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论文题目：A Direct Proof of the Prime Number Theorem using Riemann's Prime-counting Function

A Direct Proof of the Prime Number Theorem using Riemann's Prime-counting Function

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Abstract. In this paper, we develop a novel analytic method to prove the prime number theorem in de la Vallée Poussin's form:

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}(xe^{-c\sqrt{\log x}})$$

Instead of performing asymptotic expansion on Chebyshev functions as in conventional analytic methods, this new approach uses contour-integration method to analyze Riemann's prime counting function $J(x)$, which only differs from $\pi(x)$ by $\mathcal{O}(\sqrt{x}/\log x)$.

1. Introduction

The prime number theorem[1][2] has been a popular topic in analytic number theory since the 19th century. Its first proofs were independently given by Hadamard and de la Vallée Poussin in 1896 [3] using analytic methods. Since then, mathematicians such as Apostol [1], Levinson [4], Newman [5][6], Selberg [7], Stein [8], and Wright [9] have explored different approaches (e.g. elementary methods, Tauberian theorems, and contour integrations) to prove this theorem. Instead of analyzing $\pi(x)$ directly, present proofs, regardless elementary or analytic, first attacked either of the Chebyshev functions

$$\vartheta(x) = \sum_{p \le x} \log p \tag{1}$$

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{k \le \log_2 x} \vartheta(x^{1/k}) \tag{2}$$

to derive asymptotic formulae, and estimates for $\pi(x)$ were later obtained using elementary methods, such as partial summation, to $\vartheta(x)$. Although Landau [1][10] derived the prime number theorem by proving

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} \mu(n) = 0 \tag{3}$$

$$\lim_{x \to \infty} \sum_{n \le x} \frac{\mu(n)}{n} = 0 \tag{4}$$

he still justified the equivalence of (3) and (4) to the prime number theorem with Chebyshev functions. Hence, it is an interesting task to investigate a method that avoids the use of (1) and (2).

In this paper, we propose a proof of the prime number theorem that avoids the use of Chebyshev functions. In particular, we first introduce Riemann's prime counting function $J(x)$ to our problem and relate it to $\pi(x)$. Then, by inverse Mellin transform, an integral representation is obtained for $\pi(x)$. In the subsequent part of this paper, we evaluate the contour integral and obtain the prime number theorem in the version of de la Vallée Poussin. That is, there exists a positive constant c such that

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}(xe^{-c\sqrt{\log x}}) \tag{5}$$

where $\mathrm{li}(x)$ is the logarithmic integral:

$$\mathrm{li}(x) = \lim_{\varepsilon \to 0^+} \left[\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \frac{\mathrm{d}t}{\log t} \right] \tag{6}$$

In essence, the main contribution of this paper is a new proof of the prime number theorem. The significance of this proof is that it is an analytic method that avoids the use of Chebyshev functions.

2. Riemann's prime counting function $J(x)$

Let a_n be defined by

$$a_n = \begin{cases} 1/k & n = p^k, p \text{ prime} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In his paper on $\pi(x)$, Riemann [11] defined $J(x)$ as the summatory function for a_n:

$$J(x) = \sum_{n \le x} a_n = \sum_{1 < p^k \le x} \frac{1}{k} \tag{8}$$

This function has a remarkable property that allows us to connect it to the standard prime counting function $\pi(x)$:

Theorem 1. *For $x \ge 2$, we have*

$$J(x) = \pi(x) + \mathcal{O}(\sqrt{x} \log\log x) \tag{9}$$

Proof. It follows from (8) that

$$\begin{aligned} J(x) - \pi(x) &= \sum_{2 \le k \le \log_2 x} \frac{\pi(x^{1/k})}{k} \le \sum_{2 \le k \le \log_2 x} \frac{\pi(\sqrt{x})}{k} \\ &\le \sum_{2 \le k \le \log_2 x} \frac{\sqrt{x}}{k} \ll \sqrt{x} \log\log x \end{aligned}$$

□

By (9), asymptotic formula for $J(x)$ can be ported to $\pi(x)$ with ease, so we can study $\pi(x)$ simultaneously when we analyze $J(x)$.

3. Integral representation for $J(x)$ and $\pi(x)$

Riemann [11] had shown that

$$J(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{x^s}{s}\log\zeta(s)\mathrm{d}s \tag{10}$$

for noninteger $x > 2$ and $k > 1$ using Fourier analysis. However, because (10) involves infinite integrals, it would be difficult to analyze its asymptotic behavior directly, so we wish to convert (10) into a quantitative form. In other words, we prove a special case of Perron's formula [3][12].

Lemma 1. *For $\Re(s) > 1$, we have*

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s} = \log\zeta(s) \tag{11}$$

Proof. Since $\Re(s) > 1$, the left hand side converges absolutely, we can change the order of summation safely:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_n}{n^s} &= \sum_p\sum_{k=1}^{\infty}\frac{1}{kp^{ks}} = \sum_p \log\frac{1}{1-p^{-s}}\\ &= \log\prod_p\frac{1}{1-p^{-s}} = \log\zeta(s)\end{aligned}$$

□

Lemma 2. *For $k > 1$ and noninteger $x > 2$, if we set*

$$J(x) = \frac{1}{2\pi i}\int_{k-iT}^{k+iT}\frac{x^s}{s}\log\zeta(s)\mathrm{d}s + R_1(x,k,T) \tag{12}$$

then

$$R_1(x,k,T) \ll \frac{x^k}{T}\sum_{n=1}^{\infty}\frac{1}{n^k|\log x/n|} \tag{13}$$

Proof. If we define

$$h(y) = \begin{cases}0 & 0 < y < 1\\ 1 & y > 1\end{cases} \tag{14}$$

then $J(x)$ can be rewritten into

$$J(x) = \sum_{n=1}^{\infty} a_n h\left(\frac{x}{n}\right) \tag{15}$$

The right hand side converges certainly because the summand vanishes once $n > x$. Due to Titchmarsh [3], for $k > 1$ we have

$$h(y) = \frac{1}{2\pi i}\int_{k-iT}^{k+iT}\frac{y^s}{s}\mathrm{d}s + \mathcal{O}\left(\frac{y^k}{T|\log y|}\right) \tag{16}$$

Since $\log\zeta(s)$ converges absolutely whenever $\Re(s) = k > 1$, we plug (16) and (11) into (15). Finally, $|a_n| \le 1$ allows us to deduce (13). □

Lemma 3. *Using the same notation, we have*

$$R_1(x,k,T) \ll \frac{x^k}{T(k-1)} + \frac{x\log x}{T} + \frac{x}{T|x-N|} \tag{17}$$

where N denotes the nearest integer to x.

Proof. To handle the logarithms in the denominator in (13), we partition the sum into

$$\sum_n = \sum_{n<x/2, n>2x} + \sum_{x/2\le n\le N-1} + \sum_{N+1\le n\le 2x} + \frac{x^k}{Tn^k|\log x/N|}$$

In the first sum, it is evident that $|\log x/n| > \log 2$, so we have

$$\begin{aligned}\frac{x^k}{T}\sum_{n<x/2,n>2x}\frac{1}{n^k|\log x/n|} &\ll \frac{x^k}{T}\sum_{n<x/2,n>2x}\frac{1}{n^k} \le \frac{x^k}{T}\sum_{n=1}^{\infty}\frac{1}{n^k}\\ &= \frac{x^k}{T}\zeta(k) \ll \frac{x^k}{T(k-1)}\end{aligned}$$

in which the last inequality follows from the fact that $\zeta(s) \sim 1/(s-1)$ when $s \to 1$ [3].

When $x/2 \le n \le 2x$, the logarithm function satisfies that

$$\left|\log\frac{x}{n}\right| \gg \frac{|x-n|}{x} \tag{18}$$

so we have

$$\begin{aligned}\frac{x^k}{T}\sum_{x/2\le n\le N-1}\frac{1}{n^k|\log x/n|} &\ll \frac{1}{T}\sum_{x/2\le n\le N-1}\frac{1}{|\log x/n|} \ll \frac{x}{T}\sum_{x/2\le n\le N-1}\frac{1}{x-n}\\ &\ll \frac{x}{T}\int_{x/2}^{N-1}\frac{\mathrm{d}t}{x-t} = \frac{x}{T}\log\frac{x/2}{x-N+1} \ll \frac{x\log x}{T}\end{aligned}$$

in which the last part follows from the fact that $N - 1/2 \le x \le N + 1/2$. Due to symmetry, (18) can be applied to the sum over $N+1 \le n \le 2x$ to obtain the same estimate. For the last part, using (18), we have

$$\frac{x^k}{Tn^k|\log x/N|} \ll \frac{1}{T|\log x/N|} \ll \frac{x}{T|x-N|} \tag{19}$$

Combining all these estimates gives (17). □

Theorem 2. *For half-odd integer $x > 2$ and $k = 1 + 1/\log x$, we have*

$$\pi(x) = \frac{1}{2\pi i}\int_{k-iT}^{k+iT}\frac{x^s}{s}\log\zeta(s)\mathrm{d}s + R_2(x,T) \tag{20}$$

with

$$R_2(x,T) \ll \frac{x\log x}{T} + \sqrt{x}\log\log x \tag{21}$$

Proof. Plugging $k = 1 + 1/\log x$ into (17) gives

$$R_1(x, 1 + 1/\log x, T) \ll \frac{x \log x}{T} + \frac{x}{T|x - N|} \tag{22}$$

when x is half-odd, we have $|x - N| = 1/2$, so the second term of (22) can be removed. Plugging (22) into (12), we get

$$J(x) = \frac{1}{2\pi i} \int_{k-iT}^{k+iT} \frac{x^s}{s} \log \zeta(s) \mathrm{d}s + \mathcal{O}\left(\frac{x \log x}{T}\right) \tag{23}$$

Finally, we replace $J(x)$ with (23) in (9) to obtain the result. □

4. Evaluation of the integral

To handle the integral, we consider a rectangular region R with vertices $1 - \delta \pm iT$ and $k \pm iT$. This implies

$$\frac{1}{2\pi i} \int_{k-iT}^{k+iT} \frac{x^s}{s} \log \zeta(s) \mathrm{d}s = \frac{1}{2\pi i} \oint_{\partial R} \frac{x^s}{s} \log \zeta(s) \mathrm{d}s \tag{24}$$

$$+ \frac{1}{2\pi i} \left[\int_{k-iT}^{1-\delta-iT} + \int_{1-\delta-iT}^{1-\delta+iT} + \int_{1-\delta+iT}^{k+iT} \right] \tag{25}$$

where ∂R denotes the boundary of R in counterclockwise direction and δ is chosen such that $\zeta(s) \neq 0$ in R and on ∂R. This means that the only singularity of the integrand in R is at $s = 1$. However, because $s = 1$ is a logarithmic singularity, circumvention is needed to evaluate it.

Lemma 4. *If we define*

$$f(r) = \frac{1}{2\pi i} \oint_{(r+)} \frac{x^s}{s} \log \frac{1}{s/r - 1} \mathrm{d}s \tag{26}$$

where $(r+)$ denotes any counterclockwise path that encloses only the singularity at $s = r$, then

$$f'(r) = \frac{x^r}{r} \tag{27}$$

for $r \neq 0$.

Proof. Since

$$\frac{\partial}{\partial r} \log \frac{1}{s/r - 1} = \frac{\partial}{\partial r} [\log r - \log(s - r)] = \frac{1}{r} + \frac{1}{s - r}$$

we have

$$f'(r) = \frac{1}{2\pi i} \oint_{(r+)} \frac{x^s}{s} \left[\frac{1}{r} + \frac{1}{s - r}\right] \mathrm{d}s$$
$$= \lim_{s \to r} \frac{x^s}{s} \left[\frac{s - r}{r} + 1\right] = \frac{x^r}{r}$$

□

Lemma 5. *Using the same notation, for all $\eta > 0$ we have*

$$f(1 + i\eta) = \mathrm{li}(x^{1+i\eta}) - i\pi \tag{28}$$

$$f(1 - i\eta) = \mathrm{li}(x^{1-i\eta}) + i\pi \tag{29}$$

Proof. When $\eta > 0$, because (26) implies $\lim_{a\to\infty} f(a) = 0$, we have

$$f(1+i\eta) = f(1+i\eta) - f(-\infty+i\eta) = \int_{-\infty+i\eta}^{1+i\eta} f'(r)\mathrm{d}r$$
$$= \underbrace{\int_{-\infty+i\eta}^{1+i\eta} \frac{x^r}{r}\mathrm{d}r}_{u=r\log x} = \int_{(-\infty+i\eta)\log x}^{(1+i\eta)\log x} \frac{e^u}{u}\mathrm{d}u$$

By Cauchy's integral theorem [13], we can deform the path of integration of the last integral into a line segment connecting $-\infty$ and $-\varepsilon < 0$, a clockwise circular arc connecting $-\varepsilon$ and ε, and finally a line segment connecting ε and $(1+\eta i)\log x$. That is, we have

$$f(1+i\eta) = \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{(1+\delta i)\log x} + \int_{-\varepsilon}^{\varepsilon} \tag{30}$$

for all $\varepsilon > 0$. As $\varepsilon \to 0^+$, the latter integral becomes

$$\lim_{\varepsilon\to 0^+} \int_{-\varepsilon}^{\varepsilon} \frac{e^u}{u}\mathrm{d}u = \lim_{\varepsilon\to 0^+} \int_{\pi}^{0} \frac{e^{\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}}(i\varepsilon e^{i\theta})\mathrm{d}\theta = i \lim_{\varepsilon\to 0^+} \int_{\pi}^{0} e^{\varepsilon e^{i\theta}}\mathrm{d}\theta$$
$$= -i\int_{\pi}^{0} \mathrm{d}\theta = -i\pi$$

in which the interchanging of the limit operation follows from the fact that $e^{\varepsilon e^{i\theta}}$ converges to 1 uniformly:

$$|e^{\varepsilon e^{i\theta}} - 1| \leq \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} = e^{\varepsilon} - 1 \tag{31}$$

As $\varepsilon \to 0^+$, the first two integrals become

$$\lim_{\varepsilon\to 0^+} \underbrace{\left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{(1+\eta i)\log x} \frac{e^u}{u}\mathrm{d}u\right]}_{t=e^u} = \lim_{\varepsilon\to 0^+}\left[\int_{-\infty}^{e^{-\varepsilon}} + \int_{e^{\varepsilon}}^{x^{1+i\eta}} \frac{\mathrm{d}t}{\log t}\right] \tag{32}$$

By (6), it is obvious that the right hand side evaluates to $\mathrm{li}(x^{1+i\eta})$. By plugging everything into (30), we see that (28) is true. A symmetric argument can be applied to prove (29). ☐

Theorem 3. *Using the same notation, we have*

$$\frac{1}{2\pi i}\oint_{\partial R} \frac{x^s}{s}\log\zeta(s)\mathrm{d}s = \mathrm{li}(x) \tag{33}$$

Proof. Since $\zeta(s) \sim 1/(s-1)$ when s is near one [3], we can transform the integrand:

$$\frac{1}{2\pi i}\oint_{\partial R} \frac{x^s}{s}\log\zeta(s)\mathrm{d}s = \frac{1}{2\pi i}\oint_{(1+)} \frac{x^s}{s}\log\frac{1}{s-1}\mathrm{d}s \tag{34}$$

By (26), we see that the right hand side is exactly $f(1)$, which can be calculated by evaluating its Cauchy principal value from (28) and (29):

$$f(1) = \lim_{\eta\to 0^+} \frac{f(1+i\eta) + f(1-i\eta)}{2} = \mathrm{li}(x) \tag{35}$$

☐

Since Theorem 3 allows us to evaluate (24), we now move our focus to (25). To begin with, we extract some properties of $\zeta(s)$ necessary for the proof.

Lemma 6 (de la Vallée Poussin)**.** *Let $s = \sigma + it$. Then there exists a fixed constant $A > 0$ such that whenever $1 - A/\log|t| \leq \sigma \leq 2$ and $|t| \geq 2$ we have $\zeta(s) \neq 0$ and*

$$\frac{\zeta'}{\zeta}(s) \ll \log|t| \tag{36}$$

Moreover, in the same region we have

$$|\log\zeta(s)| \ll \log|t| \tag{37}$$

Corollary 1. *When $s \in \partial R$, we have*

$$|\log\zeta(s)| \ll \log T \tag{38}$$

Proof. Proof for (36) is already covered in the third chapter of [3], so we move onto proving (37). Since $|\log\zeta(2+it)| \leq \log\zeta(2)$, we can rewrite $\log\zeta(s)$ into

$$\begin{aligned}\log\zeta(s) &= \int_2^\sigma \frac{\zeta'}{\zeta}(u+it)\mathrm{d}u + \mathcal{O}(1) \\ &\ll \int_2^\sigma \log|t||\mathrm{d}u| = |\sigma - 2|\log|t|\end{aligned}$$

It is evident that σ is bounded, so (37) directly follows. □

Theorem 4. *If we choose $\delta = A/\log T$ and write*

$$\frac{1}{2\pi i}\int_{k-iT}^{k+iT} \frac{x^s}{s}\log\zeta(s)\mathrm{d}s = \mathrm{li}(x) + R_3(x,T) \tag{39}$$

then

$$R_3(x,T) \ll x\log^2 T\exp\left(-\frac{A\log x}{\log T}\right) \tag{40}$$

Proof. By Corollary 1, it is evident that the integral over the horizontal segments of ∂R satsifies

$$\left|\int_{k\pm iT}^{1-\delta\pm iT} \frac{x^s}{s}\log\zeta(s)\mathrm{d}s\right| \ll \frac{\log T}{T}\int_k^{1-\delta} x^u\mathrm{d}u \ll \frac{x\log T}{T\log x} \tag{41}$$

Using Corollary 1, we can also establish an upper bound for the left vertical segment of ∂R:

$$\left|\int_{1-\delta-iT}^{1-\delta+iT} \frac{x^s}{s}\log\zeta(s)\mathrm{d}s\right| \ll x^{1-\delta}\log T\int_{-T}^{T} \frac{\mathrm{d}u}{|1-\delta+it|} \tag{42}$$

$$\ll x^{1-\delta}\log^2 T = x\log^2 T\exp\left(-\frac{A\log x}{\log T}\right) \tag{43}$$

wherein the last inequality follows from the definition of δ. Plugging (41), (43) into (25) and applying Theorem 3 to (24) give us the desired result. □

With everything prepared, we move our attention back to the prime counting function $\pi(x)$.

5. Proof of the prime number theorem

Applying Theorem 4 to (20), we get

$$\pi(x) = \mathrm{li}(x) + R_2(x,T) + R_3(x,T) \tag{44}$$

If we set $\log T = \sqrt{\log x}$, then we have

$$R_2(x, e^{\sqrt{\log x}}) \ll x \log x e^{-\sqrt{\log x}} \tag{45}$$

$$R_3(x, e^{\sqrt{\log x}}) \ll x \log x e^{-A\sqrt{\log x}} \tag{46}$$

If we choose a proper $0 < c < \min(1, A)$, then the logarithms in (45) and (46) can be absorbed into $e^{-c\sqrt{\log x}}$. Plugging them into (44) gives us the final result:

Theorem 5 (de la Vallée Poussin)**.** *There exists a constant $c > 0$ such that*

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}(xe^{-c\sqrt{\log x}}) \tag{47}$$

Using the fact that $\mathrm{li}(x) \sim x/\log x$, we can also sharpen the remainder of (9):

Corollary 2. *Under Theorem 5, we have for $x \geq 2$ that*

$$\pi(x) = J(x) + \mathcal{O}\left(\frac{\sqrt{x}}{\log x}\right) \tag{48}$$

Proof. Similar to how Theorem 1 is proved, we just need to give a better upper bound for $J(x) - \pi(x)$. That is,

$$\begin{aligned} J(x) - \pi(x) &= \frac{\pi(x^{1/2})}{2} + \sum_{3 \leq k \leq \log_2 x} \frac{\pi(x^{1/k})}{k} \\ &\leq \frac{\pi(x^{1/2})}{2} + \pi(x^{1/3}) \sum_{3 \leq k \leq \log_2 x} \frac{1}{k} \\ &\ll \frac{x^{1/2}}{\log x} + \frac{x^{1/3} \log\log x}{\log x} \ll \frac{\sqrt{x}}{\log x} \end{aligned}$$

where the second last $\ll$ follows from the fact that Theorem 5 implies $\pi(x) \ll x/\log x$. □

6. Conclusion

In this paper, we present a new proof of the prime number theorem that avoids the use of Chebyshev functions. First, we prove (20) to transform the arithmetic problem into an analytic problem. To evaluate the integral on the right hand side of (20), we set up a counterclockwise rectangular path so that this task is divided in to (24) and (25). Subsequently, differentiation under integral is applied to evaluate (24), and classical zero-free region and upper bound for $\zeta(s)$ are used to give estimates for (25). Finally, choosing a proper T, we deduce again the prime number theorem in the version of de la Vallée Poussin.

The most significant contribution of the proof is the evaluation of (24), which gives the main term of Theorem 5. Traditionally, number theorists study the prime number theorem and its generalizations (i.e. prime number theorem for arithmetic progressions[12], prime number theorem for automorphic L-functions[14], or sums of complex numbers over primes[2]) first using the weighted sum

$$\sum_{n \leq x} c_n \Lambda(n) = \sum_{p^k \leq x} c_n \log p \tag{49}$$

and then convert it into sum over primes

$$\sum_{p\leq x} c_p \tag{50}$$

using partial summation. However, introducing the parameter r suggests a possibility to directly evaluate sum over prime powers

$$\sum_{n\leq x} a_n c_n = \sum_{p^k\leq x} \frac{c_{p^k}}{k} = \sum_{p\leq x} c_p + R \tag{51}$$

To study (50) from (49), partial summation is needed for conversion, but studying (50) from (51) only requires us to provide upper estimates for

$$R = \sum_{2\leq k\leq \log_2 x} \sum_{p\leq x^{1/k}} \frac{c_{p^k}}{k} \tag{52}$$

Consequently, the proof technique presented in this paper may inspire new proofs of generalized prime number theorems[14][2] as they are also studied using Perron's formula[3].

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