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论文题目： Crystals arising from the representations of quantum groups in the Gelfand-Tsetlin basis

# Crystals arising from the representations of quantum groups in the Gelfand-Tsetlin basis

Hongzhou Wu

In this paper, we consider irreducible finite-dimensional representations of the Drinfeld-Jimbo quantum group $U_q(\mathfrak{gl}_n)$ which are expressible in the Gelfand-Tsetlin basis. Our first result is that as $q \to 0$, the leading asymptotics of the Chevalley generators of $U_q(\mathfrak{gl}_n)$ under the representations give rise to a $\mathfrak{gl}_n$-crystal structure. Our second result is an interpretation of the known cactus group symmetry of a $\mathfrak{gl}_n$-crystal via simple involutions on the representations.

**Keywords**: quantum groups, Gelfand-Tsetlin basis, $\mathfrak{gl}_n$-crystals, cactus groups



## Contents

# 1 Introduction and main results

Quantum groups were first formulated in physics, particularly by the Leningrad school, from the inverse scattering method. The Drinfeld-Jimbo quantum groups $U_q(\mathfrak{g})$, which are deformations of the universal enveloping algebras of Lie algebras $\mathfrak{g}$, were introduced independently by Drinfeld and Jimbo around 1985. In the past 20 years, numerous applications in different branches of mathematics and mathematical physics have been found, such as solvable lattice models in statistical mechanics, the theory of knot invariants, representation theory of Lie algebras, topological quantum field theory, geometric representation theory, $\mathbb{C}^*$-algebras, and others.

The theory of crystal basis, also known as canonical basis, was introduced independently by Kashiwara [3, 4] and Lusztig [5]. Apart from being a powerful combinatorial and geometric tool in studying the representations of quantum groups and the underlying Lie algebras $\mathfrak{g}$, it is closely related to many mathematical subjects. In particular, a $\mathfrak{g}$-crystal is a finite set, equipped with crystal operators, that models a weight basis for a representation of $\mathfrak{g}$, where the crystal operators indicate the leading order behaviour of the simple root vectors on the basis under the crystal limit $q \to 0$ in the quantum group $U_q(\mathfrak{g})$.

This paper provides an elementary derivation of a $\mathfrak{gl}_n$-crystal using the $q \to 0$ leading asymptotics of a $q$-family of representation of $U_q(\mathfrak{gl}_n)$ on a vector space. The representation used here was introduced by Appel and Gautam [1], and since the $q$-family of representation has rather explicit formulae, we compute their $q \to 0$ leading asymptotics. Our first main result is to properly characterize the leading asymptotics, and to verify that it is indeed a $\mathfrak{gl}_n$-crystal. Based on this new realization of the crystals, our second main result is to interpret the known complicated cactus group action on a $\mathfrak{gl}_n$-crystal, introduced by Berenstein and Kirillov [2], using simple involution symmetry on representations.

## 1.1 Quantum group

The Drinfeld-Jimbo quantum group $U_q(\mathfrak{gl}_n)$ is a unital associative algebra with generators $q^{\pm H_i}, E_j, F_j$, where $1 \le i \le n, 1 \le j \le n-1$, and relations:

- for each $1 \le i \le n, 1 \le j \le n-1$,

$$\begin{aligned} q^{H_i} q^{-H_i} &= q^{-H_i} q^{H_i} = 1, \\ q^{H_i} E_j q^{-H_i} &= q^{\delta_{ij}} q^{-\delta_{i,j+1}} E_j, \\ q^{H_i} F_j q^{-H_i} &= q^{-\delta_{ij}} q^{\delta_{i,j+1}} F_j; \end{aligned}$$

- for each $1 \le i, j \le n-1$,

$$[E_i, F_j] = \delta_{ij} \frac{q^{H_i - H_{i+1}} - q^{-H_i + H_{i+1}}}{q - q^{-1}};$$

- for $|i-j|=1$,

$$E_i^2E_j-(q+q^{-1})E_iE_jE_i+E_jE_i^2=0,$$
$$F_i^2F_j-(q+q^{-1})F_iF_jF_i+F_jF_i^2=0;$$

- and for $|i-j|\neq 1$,

$$[E_i,E_j]=[F_i,F_j]=0.$$

### 1.2 $\mathfrak{gl}_n$-crystals

As shown by Kashiwara [3, 4], $q$ in $U_q(\mathfrak{gl}_n)$ is a parameter of temperature in the 2-dimensional solvable model, and $q=0$ corresponds to the absolute temperature zero. Because of the special nature of the absolute temperature zero, we considered a connection to crystal bases. The combinatorial structure of a crystal base is encoded by a $\mathfrak{gl}_n$-crystal, and investigation showed that the representations of $U_q(\mathfrak{gl}_n)$ have crystal bases at $q=0$.

Let $P=\{a_1V_1+\cdots+a_nV_n \mid a_i\in\mathbb{Z}\}$ denote the lattice spanned by $n$ independent vectors $V_1,\dots,V_n$, where $\langle V_i,V_j\rangle := \delta_{ij}$ are standard inner products. Thus,

**Definition 1.1.** A $\mathfrak{gl}_n$-crystal is a finite set $B$ along with maps

$$\begin{aligned} wt &: B\to P,\\ \tilde{E}_i,\tilde{F}_i &: B\to B\cup\{0\},\quad i\in I,\\ \varepsilon_i,\phi_i &: B\to\mathbb{Z}\cup\{-\infty\},\quad i\in I,\end{aligned}$$

such that for all $b,b'\in B$ and $i\in I$,

(1) $\tilde{F}_i(b)=b'$ if and only if $b=\tilde{E}_i(b')$, in which case

$$\begin{aligned} wt(b')&=wt(b)-V_i+V_{i+1},\\ \varepsilon_i(b')&=\varepsilon_i(b)+1,\\ \phi_i(b')&=\phi_i(b)-1;\end{aligned}$$

(2) $\phi_i(b)=\varepsilon_i(b)+\langle wt(b),V_i-V_{i+1}\rangle$, and if $\phi_i(b)=\varepsilon_i(b)=-\infty$, then $\tilde{E}_i(b)=\tilde{F}_i(b)=0$.

Here $wt$ is the weight map, and $\tilde{E}_i,\tilde{F}_i$ are Kashiwara operators or crystal operators.

### 1.3 Representation of quantum groups in the Gelfand-Tsetlin basis

A Gelfand-Tsetlin pattern $\Lambda$ is a collection of numbers $\{\lambda_{x,y}\}_{1\le y\le x\le n}$ satisfying the interlacing conditions

$$\lambda_{i+1,j+1}-\lambda_{i,j}\in\mathbb{Z}^+\quad\text{and}\quad\lambda_{i,j}-\lambda_{i+1,j}\in\mathbb{Z}^+\quad\text{for } i,j\in\mathbb{Z}^+, 1\le j\le i<n, \tag{1}$$

as shown in Figure 1.

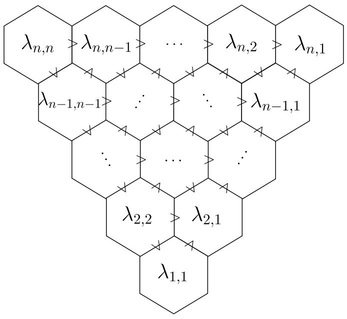



Figure 1: The Gelfand-Tsetlin pattern $\Lambda$

Given any $n$-tuple $\lambda = (\lambda_1, ..., \lambda_n)$ where $\lambda_1 > \cdots > \lambda_n$, let us consider the vector space $L(\lambda)$ spanned by the symbols $\xi_\Lambda$, where $\Lambda$ are all possible Gelfand-Tsetlin patterns with fixed $\lambda_{n,n+1-i} = \lambda_i$ where $i = 1, ..., n$. Then

**Theorem 1.2.** [1] *$\varphi$ is a representation of the algebra $U_q(\mathfrak{gl}_n)$ on the vector space $L(\lambda)$ defined by*

$$\varphi\left(E_k\right) = \frac{\hbar}{q-q^{-1}} \quad \sum_{i=1}^{k} \frac{\prod_{a=1}^{k-1} G^+\left(\zeta_i^{(k)} - \zeta_a^{(k-1)} - \frac{\hbar}{2}\right) \prod_{b=1}^{k+1} G^+\left(\zeta_i^{(k)} - \zeta_b^{(k+1)} - \frac{\hbar}{2}\right)}{\prod_{c\neq i} G^+\left(\zeta_i^{(k)} - \zeta_c^{(k)}\right) G^+\left(\zeta_i^{(k)} - \zeta_c^{(k)} - \hbar\right)} \cdot \alpha_i^{(k)},$$

$$\varphi\left(F_k\right) = \frac{\hbar}{q-q^{-1}} \quad \sum_{i=1}^{k} \frac{\prod_{a=1}^{k-1} G^-\left(\zeta_i^{(k)} - \zeta_a^{(k-1)} + \frac{\hbar}{2}\right) \prod_{b=1}^{k+1} G^-\left(\zeta_i^{(k)} - \zeta_b^{(k+1)} + \frac{\hbar}{2}\right)}{\prod_{c\neq i} G^-\left(\zeta_i^{(k)} - \zeta_c^{(k)}\right) G^-\left(\zeta_i^{(k)} - \zeta_c^{(k)} - \hbar\right)} \cdot \beta_i^{(k)},$$

$$\varphi(H_k) = h_k,$$

*where $\hbar$ is a real number such that $q = e^{\hbar/2}$, $G^\pm(x)$ are any functions such that*

$$G^-(x) = G^+(-x),$$
$$G^+(x)G^-(x) = \frac{e^{x/2} - e^{-x/2}}{x},$$

*and $h_k, \zeta_i^{(k)}, \alpha_i^{(k)}, \beta_i^{(k)} \in \mathrm{End}(L(\lambda))$ are operators acting on the basis $\xi_\Lambda$ of $L(\lambda)$,*

$$h_k \cdot \xi_\Lambda = \left(\sum_{i=1}^{k} \lambda_{k,i}(\Lambda) - \sum_{i=1}^{k-1} \lambda_{k-1,i}(\Lambda)\right) \xi_\Lambda, \tag{2}$$

$$\zeta_i^{(k)} \cdot \xi_\Lambda = \hbar\Big(\lambda_{k,k+1-i}(\Lambda) - i + (k+1)/2\Big)\xi_\Lambda, \tag{3}$$

$$\alpha_i^{(k)} \cdot \xi_\Lambda = \sqrt{-\frac{\prod_{l=1,l\neq i}^{k}(\zeta_i^{(k)} - \zeta_l^{(k)})}{\prod_{l=1}^{k+1}(\zeta_i^{(k)} - \zeta_l^{(k+1)} - \frac{1}{2})} \frac{\prod_{l=1,l\neq i}^{k}(\zeta_i^{(k)} - \zeta_l^{(k)} - 1)}{\prod_{l=1}^{k-1}(\zeta_i^{(k)} - \zeta_l^{(k-1)} - \frac{1}{2})}} \cdot \xi_{\Lambda+\delta_{k,k+1-i}} \tag{4}$$

$$\beta_i^{(k)} \cdot \xi_\Lambda = \sqrt{-\frac{\prod_{l=1,l\neq i}^{k}(\zeta_i^{(k)} - \zeta_l^{(k)})}{\prod_{l=1}^{k+1}(\zeta_i^{(k)} - \zeta_l^{(k+1)} + \frac{1}{2})} \frac{\prod_{l=1,l\neq i}^{k}(\zeta_i^{(k)} - \zeta_l^{(k)} - 1)}{\prod_{l=1}^{k-1}(\zeta_i^{(k)} - \zeta_l^{(k-1)} + \frac{1}{2})}} \cdot \xi_{\Lambda-\delta_{k,k+1-i}}. \tag{5}$$

*Here, the pattern $\Lambda \pm \delta_{k,i}$ is obtained from the pattern $\Lambda$ by replacing $\lambda_{k,i}$ with $\lambda_{k,i} \pm 1$.*

Note that $\lambda_{k,k+1-i}$ is seen as a function on the set of patterns $\Lambda$. In the following, $\lambda_{k,k+1-i}$ is understood as $\lambda_{k,k+1-i}(\Lambda)$ for the sake of simplicity.

### 1.4 $\mathfrak{gl}_n$-crystals arising from the $\hbar \to -\infty$ asymptotics of $\varphi$

In this paper, we studied the $\hbar \to -\infty$ asymptotics of the representation $\varphi$. Our first main result shows that

**Theorem 1.3.** *For each $k = 1, ..., n-1$, there exist operators $\tilde{E}_k$ and $\tilde{F}_k$ acting on the basis vectors $\{\xi_\Lambda\}$ of $L(\lambda)$, such that for any $\xi_\Lambda$ there exist two constants $c_1$, $c_2$ and*

$$\lim_{\hbar \to -\infty} e^{c_1 \hbar} \varphi(E_k) \cdot \xi_\Lambda = \tilde{E}_k(\xi_\Lambda), \tag{6}$$

$$\lim_{\hbar \to -\infty} e^{c_2 \hbar} \varphi(F_k) \cdot \xi_\Lambda = \tilde{F}_k(\xi_\Lambda). \tag{7}$$

*Furthermore, let $P_{GZ}(\lambda)$ denote the set of all basis vectors $\xi_\Lambda$ in the vector space $L(\lambda)$. Then, the finite set $P_{GZ}(\lambda)$ with the maps $\mathrm{wt}$ and $\tilde{E}_k, \tilde{F}_k, \varepsilon_k, \phi_k$ for $k = 1, ..., n-1$, is a $\mathfrak{gl}_n$-crystal. Here*

$$\mathrm{wt}\,(\xi_\Lambda) = \sum_{k=1}^{n} \left( \sum_{i=1}^{k} \lambda_{k,i} - \sum_{i=1}^{k-1} \lambda_{k-1,i} \right) \cdot V_k, \tag{8}$$

*and*

$$\varepsilon_k(\xi_\Lambda) = \max\{j \in \mathbb{Z}_{\geq 0} \mid \tilde{E}_k^j(\xi_\Lambda) \neq 0\}, \tag{9}$$

$$\phi_k(\xi_\Lambda) = \max\{j \in \mathbb{Z}_{\geq 0} \mid \tilde{F}_k^j(\xi_\Lambda) \neq 0\}. \tag{10}$$

### 1.5 Cactus group actions on $\mathfrak{gl}_n$-crystals

**Definition 1.4.** The Cactus group $Cact_n$ is generated by elements $\sigma_{ij}$, $1 \leq i < j \leq n$, subject to the relations

$$\begin{aligned} &\sigma_{ij}^2 = 1 \quad \text{if } 1 \leq i < j \leq n, \\ &\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \quad \text{if } j < k, \\ &\sigma_{ij}\sigma_{kl}\sigma_{ij} = \sigma_{i+j-l,i+j-k} \quad \text{if } i \leq k < l \leq j. \end{aligned} \tag{11}$$

Let $\sigma_i := \sigma_{1,i+1}$ for $1 \leq i \leq n-1$, so $\sigma_i^2 = 1$. Also, the elements $\sigma_1, ..., \sigma_{n-1}$ generate the Cactus group.

In [2], Berenstein and Kirillov found an action of the cactus group on the set of Gelfand-Tsetlin patterns, so

**Theorem 1.5.** [2] *Let us define the actions $t_j$ for $j = 1, ..., n-1$ on the set of Gelfand-Tsetlin patterns*

$$\begin{aligned} t_j(\lambda_{k,i}) &= \lambda_{k,i} \quad \textit{for } k \neq j, && (12) \\ t_j(\lambda_{j,i}) &= \min(\lambda_{j+1,i+1}, \lambda_{j-1,i}) + \max(\lambda_{j+1,i}, \lambda_{j-1,i-1}) - \lambda_{j,i}, && (13) \end{aligned}$$

*where $\lambda_{j,0} = -\infty$ and $\lambda_{j,j+1} = +\infty$. Then $q_j := t_1 t_2 t_1 t_3 t_2 t_1 ... t_j t_{j-1} ... t_1$ defines an action of the cactus group $Cact_n$ on the finite set of Gelfand-Tsetlin patterns.*

Since the set of patterns and the set of basis vectors have a one-to-one correspondence, we will not distinguish between a pattern $\Lambda$ and its corresponding basis vector $\xi_\Lambda$ in the rest of this paper. Similarly, we will not distinguish between the operators on the patterns $\{\Lambda\}$ and on their corresponding basis vectors $\{\xi_\Lambda\}$. For example, we can think of $t_j$ as elements in $\mathrm{End}(L(\lambda))$, defined by

$$t_j(\xi_\Lambda) := \xi_{t_j(\Lambda)} \quad \text{for any basis vector } \xi_\Lambda \in L(\lambda),$$

where the numbers of the pattern $t_j(\Lambda)$ are given on the right-hand sides of (12) and (13). Thus, equivalent to Theorem 1.5, $q_i := t_1 t_2 t_1 t_3 t_2 t_1 ... t_i t_{i-1} ... t_1$ for $i = 1, ..., n-1$ defines an action of the cactus group $Cact_n$ on the set $P_{GZ}(\lambda)$ of basis vectors.

## 1.6 The lift of cactus group actions on the representations

Motivated by Theorem 1.3, we introduce

**Definition 1.6.** Given any continuous family of representation $\rho : U_q(\mathfrak{gl}_n) \to \mathrm{End}(L(\lambda))$ for $q = e^{\frac{\hbar}{2}} \in (0, \infty)$, if for each $k = 1, ..., n-1$, there exist operators $\tilde{E}_k^\rho$ and $\tilde{F}_k^\rho$ acting on the basis vectors $\{\xi_\Lambda\}$ of $L(\lambda)$, such that for any $\xi_\Lambda$ there exist two constants $c_1, c_2$ and

$$\begin{aligned} \lim_{\hbar \to -\infty} e^{c_1 \hbar} \rho(E_k) \cdot \xi_\Lambda &= \tilde{E}_k^\rho(\xi_\Lambda), && (14) \\ \lim_{\hbar \to -\infty} e^{c_2 \hbar} \rho(F_k) \cdot \xi_\Lambda &= \tilde{F}_k^\rho(\xi_\Lambda), && (15) \end{aligned}$$

then we define the operators $\mathrm{Lead}(\rho(E_k)) := \tilde{E}_k^\rho$ and $\mathrm{Lead}(\rho(F_k)) := \tilde{F}_k^\rho$ as the leading asymptotics operators.

**Theorem 1.7.** *The maps*

$$\begin{aligned} \varphi^\tau(E_j) &:= -\varphi(F_{n-j}) \quad \textit{for } j = 1, ..., n-1, \\ \varphi^\tau(F_j) &:= -\varphi(E_{n-j}) \quad \textit{for } j = 1, ..., n-1, \\ \varphi^\tau(H_i) &:= -\varphi(H_{n+1-i}) \quad \textit{for } i = 1, ..., n, \end{aligned}$$

*define a representation of* $U_q(\mathfrak{gl}_n)$ *on the vector space* $L(\lambda)$*. Furthermore, the leading asymptotics operators* $\mathrm{Lead}(\varphi^\tau(E_j))$ *and* $\mathrm{Lead}(\varphi^\tau(F_j))$ *exist, and we have*

$$\mathrm{Lead}(\varphi(E_j))\circ q_{n-1} = q_{n-1}\circ \mathrm{Lead}(\varphi^\tau(E_j)), \tag{16}$$

$$\mathrm{Lead}(\varphi(F_j))\circ q_{n-1} = q_{n-1}\circ \mathrm{Lead}(\varphi^\tau(F_j)). \tag{17}$$

# 2 The proof of Theorem 1.3

## 2.1 The leading asymptotics of $\varphi(E_k)$ and $\varphi(F_k)$

Consider the functions

$$G^\pm(\pm x) = \sqrt{\frac{e^{x/2}-e^{-x/2}}{x}},$$

which can be easily verified to satisfy

$$G^-(x) = G^+(-x),$$

$$G^+(x)G^-(x) = \frac{e^{x/2}-e^{-x/2}}{x}.$$

By Theorem 1.2, we have an explicit representation of $U_q(\mathfrak{gl}_n)$ on the vector space $L(\lambda)$. Thus, let us directly compute the leading asymptotics of the actions of $\varphi(E_k), \varphi(F_k)$ on the basis vector $\xi_\Lambda$.

**Lemma 2.1.** *For* $i = 1, ..., k$,

$$4\lim_{\hbar\to-\infty}\frac{1}{\hbar}\ln\frac{\prod_{a=1}^{k-1}G^+\left(\zeta^{(k)}_{k+1-i}-\zeta^{(k-1)}_a-\frac{\hbar}{2}\right)\prod_{b=1}^{k+1}G^+\left(\zeta^{(k)}_{k+1-i}-\zeta^{(k+1)}_b-\frac{\hbar}{2}\right)}{\prod_{c\neq k+1-i}G^+\left(\zeta^{(k)}_{k+1-i}-\zeta^{(k)}_c\right)G^+\left(\zeta^{(k)}_{k+1-i}-\zeta^{(k)}_c-\hbar\right)}\cdot\xi_{\Lambda+\delta_{k,i}}$$
$$=-\left(\sum_{j=i}^{k-1}\lambda_{k-1,j}-\sum_{j=1}^{i-1}\lambda_{k-1,j}+\sum_{j=i+1}^{k+1}\lambda_{k+1,j}-\sum_{j=1}^{i}\lambda_{k+1,j}-2\sum_{j=i+1}^{k}\lambda_{k,j}+2\sum_{j=1}^{i-1}\lambda_{k,j}+1\right),$$

*and*

$$4\lim_{\hbar\to-\infty}\frac{1}{\hbar}\ln\frac{\prod_{a=1}^{k-1}G^+\left(\zeta^{(k)}_{k+1-i}-\zeta^{(k-1)}_a+\frac{\hbar}{2}\right)\prod_{b=1}^{k+1}G^+\left(\zeta^{(k)}_{k+1-i}-\zeta^{(k+1)}_b+\frac{\hbar}{2}\right)}{\prod_{c\neq k+1-i}G^+\left(\zeta^{(k)}_{k+1-i}-\zeta^{(k)}_c\right)G^+\left(\zeta^{(k)}_{k+1-i}-\zeta^{(k)}_c-\hbar\right)}\cdot\xi_{\Lambda-\delta_{k,i}}$$
$$=-\left(\sum_{j=i}^{k-1}\lambda_{k-1,j}-\sum_{j=1}^{i-1}\lambda_{k-1,j}+\sum_{j=i+1}^{k+1}\lambda_{k+1,j}-\sum_{j=1}^{i}\lambda_{k+1,j}-2\sum_{j=i+1}^{k}\lambda_{k,j}+2\sum_{j=1}^{i-1}\lambda_{k,j}+1\right).$$

**Proof.** Since $G^\pm(x) = \sqrt{\frac{e^{\hbar x/2}-e^{-\hbar x/2}}{\hbar x}}$,

$$4\lim_{\hbar\to-\infty}\frac{1}{\hbar}\ln G^\pm(\hbar x) = 4\lim_{\hbar\to-\infty}\frac{1}{\hbar}\ln\sqrt{\frac{e^{\hbar x/2}-e^{-\hbar x/2}}{\hbar x}} = -|x|$$

Thus, by the defining identities (3), (4), (5), direct computation yields

$$
\begin{aligned}
&4\lim_{\hbar\to-\infty}\frac{1}{\hbar}\ln\frac{\prod_{a=1}^{k-1}G^{+}\left(\zeta_i^{(k)}-\zeta_a^{(k-1)}-\frac{\hbar}{2}\right)\prod_{b=1}^{k+1}G^{+}\left(\zeta_i^{(k)}-\zeta_b^{(k+1)}-\frac{\hbar}{2}\right)}{\prod_{c\neq k+1-i}G^{+}\left(\zeta_i^{(k)}-\zeta_c^{(k)}\right)G^{+}\left(\zeta_i^{(k)}-\zeta_c^{(k)}-\hbar\right)}\cdot\xi_{\Lambda+\delta_{k,k+1-i}}\\
=&-\sum_{a=1}^{k-1}\left|\lambda_{k,k+1-i}-i-\lambda_{k-1,k-a}+a-\frac{1}{2}\right|-\sum_{b=1}^{k+1}\left|\lambda_{k,k+1-i}-i-\lambda_{k+1,k+2-b}+b-\frac{1}{2}\right|\\
&+\sum_{c=1}^{k}\left|\lambda_{k,k+1-i}-i-\lambda_{k,k+1-c}+c\right|+\sum_{c=1}^{k}\left|\lambda_{k,k+1-i}-i-\lambda_{k,k+1-c}+c-1\right|\\
=&-\sum_{a=1}^{i-1}\left(\lambda_{k-1,k-a}-a-\lambda_{k,k+1-i}+i+\frac{1}{2}\right)-\sum_{a=i}^{k-1}\left(\lambda_{k,k+1-i}-i-\lambda_{k-1,k-a}+a-\frac{1}{2}\right)\\
&-\sum_{b=1}^{i}\left(\lambda_{k+1,k+2-b}-b-\lambda_{k,k+1-i}+i+\frac{1}{2}\right)-\sum_{b=i+1}^{k+1}\left(\lambda_{k,k+1-i}-i-\lambda_{k+1,k+2-b}+b-\frac{1}{2}\right)\\
&+2\sum_{c=1}^{i-1}\left(\lambda_{k,k+1-c}-c-\lambda_{k,k+1-i}+i-\frac{1}{2}\right)+2\sum_{c=i+1}^{k}\left(\lambda_{k,k+1-i}-i-\lambda_{k,k+1-c}+c-\frac{1}{2}\right)\\
=&\sum_{j=1}^{k-i}\lambda_{k-1,j}-\sum_{j=k-1+i}^{k-1}\lambda_{k-1,j}-\sum_{j=k+2-i}^{k+1}\lambda_{k+1,j}+\sum_{j=1}^{k+1-i}\lambda_{k+1,j}+2\sum_{j=k+2-i}^{k}\lambda_{k,j}-2\sum_{j=1}^{k-i}\lambda_{k,j}-1,
\end{aligned}
$$

where the second identity follows from the interlacing conditions (1). Similarly,

$$
\begin{aligned}
&4\lim_{\hbar\to-\infty}\frac{1}{\hbar}\ln\frac{\prod_{a=1}^{k-1}G^{+}\left(\zeta_{k+1-i}^{(k)}-\zeta_a^{(k-1)}+\frac{\hbar}{2}\right)\prod_{b=1}^{k+1}G^{+}\left(\zeta_{k+1-i}^{(k)}-\zeta_b^{(k+1)}+\frac{\hbar}{2}\right)}{\prod_{c\neq k+1-i}G^{+}\left(\zeta_{k+1-i}^{(k)}-\zeta_c^{(k)}\right)G^{+}\left(\zeta_{k+1-i}^{(k)}-\zeta_c^{(k)}-\hbar\right)}\cdot\xi_{\Lambda-\delta_{k,k+1-i}}\\
=&\sum_{j=1}^{k-i}\lambda_{k-1,j}-\sum_{j=k-1+i}^{k-1}\lambda_{k-1,j}-\sum_{j=k+2-i}^{k+1}\lambda_{k+1,j}+\sum_{j=1}^{k+1-i}\lambda_{k+1,j}+2\sum_{j=k+2-i}^{k}\lambda_{k,j}-2\sum_{j=1}^{k-i}\lambda_{k,j}-1.
\end{aligned}
$$

■

To simplify, we introduce

**Definition 2.2.** For $i=1,\ldots,k$, define function $g_\Lambda(k,i)$ as

$$
g_\Lambda(k,i)=\sum_{j=i}^{k-1}\lambda_{k-1,j}-\sum_{j=1}^{i-1}\lambda_{k-1,j}+\sum_{j=i+1}^{k+1}\lambda_{k+1,j}-\sum_{j=1}^{i}\lambda_{k+1,j}-2\sum_{j=i+1}^{k}\lambda_{k,j}+2\sum_{j=1}^{i-1}\lambda_{k,j}+1.
$$

The function $g_\Lambda(k,i)$ is a linear combination of the numbers in rows $k+1,k,k-1$ of $\Lambda$, where the sign of each number is determined by its position relative to $\lambda_{k,i}$, as shown in Figure 2.

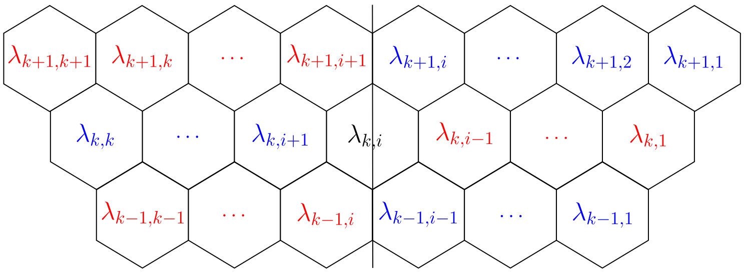



Figure 2: A graphical representation of $g_\Lambda(k,i)$ (Red:$+$; Blue:$-$)

**Proposition 2.3.** *If $l$ is the only index such that $g_\Lambda(k,l)$ is maximized:*

$$g_\Lambda(k,l) = \max\{g_\Lambda(k,k), ..., g_\Lambda(k,1)\},$$

*then*

$$\lim_{\hbar\to-\infty} e^{\hbar g_\Lambda(k,l)/4}\varphi(E_k)\cdot\xi_\Lambda = \xi_{\Lambda+\delta_{kl}}, \tag{18}$$

$$\lim_{\hbar\to-\infty} e^{\hbar g_\Lambda(k,l)/4}\varphi(F_k)\cdot\xi_\Lambda = \xi_{\Lambda-\delta_{kl}}. \tag{19}$$

**Proof.** By Theorem 1.2 and Lemma 2.1, as $\hbar\to-\infty$, we have

$$\begin{aligned}\varphi(E_k)\xi_\Lambda &\to \sum_{i=1}^{k} e^{-\hbar g_\Lambda(k,i)/4}\xi_{\Lambda+\delta_{ki}},\\ \varphi(F_k)\xi_\Lambda &\to \sum_{i=1}^{k} e^{-\hbar g_\Lambda(k,i)/4}\xi_{\Lambda-\delta_{ki}}.\end{aligned} \tag{20}$$

If $g_\Lambda(k,i) < g_\Lambda(k,j)$, then as $\hbar\to-\infty$,

$$\frac{e^{-\hbar g_\Lambda(k,i)/4}}{e^{-\hbar g_\Lambda(k,j)/4}} = e^{\hbar(g_\Lambda(k,j)-g_\Lambda(k,i))/4} \to 0. \tag{21}$$

Therefore, the identities (18) and (19) follow from (20) and the assumption that $l$ is the only index such that $g_\Lambda(k,l)$ is maximized. ■

Motivated by the above proposition, let us define the function $g_k$, where $k = 1, ..., n-1$, on the finite set $P_{GZ}(\lambda)$ as

$$g_k(\Lambda) = \max\{g_\Lambda(k,k), ..., g_\Lambda(k,1)\}.$$

**Definition 2.4.** If there only exists one index $l_k$ such that $g_\Lambda(k,l_k) = g_k(\Lambda)$ for $k = 1, ..., n-1$, then a pattern $\Lambda \in P_{GZ}(\lambda)$ is referred to as generic. Then, the leading asymptotics of $\varphi(E_k)$ and $\varphi(F_k)$ given in Proposition 2.3 define operators $\tilde{E}_k$ and $\tilde{F}_k$ on the set of generic patterns as

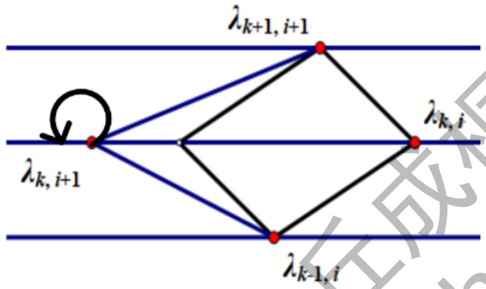
$$\tilde{E}_k(\xi_\Lambda) = \xi_{\Lambda+\delta_{k,l_k}}, \tag{22}$$
$$\tilde{F}_k(\xi_\Lambda) = \xi_{\Lambda-\delta_{k,l_k}}. \tag{23}$$

## 2.2 Extensions of the operators $\tilde{E}_k, \tilde{F}_k$ to the whole set $P_{GZ}(\lambda)$

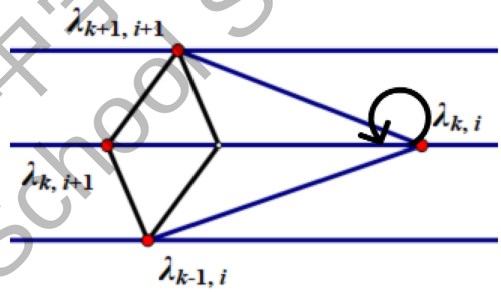
Consider the difference between $g_\Lambda(k,i)$ and $g_\Lambda(k,i+1)$:

$$g_\Lambda(k,i+1) - g_\Lambda(k,i) = -2\lambda_{k-1,i} + 2\lambda_{k,i} + 2\lambda_{k,i+1} - 2\lambda_{k+1,i+1}.$$

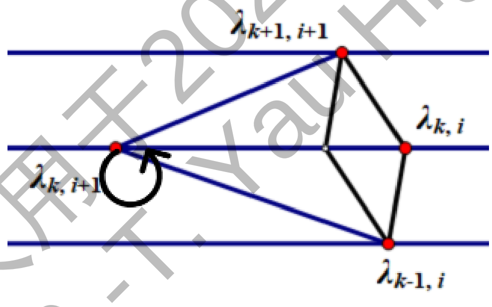
Thus, $g_\Lambda(k,i+1) = g_k(\Lambda) \geq g_\Lambda(k,i)$ only if $\lambda_{k,i} + \lambda_{k,i+1} \geq \lambda_{k-1,i} + \lambda_{k+1,i+1}$. Also, if $g_\Lambda(k,i) < g_k(\Lambda)$, then $g_\Lambda(k,i) \leq g_k(\Lambda) - 2$, because $\lambda_{k,i+1} - \lambda_{k+1,i+1} \in \mathbb{Z}^+$ and $\lambda_{k-1,i} - \lambda_{k,i} \in \mathbb{Z}^+$. When either $g_\Lambda(k,i+1) = g_k(\Lambda)$ or $g_\Lambda(k,i) = g_k(\Lambda)$, consider the effects of $\tilde{E}_k$ and $\tilde{F}_k$ on the pattern $\Lambda$, as shown in Figure 3.



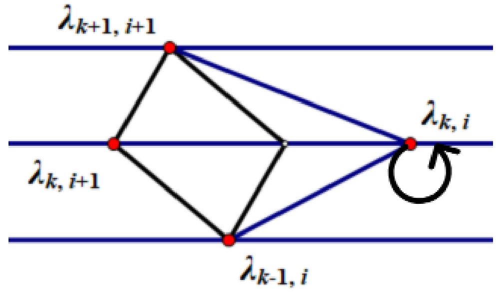

(a) The effect of $\tilde{E}_k$ when $g_\Lambda(k,i+1) = g_k(\Lambda)$




(b) The effect of $\tilde{E}_k$ when $g_\Lambda(k,i) = g_k(\Lambda)$




(c) The effect of $\tilde{F}_k$ when $g_\Lambda(k,i+1) = g_k(\Lambda)$




(d) The effect of $\tilde{F}_k$ when $g_\Lambda(k,i) = g_k(\Lambda)$

Figure 3: The effects of $\tilde{E}_k, \tilde{F}_k$ on the pattern $\Lambda$ (the number line $\longleftarrow$ goes from right to left)

Suppose that $g_\Lambda(k,x) = g_\Lambda(k,y) = g_k(\Lambda)$ for some $x, y \in \mathbb{Z}^+, 1 \leq x < y \leq k$. Thus, consider the constituent patterns $\Lambda_1 = \Lambda + \delta_{k,y}$ and $\Lambda_2 = \Lambda + \delta_{k,x}$ of $\tilde{E}_k(\xi_\Lambda)$.

For both $\Lambda_1$ and $\Lambda_2$,

$$\begin{aligned} g_{\Lambda_1}(k,y) &= g_\Lambda(k,y), \quad g_{\Lambda_1}(k,x) = g_\Lambda(k,x) - 2 &\implies& \quad g_{\Lambda_1}(k,y) > g_{\Lambda_1}(k,x), \\ g_{\Lambda_2}(k,y) &= g_\Lambda(k,y) + 2, \quad g_{\Lambda_2}(k,y) = g_\Lambda(k,y) &\implies& \quad g_{\Lambda_2}(k,y) > g_{\Lambda_2}(k,x), \end{aligned}$$

so $\tilde{E}_k(\xi_{\Lambda_1})$ and $\tilde{E}_k(\xi_{\Lambda_2})$ will move ($+\delta_{k,y}$ or some other numbers) together. Thus, to maintain stability, let $\tilde{E}_k(\xi_\Lambda) = \xi_{\Lambda+\delta_{k,l}}$ where $l$ is the largest index such that $g_\Lambda(k,l)$ is maximized.

Similarly, consider the constituent patterns $\Lambda_3 = \Lambda - \delta_{k,y}$ and $\Lambda_4 = \Lambda - \delta_{k,x}$ of $\tilde{F}_k\xi_\Lambda$. For both $\Lambda_3$ and $\Lambda_4$,

$$\begin{aligned} g_{\Lambda_3}(k,x) &= g_\Lambda(k,x) + 2, \quad g_{\Lambda_3}(k,y) = g_\Lambda(k,y) &\implies& \quad g_{\Lambda_3}(k,x) > g_{\Lambda_3}(k,y), \\ g_{\Lambda_4}(k,x) &= g_\Lambda(k,x), \quad g_{\Lambda_4}(k,y) = g_\Lambda(k,y) - 2 &\implies& \quad g_{\Lambda_4}(k,x) > g_{\Lambda_4}(k,y), \end{aligned}$$

so $\tilde{F}_k(\xi_{\Lambda_3})$ and $\tilde{F}_k(\xi_{\Lambda_4})$ will move ($-\delta_{k,x}$ or some other numbers) together. Thus, to maintain stability, let $\tilde{F}_k(\xi_\Lambda) = \xi_{\Lambda-\delta_{k,r}}$ where $r$ is the smallest index such that $g_\Lambda(k,r)$ is maximized.

Therefore, for the sake of stability,

**Definition 2.5.** Extend the operators given in Definition 2.4 to the whole set $P_{GZ}(\lambda)$ as

$$\begin{aligned} \tilde{E}_k(\xi_\Lambda) &= \xi_{\Lambda+\delta_{k,l}}, \quad \text{where } l \text{ is the largest index such that } g_\Lambda(k,l) \text{ is maximized}, \\ \tilde{F}_k(\xi_\Lambda) &= \xi_{\Lambda-\delta_{k,r}}, \quad \text{where } r \text{ is the smallest index such that } g_\Lambda(k,r) \text{ is maximized}. \end{aligned}$$

## 2.3 The datum $(P_{GZ}(\lambda) = \{\xi_\Lambda\}, \mathrm{wt}, \tilde{E}_k, \tilde{F}_k, \varepsilon_k, \phi_k)$ is a $\mathfrak{gl}_n$-crystal

Recall that $\tilde{E}_k$ and $\tilde{F}_k$ are given in Definition 2.5, and $\varepsilon_k, \phi_k$ and wt are given by

$$\begin{aligned} \varepsilon_k(\xi_\Lambda) &= \max\{j \in \mathbb{Z}_{\geq 0} \mid \tilde{E}_k^j(\xi_\Lambda) \neq 0\}, \\ \phi_k(\xi_\Lambda) &= \max\{j \in \mathbb{Z}_{\geq 0} \mid \tilde{F}_k^j(\xi_\Lambda) \neq 0\}, \\ \mathrm{wt}\,(\xi_\Lambda) &= \sum_{k=1}^{n} \left( \sum_{i=1}^{k} \lambda_{k,i} - \sum_{i=1}^{k-1} \lambda_{k-1,i} \right) \cdot V_k. \end{aligned}$$

Then, we will prove that they satisfy the conditions (1) and (2) in Definition 1.1.

## 2.4 The proof of Condition (1) in Definition 1.1

For pattern $X$, consider a pattern $Y$ such that $\xi_Y = \tilde{E}_k(\xi_X)$. Let $l \in \mathbb{Z}^+, 1 \leq l \leq k$ be the largest index where $g_X(k,l)$ is maximized, so $y_{k,l} = x_{k,l} + 1$. Thus, $g_Y(k,l) = g_X(k,l)$, and for $i = 1, ..., k, i \neq l$,

$$g_Y(k,i) = \begin{cases} g_X(k,i) + 2 & \text{if } i > l, \\ g_X(k,i) - 2 & \text{if } i < l. \end{cases}$$

Also,

$$\begin{cases} g_X(k,i) < g_X(k,l) & \text{if } i > l, \\ g_X(k,i) \leq g_X(k,l) & \text{if } i < l, \end{cases} \implies \begin{cases} g_X(k,i) + 2 \leq g_X(k,l) & \text{if } i > l, \\ g_X(k,i) - 2 < g_X(k,l) & \text{if } i < l, \end{cases}$$

so

$$\begin{cases} g_Y(k,i) = g_X(k,i) + 2 \leq g_X(k,l) = g_Y(k,l) & \text{if } i > l, \\ g_Y(k,i) = g_X(k,i) - 2 < g_X(k,l) = g_Y(k,l) & \text{if } i < l. \end{cases}$$

Thus, $i = l$ is the smallest index where $g_Y(k,i)$ is maximized, so $\tilde{F}_k(\xi_Y) = \xi_X$.

Similarly, consider a pattern $Z$ such that $\tilde{F}_k(\xi_Z) = \xi_X$. Let $r \in \mathbb{Z}^+, 1 \leq r \leq k$ be the smallest index where $g_Z(k,r)$ is maximised, so $x_{k,r} = z_{k,r} - 1$. Thus, $g_X(k,r) = g_Z(k,r)$, and for $i = 1, ..., k, i \neq r$,

$$g_X(k,i) = \begin{cases} g_Z(k,i) - 2 & \text{if } i > r, \\ g_Z(k,i) + 2 & \text{if } i < r. \end{cases}$$

Also,

$$\begin{cases} g_Z(k,i) \leq g_Z(k,r) & \text{if } i > r, \\ g_Z(k,i) < g_Z(k,r) & \text{if } i < r, \end{cases} \implies \begin{cases} g_Z(k,i) - 2 < g_Z(k,r) & \text{if } i > r, \\ g_Z(k,i) + 2 \leq g_Z(k,r) & \text{if } i < r, \end{cases}$$

so

$$\begin{cases} g_X(k,i) = g_Z(k,i) - 2 < g_Z(k,r) = g_X(k,r) & \text{if } i > r, \\ g_X(k,i) = g_Z(k,i) + 2 \leq g_Z(k,r) = g_X(k,r) & \text{if } i < r. \end{cases}$$

Thus, $i = r$ is the largest index where $g_X(k,i)$ is maximized, so $\tilde{E}_k(\xi_X) = \xi_Z$.

Both $i = l$ and $i = r$ are the largest index where $g_X(k,i)$ is maximized, so $l = r$. Thus, $y_{k,l} = x_{k,l} + 1 = x_{k,r} + 1 = z_{k,r}$, so $Y = Z$. Therefore, $\tilde{E}_k(\xi_X) = \xi_Y \iff \tilde{F}_k(\xi_Y) = \xi_X$.

## 2.5 The proof of Condition (2) in Definition 1.1

First, by the definition of $\mathrm{wt}(\xi_\Lambda)$ given in (8),

$$\langle \mathrm{wt}\,(\xi_\Lambda)\,, V_k - V_{k+1} \rangle = 2\sum_{i=1}^{k} \lambda_{k,i} - \sum_{i=1}^{k-1} \lambda_{k-1,i} - \sum_{i=1}^{k+1} \lambda_{k+1,i}.$$

Then, let $\lambda_{i,0} = -\infty$ and $\lambda_{i,i+1} = +\infty$ for $i = 1, ..., n$, and let $x_j = \min(\lambda_{k+1,j+1}, \lambda_{k-1,j})$ and $y_{j+1} = \max(\lambda_{k+1,j+1}, \lambda_{k-1,j})$ for $j = 0, ..., k$. Since $x_j + y_{j+1} = \lambda_{k+1,j+1} + \lambda_{k-1,j}$,

$$\begin{aligned}&g_\Lambda(k, j+1) - g_\Lambda(k, j)\\=&2(-\lambda_{k-1,j} + \lambda_{k,j} + \lambda_{k,j+1} - \lambda_{k+1,j+1})\\=&2\Big((\lambda_{k,j+1} - y_{j+1}) - (x_j - \lambda_{k,j})\Big).\end{aligned}$$

Also, $\lambda_{k+1,j+1} - \lambda_{k,j}, \lambda_{k,j+1} - \lambda_{k-1,j} \in \mathbb{Z}^+$ and $\lambda_{k,j+1} - \lambda_{k+1,j+1}, \lambda_{k-1,j} - \lambda_{k,j} \in \mathbb{Z}^+$, so $\lambda_{k,j+1} - y_{j+1} \geq 1$ and $x_j - \lambda_{k,j} \geq 1$. Thus, if $\lambda_{k,j} = x_j - 1$, then $g_\Lambda(k, j+1) - g_\Lambda(k, j) \geq 0$, so $\tilde{E}_k(\xi_\Lambda) = 0$ if and only if $g_\Lambda(k, k) = g_k(\Lambda)$ and $\lambda_{k,k} = \lambda_{k+1,k+1} - 1$. Similarly, if $\lambda_{k,j+1} = y_j + 1$, then $g_\Lambda(k, j+1) - g_\Lambda(k, j) \leq 0$, so $\tilde{F}_k(\xi_\Lambda) = 0$ only if $g_\Lambda(k, 1) = g_k(\Lambda)$ and $\lambda_{k,1} = \lambda_{k+1,1} + 1$.

Let $l_1$ be the largest index such that $g_\Lambda(k, l_1) = g_k(\Lambda)$, and let $r_1$ be the smallest index such that $g_\Lambda(k, r_1) = g_k(\Lambda)$. For $i \in \mathbb{Z}^+$, let $l_{i+1} \in [k, l_i)$ be the largest index such that

$$g_\Lambda(k, l_{i+1}) = \max\{g_\Lambda(k, k), ..., g_\Lambda(k, l_i + 1)\},$$

and let $r_{i+1} \in (r_i, 1]$ be the smallest index such that

$$g_\Lambda(k, r_{i+1}) = \max\{g_\Lambda(k, r_i - 1), ..., g_\Lambda(k, 1)\}.$$

Thus, let $X^{(j+1)}$ be the pattern such that $\xi_{X^{(j+1)}} = \tilde{E}_k(\xi_{X^{(j)}})$, where $X^{(0)} = \Lambda$. Consider row $k$ of pattern $X^{(j)}$: $x^{(j)}_{k,k}, ..., x^{(j)}_{k,2}, x^{(j)}_{k,1}$, and row $k$ of pattern $X^{(j+1)}$: $x^{(j+1)}_{k,k}, ..., x^{(j+1)}_{k,1}$. If $x^{(j+1)}_{k,l_i} = x^{(j)}_{k,l_i} + 1$, then for $t = 1, ..., k$,

$$g_{X^{(j+1)}}(k, t) = \begin{cases} g_{X^{(j)}}(k, t) + 2 & \text{if } t > l_i, \\ g_{X^{(j)}}(k, t) & \text{if } t = l_i, \\ g_{X^{(j)}}(k, t) - 2 & \text{if } t < l_i, \end{cases}$$

so $g_Z(k, l_{i+1}) = g_Z(k, l_i)$ for a pattern $Z$ such that $\xi_Z = \tilde{E}_k^{(g_{X^{(j)}}(k,l_i) - g_{X^{(j)}}(k,l_{i+1}))/2}(\xi_{X^{(j)}})$. Since $l_{m+1}$ exists if $l_m < k$ and $m$ is finite, $l_m = k$ for some $m \in \mathbb{Z}^+$, so

$$\begin{aligned}\varepsilon_k(\xi_\Lambda) &= \lambda_{k+1,k+1} - \lambda_{k,k} + \frac{1}{2}\sum_{i=1}^{m-1}\Big(g_\Lambda(k, l_i) - g_\Lambda(k, l_{i+1})\Big)\\&= \frac{1}{2}\Big(g_k(\Lambda) - g_\Lambda(k, k)\Big) - \lambda_{k,k} + \lambda_{k+1,k+1}.\end{aligned}$$

Similarly, let $Y^{(j+1)}$ be the pattern such that $\xi_{Y^{(j+1)}} = \tilde{F}_k(\xi_{Y^{(j)}})$, where $Y^{(0)} = \Lambda$. Consider row $k$ of pattern $Y^{(j)}$: $y^{(j)}_{k,k}, ..., y^{(j)}_{k,2}, y^{(j)}_{k,1}$, and row $k$ of pattern $Y^{(j+1)}$: $y^{(j+1)}_{k,k}, ..., y^{(j+1)}_{k,1}$.

If $y_{k,r_i}^{(j+1)} = y_{k,r_i}^{(j)} + 1$, then for $t = 1, ..., k$,

$$g_{Y^{(j+1)}}(k,t) = \begin{cases} g_{Y^{(j)}}(k,t) + 2 & \text{if } t > r_i, \\ g_{Y^{(j)}}(k,t) & \text{if } t = r_i, \\ g_{Y^{(j)}}(k,t) - 2 & \text{if } t < r_i, \end{cases}$$

so $g_Z(k, r_{i+1}) = g_Z(k, r_i)$ for a pattern $Z$ such that $\xi_Z = \tilde{F}_k^{(g_{Y^{(j)}}(k,r_i) - g_{Y^{(j)}}(k,r_{i+1}))/2}(\xi_{Y^{(j)}})$. Since $r_{m+1}$ exists if $r_m > 1$ and $m$ is finite, $r_m = 1$ for some $m \in \mathbb{Z}^+$, so

$$\begin{aligned} \phi_k(\xi_\Lambda) &= \lambda_{k,1} - \lambda_{k+1,1} + \frac{1}{2}\sum_{i=1}^{m-1}\Big(g_\Lambda(k,r_i) - g_\Lambda(k,r_{i+1})\Big) \\ &= \frac{1}{2}\Big(g_k(\Lambda) - g_\Lambda(k,1)\Big) + \lambda_{k,1} - \lambda_{k+1,1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi_k(\xi_\Lambda) - \varepsilon_k(\xi_\Lambda) &= \frac{1}{2}\Big(g_\Lambda(k,k) - g_\Lambda(k,1)\Big) + \lambda_{k,1} + \lambda_{k,k} - \lambda_{k+1,1} - \lambda_{k+1,k+1} \\ &= 2\sum_{i=1}^{k}\lambda_{k,i} - \sum_{i=1}^{k-1}\lambda_{k-1,i} - \sum_{i=1}^{k+1}\lambda_{k+1,i}, \end{aligned}$$

so $\phi_k(\xi_\Lambda) - \varepsilon_k(\xi_\Lambda) = \langle \mathrm{wt}(\xi_\Lambda), V_k - V_{k+1}\rangle$.

# 3 The proof of Theorem 1.7

Recall that for $i = 1, ..., n-1$, we have defined action $t_i$ on the set of patterns $\Lambda$ (equivalently on the set $P_{GZ}(\lambda)$ of basis vectors $\xi_\Lambda$):

$$\begin{aligned} &t_i(\lambda_{k,j}) = \lambda_{k,j} \quad \text{for } k \neq i, \\ &t_i(\lambda_{i,j}) = \min(\lambda_{i+1,j+1}, \lambda_{i-1,j}) + \max(\lambda_{i+1,j}, \lambda_{i-1,j-1}) - \lambda_{i,j}. \end{aligned}$$

Before giving a proof of Theorem 1.7, let us study the commutative relations between $t_k, q_k$ and $\tilde{E}_j, \tilde{F}_j$.

## 3.1 The commutative relation between $t_k$ and $\tilde{E}_j$

In this subsection, we prove that

**Proposition 3.1.** *The operators satisfy*

$$t_{k-1}\tilde{E}_k t_{k-1} = t_k \tilde{E}_{k-1} t_k.$$

**Proof.** For $k = 2, ..., n-1$, consider rows $k+1, k, k-1, k-2$ of pattern $\Lambda$, as shown in Figure 4.

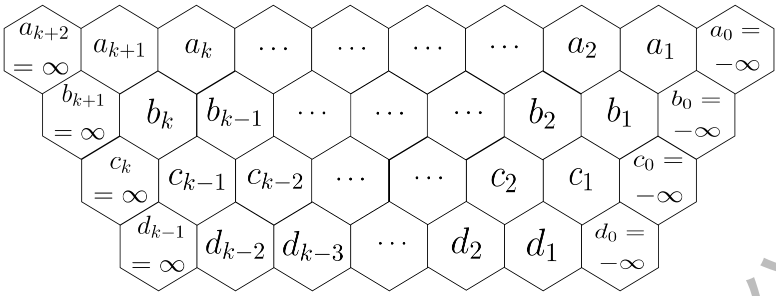



Figure 4: Rows $k+1, k, k-1, k-2$ of the pattern $\Lambda$

In the following, we will compute $t_k \tilde{E}_{k-1} t_k(\xi_\Lambda)$ and $t_{k-1} \tilde{E}_k t_{k-1}(\xi_\Lambda)$ respectively, and then show that the results coincide.

**Step One: the computation of $t_k \tilde{E}_{k-1} t_k(\xi_\Lambda)$**

First, consider $t_k(\xi_\Lambda)$. Let $Z_{k,1}$ be the pattern such that $\xi_{Z_{k,1}} = t_k(\xi_\Lambda)$, as shown in Figure 5. For $j = 1, ..., k$, let $\beta_j = \min(a_{j+1}, c_j)$ and $\gamma_j = \max(a_j, c_{j-1})$. Thus, $\gamma_{j+1} = \max(a_{j+1}, c_j)$, so $\beta_j + \gamma_{j+1} = a_{j+1} + c_j$.

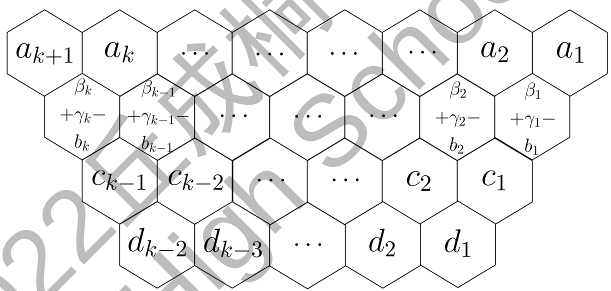



Figure 5: Rows $k+1, k, k-1, k-2$ of the pattern $Z_{k,1}$

Second, consider $\tilde{E}_{k-1} t_k(\xi_\Lambda)$, as shown in Figure 6. Let $i \in \mathbb{Z}^+, 1 \le i \le k-1$ be the largest index where $g_{Z_{k,1}}(k-1, i)$ is maximised, so $c_i' = c_i + 1$ and $c_j' = c_j$ for $j \ne i$.

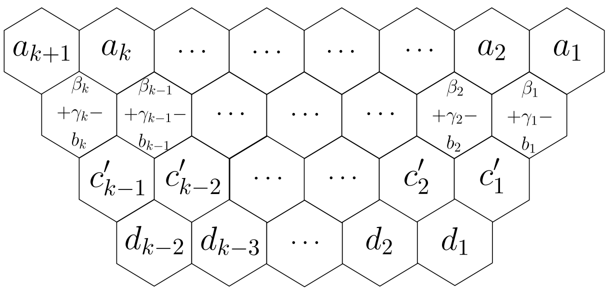



Figure 6: Rows $k+1, k, k-1, k-2$ of the pattern of $\tilde{E}_{k-1} t_k(\xi_\Lambda)$

Let $\beta_j' = \min(a_{j+1}, c_j')$ and $\gamma_j' = \max(a_j, c_{j-1}')$.

- If $j \neq i$, then $\beta'_j = \beta_j$ and $\gamma'_{j+1} = \gamma_{j+1}$.
- If $a_{i+1} > c_i$, then $\beta'_i = c'_i = c_i + 1 = \beta_i + 1$ and $\gamma'_{i+1} = a_{i+1} = \gamma_{i+1}$.
- If $a_{i+1} \leq c_i$, then $\beta'_i = a_{i+1} = \beta_i$ and $\gamma'_{i+1} = c'_i = c_i + 1 = \gamma_{i+1} + 1$.

Third, consider $t_k \tilde{E}_{k-1} t_k(\xi_\Lambda)$, as shown in Figure 7.

$$\begin{matrix} b_k + (\beta'_k - \beta_k) & \cdots & b_2 + (\beta'_2 - \beta_2) & b_1 + (\beta'_1 - \beta_1) \\ +(\gamma'_k - \gamma_k) & & +(\gamma'_2 - \gamma_2) & +(\gamma'_1 - \gamma_1) \end{matrix}$$

Figure 7: Row $k$ of the pattern of $t_k \tilde{E}_{k-1} t_k(\xi_\Lambda)$

Let $X$ be the pattern such that $\xi_X = t_k \tilde{E}_{k-1} t_k(\xi_\Lambda)$, so

$$x_{k-1,j} = c'_j \quad \text{for } j = 1, ..., k-1,$$
$$x_{k,j} = \lambda_{k,j} + (\beta'_j - \beta_j) + (\gamma'_j - \gamma_j) \quad \text{for } j = 1, ..., k.$$

**Step Two: the computation of $t_{k-1} \tilde{E}_k t_{k-1}(\xi_\Lambda)$**

First, consider $t_{k-1}(\xi_\Lambda)$. Let $Z_{k,2}$ be the pattern such that $\xi_{Z_{k,2}} = t_{k-1}(\xi_\Lambda)$, as shown in Figure 8. For $j = 1, ..., k-1$, let $\eta_j = \min(b_{j+1}, d_j)$ and $\theta_i = \max(b_j, d_{j-1})$. Thus, $\theta_{j+1} = \max(b_{j+1}, d_j)$, so $\eta_j + \theta_{j+1} = b_{j+1} + d_i$.

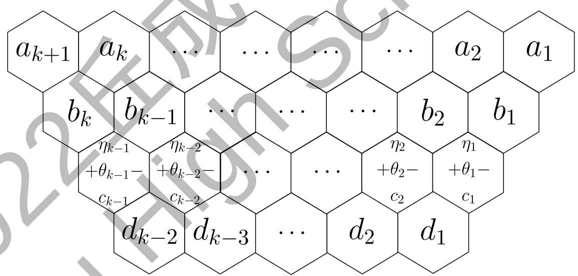



Figure 8: Rows $k+1, k, k-1, k-2$ of the pattern $Z_{k,2}$

Second, consider $\tilde{E}_k t_{k-1}(\xi_\Lambda)$, as shown in Figure 9. Let $i \in \mathbb{Z}^+, 1 \leq i \leq k$ be the largest index where $g_{Z_{k,2}}(k, i)$ is maximised, so $b'_i = b_i + 1$ and $b'_j = b_j$ for $j \neq i$.

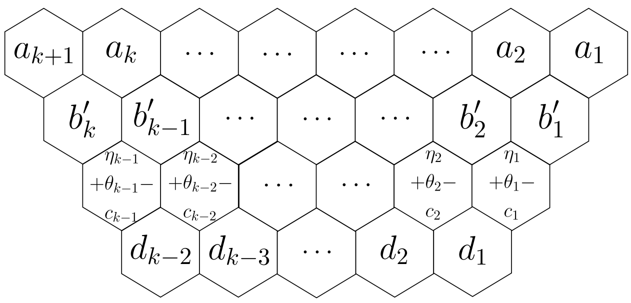



Figure 9: Rows $k+1, k, k-1, k-2$ of the pattern of $\tilde{E}_k t_{k-1}(\xi_\Lambda)$

Let $\eta'_j = \min(b'_{j+1}, d_j)$ and $\theta'_j = \max(b'_j, d_{j-1})$.

- If $j \neq i$, then $\eta'_j = \eta_j$ and $\theta'_{j+1} = \theta_{j+1}$.
- If $b_{i+1} \geq d_i$, then $\eta'_i = d_i = \eta_i$ and $\theta'_{i+1} = b'_{i+1} = b_{i+1} + 1 = \theta_{i+1} + 1$.
- If $b_{i+1} < d_i$, then $\eta'_i = b'_{i+1} = b_{i+1} + 1 = \eta_i + 1$ and $\theta'_{i+1} = d_i = \theta_{i+1}$.

Third, consider $t_{k-1}\tilde{E}_k t_{k-1}(\xi_\Lambda)$, as shown in Figure 10.

$$\begin{matrix} c_{k-1} + (\eta'_{k-1} - \eta_{k-1}) \\ +(\theta'_{k-1} - \theta_{k-1}) \end{matrix} \quad \cdots \quad \begin{matrix} c_2 + (\eta'_2 - \eta_2) \\ +(\theta'_2 - \theta_2) \end{matrix} \quad \begin{matrix} c_1 + (\eta'_1 - \eta_1) \\ +(\theta'_1 - \theta_1) \end{matrix}$$

Figure 10: Row $k-1$ of the pattern of $t_{k-1}\tilde{E}_k t_{k-1}(\xi_\Lambda)$

Let $Y$ be the pattern such that $\xi_Y = t_{k-1}\tilde{E}_k t_{k-1}(\xi_\Lambda)$, so

$$y_{k-1,j} = \lambda_{k-1,j} + (\eta'_j - \eta_j) + (\theta'_j - \theta_j) \quad \text{for } j = 1, ..., k-1,$$
$$y_{k,j} = b'_j \quad \text{for } j = 1, ..., k.$$

**Step Three:** $X = Y$

For $m = 1, ..., k$, let

$$S_1(k,m) = \sum_{j=m+1}^{k+1} a_j - \sum_{j=1}^{m} a_j - \sum_{j=m+1}^{k} b_j + \sum_{j=1}^{m} b_j - \sum_{j=m}^{k-1} c_j + \sum_{j=1}^{m-1} c_j + \sum_{j=m}^{k-2} d_j - \sum_{j=1}^{m-1} d_j,$$
$$S_2(k,m) = \sum_{j=m+1}^{k+1} a_j - \sum_{j=1}^{m} a_j - \sum_{j=m}^{k} b_j + \sum_{j=1}^{m-1} b_j - \sum_{j=m}^{k-1} c_j + \sum_{j=1}^{m-1} c_j + \sum_{j=m-1}^{k-2} d_j - \sum_{j=1}^{m-2} d_j.$$

For $i = 1, ..., k-1$ and $j = 1, ..., k$,

$$g_{Z_{k,1}}(k-1, i) = \max\{S_1(k,i), S_2(k,i+1)\} = \begin{cases} S_1(k,i) & \text{if } a_{i+1} > c_i, \\ S_2(k,i+1) & \text{if } a_{i+1} \leq c_i, \end{cases}$$

$$g_{Z_{k,2}}(k, j) = \max\{S_1(k,j), S_2(k,j)\} = \begin{cases} S_1(k,j) & \text{if } b_j \geq d_{j-1}, \\ S_2(k,j) & \text{if } b_j < d_{j-1}. \end{cases}$$

Also, $d_{k-1} = \infty$ and $d_0 = -\infty$, so $g_{Z_{k,2}}(k,k) = S_2(k,k)$ and $g_{Z_{k,2}}(k,1) = S_1(k,1)$.

For the pattern $X$, consider $l \in \mathbb{Z}^+, 1 \leq l \leq k-1$ such that $x_{k-1,l} = \lambda_{k-1,l} + 1$. Thus, $g_{Z_{k,1}}(k-1,l)$ is maximized, so $g_{Z_{k,1}}(k-1,l) = \max\{g_{Z_{k,1}}(k-1,k-1), ..., g_{Z_{k,1}}(k-1,1)\}$. For the pattern $Y$, consider $r \in \mathbb{Z}^+, 1 \leq r \leq k$ such that $y_{k,r} = \lambda_{k,r} + 1$. Thus, $g_{Z_{k,2}}(k,r)$ is maximized, so $g_{Z_{k,2}}(k,r) = \max\{g_{Z_{k,2}}(k,k), ..., g_{Z_{k,2}}(k,1)\}$.

- If $a_{l+1} > c_l$, then $\beta'_l - \beta_l = 1$, so $x_{k,l} = \lambda_{k,l} + 1$.

- If $a_{l+1} \leq c_l$, then $\gamma'_{l+1} - \gamma_{l+1} = 1$, so $x_{k,l+1} = \lambda_{k,l+1} + 1$.
- If $b_r \geq d_{r-1}$, then $\theta'_r - \theta_r = 1$, so $y_{k-1,r} = \lambda_{k-1,r} + 1$.
- If $b_r < d_{r-1}$, then $\eta'_{r-1} - \eta_{r-1} = 1$, so $y_{k-1,r-1} = \lambda_{k-1,r-1} + 1$.

Also,

$$\begin{aligned} & g_{Z_{k,1}}(k-1,l) = \max\{g_{Z_{k,1}}(k-1,k-1), ..., g_{Z_{k,1}}(k-1,1)\} \\ =& \max\{S_1(k,k-1), S_2(k,k), ..., S_1(k,1), S_2(k,2)\} \\ =& \max\{S_2(k,k), S_1(k,k-1), S_2(k,k-1), ..., S_1(k,2), S_2(k,2), S_1(k,1)\} \\ =& \max\{g_{Z_{k,2}}(k,k), ..., g_{Z_{k,2}}(k,1)\} = g_{Z_{k,2}}(k,r). \end{aligned}$$

- If $a_{l+1} > c_l$, then $S_1(l)$ is the maximum of $S_1(i), S_2(j)$ for $i = 1, ..., k-1, j = 2, ..., k$, so $r = l$ and $b_r \geq d_{r-1}$. Let $z = l = r$, so $l = z, r = z$. Thus,

$$\begin{matrix} x_{k-1,z} = \lambda_{k-1,z} + 1 \\ x_{k,z} = \lambda_{k,z} + 1 \end{matrix}, \begin{matrix} y_{k,z} = \lambda_{k,z} + 1 \\ y_{k-1,z} = \lambda_{k-1,z} + 1 \end{matrix} \implies \begin{matrix} x_{k-1,z} = y_{k-1,z} \\ x_{k,z} = y_{k,z} \end{matrix}.$$

- If $a_{l+1} \leq c_l$, then $S_2(l+1)$ is the maximum of $S_1(i), S_2(j)$ for $i = 1, ..., k-1, j = 2, ..., k$, so $r = l+1$ and $b_r < d_{r-1}$. Let $z = l = r-1$, so $l = z, r = z+1$. Thus,

$$\begin{matrix} x_{k-1,z} = \lambda_{k-1,z} + 1 \\ x_{k,z+1} = \lambda_{k,z+1} + 1 \end{matrix}, \begin{matrix} y_{k,z+1} = \lambda_{k,z+1} + 1 \\ y_{k-1,z} = \lambda_{k-1,z} + 1 \end{matrix} \implies \begin{matrix} x_{k-1,z} = y_{k-1,z} \\ x_{k,z+1} = y_{k,z+1} \end{matrix}.$$

Therefore, $X = Y$, so $t_k\tilde{E}_{k-1}t_k(\xi_\Lambda) = t_{k-1}\tilde{E}_k t_{k-1}(\xi_\Lambda)$ for any $\xi_\Lambda$ and $k = 2, ..., n-1$.

■

### 3.2 The commutative relation between $q_k$ and $\tilde{E}_j$, $\tilde{F}_j$

By Theorem 1.5, $t_k t_k(\Lambda) = \Lambda$ and $t_k t_k(\xi_\Lambda) = \xi_\Lambda$, so $t_k t_k = 1$. Because $t_k$ only affects row $k$ of $\Lambda$, if $|i-j| \neq 1$, then $t_i t_j = t_j t_i$ because $t_i t_j(\Lambda) = t_j t_i(\Lambda)$ and $t_i t_j(\xi_\Lambda) = t_j t_i(\xi_\Lambda)$. For $m = 1, ..., n-1$, let $p_m = t_m t_{m-1} ... t_2 t_1$, so

$$\begin{aligned} t_m p_m = t_m t_m t_{m-1} ... t_2 t_1 = t_{m-1} ... t_2 t_1 = p_{m-1}, \\ q_m = t_1 t_2 t_1 ... t_m ... t_2 t_1 = p_1 p_2 ... p_{m-1} p_m = q_{m-1} p_m. \end{aligned}$$

The following lemma will be used in the proof of Proposition 3.3.

**Lemma 3.2.** *We have that* $q_m q_m = 1$ *for all* $m \in \mathbb{Z}^+$.

**Proof.** When $m = 1$ or $m = 2$, $q_m q_m = 1$:

$$q_1 q_1 = p_1 p_1 = t_1 t_1 = 1,$$
$$q_2 q_2 = t_1 t_2 t_1 t_1 t_2 t_1 = t_1 t_2 t_2 t_1 = t_1 t_1 = 1.$$

By induction, suppose that $q_m q_m = 1$ for $m < r$, where $r \in \mathbb{Z}^+, r \geq 3$. Then, for $i = 3, ..., r$,

$$q_{r-1}(t_r...t_3)p_1\left(\prod_{j=3}^{r} p_j\right) = \cdots = q_{r-1}(t_r...t_i)\left(\prod_{j=1}^{i-2} p_j\right)\left(\prod_{j=i}^{r} p_j\right) = \cdots = q_{r-1}t_r\left(\prod_{j=1}^{r-2} p_j\right)p_r,$$

because

$$\begin{aligned} &q_{r-1}(t_r...t_i)\left(\prod_{j=1}^{i-2} p_j\right)\left(\prod_{j=i}^{r} p_j\right) = q_{r-1}(t_r...t_{i+1})t_i\left(\prod_{j=1}^{i-2} p_j\right)p_i\left(\prod_{j=i+1}^{r} p_j\right) \\ =&q_{r-1}(t_r...t_{i+1})\left(\prod_{j=1}^{i-2} p_j\right)t_i p_i\left(\prod_{j=i+1}^{r} p_j\right) = q_{r-1}(t_r...t_{i+1})\left(\prod_{j=1}^{i-2} p_j\right)p_{i-1}\left(\prod_{j=i+1}^{r} p_j\right) \\ =&q_{r-1}(t_r...t_{i+1})\left(\prod_{j=1}^{i-1} p_j\right)\left(\prod_{j=i+1}^{r} p_j\right) = q_{r-1}(t_r...t_{(i+1)})\left(\prod_{j=1}^{(i+1)-2} p_j\right)\left(\prod_{j=(i+1)}^{r} p_j\right). \end{aligned}$$

Thus, when $m = r$,

$$\begin{aligned} q_m q_m &= q_r q_r = (q_{r-1}p_r)(p_1 p_2...p_r) = q_{r-1}(t_r...t_3 t_2 t_1)(t_1)(t_2 t_1)(p_3...p_r) \\ &= q_{r-1}(t_r...t_3 t_2)(t_2 t_1)(p_3...p_r) = q_{r-1}(t_r...t_3)(t_1)(p_3...p_r) \\ &= q_{r-1}(t_r...t_3)\left(\prod_{j=1}^{1} p_j\right)\left(\prod_{j=3}^{r} p_j\right) = q_{r-1}(t_r)\left(\prod_{j=1}^{r-2} p_j\right)\left(\prod_{j=r}^{r} p_j\right) \\ &= q_{r-1}t_r p_1...p_{r-2}p_r = q_{r-1}p_1...p_{r-2}t_r p_r = q_{r-1}p_1...p_{r-2}p_{r-1} = q_{r-1}q_{r-1} = 1. \end{aligned}$$

Therefore, $q_m q_m = 1$ for all $m \in \mathbb{Z}^+$.

**Proposition 3.3.** *For $m = 1, ..., n-1$, we have*

$$\tilde{E}_m q_{n-1} = q_{n-1}\tilde{F}_{n-m} \text{ and } \tilde{F}_m q_{n-1} = q_{n-1}\tilde{E}_{n-m}.$$

**Proof.** We prove it by induction. For $n = 2$, $\tilde{E}_1 q_1 = q_1 \tilde{F}_1$ and $\tilde{F}_1 q_1 = q_1 \tilde{E}_1$, as shown in Figure 11.

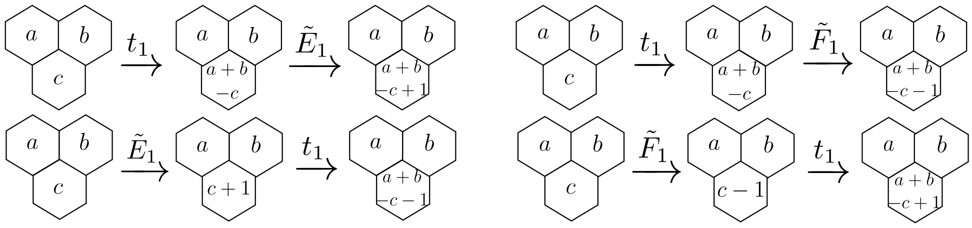



Figure 11: $\tilde{E}_1 q_1 = \tilde{E}_1 t_1, \tilde{E}_1 q_1 = \tilde{F}_1 t_1, t_1\tilde{E}_1 = q_1\tilde{E}_1, t_1\tilde{F}_1 = q_1\tilde{F}_1$

For the induction step, first suppose that $\tilde{E}_m q_{n-1} = q_{n-1}\tilde{F}_{n-m}, \tilde{F}_m q_{n-1} = q_{n-1}\tilde{E}_{n-m}$ for $m = 1, ..., n-1$ when $n < r$, where $r \in \mathbb{Z}^+, r \geq 2$. Then, consider $\tilde{E}_m q_{n-1} = q_{n-1}\tilde{F}_{n-m}$ and $\tilde{F}_m q_{n-1} = q_{n-1}\tilde{E}_{n-m}$ when $n = r$.

By Proposition 3.1, $t_k t_{k-1}\tilde{E}_k = t_k t_{k-1}\tilde{E}_k t_{k-1} t_{k-1} = t_k t_k \tilde{E}_{k-1} t_k t_{k-1} = \tilde{E}_{k-1} t_k t_{k-1}$. Thus,

$$\begin{aligned} p_{r-1}\tilde{E}_k &= t_{r-1}...t_2 t_1 \tilde{E}_k = t_{r-1}...t_{k+1} t_k t_{k-1}\tilde{E}_k t_{k-2}...t_2 t_1 \\ &= t_{r-1}...t_{k+1}\tilde{E}_{k-1} t_k t_{k-1} t_{k-2}...t_2 t_1 = \tilde{E}_{k-1} t_{r-1}...t_2 t_1 = \tilde{E}_{k-1} p_{r-1}, \end{aligned}$$

so $q_{r-1}\tilde{E}_k = q_{r-2}p_{r-1}\tilde{E}_k = q_{r-2}\tilde{E}_{k-1}p_{r-1}$. Also, $\tilde{F}_m q_{n-1} = q_{n-1}\tilde{E}_{n-m}$ holds when $n = r-1 < r$, so $q_{r-2}\tilde{E}_{k-1} = \tilde{F}_{r-k} q_{r-2}$, and

$$q_{r-1}\tilde{E}_k = q_{r-2}\tilde{E}_{k-1}p_{r-1} = \tilde{F}_{r-k} q_{r-2} p_{r-1} = \tilde{F}_{r-k} q_{r-1} \quad \text{for } k = 2, ..., r-1.$$

Although $k \neq 1$, by Condition (1) of Definition 1.1 and Proposition 3.2, we have the following commutative diagram,

$$\begin{array}{ccc} \tilde{F}_{r-k} q_{r-1} = q_{r-1}\tilde{E}_k & \xrightarrow{\tilde{E}_{r-k}\tilde{F}_{r-k} q_{r-1}\tilde{F}_k = \tilde{E}_{r-k} q_{r-1}\tilde{E}_k\tilde{F}_k} & q_{r-1}\tilde{F}_k = \tilde{E}_{r-k} q_{r-1} \\ {\scriptstyle q_{r-1}\tilde{F}_{r-k} q_{r-1} q_{r-1} = q_{r-1} q_{r-1}\tilde{E}_k q_{r-1}} \big\downarrow & & \big\downarrow {\scriptstyle q_{r-1} q_{r-1}\tilde{F}_k q_{r-1} = q_{r-1}\tilde{E}_{r-k} q_{r-1} q_{r-1}} \\ q_{r-1}\tilde{F}_{r-k} = \tilde{E}_k q_{r-1} & \xrightarrow{\tilde{F}_k q_{r-1}\tilde{F}_{r-k}\tilde{E}_{r-k} = \tilde{F}_k\tilde{E}_k q_{r-1}\tilde{E}_{r-k}} & \tilde{F}_k q_{r-1} = q_{r-1}\tilde{E}_{r-k}, \end{array}$$

so $\tilde{E}_m q_{r-1} = q_{r-1}\tilde{F}_{r-m}$ and $\tilde{F}_m q_{r-1} = q_{r-1}\tilde{E}_{r-m}$ for $m = 1 \cdots, r-1$.

Therefore, the proof follows from induction. ■

### 3.3 An interpretation of the actions of the cactus group

Let us prove the first part that

**Lemma 3.4.** *The maps*

$$\varphi^\tau(E_j) := -\varphi(F_{n-j}) \quad \textit{for } j = 1, ..., n-1,$$
$$\varphi^\tau(F_j) := -\varphi(E_{n-j}) \quad \textit{for } j = 1, ..., n-1,$$
$$\varphi^\tau(H_i) := -\varphi(H_{n+1-i}) \quad \textit{for } i = 1, ..., n,$$

*define a representation of* $U_q(\mathfrak{gl}_n)$ *on the vector space* $L(\lambda)$.

**Proof.** To show that $\varphi^\tau$ defines a representation, we need to verify the following identity, which can be deduced from the fact that $\varphi$ is a representation.

- for each $1 \le i \le n$, $1 \le j \le n-1$,

$$q^{\varphi^\tau(H_i)} q^{-\varphi^\tau(H_i)} = q^{-\varphi^\tau(H_i)} q^{\varphi^\tau(H_i)} = 1,$$
$$q^{\varphi^\tau(H_i)} \varphi^\tau(E_j) q^{-\varphi^\tau(H_i)} = q^{\delta_{ij}} q^{-\delta_{i,j+1}} \varphi^\tau(E_j),$$
$$q^{\varphi^\tau(H_i)} \varphi^\tau(F_j) q^{-\varphi^\tau(H_i)} = q^{-\delta_{ij}} q^{\delta_{i,j+1}} \varphi^\tau(F_j);$$

- for each $1 \le i, j \le n-1$,

$$[\varphi^\tau(E_i), \varphi^\tau(F_j)] = \delta_{ij} \frac{q^{\varphi^\tau(H_i) - \varphi^\tau(H_{i+1})} - q^{-\varphi^\tau(H_i) + \varphi^\tau(H_{i+1})}}{q - q^{-1}};$$

- for $|i - j| = 1$,

$$\varphi^\tau(E_i)^2 \varphi^\tau(E_j) - (q + q^{-1}) \varphi^\tau(E_i) \varphi^\tau(E_j) \varphi^\tau(E_i) + \varphi^\tau(E_j) \varphi^\tau(E_i)^2 = 0,$$

$$\varphi^\tau(F_i)^2 \varphi^\tau(F_j) - (q + q^{-1}) \varphi^\tau(F_i) \varphi^\tau(F_j) \varphi^\tau(F_i) + \varphi^\tau(F_j) \varphi^\tau(F_i)^2 = 0;$$

- and for $|i - j| \neq 1$,

$$[\varphi^\tau(E_i), \varphi^\tau(E_j)] = [\varphi^\tau(F_i), \varphi^\tau(F_j)] = 0.$$

■

**Lemma 3.5.** *The leading asymptotics operators* $\mathrm{Lead}(\varphi^\tau(E_j))$ *and* $\mathrm{Lead}(\varphi^\tau(F_j))$ *exist and are given by*

$$\mathrm{Lead}(\varphi^\tau(E_j)) := -\tilde{F}_{n-j},$$
$$\mathrm{Lead}(\varphi^\tau(F_j)) := -\tilde{E}_{n-j}.$$

**Proof of Theorem 1.7.** First, by the above lemma, the leading asymptotics operators $\mathrm{Lead}(\varphi^\tau(E_j))$ and $\mathrm{Lead}(\varphi^\tau(F_j))$ exist and can be expressed by $\tilde{E}_i$ and $\tilde{F}_i$. Following from the commutative relations given in Proposition 3.3, we get

$$\begin{aligned}
\mathrm{Lead}(\varphi(E_j)) \circ q_{n-1} &= q_{n-1} \circ \mathrm{Lead}(\varphi^\tau(E_j)), && (24)\\
\mathrm{Lead}(\varphi(F_j)) \circ q_{n-1} &= q_{n-1} \circ \mathrm{Lead}(\varphi^\tau(F_j)). && (25)
\end{aligned}$$

■

# 4 Conclusion

The main theorems, i.e., Theorem 1.3 and 1.7, can be summarized in the following commutative diagram,

$$\begin{array}{ccc}
\Big(\varphi(E_k), \varphi(F_k)\Big) & \xrightarrow{\text{Involution } \tau} & \Big(\varphi^\tau(E_k), \varphi^\tau(F_k)\Big) \\
\text{Asymptotics } \hbar \to -\infty \Big\downarrow & & \text{Asymptotics } \hbar \to -\infty \Big\downarrow \\
\Big(\{\xi_\Lambda\}, \tilde{E}_k, \tilde{F}_k\Big) & \xrightarrow{\text{Action by the generator } q_{n-1} \text{ of } Cact_n} & \Big(\{\xi_\Lambda\}, \tilde{E}_k, \tilde{F}_k\Big),
\end{array}$$

whose left down-arrow states that the $\hbar \to -\infty$ leading asymptotics of the $\hbar$-family of representation $\varphi$ of $U_q(\mathfrak{gl}_n)$ on the vector space $L(\lambda)$ gives rise to a $\mathfrak{gl}_n$-crystal. Furthermore, the two right-arrows and the right down-arrow state that the leading asymptotics of the $\hbar$-family of representation $\varphi^\tau$ obtained by certain simple involution recovers the actions of (the generators of) the cactus group $Cact_n$ on the $\mathfrak{gl}_n$-crystal. In particular, the identities in Theorem 1.7 can be rewritten as

$$\begin{aligned}
\mathrm{Lead}(\varphi^\tau(E_j)) &= q_{n-1}^{-1} \circ \mathrm{Lead}(\varphi(E_j)) \circ q_{n-1}, && (26)\\
\mathrm{Lead}(\varphi^\tau(F_j)) &= q_{n-1}^{-1} \circ \mathrm{Lead}(\varphi(F_j)) \circ q_{n-1}. && (27)
\end{aligned}$$

Apart from Type A Classical Lie algebras, there are also quantum groups $U_q(\mathfrak{g})$ for Types B, C, D classical Lie algebras $\mathfrak{g}$, which should also have explicit representations of $U_q(\mathfrak{g})$ in the Gelfand-Tsetlin basis of the corresponding classical Lie algebras. (See [6] for the Gelfand-Tsetlin basis of classical Lie algebras.) It would be interesting to study the asymptotics as $\hbar \to -\infty$ of the explicit representations, and to recover their $\mathfrak{g}$-crystals and cactus group actions in the same way as the present paper.

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