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论文题目：A study of the group of magic pyramid puzzle

A STUDY OF THE GROUP OF MAGIC PYRAMID PUZZLE

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Abstract. We will explore the group theory of a magic pyramid puzzle by a group action. We will show the group of a magic pyramid puzzle is isomorphic to a subgroup of Coxeter group of type D. We will also consider generators and relations of the group of a magic pyramid puzzle and its subgroups.

Keywords: Coxeter group, Even-signed permutation, Length function

Contents

1. Introduction

The magic pyramid puzzle consists of corner pieces, edge pieces and face pieces. There are many tutorials explaining how to recover a scrambled magic pyramid puzzle. One popular tutorial mentions a type of series of moves: turn downward on the left, turn downward on the right, turn upward on the left and turn upward on the right. In this paper, we explained why these series

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of moves definitely recover a scrambled magic pyramid puzzle. Moreover, we explained why some cases are impossible for a magic pyramid puzzle.

We refer to any 120° turn on any plane by a move. If we apply several moves one by one, we call it a series of moves. It is not hard to see that all the series of moves form a group. The operation of this group is composition of series of moves. For any two series of moves α and β, their composition $\alpha \circ \beta$ refers to another series of moves which applies β first and then α. The identity of the group is no move. For any series of moves, the inverse is to apply inverse move in a reverse order.

In this paper, we explored the group theory of the group of magic pyramid puzzle. In section 3, we specified the notations we used through this paper. In section 4, we defined a group action by which we obtained an important subgroup $\mathbb{E}$. The group $\mathbb{E}$ is isomorphic to a subgroup of S_n^B, the Coxeter group of type B. And we obtained a generating set of the subgroup $\mathbb{E}$ in section 5, which explained why the popular tutorial works. In section 6, we verified two important properties of series of moves of $\mathbb{E}$, which explained why some cases never show up in a magic pyramid puzzle. We talked about a small subgroup, a group of one face, in section 7. In section 8, we listed the generators and relations of the group $\mathbb{E}$.

2. Preliminary knowledge

2.1. **Group action.** Let $(G, \circ)$ be a group and S be a set. The group action of G on S is defined as follows:
For any element $g \in G$ and any element $x \in S$, let $g.x \in S$ satisfying

$$id.x = x$$

and

$$h.(g.x) = (h.g).x$$

for any $h, g \in G$ and $x \in S$.

[1] gave an equivalent definition of group action. Let $Perm(S)$ be the collection of all the bijections on set S. With composition $\circ$ of bijections, $(Perm(S), \circ)$ forms a group. Equivalently, the action of group G on a set S is a homomorphism of group from group G to group $Perm(S)$

$$\tau : G \to Perm(S),$$

with $\tau(g)(x) = g.x$ for any $g \in G$ and $x \in S$. If the kernel of the group homomorphism τ is trivial, then we say that G acts faithfully on the set S.

2.2. **Coxeter group and length function.** [2] and [3] are good references for Coxeter group.

Definition 2.1. *A Coxeter group is a group admitting a finite generating set* $\mathscr{W} = \{s_1, s_2, \cdots, s_n\}$ *and relations*

$$(s_i s_j)^{m_{ij}} = 1,$$

where 1 *is the identity of the group and* m_{ij} *is an integer such that*

$$m_{ii} = 1$$

and for $i \neq j$,

$$m_{ij} \in \{2, 3, \cdots\} \cup \{\infty\}.$$

For any ω in a Coxeter group, ω has expressions or words in the Coxeter generators. The expressions that can not be further simplified according to the relations or words are called reduced expressions or reduced words. In 1.2.1 of [2], the length function $\ell(\omega)$ is defined to be the number of generators in a reduced expression of ω. Proposition 1.2 in [2] includes an important fact about the length function.

Proposition 2.2. [2] *For an arbitrary element* ω *in the Coxeter group and* $s_i \in \mathscr{W}$

$$\ell(s_i \omega) = \ell(\omega) \pm 1.$$

2.3. **Signed permutation group.** Section 8.1 of [3] gave a detailed definition of the signed permutation group. Let ω be a bijection or a permutation on $\{\pm 1, \pm 2, \cdots, \pm n\}$ such that

$$\omega(-i) = -\omega(i)$$

for $i \in \{\pm 1, \pm 2, \cdots, \pm n\}$. Then we call ω is a signed permutation of $\{1, 2, \cdots, n\}$. Let S_n^B be the group of all the signed permutations on $\{1, 2, \cdots, n\}$. For an integer i such that $1 \leq i < n$, let $s_i = (i, i+1)$, the transposition of i and $i+1$. Let $s_0 = (-1, 1)$, the transposition of -1 and 1. Then $\{s_0, s_1, \cdots, s_{n-1}\}$ is a generating set of S_n^B with relations

$$s_i^2 = 1 (i = 0, 1, \cdots, n-1),$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} (i = 1, 2, \cdots, n-1)$$

and

$$s_i s_j = s_j s_i (|i - j > 1|).$$

Hence the structure above is a Coxeter structure. The group S_n^B is called the Coxeter group of type B.
Moreover, for each signed permutation, we have a window notation [3]. If the number of negative signs in the window notation is an even number, then we say this permutation is an even-signed permutation. The collection of all the even-signed permutations is a subgroup of S_n^B, which is the Coxeter group of type D and is denoted by S_n^D.

3. Facts and notations

For this paper, we use a magic pyramid puzzle with four colors, yellow, blue, pink and green. If we take the yellow face as the bottom and check from above, the colors faces are blue, pink and green counterclockwise. To refer a face color, we use a capitalized alphabet. Namely, Y refers to yellow, B refers to blue, P refers to pink and G refers to green.

There are three types of pieces in a magic pyramid puzzle. A corner piece involves 3 colors, there are 4 corner pieces in a magic pyramid puzzle. One edge piece lies on each edge and an edge piece involves 2 colors. There are 6 edge pieces since there are six edges. Moreover, for each face, there are 3 face pieces, each of which involves only one color. There are 12 face pieces in total.

An easy fact is that, for a fixed corner piece, the color of face opposite to this corner piece is fixed. Namely, the face opposite to the corner piece BPG is the yellow face.

The identity of the group of magic pyramid puzzle is no move, which we denote by $\mathbb{1}$. A move turning only one corner piece is trivial, which commutes with all other moves. So in this paper, by the group of magic pyramid puzzle, we only consider the moves turning top two layers. Namely a move turns a corner piece and the middle layer below it. We refer this kind of moves by moves of big corners.

We use the following notations to denote this kind of moves. For the corner piece BPG opposite to the yellow face, let y denote the move turning this corner and the middle layer beneath by $120°$ clockwise. Since $y^3 = \mathbb{1}$, y^2 is the inverse move of y, i.e. the move of this big corner counterclockwise. For the corner opposite to the green face, let g denote the move of this corner clockwise and g^2 is the move of the same big corner counterclockwise. For corner opposite to the blue face, let b denote the move of this corner clockwise and b^2 is the move of the same big corner counterclockwise. For corner opposite to the pink face, let p denote the move of this corner clockwise and p^2 is the move of the same big corner counterclockwise.

It is easy to see that the group of magic pyramid puzzle is generated by y, b, p, g. Let $(\mathbb{P}, \circ)$ denote the group of magic pyramid puzzle. We call each of y, b, p, g a move. By yb we mean apply move b first and then y. And we call any word using y, b, p, g a series of moves. Let α and β be two series of moves. Namely α and β are two words. The product $\alpha \circ \beta$ is the new word obtained by concatenation of the two words, which is a new series of moves. For a series of moves g^2yb, the inverse is b^2y^2g.

4. Action of the group of magic pyramid puzzle

The way we express a series of moves is to consider the action of the group of magic pyramid puzzle on the signed edge set defined below.

Let Y denote the yellow face, B denote the blue face, P denote the pink face and G denote the green face. We will refer to Y-face, B-face, P-face and G-face in this paper. Let PY denote the edge piece position involving pink and yellow colors. The notation YP refers to the same position, but we will keep both notations and set $YP = -PY$ (or $PY = -YP$). The signed edge position set $\mathscr{SE}$ is

$$\{\pm PY, \pm GP, \pm BP, \pm BG, \pm YB, \pm GY\}$$

Consider the following action: Let $\alpha \in \mathbb{P}$ and $XU \in \mathscr{SE}$, where $X, U \in \{Y, B, P, G\}$. Apply α on the magic pyramid puzzle, the resulting position of the edge piece originally lying at the position XU is represented by a unique element in $\mathscr{SE}$. Set

$$\alpha.(XU) = WZ,$$

if the resulting position after applying α is the edge piece position WZ, the face originally on the X-face is moved to W-face and the face originally on the U-face is moved to Z-face. By this definition, it follows that

$$\alpha.(-XU) = -\alpha.(XU).$$

For instance, for $y \in \mathbb{P}$ and $BP \in \mathscr{SE}$,

$$y.(BP) = GB = -BG,$$

since the piece originally lying at the edge piece position BP, after applying y, is moved to the edge piece position GB with the face on B-face moved to G-face and the face on P-face moved to B-face.
The identity $\mathbb{1}$ of the magic pyramid group $\mathbb{P}$ is no move, which keeps every piece at original position. Namely, for any element $XU \in \mathscr{SE}$,

$$\mathbb{1}.(XU) = XU.$$

Let $\alpha, \beta \in \mathbb{P}$ and $XU, WZ, SV \in \mathscr{SE}$, if

$$\alpha.(XU) = WZ$$

and

$$\beta.(WZ) = SV,$$

i.e. α bring the piece originally at XU to the edge piece position WZ with the face originally at X-face moved to W-face and the face originally at U-face moved to Z-face; β bring the piece originally at WZ to the edge piece position SV with the face originally at W-face moved to S-face and the face originally at Z-face moved to V-face. So applying α first and then β brings the piece originally at the edge piece position XU to the edge piece position SV with the face originally at X-face moved to S-face and the face originally at U-face moved to V-face, i.e.

$$(\beta \circ \alpha).(XU) = SV = \beta.(\alpha.(XU)).$$

Hence the action of $\mathbb{P}$ defined on the signed edge position set $\mathscr{S}\mathscr{E}$ is well-defined. Equivalently, this action is expressed by the group homomorphism:

$$F : \mathbb{P} \to Perm(\mathscr{S}\mathscr{E}),$$

for any $\alpha \in \mathbb{P}$ and $XU \in \mathscr{S}\mathscr{E}$,

$$F(\alpha)(XU) = \alpha.(XU).$$

We express a bijection in $Perm(\mathscr{S}\mathscr{E})$ by two-line notation. The first line is $\{BP, BG, GP, PY, YB, GY\}$ and each entry on the second line is the image of the entry above it in the first line. The order of the columns doesn't matter. For instance, the move y is represented as follows,

$$F(y) = \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ -BG & PY & -BP & GP & YB & GY \end{pmatrix}.$$

Let $\mathbb{F}$ denote the collection of all series of moves fixing every edge piece. Obviously, $\mathbb{F}$ is a subgroup of the magic pyramid group $\mathbb{P}$. Moreover, $\mathbb{F}$ is the kernel of the group homomorphism F. Thus $\mathbb{F}$ is a normal subgroup of $\mathbb{P}$. Hence we could consider the quotient group $\mathbb{P}/\mathbb{F}$.

Let $\mathbb{E}$ be the collection of all series of moves fixing every face piece. It is easy to see that $\mathbb{E}$ is also a subgroup of $\mathbb{P}$. Furthermore, since elements of $\mathbb{E}$ and $\mathbb{F}$ moves two disjoint subsets of pieces of all the pieces of a magic pyramid puzzle, we have the following conclusion.

Proposition 4.1. *For any $\alpha \in \mathbb{F}$ and $\beta \in \mathbb{E}$, α and β commute, i.e.*

$$\alpha \circ \beta = \beta \circ \alpha.$$

Moreover, a stronger fact follows.

Proposition 4.2. *We have the following splitting short exact sequence*

$$\mathbb{1} \to \mathbb{F} \xrightarrow{i} \mathbb{P} \xrightarrow{q} \mathbb{P}/\mathbb{F} \to \mathbb{1}$$

Proof. For any element $\alpha\mathbb{F} \in \mathbb{P}/\mathbb{F}$, it is a coset in $\mathbb{P}$. There is an element $\sigma \in F$ such that $\epsilon = \alpha \circ \sigma$ is an element in $\mathbb{P}$ fixing every face piece, i.e. $\epsilon \in \mathbb{E}$. Moreover, for any $\epsilon' \in E$ such that ϵ' lies in the coset $\alpha\mathbb{F}$, $\epsilon' = \epsilon \circ \gamma$ for some $\gamma \in \mathbb{F}$. If $\gamma \neq \mathbb{1}$, this implies γ must move some of the face pieces since $\gamma \in \mathbb{F}$ fixes every edge piece, which forces $\gamma = \mathbb{1}$ considering both ϵ and ϵ' are in $\mathbb{E}$. Hence, we have $\epsilon = \epsilon'$, i.e. in a coset $\alpha\mathbb{F}$, there exists a unique element $\tilde{\alpha} \in \mathbb{E}$. Define the following map

$$h : \mathbb{P}/\mathbb{F} \to \mathbb{E}$$
$$\alpha \mapsto \tilde{\alpha}.$$

For any two cosets $\alpha\mathbb{F}$ and $\beta\mathbb{F}$, let $\tilde{\alpha}$ and $\tilde{\beta}$ be the images of $\alpha\mathbb{F}$ and $\beta\mathbb{F}$ respectively. Then $\alpha = \tilde{\alpha} \circ \sigma$ and $\beta = \tilde{\beta} \circ \tau$, for some $\sigma, \tau \in \mathbb{F}$. By Proposition

4.1,

$$\begin{aligned} \alpha \circ \beta &= \tilde{\alpha} \circ \sigma \circ \tilde{\beta} \circ \tau \\ &= \tilde{\alpha} \circ \tilde{\beta} \circ \sigma \circ \tau \end{aligned}$$

i.e. $(\alpha\mathbb{F})(\beta\mathbb{F}) = (\tilde{\alpha} \circ \tilde{\beta})\mathbb{F}$. Hence $h((\alpha\mathbb{F})(\beta\mathbb{F})) = \tilde{\alpha} \circ \tilde{\beta} = h(\alpha) \circ h(\beta)$. The map h is a group homomorphism.
For any $\alpha\mathbb{F} \in \mathbb{P}/\mathbb{F}$, $q(h(\alpha\mathbb{F})) = q(\tilde{\alpha}) = \tilde{\alpha}\mathbb{F} = \alpha\mathbb{F}$, i.e. $q \circ h$ is the identity map. Hence h is an isomorphism

$$\mathbb{P}/\mathbb{F} \cong \mathbb{E}$$

and, by Theorem 2.1 in [4], the short exact sequence splits.
□

Corollary 4.3. *The group of magic pyramid puzzle $\mathbb{P}$ is the direct sum of the subgroups $\mathbb{E}$ and $\mathbb{F}$, i.e.*

$$\mathbb{P} \cong \mathbb{E} \oplus \mathbb{F}.$$

The group homomorphism $F : \mathbb{P} \to Perm(\mathscr{SE})$ induces a group homomorphism

$$\tilde{F} : \mathbb{P}/\mathbb{F} \to Perm(\mathscr{SE}),$$

where $\tilde{F}(\alpha\mathbb{F}) = F(\alpha)$. According to the first isomorphism theorem 2.12.10 and its corollary 2.12.11 in [5], $\tilde{F}$ is injective. Let $f = \tilde{F} \circ h^{-1}$. Namely,

$$f : \mathbb{E} \to Perm(\mathscr{SE}),$$

where $f(\alpha) = F(\alpha)$ for any $\alpha \in \mathbb{E}$.

Corollary 4.4. *The induced group homomorphism $f : \mathbb{E} \to perm(\mathscr{SE})$ is injective. Hence the action of $\mathbb{P}$ on set $\mathscr{SE}$ induces a faithful action of $\mathbb{E}$ on the signed edge piece position set $\mathscr{SE}$.*

Since $\{y, b, p, g\}$ generates the group $\mathbb{P}$, we get a generating set of the subgroup $\mathbb{E}$.

Corollary 4.5. *Let $\tilde{y}, \tilde{b}, \tilde{p}, \tilde{g}$ be the images of y, b, p, g respectively under the group homomorphism h. Then $\{\tilde{y}, \tilde{b}, \tilde{p}, \tilde{g}\}$ generates the subgroup $\mathbb{E}$.*

5. Generators of the subgroups $\mathbb{E}$

In many tutorials about the magic pyramid puzzle, a popular way to recover a magic pyramid puzzle is to repeat a series of moves: take a face of the magic pyramid puzzle as bottom and face another face of the puzzle, apply four 120° moves, left down, right down, left up and right up. In this section, we will explain why this type of moves can be applied to recover the magic pyramid puzzle. Let P_Y denote the series of moves

$$P_Y = bg^2b^2g.$$

The series P_Y of moves is obtained as follows: take the yellow face as the bottom and face the pink face, turn downward on the left by 120°, turn downward on the right by 120°, turn upward on the left by 120° and turn upward on the right by 120°. So the series P_Y of moves is the series of moves we mentioned above. Similarly, define other eleven series of moves, let

$$P_B = gy^2g^2y, \qquad P_G = yb^2y^2b,$$

$$B_P = yg^2y^2g, \qquad B_Y = gp^2g^2p, \qquad B_G = py^2p^2y,$$

$$G_P = by^2b^2y, \qquad G_Y = pb^2p^2b, \qquad G_B = yp^2y^2p,$$

$$Y_P = gb^2g^2b, \qquad Y_B = pg^2p^2g, \qquad Y_G = pb^2p^2b.$$

We denote the set consisting of these twelve series of moves by

$$\mathscr{G} = \{X_I | each\ of\ X, I\ is\ one\ of\ the\ four\ colors\ Y, B, P, G\ and\ X \neq I\}$$

Lemma 5.1. *Each X_I is an element in the subgroup $\mathbb{E}$.*

Proof. According to the definitions above,

$$X_I = wz^2w^2z,$$

where $w, z \in \{y, b, p, g\}$ and each of w, z is rotate some big corner, a corner piece and middle layer below it, by 120° clockwise. Then w and w^2 rotate the corresponding big corner by a full circle and thus keep the face pieces involved in original positions. Similarly, z and z^2 rotate the corresponding big corner by a full circle. Hence $X_I = wz^2w^2z$ keeps every face piece in its original position, i.e. $X_I \in \mathbb{E}$.
□

Now we have a new generating set of the subgroup $\mathbb{E}$ as follows.

Theorem 5.2. *The subgroup $\mathbb{E}$ is generated by $\mathscr{G}$.*

Proof. Now we show that each of $\tilde{y}, \tilde{b}, \tilde{p}, \tilde{g}$ is generated by $\mathscr{G}$. By the action of $\mathbb{P}$ on $\mathscr{SE}$ defined in section 4, we have

$$f(\tilde{g}) = \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ PY & YB & GP & BG & BP & GY \end{pmatrix}.$$

Moreover, we have

$$f(Y_B) = \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ BP & YB & GP & BG & GY & PY \end{pmatrix}$$

and

$$f(B_Y) = \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ BG & PY & GP & YB & BP & GY \end{pmatrix}.$$

Then we have $f(\tilde{g}) = f(B_Y)f^2(Y_B)f^2(B_Y)f(Y_B)$ and thus

$$\tilde{g} = B_Y Y_B^2 B_Y^2 Y_B,$$

since the group homomorphism f is injective. Similarly, we could show $\tilde{y}, \tilde{b}, \tilde{p}$ are also generated by $\mathscr{G}$.
So $\mathscr{G}$ is a generating set of the subgroup $\mathbb{E}$.
□

6. Properties of elements in $\mathbb{E}$

From the action of $\mathbb{E}$ on the signed edge piece position set $\mathscr{S}\mathscr{E}$, it is easy to observe that $\mathbb{E}$ is a subgroup of the signed permutation group.

Proposition 6.1. *The group $\mathbb{E}$, consisting of all series of moves fixing corner pieces and face pieces, is isomorphic to a subgroup of the Coxeter group S_6^B of type B.*

Proof. Set $\pm BP = \pm 1$, $\pm PY = \pm 2$, $\pm GP = \pm 3$, $\pm BG = \pm 4$, $\pm YB = \pm 5$ and $\pm GY = \pm 6$. Then the injective group homomorphism $f : \mathbb{E} \to Perm(\mathscr{S}\mathscr{E})$ becomes an injective group homomorphism

$$f : \mathbb{E} \to Perm(\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\}).$$

Moreover, for any $\alpha \in \mathbb{E}$ and $XU \in \mathscr{S}\mathscr{E}$, the fact $\alpha.(-XU) = -\alpha(XU)$ implies that

$$\begin{aligned} f(\alpha)(-XU) &= F(\alpha)(-XU) \\ &= \alpha.(-XU) \\ &= -\alpha.(XU) \\ &= -F(\alpha)(XU) \\ &= -f(\alpha)(XU) \end{aligned}$$

and hence $f(\alpha)(-i) = -f(\alpha)(i)$ for $i \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\}$.
So the map h is an injective group homomorphism from $\mathbb{E}$ to the Coxeter group S_6^B of type B.
□

Now let us explore more properties of elements in $\mathbb{E}$.

6.1. **Even-signed permutations.** We want to show that the image of any element in $\mathbb{E}$ is an even-signed permutation.
Let α be an arbitrary element in $\mathbb{E}$. We have $f(\alpha)$ is a signed permutation of

$$\mathscr{E} = \{BP, PY, GP, BG, YB, GY\}.$$

If $f(\alpha)(XU) = WZ$, this means the series of moves take the edge piece originally at XU to the edge piece position WZ. And we have $UX = -XU$. Hence, if we express $f(\alpha)$ in a two-line notation, the second line of the notation may contains negative signs.
If we change the order of the alphabets of one element of $\mathscr{E}$, for instance, replace BP by PB, we will get a new two-line notation of $f(\alpha)$. The number of negative signs of the new second line differs by two, or keeps unchanged. If $f(\alpha)(BP) \neq \pm BP$, then sign of $f(\alpha)(BP)$ changes and the $\pm BP$ in the original second line changes to $\mp PB$. For example,

$$\begin{aligned} f(B_Y) &= \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ BG & PY & GP & YB & BP & GY \end{pmatrix} \\ &= \begin{pmatrix} PB & PY & GP & BG & YB & GY \\ -BG & PY & GP & YB & -PB & GY \end{pmatrix}. \end{aligned}$$

If $f(\alpha)(BP) = \pm BP$, then the number of negative signs of the second line keeps unchanged. For example,

$$\begin{aligned} f(Y_B) &= \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ BP & YB & GP & BG & GY & PY \end{pmatrix} \\ &= \begin{pmatrix} PB & PY & GP & BG & YB & GY \\ PB & YB & GP & BG & GY & PY \end{pmatrix}. \end{aligned}$$

Proposition 6.2. *The image of $\mathbb{E}$ under the group homomorphism f is a subgroup of the Coxeter group S_6^D of type D.*

Proof. For an arbitrary element $\alpha \in \mathbb{E}$, we could find a set $\mathscr{E}_0$, consisting of all six edge piece positions, such that for each edge piece position XU, either $XU \in \mathscr{E}_0$ or $UX \in \mathscr{E}_0$, and there is no negative sign on the second line of the notation of $f(\alpha)$ in terms of $\mathscr{E}_0$.
Start with an edge piece position XU and let $XU \in \mathscr{E}_0$. If $f(\alpha)(XU) = WZ$, then let $WZ \in \mathscr{E}_0$ and repeat the previous step. Let $f(\alpha)_0$ denote the two line notation of $f(\alpha)$ in terms of $\mathscr{E}_0$. Then the second line of $f(\alpha)_0$ contains no negative sign.
For each $XU \in \mathscr{E}_0$, if $XU \notin \mathscr{E}$, replace XU by UX. Denote the new set by $\mathscr{E}_1$ and denote by $f(\alpha)_1$ the two-line notation of $f(\alpha)$ in terms of $\mathscr{E}_1$. Then the number of negative signs of $f(\alpha)_1$ differs from that of $f(\alpha)_0$ by 2. Repeat this process until $\mathscr{E}_k = \mathscr{E}$ for some integer k. For any $0 < i \leq k$, the number of negative signs of $f(\alpha)_{i-1}$ differs from that of $f(\alpha)_i$ by 2. Hence the number of negative signs of $f(\alpha)_k$ differs from that of $f(\alpha)_0$ by an even number. Note that $f(\alpha)_k$ is the two-line notation of $f(\alpha)$ in terms of $\mathscr{E}$. So we conclude that, in terms of $\mathscr{E}$, any element $\alpha \in \mathbb{E}$ corresponds to an even signed permutation $f(\alpha)$.
□

Therefore, we showed that the group $\mathbb{E}$ is isomorphic to a subgroup of S_6^D, the even-signed permutation group or the Coxeter group of type D.

6.2. **Length of a signed permutation.** Fix the setting $\pm BP = \pm 1$, $\pm PY = \pm 2$, $\pm GP = \pm 3$, $\pm BG = \pm 4$, $\pm YB = \pm 5$ and $\pm GY = \pm 6$. Consider the Coxeter structure of S_6^B, for an arbitrary element $\alpha \in \mathbb{E}$, $f(\alpha)$ has a reduced expression and a length. We are going to show $f(\alpha)$ has an even length.

Lemma 6.3. *For any two signed permutations with even lengths, their product also has a reduce expression with even length.*

Proof. Let ω be a signed permutations of even length and admit the reduced expression

$$\underline{\omega} = s_{i_1} s_{i_2} \cdots s_{i_k},$$

where k is an even positive number. Let τ be a signed permutation of length 2 and with reduced expression $\underline{\tau} = s_{j_1} s_{j_2}$.
Let $\sigma = s_{j_2}\omega$. By Proposition 2.3 [2] the length function, $\ell(\sigma) = \ell(\omega) \pm 1$ and thus σ is of odd length.
Similarly, $\tau \circ \omega = s_{j_1}\sigma$. By Proposition 2.3 [2] the length function, $\ell(\tau \circ \omega) = \ell(\sigma) \pm 1$. So $\tau \circ \omega$ is of even length.
□

Proposition 6.4. *For any $\alpha \in \mathbb{E}$, the reduced expression of $f(\alpha)$ has an even length.*

Proof. Consider the generators in $\mathscr{G}$. It is sufficient to consider B_Y, P_B, P_Y, Y_B. In Theorem 8.1, we prove these four elements generate the group $\mathbb{E}$.

$$\begin{aligned} f(B_Y) &= \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ BG & PY & GP & YB & BP & GY \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 3 & 5 & 1 & 6 \end{pmatrix} \end{aligned}$$

The reduced expression of $f(B_Y)$ is $s_3 s_2 s_1 s_2 s_3 s_4$.

$$\begin{aligned} f(P_Y) &= \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ PY & GP & BP & BG & YB & GY \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 5 & 6 \end{pmatrix} \end{aligned}$$

The reduced expression of $f(P_Y)$ is s_1s_2.

$$f(P_B) = \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ PY & -GP & -BP & BG & YB & GY \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & -3 & -1 & 4 & 5 & 6 \end{pmatrix}$$

The reduced expression of $f(P_B)$ is $s_0s_1s_2s_1s_0s_1$.

$$f(Y_B) = \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ BP & YB & GP & BG & GY & PY \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 4 & 6 & 2 \end{pmatrix}$$

The reduced expression of $f(Y_B)$ is $s_4s_3s_2s_3s_4s_5$.

We could check that all these generators correspond to signed permutations with even lengths. By Lemma 6.3, for any $\alpha \in \mathbb{E}$, α is a product of generators and thus is of even length.
□

6.3. **Impossible arrangements.** Using Proposition 6.2 and Proposition 6.4, we could show that some arrangements are impossible in a magic pyramid puzzle. For instance, it is impossible to have three edge pieces on the pink face in the order BP, GP, PY and with other pieces in place, which is expressed by

$$\begin{pmatrix} BP & PY & GP & BG & YB & GY \\ BP & GP & PY & BG & YB & GY \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{pmatrix}$$
$$= s_2.$$

It is a permutation of odd length. Hence it is impossible for this arrangement to show up in a magic pyramid puzzle.

7. The Pink Face Subgroup

Consider the three series of moves P_B, P_Y, P_G.

$$f(P_B) = \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ PY & -GP & -BP & BG & YB & GY \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & -3 & -1 & 4 & 5 & 6 \end{pmatrix}$$

$$
\begin{aligned}
f(P_Y) &= \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ PY & GP & BP & BG & YB & GY \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 5 & 6 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
f(P_G) &= \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ -PY & GP & -BP & BG & YB & GY \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -2 & 3 & -1 & 4 & 5 & 6 \end{pmatrix}
\end{aligned}
$$

Note that P_B, P_Y, P_G move merely the edge pieces BP, PY, GP on the pink face and fix edge pieces at other three edge piece positions BG, GY, YB. So $f(P_B), f(P_Y), f(P_G)$ are also signed permutations of $\{BP, PY, GP\}$ or $\{1, 2, 3\}$. Hence $\{P_B, P_Y, P_G\}$ generates a group isomorphic to a subgroup of Coxeter group S_3^B. Since this group corresponds to the permutations of the edge pieces on the pink face; we call this group by the pink face subgroup of $\mathbb{E}$ in this paper.

7.1. **Generators and relations of the pink face subgroup of $\mathbb{E}$.** The pink face subgroup of $\mathbb{E}$ is generated by $\{P_B, P_Y, P_G\}$. But we could reduce to two generators.

Proposition 7.1. *The pink face subgroup of $\mathbb{E}$ is generated by P_B and P_Y.*

Proof. From the computation

$$
\begin{aligned}
f(P_G^2) = f(P_G)^2 &= \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ -PY & GP & -BP & BG & YB & GY \end{pmatrix}^2 \\
&= \begin{pmatrix} BP & PY & GP & BG & YB & GY \\ -GP & -BP & PY & BG & YB & GY \end{pmatrix} \\
&= f(P_B)f(P_Y) = f(P_B P_Y),
\end{aligned}
$$

it follows that $(P_G)^2 = P_B P_Y$. So the pink face subgroup is generated by P_B, P_Y.
□

Next we explore the relations of these two generators. First, each of these two generators is of order 3, i.e.

$$(P_B)^3 = \mathbb{1} \textit{ and } (P_Y)^3 = \mathbb{1}.$$

Next, we could check that

$$P_B = P_Y (P_B)^2 P_Y,$$

which is equivalent to the relation

$$P_Y = P_B (P_Y)^2 P_B.$$

The last relation is

$$P_Y P_B P_Y = P_B P_Y P_B.$$

The pink face subgroup is

$$\langle P_B, P_Y | (P_B)^3 = (P_Y)^3 = \mathbb{1}, P_B = P_Y (P_B)^2 P_Y, P_Y P_B P_Y = P_B P_Y P_B \rangle.$$

7.2. **A list of all the elements of the pink face subgroup.** By the generators and relations above, we list all the elements of the pink face subgroup. First, we have the identity $\mathbb{1}$ in the subgroup

$$\begin{aligned} f(\mathbb{1}) &= \begin{pmatrix} BP & PY & GP \\ BP & PY & GP \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}. \end{aligned}$$

We also have P_B, P_Y

$$\begin{aligned} f(P_B) &= \begin{pmatrix} BP & PY & GP \\ PY & -GP & -BP \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & -1 \end{pmatrix} \\ &= s_0 s_1 s_2 s_1 s_0 s_1, \end{aligned}$$

$$\begin{aligned} f(P_Y) &= \begin{pmatrix} BP & PY & GP \\ PY & GP & BP \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ &= s_1 s_2, \end{aligned}$$

and their inverses $(P_B)^2, (P_Y)^2$

$$\begin{aligned} f(P_B^2) &= \begin{pmatrix} BP & PY & GP \\ -GP & BP & -PY \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ -3 & 1 & -2 \end{pmatrix} \\ &= s_1 s_0 s_1 s_2 s_1 s_0, \end{aligned}$$

$$\begin{aligned} f(P_Y^2) &= \begin{pmatrix} BP & PY & GP \\ GP & BP & PY \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ &= s_2 s_1. \end{aligned}$$

We also have $P_B P_Y$

$$\begin{aligned} f(P_B P_Y) &= \begin{pmatrix} BP & PY & GP \\ GP & -BP & -PY \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & -1 & -2 \end{pmatrix} \\ &= s_1 s_0 s_1 s_0 s_2 s_1, \end{aligned}$$

and $P_Y P_B$ in the pink face subgroup

$$\begin{aligned} f(P_Y P_B) &= \begin{pmatrix} BP & PY & GP \\ -GP & -BP & PY \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ -3 & -1 & 2 \end{pmatrix} \\ &= s_0 s_2 s_1 s_0. \end{aligned}$$

The series of moves $P_B P_Y P_B P_Y$ is as follows.

$$\begin{aligned} f(P_B P_Y P_B P_Y) &= \begin{pmatrix} BP & PY & GP \\ -PY & -GP & BP \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \end{pmatrix} \\ &= s_1 s_0 s_2 s_1 s_0 s_1. \end{aligned}$$

The series of moves $P_Y P_B P_Y P_B$ is as follows.

$$\begin{aligned} f(P_Y P_B P_Y P_B) &= \begin{pmatrix} BP & PY & GP \\ -PY & GP & -BP \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & -1 \end{pmatrix} \\ &= s_0 s_1 s_2 s_0. \end{aligned}$$

The elements $P_B P_Y, P_Y P_B, P_B P_Y P_B P_Y, P_Y P_B P_Y P_B$ are of order 3. Moreover,

$$(P_B P_Y)^{-1} = P_B P_Y P_B P_Y$$

and

$$(P_Y P_B)^{-1} = P_Y P_B P_Y P_B.$$

The series of moves $P_B P_Y P_B$ is as follows.

$$\begin{aligned} f(P_B P_Y P_B) &= \begin{pmatrix} BP & PY & GP \\ -BP & -PY & GP \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & 3 \end{pmatrix} \\ &= s_1 s_0 s_1 s_0. \end{aligned}$$

The series of moves $(P_B)^2 P_Y$ is as follows.

$$\begin{aligned} f(P_B^2 P_Y) &= \begin{pmatrix} BP & PY & GP \\ -BY & PY & -BP \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -3 \end{pmatrix} \\ &= s_0 s_2 s_1 s_2 s_0 s_1. \end{aligned}$$

The series of moves $P_B(P_Y)^2$ is as follows.

$$\begin{aligned} f(P_B P_Y^2) &= \begin{pmatrix} BP & PY & GP \\ BP & -PY & -GP \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & -3 \end{pmatrix} \\ &= s_1 s_2 s_0 s_1 s_2 s_0 s_1 s_2. \end{aligned}$$

The three elements, $P_B P_Y P_B, (P_B)^2 P_Y, P_B(P_Y)^2$, are of order 2.

In summary, the pink face subgroup consists of 12 elements, which are all the even-signed permutations of $\{1, 2, 3\}$ of even length.

8. Generators and Relations of $\mathbb{E}$

By Theorem 5.1, we showed that the subgroup $\mathbb{E}$ is generated by $\mathscr{G}$. In this section, we are going to reduce the number of generators to four.

Theorem 8.1. *The group $\mathbb{E}$ is generated by $\{B_Y, P_B, P_Y, Y_B\}$.*

Proof. Let $X, Z, U, W \in \{Y, B, P, G\}$ and X, Z, U, W are distinct. From section 7, we know $\{X_Z, X_U, X_W\}$ generates the X face subgroup of $\mathbb{E}$. For each color X, it is sufficient to use two generators of $\{X_Z, X_U, X_W\}$ to generate the third one.
Moreover, we have

(1) $$B_G = P_B P_Y P_B B_Y P_B P_Y P_B,$$

(2) $$Y_G = P_Y P_B P_Y Y_B P_Y P_B P_Y$$

and

(3) $$G_B = P_Y (B_Y)^2 (Y_B)^2 B_Y (P_Y)^2.$$

In section 7, we verified $P_G = P_B P_Y P_B P_Y$. Hence we have

(4) $$G_Y = P_G P_B P_G G_B P_G P_B P_G.$$

Now we verified that $\{B_Y, P_B, P_Y, Y_B\}$ generates two generators for each color and thus generates the group $\mathbb{E}$.
□

We explore the relations among $\{B_Y, P_B, P_Y, Y_B\}$ in the following. We will use $X, W, I, J \in \{Y, B, P, G\}$

(i) We have the elements in the following form are of order 2. For $X \neq W$, $X \neq I$ and $W \neq J$,

$$((X_I)^{-1} W_J X_I W_J)^2 = \mathbb{1}.$$

(ii) We have relations in the following form, for $X \neq W$, $X \neq I$ and $W \neq J$,

$$X_I (W_J)^{\pm} X_I (W_J)^{\pm} X_I = (W_J)^{\pm} X_I (W_J)^{\pm} X_I (W_J)^{\pm}.$$

For example, we have the relations

$$P_Y (B_Y)^{\pm} P_Y (B_Y)^{\pm} P_Y = (B_Y)^{\pm} P_Y (B_Y)^{\pm} P_Y (B_Y)^{\pm}$$
$$P_B (B_Y)^{\pm} P_B (B_Y)^{\pm} P_B = (B_Y)^{\pm} P_B (B_Y)^{\pm} P_B (B_Y)^{\pm}.$$

(iii) We have relations in the following form, for $X \neq W$, $X \neq I$ and $W \neq J$,

$$(X_I)^{\pm} W_J (X_I)^{\mp} W_J (X_I)^{\pm} = (W_J)^{-1} (X_I)^{\pm} (W_J)^{-1}.$$

For example, we have the relation

$$B_Y (P_B)^{-1} (B_Y)^{-1} (P_B)^{-1} B_Y = P_B B_Y P_B.$$

(iv) We have

$$P_B B_Y (P_B)^2 = P_Y B_Y (P_Y)^2 = (Y_B)^2 (B_Y)^2 Y_B B_Y,$$

and

$$(P_B)^2 Y_B P_B = (P_Y)^2 Y_B P_Y = Y_B B_Y (Y_B)^2 (B_Y)^2.$$

(v) In addition, we have the relations

$$B_Y (Y_B)^2 (B_Y)^2 Y_B = P_Y (B_Y)^2 (P_Y)^2 B_Y = Y_B (P_B)^2 (Y_B)^2 P_B$$

and

$$B_Y Y_B B_Y Y_B B_Y Y_B = Y_B P_B Y_B P_B.$$

9. Future Work

We have showed that the pink face subgroup is isomorphic to the subgroup of Coxeter group S_3^B, which consists of all the even-signed permutations of even length. Coxeter groups of type B and type D are related to many topics, for instance Weyl groups of Lie algebras and Hecke algebras [6]. We will explore the relations between the group of magic pyramid group or the pink face subgroup and other areas of mathematics.

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