Big|_{A_6=2} = -13440\alpha(19\alpha^5 - 41\alpha^4 - 96\alpha^3 - 308\alpha^2 - 67\alpha + 621)(-1+\alpha) > 0,$$

$$\frac{\partial^2 \Delta_{29}}{\partial A_6^2}\Big|_{A_6=2} = -1792\alpha(435\alpha^5 + 475\alpha^4 + 1054\alpha^3 - 600\alpha^2 - 3765\alpha + 3105)(-1+\alpha) > 0,$$

$$\frac{\partial \Delta_{29}}{\partial A_6}\Big|_{A_6=2} = 1809360\alpha^7 - 1085648\alpha^6 - 1346480\alpha^5 - 2415184\alpha^4 + 10085360\alpha^3 - 7677168\alpha^2 - 810000\alpha + 2422800 > 0.$$

This confirms that Δ_{29} is increasing. We evaluate Δ_{29} at its minimum:

$$\Delta_{29}\Big|_{A_6=2} = 574961632\alpha^7 - 573832800\alpha^6 + 678816\alpha^5 - 5935904\alpha^4 + 7521568\alpha^3 + 457056\alpha^2 - 7712928\alpha + 4845600 > 0,$$

for $\alpha \in (\frac{1}{3}, \frac{3}{7}]$. This completes the subcase.

3.6.5. *Case VI(e).* We know that $P_6(x_7=5)>0$, so

$$(x_1,x_2,x_3,x_4,x_5,x_6,x_7)=(1,1,1,1,1,1,5)$$

is a solution to the inequality in Theorem 3.6. If

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5}+\frac{1}{a_6}\le 1-\frac{5}{a_7}:=\alpha,$$

then $\alpha\in(\frac{1}{6},\frac{2}{7}]$ because $a_7\in(6,7]$. Let $A_i=a_i\cdot\alpha$ for $i=1,2,3,4,5,6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1}+\frac{1}{A_2}+\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_5}+\frac{1}{A_6}\le 1.$$

Thus, by the Yau Number Theoretic Conjecture for $n=6$, we have

$$\begin{aligned}7!\,P_6(x_7=5)\le&6[(A_1-1)(A_2-1)(A_3-1)(A_4-1)(A_5-1)(A_6-1)-(A_6-1)^6+A_6\\&(A_6-1)(A_6-2)(A_6-3)(A_6-4)(A_6-5)]:=B_1.\end{aligned}$$

For the $x_7=4$ layer

$$\begin{aligned}7!\,P_6(x_7=4)\le&7[(A_1\cdot\frac{1+4\alpha}{5\alpha}-1)(A_2\cdot\frac{1+4\alpha}{5\alpha}-1)(A_3\cdot\frac{1+4\alpha}{5\alpha}-1)(A_4\cdot\frac{1+4\alpha}{5\alpha}-1)\\&(A_5\cdot\frac{1+4\alpha}{5\alpha}-1)(A_6\cdot\frac{1+4\alpha}{5\alpha}-1)-(A_6\cdot\frac{1+4\alpha}{5\alpha}-1)^6+(A_6\cdot\frac{1+4\alpha}{5\alpha})\\&(A_6\cdot\frac{1+4\alpha}{5\alpha}-1)(A_6\cdot\frac{1+4\alpha}{5\alpha}-2)(A_6\cdot\frac{1+4\alpha}{5\alpha}-3)(A_6\cdot\frac{1+4\alpha}{5\alpha}-4)(A_6\\&\cdot\frac{1+4\alpha}{5\alpha}-5)]:=B_2,\end{aligned}$$

for the $x_7=3$ layer

$$\begin{aligned}7!\,P_6(x_7=3)\le&7[(A_1\cdot\frac{2+3\alpha}{5\alpha}-1)(A_2\cdot\frac{2+3\alpha}{5\alpha}-1)(A_3\cdot\frac{2+3\alpha}{5\alpha}-1)(A_4\cdot\frac{2+3\alpha}{5\alpha}-1)\\&(A_5\cdot\frac{2+3\alpha}{5\alpha}-1)(A_6\cdot\frac{2+3\alpha}{5\alpha}-1)-(A_6\cdot\frac{2+3\alpha}{5\alpha}-1)^6+(A_6\cdot\frac{2+3\alpha}{5\alpha})\\&(A_6\cdot\frac{2+3\alpha}{5\alpha}-1)(A_6\cdot\frac{2+3\alpha}{5\alpha}-2)(A_6\cdot\frac{2+3\alpha}{5\alpha}-3)(A_6\cdot\frac{2+3\alpha}{5\alpha}-4)(A_6\\&\cdot\frac{2+3\alpha}{5\alpha}-5)]:=B_3,\end{aligned}$$

for the $x_7=2$ layer

$$\begin{aligned}7!\,P_6(x_7=2)\le&7[(A_1\cdot\frac{3+2\alpha}{5\alpha}-1)(A_2\cdot\frac{3+2\alpha}{5\alpha}-1)(A_3\cdot\frac{3+2\alpha}{5\alpha}-1)(A_4\cdot\frac{3+2\alpha}{5\alpha}-1)\\&(A_5\cdot\frac{3+2\alpha}{5\alpha}-1)(A_6\cdot\frac{3+2\alpha}{5\alpha}-1)-(A_6\cdot\frac{3+2\alpha}{5\alpha}-1)^6+(A_6\cdot\frac{3+2\alpha}{5\alpha})\\&(A_6\cdot\frac{3+2\alpha}{5\alpha}-1)(A_6\cdot\frac{3+2\alpha}{5\alpha}-2)(A_6\cdot\frac{3+2\alpha}{5\alpha}-3)(A_6\cdot\frac{3+2\alpha}{5\alpha}-4)(A_6\\&\cdot\frac{3+2\alpha}{5\alpha}-5)]:=B_4,\end{aligned}$$

and for the $x_7=1$ layer

$$\begin{aligned}7!\,P_6(x_7=1)\le&7[(A_1\cdot\frac{4+\alpha}{5\alpha}-1)(A_2\cdot\frac{4+\alpha}{5\alpha}-1)(A_3\cdot\frac{4+\alpha}{5\alpha}-1)(A_4\cdot\frac{4+\alpha}{5\alpha}-1)\\&(A_5\cdot\frac{4+\alpha}{5\alpha}-1)(A_6\cdot\frac{4+\alpha}{5\alpha}-1)-(A_6\cdot\frac{4+\alpha}{5\alpha}-1)^6+(A_6\cdot\frac{4+\alpha}{5\alpha})\end{aligned}$$

$$(A_6 \cdot \frac{4+\alpha}{5\alpha} - 1)(A_6 \cdot \frac{4+\alpha}{5\alpha} - 2)(A_6 \cdot \frac{4+\alpha}{5\alpha} - 3)(A_6 \cdot \frac{4+\alpha}{5\alpha} - 4)(A_6 \cdot \frac{4+\alpha}{5\alpha} - 5)] := B_5.$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.6 and rid the denominator. Let this difference be Δ_{30}. Note the relation between Δ_{30} and Δ_{22}, $\Delta_{30} = \Delta_{22} - 630659375\alpha^6(1-\alpha)$ Δ_{30} and Δ_{22} also have the same domain for A_1, A_2, A_3, A_4, A_5 and A_6

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 \geq \frac{5\alpha}{1-\alpha} > 1.$$

Since we already applied the partial differentiation test for Δ_{22}, we know that the partial derivatives for Δ_{30} are also all positive. Thus, we only need to test Δ_{30} at its minimum.

$$\Delta_{30}\Big|_{A_1=6,A_2=5,A_3=4,A_4=3,A_5=2,A_6=1} = 873622855\alpha^7 - 873870440\alpha^6 + 3179120\alpha^5 - 6094550\alpha^4 + 3999625\alpha^3 + 303870\alpha^2 - 2961360\alpha + 1820880,$$

over the domain of $\alpha \in (\frac{1}{6}, \frac{2}{7}]$. This completes the subcase.

3.6.6. *Case VI(f).* We know that $P_6(x_7 = 6) > 0$, so

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, 1, 1, 1, 1, 6)$$

is a solution to the inequality in Theorem 3.6. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6} \leq 1 - \frac{6}{a_7} := \alpha,$$

then $\alpha \in (0, \frac{1}{7}]$ because $a_7 \in (6, 7]$. Let $A_i = a_i \cdot \alpha$ for $i = 1, 2, 3, 4, 5, 6$. Rewriting the equation by substituting A_i, we yield

$$\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} \leq 1.$$

Thus, by the Yau Number Theoretic Conjecture for n=6, we have

$$7!\, P_6(x_7 = 6) \leq 6[(A_1 - 1)(A_2 - 1)(A_3 - 1)(A_4 - 1)(A_5 - 1)(A_6 - 1) - (A_6 - 1)^6 + A_6 (A_6 - 1)(A_6 - 2)(A_6 - 3)(A_6 - 4)(A_6 - 5)] := B_1,$$

for the $x_7 = 5$ layer

$$7!\, P_6(x_7 = 5) \leq 7[(A_1 \cdot \frac{1+5\alpha}{6\alpha} - 1)(A_2 \cdot \frac{1+5\alpha}{6\alpha} - 1)(A_3 \cdot \frac{1+5\alpha}{6\alpha} - 1)(A_4 \cdot \frac{1+5\alpha}{6\alpha} - 1)(A_5 \cdot \frac{1+5\alpha}{6\alpha} - 1)(A_6 \cdot \frac{1+5\alpha}{6\alpha} - 1) - (A_6 \cdot \frac{1+5\alpha}{6\alpha} - 1)^6 + (A_6 \cdot \frac{1+5\alpha}{6\alpha}) (A_6 \cdot \frac{1+5\alpha}{6\alpha} - 1)(A_6 \cdot \frac{1+5\alpha}{6\alpha} - 2)(A_6 \cdot \frac{1+5\alpha}{6\alpha} - 3)(A_6 \cdot \frac{1+5\alpha}{6\alpha} - 4)(A_6 \cdot \frac{1+5\alpha}{6\alpha} - 5)] := B_2,$$

for the $x_7 = 4$ layer

$$7!\, P_6(x_7 = 4) \leq 7[(A_1 \cdot \frac{2+4\alpha}{6\alpha} - 1)(A_2 \cdot \frac{2+4\alpha}{6\alpha} - 1)(A_3 \cdot \frac{2+4\alpha}{6\alpha} - 1)(A_4 \cdot \frac{2+4\alpha}{6\alpha} - 1)(A_5 \cdot \frac{2+4\alpha}{6\alpha} - 1)(A_6 \cdot \frac{2+4\alpha}{6\alpha} - 1) - (A_6 \cdot \frac{2+4\alpha}{6\alpha} - 1)^6 + (A_6 \cdot \frac{2+4\alpha}{6\alpha}) (A_6 \cdot \frac{2+4\alpha}{6\alpha} - 1)(A_6 \cdot \frac{2+4\alpha}{6\alpha} - 2)(A_6 \cdot \frac{2+4\alpha}{6\alpha} - 3)(A_6 \cdot \frac{2+4\alpha}{6\alpha} - 4)(A_6$$

$$\cdot \frac{2+4\alpha}{6\alpha} - 5)] := B_3,$$

for the $x_7 = 3$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 3) \leq & 7[(A_1 \cdot \frac{3+3\alpha}{6\alpha} - 1)(A_2 \cdot \frac{3+3\alpha}{6\alpha} - 1)(A_3 \cdot \frac{3+3\alpha}{6\alpha} - 1)(A_4 \cdot \frac{3+3\alpha}{6\alpha} - 1)(A_5 \\ & \cdot \frac{3+3\alpha}{6\alpha} - 1)(A_6 \cdot \frac{3+3\alpha}{6\alpha} - 1) - (A_6 \cdot \frac{3+3\alpha}{6\alpha} - 1)^6 + (A_6 \cdot \frac{3+3\alpha}{6\alpha}) \\ & (A_6 \cdot \frac{3+3\alpha}{6\alpha} - 1)(A_6 \cdot \frac{3+3\alpha}{6\alpha} - 2)(A_6 \cdot \frac{3+3\alpha}{6\alpha} - 3)(A_6 \cdot \frac{3+3\alpha}{6\alpha} - 4)(A_6 \\ & \cdot \frac{3+3\alpha}{6\alpha} - 5)] := B_4, \end{aligned}$$

for the $x_7 = 2$ layer,

$$\begin{aligned} 7!\, P_6(x_7 = 2) \leq & 7[(A_1 \cdot \frac{4+2\alpha}{6\alpha} - 1)(A_2 \cdot \frac{4+2\alpha}{6\alpha} - 1)(A_3 \cdot \frac{4+2\alpha}{6\alpha} - 1)(A_4 \cdot \frac{4+2\alpha}{6\alpha} - 1)(A_5 \\ & \cdot \frac{4+2\alpha}{6\alpha} - 1)(A_6 \cdot \frac{4+2\alpha}{6\alpha} - 1) - (A_6 \cdot \frac{4+2\alpha}{6\alpha} - 1)^6 + (A_6 \cdot \frac{4+2\alpha}{6\alpha}) \\ & (A_6 \cdot \frac{4+2\alpha}{6\alpha} - 1)(A_6 \cdot \frac{4+2\alpha}{6\alpha} - 2)(A_6 \cdot \frac{4+2\alpha}{6\alpha} - 3)(A_6 \cdot \frac{4+2\alpha}{6\alpha} - 4)(A_6 \\ & \cdot \frac{4+2\alpha}{6\alpha} - 5)] := B_5, \end{aligned}$$

and for the $x_7 = 1$ layer

$$\begin{aligned} 7!\, P_6(x_7 = 1) \leq & 7[(A_1 \cdot \frac{5+\alpha}{6\alpha} - 1)(A_2 \cdot \frac{5+\alpha}{6\alpha} - 1)(A_3 \cdot \frac{5+\alpha}{6\alpha} - 1)(A_4 \cdot \frac{5+\alpha}{6\alpha} - 1)(A_5 \\ & \cdot \frac{5+\alpha}{6\alpha} - 1)(A_6 \cdot \frac{5+\alpha}{6\alpha} - 1) - (A_6 \cdot \frac{5+\alpha}{6\alpha} - 1)^6 + (A_6 \cdot \frac{5+\alpha}{6\alpha}) \\ & (A_6 \cdot \frac{5+\alpha}{6\alpha} - 1)(A_6 \cdot \frac{5+\alpha}{6\alpha} - 2)(A_6 \cdot \frac{5+\alpha}{6\alpha} - 3)(A_6 \cdot \frac{5+\alpha}{6\alpha} - 4)(A_6 \\ & \cdot \frac{5+\alpha}{6\alpha} - 5)] := B_6. \end{aligned}$$

Similar to previous cases, we take the difference obtained by subtracting the RHS of the above inequality from the RHS of Theorem 3.6 and rid the denominator. Let this difference be Δ_{31}.

$$\begin{aligned} \Delta_{31} := & [(a_1 - 1)(a_2 - 1)(a_3 - 1)(a_4 - 1)(a_5 - 1)(a_6 - 1)(a_7 - 1) - 279936 - (B_1 + B_2 + B_3 \\ & + B_4 + B_5 + B_6)]46656\alpha^6(1 - \alpha). \end{aligned}$$

Here we need to show Δ_{31} is positive. First, we must determine the domain of Δ_{31}. Using the same logic as previous cases, we have

$$A_1 \geq 6, A_2 \geq 5, A_3 \geq 4, A_4 \geq 3, A_5 \geq 2, A_6 > 1.$$

Now, we can apply the partial differentiation test

$$\begin{aligned} \frac{\partial^6 \Delta_{31}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6} = & 470197\alpha^7 - 216727\alpha^6 + 25725\alpha^5 + 1225\alpha^4 - 1225\alpha^3 - 25725\alpha^2 \\ & - 63209\alpha + 89675 > 0. \end{aligned}$$

For all $\alpha \in (0, \frac{1}{7}]$ Thus, the partial derivative of Δ_{31} with respect to $A_1, A_2, A_3, A_4, A_5, A_6$ is positive and minimized at $A_6 = 1$.

$$\frac{\partial^5 \Delta_{31}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=1} = -42245\alpha^7 - 8575\alpha^6 + 3675\alpha^5 + 1225\alpha^4 + 20825\alpha^3$$

$$+46059\alpha^2 - 110639\alpha + 89675 > 0.$$

The partial is again positive with respect to A_1, A_2, A_3, A_4, A_6 for all $A_5 \geq 1$, $\alpha \in (0, \frac{1}{7}]$ by symmetry. Hence, the next partial derivative is increasing, and we again take the minimum value

$$\frac{\partial^4 \Delta_{31}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=1, A_6=1} = (-1+\alpha)^2(18613\alpha^5 + 37471\alpha^4 + 55594\alpha^3 + 57302\alpha^2 + 21281\alpha + 89675) > 0.$$

For the same reason as above, using symmetry, we can evaluate the minimum value of the partial derivative with $A_i = 1$ for $i = 2, 3, 4$,

$$\frac{\partial^3 \Delta_{31}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^3(14021\alpha^4 + 41818\alpha^3 + 70896\alpha^2 + 63526\alpha + 89675) > 0,$$

$$\frac{\partial^2 \Delta_{31}}{\partial A_1 \partial A_2}\Big|_{A_3=1, A_4=1, A_5=1, A_6=1} = (-1+\alpha)^4(18613\alpha^3 + 65877\alpha^2 + 105771\alpha + 89675) > 0,$$

$$\frac{\partial \Delta_{31}}{\partial A_1}\Big|_{A_2=1, A_3=1, A_4=1, A_5=1, A_6=1} = -(-1+\alpha)^5(42245\alpha^2 + 148016\alpha + 89675) > 0,$$

over $\alpha \in (0, \frac{1}{7}]$. By symmetry of Δ_{31} in A_1, A_2, A_3, A_4, and A_5, all $\frac{\partial \Delta_{31}}{\partial A_2}, \frac{\partial \Delta_{31}}{\partial A_3}, \frac{\partial \Delta_{31}}{\partial A_4}, \frac{\partial \Delta_{31}}{\partial A_5}$ are positive over the given domain. We then plug in the minimum values for A_1, A_2, A_3, A_4, and A_5 to get a polynomial in terms of A_6 and α, and we want to show that it is positive. We define

$$\Delta_{32} := \Delta_{31}\Big|_{A_1=6, A_2=5, A_3=4, A_4=3, A_5=2}.$$

We must show Δ_{31} is positive over its domain. To do so, we normally apply the partial differentiation test

$$\frac{\partial^5 \Delta_{31}}{\partial A_6^5} = -553437360\alpha^7 + 224804160\alpha^6 - 23814000\alpha^5 + 23814000\alpha^3 + 127915200\alpha^2 + 200718000\alpha > 0,$$

$$\frac{\partial^4 \Delta_{31}}{\partial A_6^4}\Big|_{A_6=1} = 15120\alpha(27097\alpha^5 + 19817\alpha^4 + 20202\alpha^3 + 18242\alpha^2 + 5677\alpha - 13275)(-1+\alpha) > 0,$$

$$\frac{\partial^3 \Delta_{31}}{\partial A_6^3}\Big|_{A_6=1} = -7560\alpha(17689\alpha^5 + 16121\alpha^4 + 13776\alpha^3 - 4444\alpha^2 - 33089\alpha + 13275)(-1+\alpha) > 0,$$

$$\frac{\partial^2 \Delta_{31}}{\partial A_6^2}\Big|_{A_6=1} = -1512\alpha(2107\alpha^5 - 1645\alpha^4 + 37800\alpha^3 + 107840\alpha^2 - 100835\alpha + 22125)(-1+\alpha) > 0,$$

$$\frac{\partial \Delta_{31}}{\partial A_6}\Big|_{A_6=1} = 72794358\alpha^7 - 19081224\alpha^6 + 31063950\alpha^5 - 180513360\alpha^4 + 166542930\alpha^3 - 15116184\alpha^2 - 86664150\alpha + 64566000 > 0.$$

This confirms that Δ_{31} is increasing. We evaluate Δ_{31} at its minimum:

$$\Delta_{31}\Big|_{A_6=1} = 13039083414\alpha^7 - 13045609872\alpha^6 + 63741510\alpha^5 - 111101760\alpha^4 + 46730754\alpha^3 + 70094304\alpha^2 - 127504350\alpha + 64566000 > 0,$$

for $\alpha \in (0, \frac{1}{7}]$. This completes the subcase, therefore completing the proof for Case VI. □

3.7. **Case VII.** In this case, $a_7 > 7$, and we are trying to prove:

Theorem 3.7. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 > 7$ be real numbers and let P_7 be the number of positive integral solutions of $\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_5}{a_5} + \frac{x_6}{a_6} + \frac{x_7}{a_7} \leq 1$. If $P_7 > 0$ and $6 < a_7 \leq 7$, then*

$$7!\,P_7 \leq (a_1-1)(a_2-1)(a_3-1)(a_4-1)(a_5-1)(a_6-1)(a_7-1) - (14v^6 - 154v^5 + 700v^4 - 1589v^3 + 1743v^2 - 713v - 1)\Big|_{v=a_7-\beta+1} := B_g.$$

Let the fractional part of a_7 be β. Note that it must be one of $\frac{a_7}{a_6}, \frac{a_7}{a_5}, \frac{a_7}{a_4}, \frac{a_7}{a_3}, \frac{a_7}{a_2}, \frac{a_7}{a_1}$. Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 \in \mathbb{Z}$.

Proof. For the proof to this case, we need to use the sharp GLY Conjecture for $n = 7$. Although it wasn't proven, it is modified to fit the conditions to this question, as explained in the introduction:

Theorem 3.8. *Let $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 \geq 7$ be real numbers. Then,*

$$\begin{aligned} 7!\,P_7 \leq\, & a_1a_2a_3a_4a_5a_6a_7 - 3(a_1a_2a_3a_4a_5a_6 + a_1a_2a_3a_4a_5a_7 + a_1a_2a_3a_4a_6a_7 + a_1a_2a_3a_5a_6a_7 + \\ & a_1a_2a_4a_5a_6a_7 + a_1a_3a_4a_5a_6a_7 + a_2a_3a_4a_5a_6a_7) + \frac{175}{6}(a_1a_2a_3a_4a_5 + a_1a_2a_3a_4a_6 + \\ & a_1a_2a_3a_5a_6 + a_1a_2a_4a_5a_6 + a_1a_3a_4a_5a_6 + a_2a_3a_4a_5a_6) - 49(a_1a_2a_3a_4 + a_1a_2a_3a_5 + \\ & a_1a_2a_3a_6 + a_1a_2a_4a_5 + a_1a_2a_4a_6 + a_1a_2a_5a_6 + a_1a_3a_4a_5 + a_1a_3a_4a_6 + a_1a_3a_5a_6 + \\ & a_1a_4a_5a_6 + a_2a_3a_4a_5 + a_2a_3a_4a_6 + a_2a_3a_5a_6 + a_2a_4a_5a_6 + a_3a_4a_5a_6) + \frac{406}{5}(a_1a_2a_3 + \\ & a_1a_2a_4 + a_1a_2a_5 + a_1a_2a_6 + a_1a_3a_4 + a_1a_3a_5 + a_1a_3a_6 + a_1a_4a_5 + a_1a_4a_6 + a_1a_5a_6 + \\ & a_2a_3a_4 + a_2a_3a_5 + a_2a_3a_6 + a_2a_4a_5 + a_2a_4a_6 + a_2a_5a_6 + a_3a_4a_5 + a_3a_4a_6 + a_3a_5a_6 + \\ & a_4a_5a_6) - \frac{588}{5}(a_1a_2 + a_1a_3 + a_1a_4 + a_1a_5 + a_1a_6 + a_2a_3 + a_2a_4 + a_2a_5 + a_2a_6 + a_3a_4 \\ & + a_3a_5 + a_3a_6 + a_4a_5 + a_4a_6 + a_5a_6) + 120(a_1 + a_2 + a_3 + a_4 + a_5 + a_6) := B_s. \end{aligned}$$

Equality holds if and only if $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 \in \mathbb{Z}$.

We want to show that the RHS of the inequality in Theorem 3.8 is greater than that of Theorem 3.7.Therefore we take their difference and substitute in $a_i = A_i \cdot a_7$ for $i = 1, 2, 3, 4, 5, 6$ and $\beta = \frac{a_7}{a_6}$ without loss of generality.We eliminate the denominator of the difference, and define it as Δ_{32}

$$\Delta_{32} := (B_g - B_s)A_6^6.$$

Now, we apply the partial differentiation test over the interval $A_1 \geq A_2 \geq A_3 \geq A_4 \geq A_5 \geq A_6 \geq 1$ and $a_7 > 7$:

$$\frac{\partial^{12}\Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6^7} = 10080a_7^6 > 0,$$

$$\frac{\partial^{11}\Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6^6}\Big|_{A_6=1} = 11520a_7^6 - 20280a_7^5 > 0,$$

$$\frac{\partial^{10}\Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6^5}\Big|_{A_6=1} = 6480a_7^6 - 20280a_7^5 > 0,$$

$$\frac{\partial^{9}\Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6^4}\Big|_{A_6=1} = 2400a_7^6 - 10140a_7^5 > 0,$$

$$\frac{\partial^8 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6^3}\Big|_{A_6=1} = 660a_7^6 - 3380a_7^5 > 0,$$
$$\frac{\partial^7 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6^2}\Big|_{A_6=1} = 144a_7^6 - 845a_7^5 > 0,$$
$$\frac{\partial^6 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5 \partial A_6}\Big|_{A_6=1} = 26a_7^6 - 169a_7^5 > 0,$$
$$\frac{\partial^5 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4 \partial A_5}\Big|_{A_6=1} = 4a_7^6 - \frac{159}{6}a_7^5 > 0,$$
$$\frac{\partial^4 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3 \partial A_4}\Big|_{A_5=1, A_6=1} = 6a_7^6 - \frac{136}{3}a_7^5 + 48a_7^4 > 0,$$
$$\frac{\partial^3 \Delta_{32}}{\partial A_1 \partial A_2 \partial A_3}\Big|_{A_4=1, A_5=1, A_6=1} = 8a_7^6 - \frac{145}{2}a_7^5 + 143a_7^4 - \frac{401}{5}a_7^3 > 0,$$
$$\frac{\partial^2 \Delta_{32}}{\partial A_1 \partial A_2}\Big|_{A_3=1, A_4=1, A_5=1, A_6=1} = 10a_7^6 - \frac{320}{3}a_7^5 + 284a_7^4 - \frac{1599}{5}a_7^3 + \frac{583}{5}a_7^2 > 0,$$
$$\frac{\partial \Delta_{32}}{\partial A_1}\Big|_{A_2=1, A_3=1, A_4=1, A_5=1, A_6=1} = 12a_7^6 - \frac{785}{6}a_7^5 + 470a_7^4 - 797a_7^3 + 582a_7^2 - 119a_7 > 0.$$

Thus, all that it remains is for us to test the minimum equality case, where $A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = 1$:

$$\Delta_{32}\Big|_{A_1=A_2=A_3=A_4=A_5=A_6=1} = 0.$$

This shows the equality case, and thus Δ_{32} is non-negative. This case is complete. □

4. Conclusion

Thus, we have completed the proof to the Yau Geometric Conjecture, giving a sharp upper estimate of the geometric genus. The main theorem states that $P_7 > 0$ for the inequality to hold. It has already been proven by Yau and Zuo [36] that the Yau Geometric Conjecture holds when the geometric genus of the singularity is 0. Thus this paper completely proves the conjecture in the seventh-dimensional case.

The method that has been applied in this paper provides possible insights to proving the conjecture in the general n case. While this paper approaches the proof numerically, is it possible to generalize the results from the partial differentiation test to all n in an algebraic sense? If so, how can it be adapted and how to we split an n-dimensional simplex into layers as we did in the paper? If these questions are solved, we might be able to make certain advancements in proving the general-n case.

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