ON THE INSTANTON KNOT HOMOLOGY OF SOME 3-STRANDS PRETZEL KNOTS

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Abstract. In this paper we study the instanton knot homology of some families of some 3-strands Pretzel knots $P(1,m,n)$. For the families we study, we found upper bounds for the dimensions of their instanton knot homology in terms of a quadratic function in m and n. We also give some examples where this upper bound is sharp.

Contents

1. Introduction

Knot theory is a central topic of low dimensional topology. Over the past decades, there have been many versions of knot homologies introduced. Among them three versions are of the most interests: knot (Heegaard) Floer homology [OS04], monopole knot homology and instanton knot homology [KM10b]. It has been known that knot Floer homology is computable and monopole knot homology and knot Floer homology are isomorphic [Lek13] or [BS20]. However, instanton knot homology remains isolated. Instanton knot homology is closed related to the representation varieties of the knot group, and has many important applications in knot theory. By definition instanton knot homology is built from a set of partial differential equations related to the knot, which is almost impossible to be solved explicitly. This makes instanton knot homology extremely difficult to compute. For these reason, the computation of instanton knot homology would be an interesting problem to study. In this paper we use the algorithm introduced in [LY22] to find upper bounds for the instanton knot homology of two families of Pretzel knots: $P(n,-m,0)$ and $P(1,-m,-n)$. We use an induction on m and n to find the upper bound as a quadratic function in terms of m and n. We also find that in special cases this upper bound is sharp, i.e., coincide with the lower bound coming from the Alexadner polynomial of the knot as in [KM10a]. However, in general this bound is not sharp.

2. Preliminary

The basic concepts involved in the paper are the following.

Definition 2.1. A **knot** is a smooth embedded circle in $\mathbb{R}^3$.

Definition 2.2. Two knots are **equivalent** if they can transform from one to another in $\mathbb{R}^3$

Definition 2.3. A **knot class** is the collection of equivalent knots.

To describe the positioning of the embedded circles in $\mathbb{R}^3$ more clearly, we project knots onto a plane in $\mathbb{R}^3$ complying certain rules such that only two strands are crossed in each crossing.

Definition 2.4. A **knot diagram** is a projection of a knot onto a plane in $\mathbb{R}^3$ so that only double intersections are allowed. For each double crossing in the knot diagram, we have an **over strand** and an **under strand**.

To find the equivalence of knots, we introduce following operations on knot diagrams.

Theorem 2.5. *Two knot diagrams represent the same knot class if and only if they are related by a finite sequence of the following four types of moves as in Figure 1. The four types of moves are called* **planar isotopy** *and* **Reidmeister move I, II, and III** *respectively.*

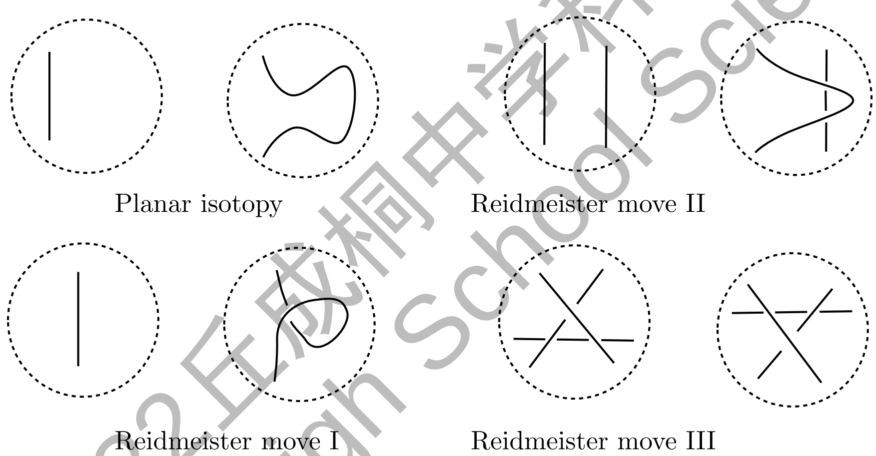


FIGURE 1. Planar isotopy and three types of Reidmeister moves

Definition 2.6. [[KM10b]] *Instanton knot homology* For any knot $K \subset S^3$, there is a well-defined finite dimensional complex vector space, called the instanton knot homology and is denoted by $KHI(K)$, associated to K

To obtain a lower bound of the dimension of the instanton knot homology of a knot, we first compute its Alexander polynomial:

Theorem 2.7. *For any knot $K \subset S^3$, there exists a unique Laurent polynomial, called the* **Alexander polynomial** *of the knot and is denoted by $\Delta_K(t)$, that satisfies the following two axioms:*

- *If K is the unknot, then $\Delta_K(t) = 1$.*
- *$\Delta_{K_+}(t) - \Delta_{K_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \Delta_{K_0}$, where K_+, K_-, and K_0 are three knots which only differ near a crossing and near that crossing, the three knots are depicted as in Figure 2:*

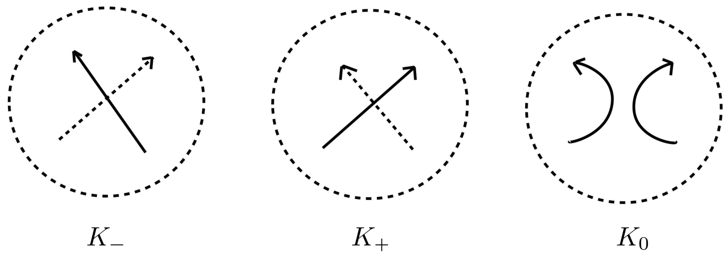


FIGURE 2. K_-, K_+, K_0

Theorem 2.8. *[[KM10a]] Suppose $K \subset S^3$ is a knot. Let $\Delta_K(t)$ be the symmetrized Alexander polynomial of K. Suppose*

$$\Delta_K(t) = \sum_{i \in \mathbb{Z}} a_i \cdot t^i.$$

Then we have an inequality

$$\sum_{i \in \mathbb{Z}} |a_i| \leqslant \dim KHI(K). \tag{2.1}$$

We denote the pretzel knot we study by $P(a, b, c)$ for some $a \in \mathbb{N}, b, c \in \mathbb{Z}$.
Then, we denote the lower bound of $\dim KHI(P(a, b, c))$ obtained in Theorem 2.7 by $L(a, b, c)$.

3. ALGORITHM

In Section 2, we introduced instanton knot homology in Theorem **??**. In this section, we study the dimension of instanton knot homology for some special family of knots. In particular, a lower bound of the dimension can be found in formula (2.1) so we focus on computing an upper bound in this section.

We now describe an algorithm to find possible upper bounds.

Given a knot $K \subset S^3$, we fix any diagram D of K (see an example in Figure 3). We want to construct a triple (Σ, α, γ) and apply the algorithm described below to find an integer d associated to the triple. To construct a triple (Σ, α, γ), first, we draw a diagram of singular knot, projecting the diagram of K to the plane (see an example in Figure 4), and then pick the boundary of the tubular neighborhood of this singular knot, denoting it by Σ (see an example in Figure 5).

In particular, if there are c many crossings in the original knot diagram D, then Σ has genus $(c+1)$. Then, we pick the α curves and γ curves on Σ as follows. We pick $(c+1)$ many α curves (see an example in Figure 6), and one around each hole of Σ. We pick $(c+2)$ many γ curves in three steps:

- First, we pick a meridian of K on Σ as the first γ curve (see an example in Figure 7);
- Second, for each crossing of D, we pick one γ curve around each crossing so that the above curve is in the same direction with the over strand (see an example in Figure 7 and 8);
- Third, noting we have already picked $(c+1)$ γ curves, we pick a parallel copy of each and take c many band sums on the parallel copies to make a connected curve disjoint from all existing γ curves. Thus we obtain the last γ curve (see an example in Figure 9 and 10).

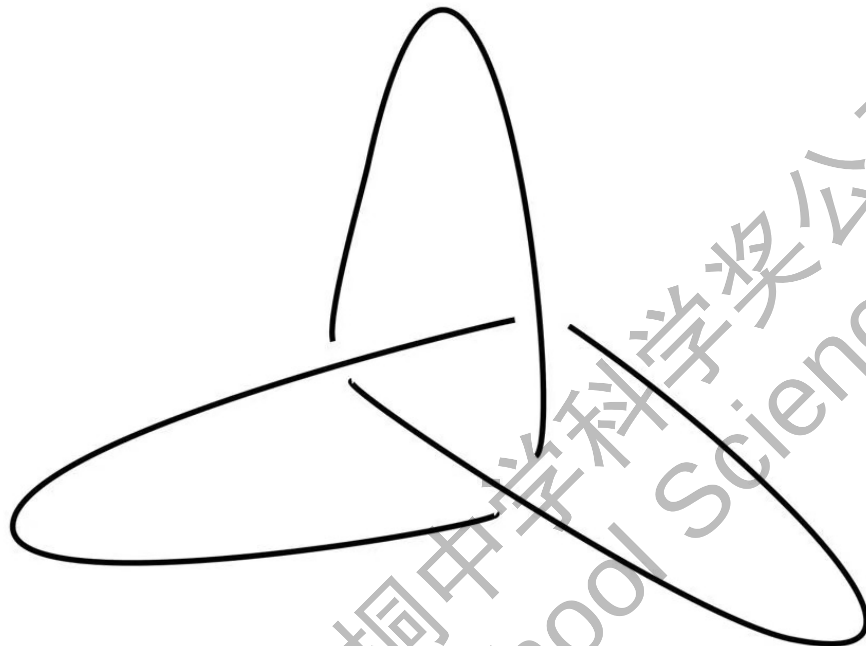


Figure 3. Trefoil

From a knot diagram D with c crossings, we can construct a triple (Σ, α, γ) as above. In [LY22], the triple (Σ, α, γ) corresponds to a sutured handle body as follows: we glue 3-dimensional 2-handle to thickened Σ along α, and then cap off the spherical boundary by a 3-ball. Then the resulting manifold is a handlebody H of genus $c+1$. The curve γ becomes a suture on H making the pair (H, γ) a balanced sutured manifold (*c.f.* [KM10b]). Then we define

$$d(\Sigma, \alpha, \gamma) = \dim SHI(H, \gamma), \tag{3.1}$$

where SHI denotes the sutuerd isntanton Floer homology of a balanced sutured manifold introduced by Kronheimer and Mrowka [KM10b]. We have the following.

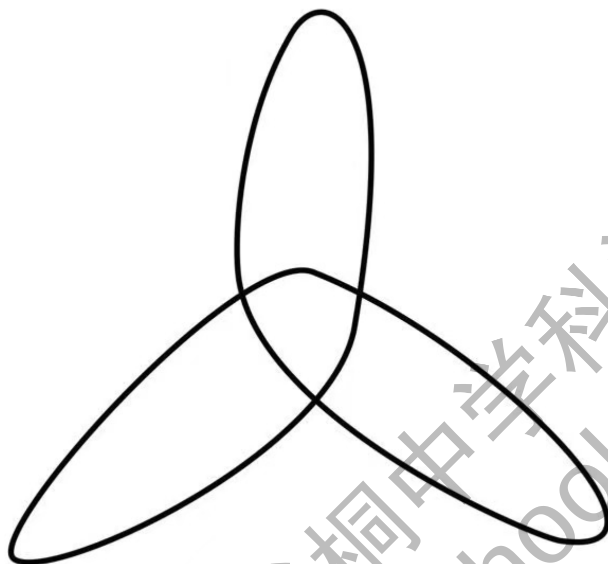


FIGURE 4. Singular projection of trefoil

Theorem 3.1 ([LY22]). *Suppose $K \subset S^3$ is a knot. Suppose we construct a triple (Σ, α, γ) out of a diagram of K. Then there is an inequality*

$$\dim KHI(K) \leqslant d(\Sigma, \alpha, \gamma).$$

Note in practice, we apply isotopy for the last γ curve on the surface, avoiding intersecting with other γ curves, and reduce it to the simplest form.

Then, we can compute an upper bound d and hence for $\dim KHI(K)$, through the algorithm:

- **i** When genus of Σ is one, suppose $n = a, p = b$ for some $a, b \in \mathbb{R}$, where n is the number of γ curves intersected with the α curve and p is the number of intersecting points between γ

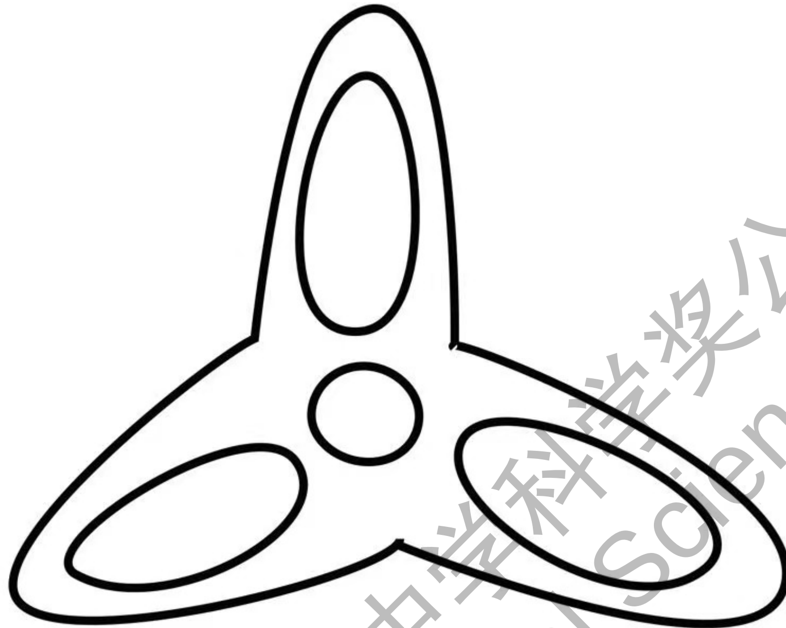


FIGURE 5. The surface Σ constructed out of a diagram of trefoil

curves and the α curve,

$$d = 2^{\frac{a}{2}-1} \times b. \tag{3.2}$$

- **ii** When genus of Σ is larger than one, we can use the following steps iv and v to either reduce the intersection $|\alpha \cap \gamma|$ by two or reduce the genus of Σ by one, and hence finally reduce to i.
- **iii** If there exists a γ curve that can contract into a point on the complement of α curve, then $d = 0$.
- **iv Reduction 1** When an α curve intersects with γ curves twice, we can reduce genus of Σ, as well as the number of α curve, by one, and alter γ curves as well as in Figure 10. Let $(\Sigma^*, \alpha^*, \gamma^*)$ be the triple after the reduction. Then, we have the following equality:

$$d(\Sigma, \alpha, \gamma) = d(\Sigma^*, \alpha^*, \gamma^*) \tag{3.3}$$

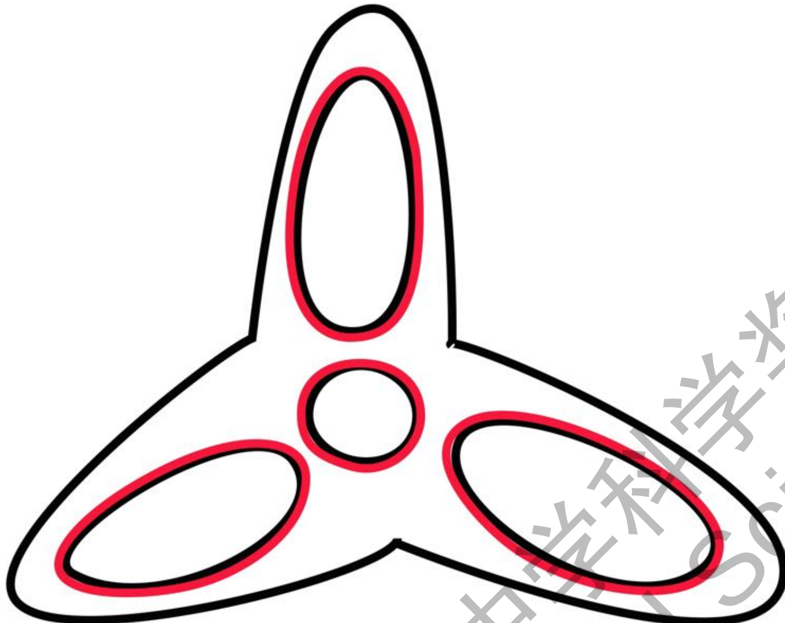


FIGURE 6. α curve(red) of the singular projection of trefoil

- **v Reduction 2** When an α curve intersects with γ curves **at least four times?**, we can apply the reduction in Figure 11. In this case, Σ and α is unchanged, and we can cut γ and reglue in two different ways to obtain two new curves γ' and γ'' as shown in Figure 11. Note it is clear after isotopy we have:

$$|\alpha \cap \gamma'|, |\alpha \cap \gamma''| \leqslant |\alpha \cap \gamma| - 2 \tag{3.4}$$

Then, we have the following equation:

$$d(\Sigma, \alpha, \gamma) \leqslant d(\Sigma, \alpha, \gamma') + d(\Sigma, \alpha, \gamma'') \tag{3.5}$$

To easily represent the upper bound obtained for each $P(a, b, c)$, we denote d for each $P(a, b, c)$ by $U(a, b, c)$ for some $a \in \mathbb{N}, b, c \in \mathbb{Z}$.

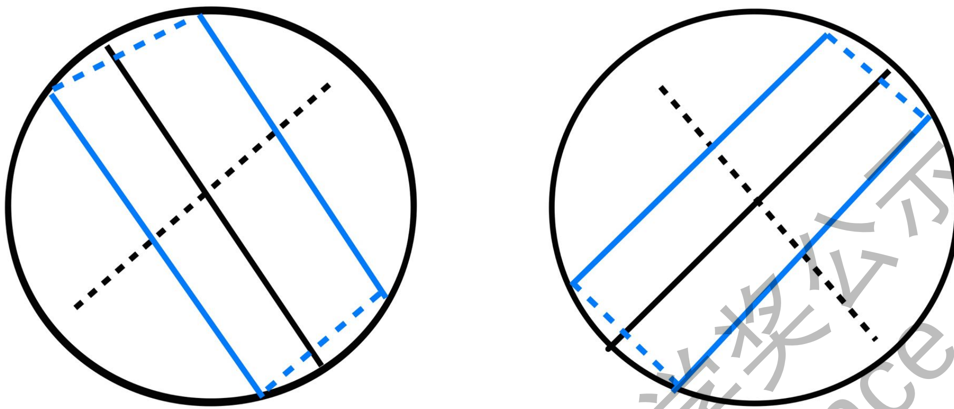


FIGURE 7. The relationship between crossing and second γ curve(blue)

4. COMPUTATIONS

Applying the above algorithms, we study the dimension of the instanton knot homology of some three-strand pretzel knots.

Theorem 4.1. *Suppose* $K = P(1, -n, -m)$ *for some* $n, m \in \mathbb{N}$,

$$dimKHI(K) \leqslant 8mn + 4n + 2m + 3 \tag{4.1}$$

Proof. For the triple of the singular projection of P(1,-1,-1), we can reduce the last γ to the simplest form by simple planar isotopy as in Figure 13.

Noticing in reducing the last γ curve, there are only two kinds of relationships between the α and the last γ curve around each α curve as in Figure 14, and in both situations, we can reduce to Figure 15.

Then, it is easy to tell that, to reduce the last γ curve on the singular projection of $P(1, -n, -m)$ for some $n, m \in \mathbb{N}$, we only need to tackle the two situations around each α curve.

Similarly, for general m and n, we can always reduce the last γ to a simple form as in Figure 16. Then, we compute the lower bound, $U(1, -n, -m)$ inductively.

First, we apply the Reduction 1 on every curve in Figure 17 as we could.

We apply the Reduction 2 as in Figure 18.

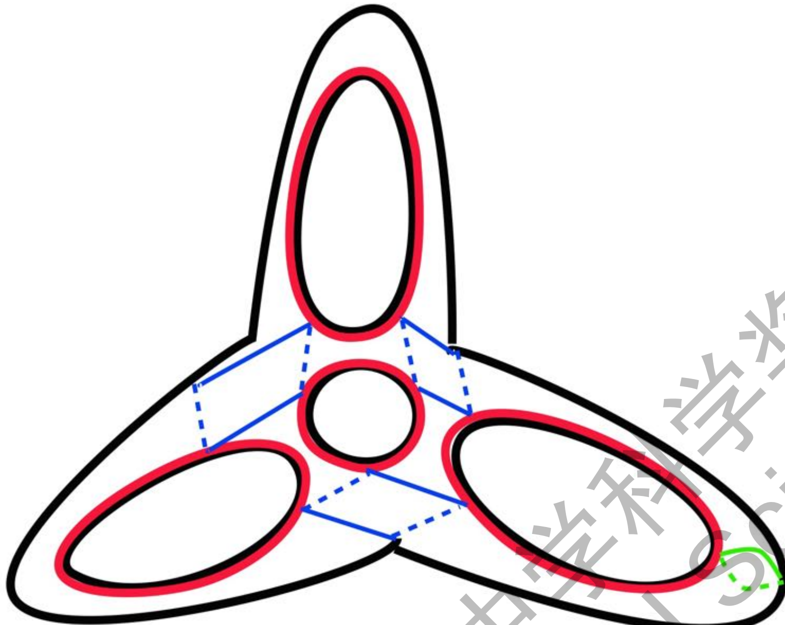


FIGURE 8. Meridian (green) and the second γ curve(blue) of the singular projection of trefoil

Then, we can tell that the number d of the original triple is no larger than the sum of the d of two new triples as in Figure 19.

For the triple on the left in Figure 19, we can do the following planar isotopy as in Figure 20 such that we can show it is unrelated to $m \in \mathbb{N}$.
Since one (right) of the triples is the triple constructed out of $P(1, -n+1, -m)$ and the other (left) one is a triple unrelated to $n \in \mathbb{N}$, we then study the latter triple to derive the final deduction.

To describe the latter triple based on m,

Definition 4.2. $r(m)$ is the triple where the number of crossing of second γ curves in the black block is denoted by $2m+1$ as in Figure 21.

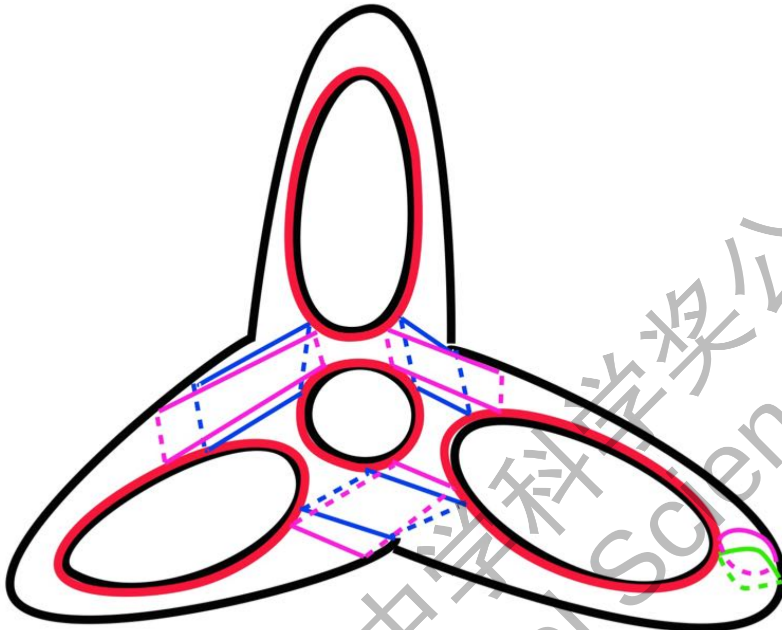


FIGURE 9. Parallel curves(purple) around every second γ curve

By applying Reduction 2 on the curves in black block as in Figure 21, we can tell that the d for the $r(m)$ is no larger than the sum of d of two triples in Figure 22.

Still, noticing that one of the triples is $r(m-1)$, and the other one is a triple unrelated to both n and m, we compute the d for the latter triple as follow.

First, we apply reduction 2 on the blocked γ curves as in Figure 23.

Then, after obtaining the two triples derived from the above reduction, we apply reduction 1 on the curves in black block as in Figure 24.

For the two triples above, we apply reduction 2 on each of them as in Figure 25.

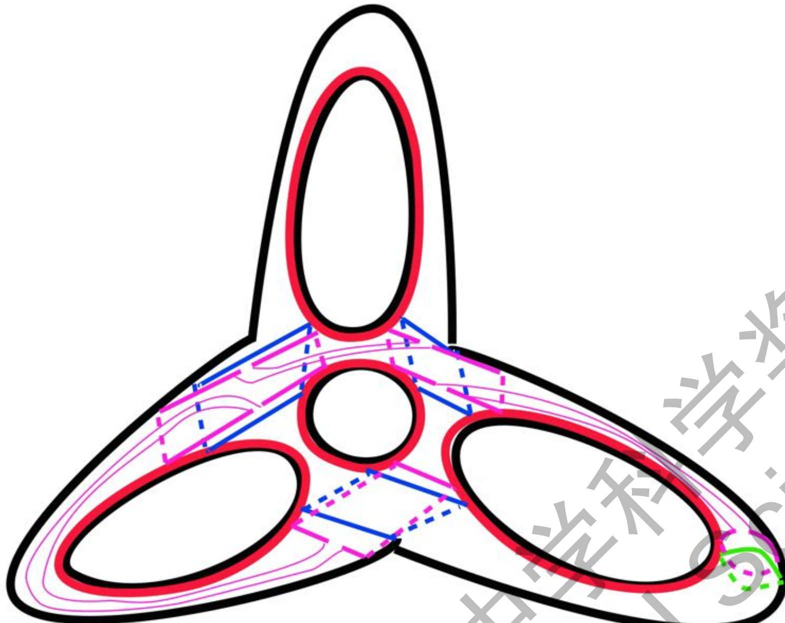


FIGURE 10. The third γ curve(purple) of the singular projection of trefoil

After the reduction 2 above conducted, we obtain four triples, where we choose two of them and do reduction 1 as in Figure 26.

Noticing in one of the triples in Figure 28, one γ curve does not intersect with any α curves. Therefore, based on the algorithm, the d of this triple is 0.

For the other triple in Figure 28, we apply reduction 1 as, count the number of intersecting points between each γ curve and α curve and the number of γ curve intersected with α curve, and compute the d through$d = 2^{\frac{a}{2}-1} \times b$. Hence, we obtain that the d of this triple is equal to 2.

For the other triple in Figure 25, after the reduction 2, we can also obtain two triples as in Figure 29, on which we apply the reduction 1 on curves in black block.

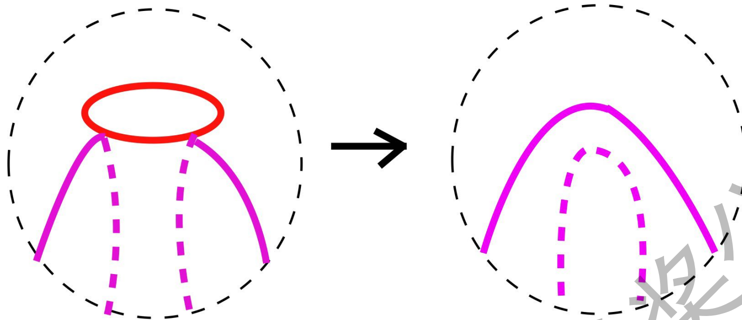


FIGURE 11. Reduction 1

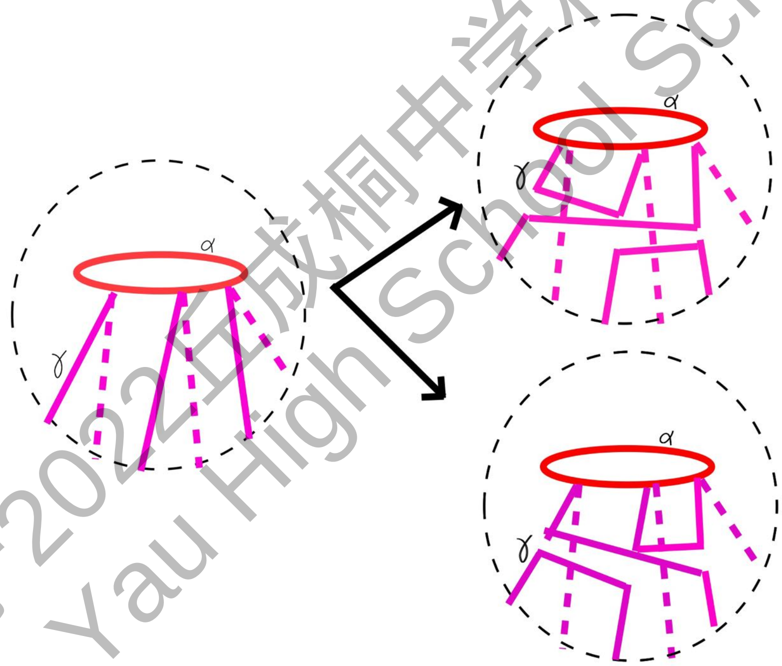
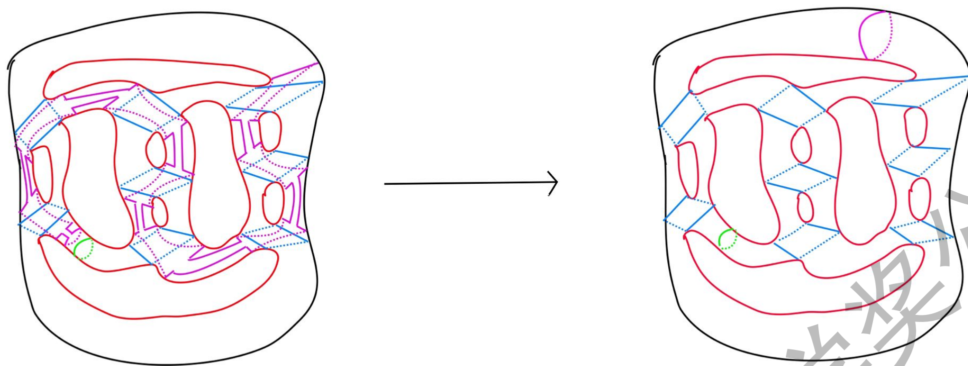


FIGURE 12. Reduction 2

Then, for the triple on the left of the Figure 29, after a reduction 1 we can compute that its d is equal to 2.



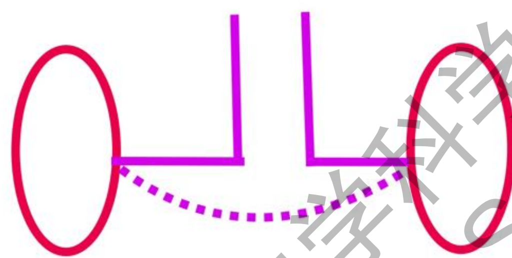
α curve (red); meridian (green); last γ curve (purple)

FIGURE 13. Reudction of last γ curve for $n, m = 1$

α curve (red); last γ curve (purple)

FIGURE 14. Two situations

For the triple on the right of the Figure, we apply another reduction 2 as in Figure 30; then, by a simple reduction 1 we can compute that its d is equal to 2.



α curve (red); last γ curve (purple)

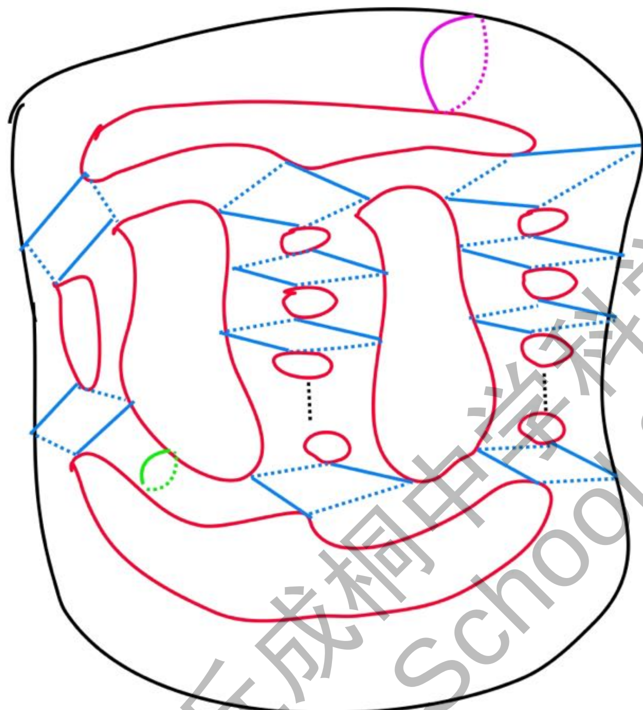
FIGURE 15. Reduction of the two situations

As for each time applying the reduction 2, we have

$$d(\Sigma, \alpha, \gamma) \leqslant d(\Sigma, \alpha, \gamma') + d(\Sigma, \alpha, \gamma'') \tag{4.2}$$

; therefore, after computing the d for above five triples, we obtain that

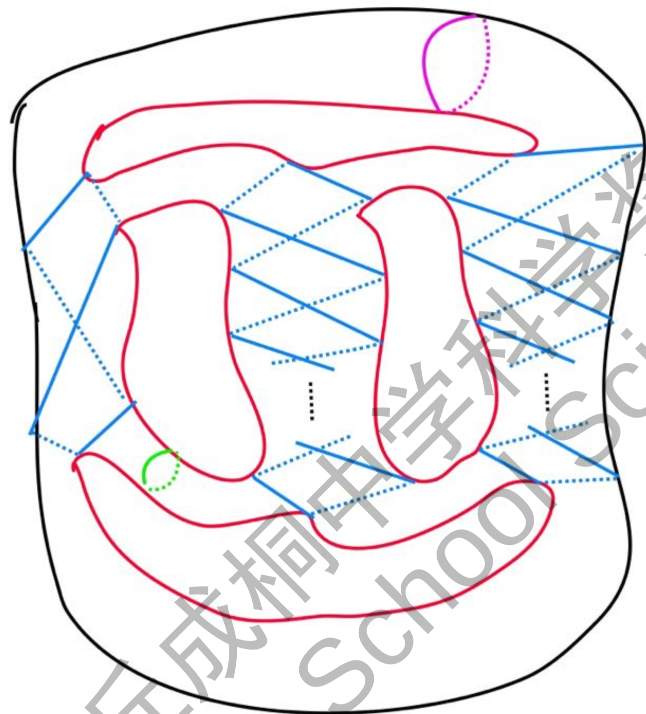
$$r(m) \leqslant r(m-1) + 8 \tag{4.3}$$



α curve (red); last γ curve (purple)

FIGURE 16. Reduction of the last γ curve for all the n, m

Similarly, we compute that for $r(0)$ in Figure 31, its d is equal to 4.



α curve (red); meridian (green); second γ curve (blue); last γ curve (purple)

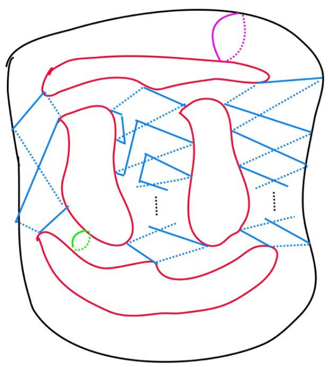
FIGURE 17.

In this way, we have the following inequality that

$$U(1,-n,-m) \leqslant U(1,-n+1,-m)+4+8m \tag{4.4}$$

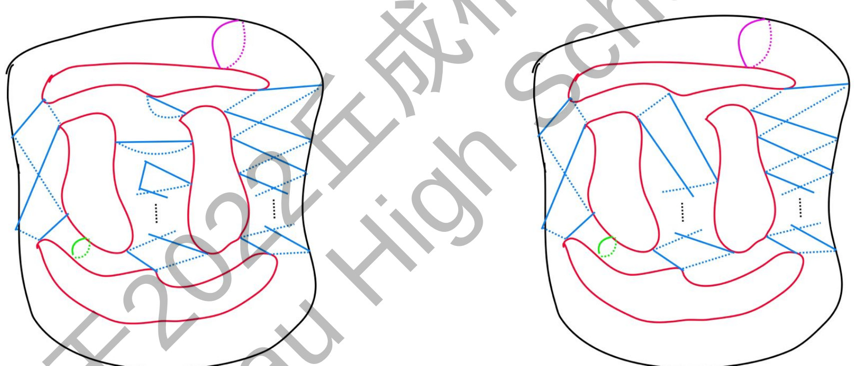
Therefore, we can deduce that

$$U(1,-n,-m) \leqslant U(1,-n+1,-m)+8m+4 \tag{4.5}$$



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

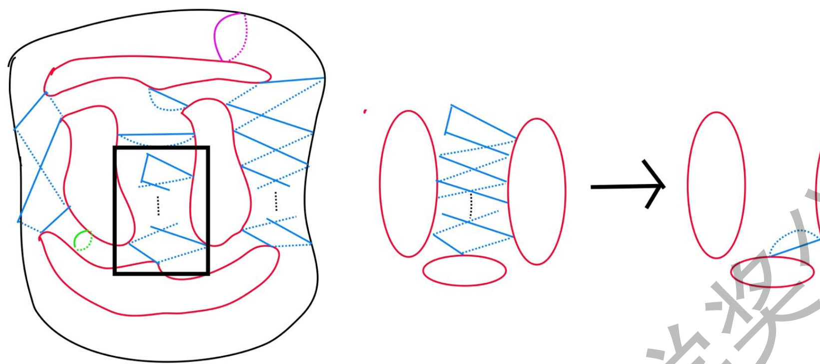
FIGURE 18.



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 19.

Then, we deductively study the upper bound obtained through the algorithm for the dimension of instanton knot homology of $P(1, 0, -m)$.



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 20.

For the triple constructed out of $P(1,0,-m)$, we do the reduction 2 as in the Figure. Similarly to the previous computation, the d of the triple on the left of the Figure 32 is a constant, which, by the simple computation, is found to be 2.

Then, since we can easily compute that the dimension of the instanton knot homology of $P(1,0,0)$ is 3, we have the following inequality:

$$U(1,0,-m) \leqslant 3+2m \tag{4.6}$$

Based on the two inequalities above, we then derive that

$$U(1,-n,-m) \leqslant 8mn+4n+2m+3 \tag{4.7}$$

Hence, we proved that

$$dimKHI(K) \leqslant 8mn+4n+2m+3 \tag{4.8}$$

□

Then, we further study the exact dimension of instanton knot homology of $P(n,m,0)$ for some $n \in \mathbb{N}, m \in \mathbb{Z}$.

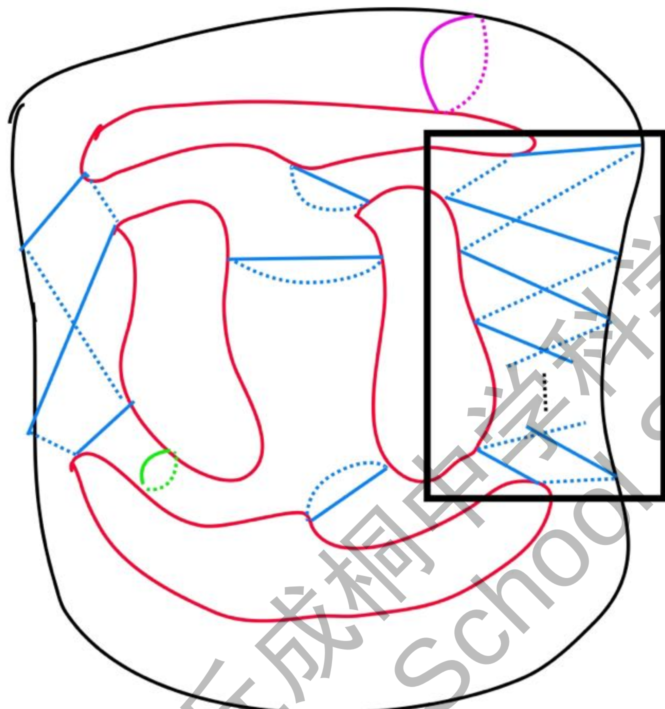
Theorem 4.3. *Suppose $K = P(n,-m,0)$ for some $n,m \in \mathbb{N}$. Then*

$$\dim KHI(K) = 4mn+4n-1 \tag{4.9}$$

Proof. . We first compute an upper bound $U(n,-m,0)$ of $\dim KHI(K)$

Proposition 4.4. $U(n,-m,0) \leqslant 4mn+4n-1$

Proof. For $P(1,-1,0)$, we can transform it to another knot as in Figure 33.

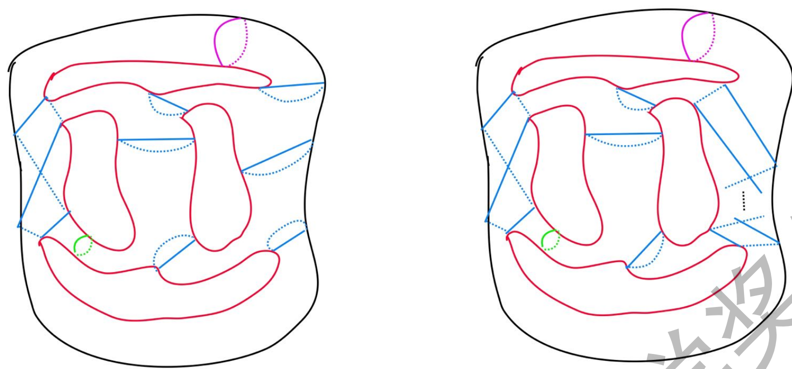


α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 21.

Similarly, for $P(n, -m, 0)$, we can also transform it to another knot as in Figure 34.

Noticing that the number of crossing inside the blue block in Figure 34 is reduced by one from that in the black block through the transformation, to clearly represent the type of knot on the right of Figure 34, we define $P'(n, -m)$ as the knot transformed as above from $P(n, -m, 0)$, where the number of crossing in the blue block is $2n - 1$ and the number of crossing in the red block is $2m + 1$.



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 22.

As the reduction of the last γ curve is similar to the reduction in Figure 14, 15, and 16, we construct the triple out of the singular projection of $P'(n,-m)$ as in Figure 35.

Then, we compute the upper bound, $U(n,-m,0)$, inductively.

First, we apply Reduction 1 on every curve in Figure 35 as we could as in Figure 36.

We apply reduction 2 as in Figure 38.

Then, by the similar process in Figure 20, we can obtain to triples as in Figure 39, and have the following inequality.

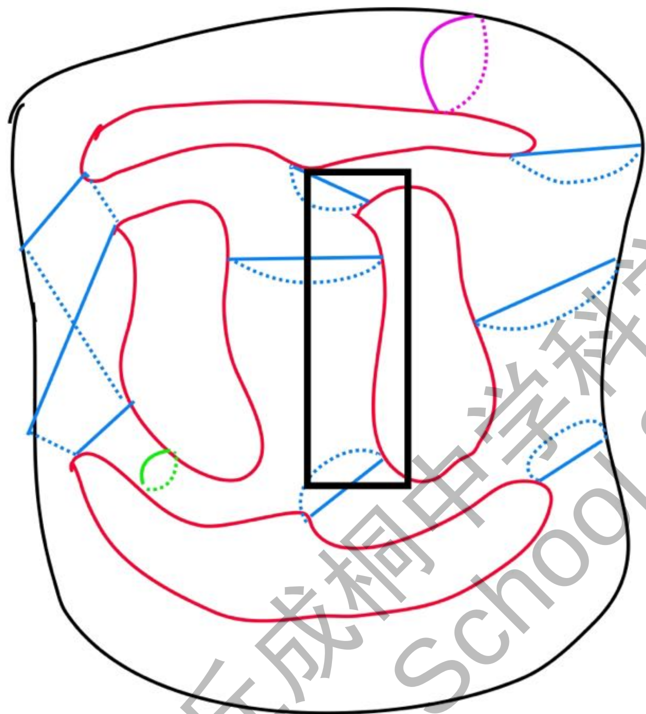
Definition 4.5. $r'(-m)$ is the triple in the left of Figure 40, whose number of crossing in the black block is $2m+1$.

Noticing the right one is the triple for $P'(n-1,-m)$, we further deductively study the $r'(-m)$.

By applying reduction 2 on as in Figure 41, we obtain two triples as in Figure 42.
Between the two triples obtained, it is easy to tell that the left one is independent with both n and m, and the right one is $r'(-m+1)$.
Then, we compute the d of the left triple.

First, we do the reduction 1 as in Figure 42.

Then, we do the reduction 2 as in Figure 43, and obtain two triples as in Figure 44.



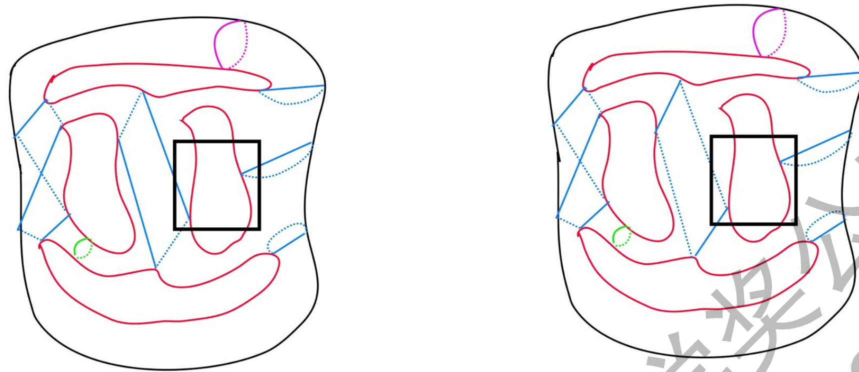
α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 23.

Then, by doing reduction 1 on each of them, we can compute that both of their d are equal to 2.

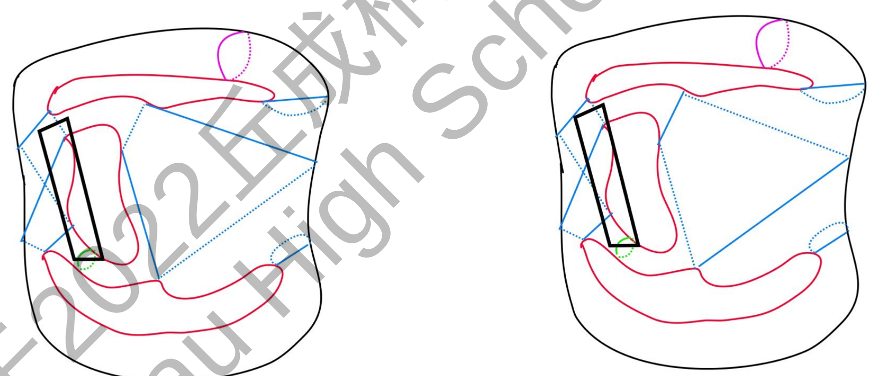
Similarly, we can compute that $r'(0)$ is equal to 4 and have the following inequality that

$$r'(-m) \leqslant 4(m+1) \tag{4.10}$$



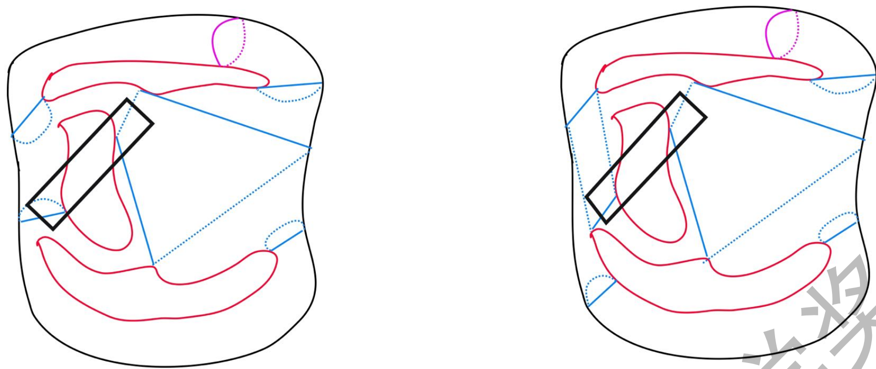
α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

Figure 24.



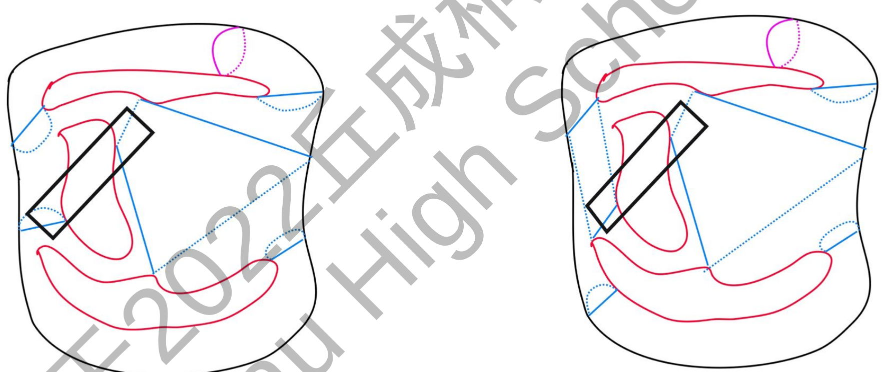
α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

Figure 25.



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 26.

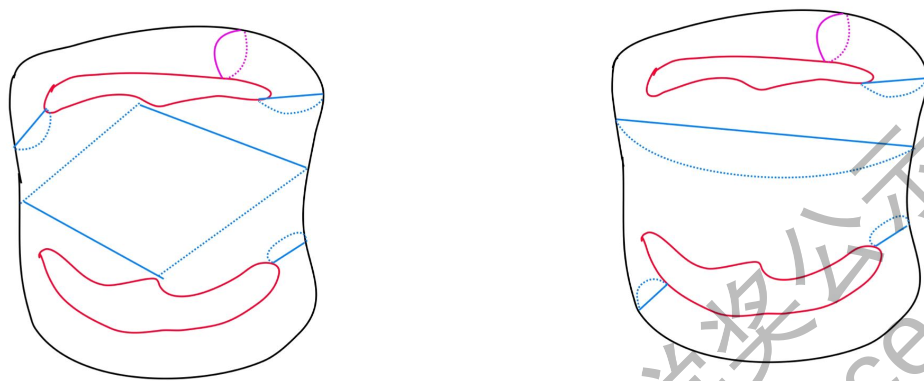


α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 27.

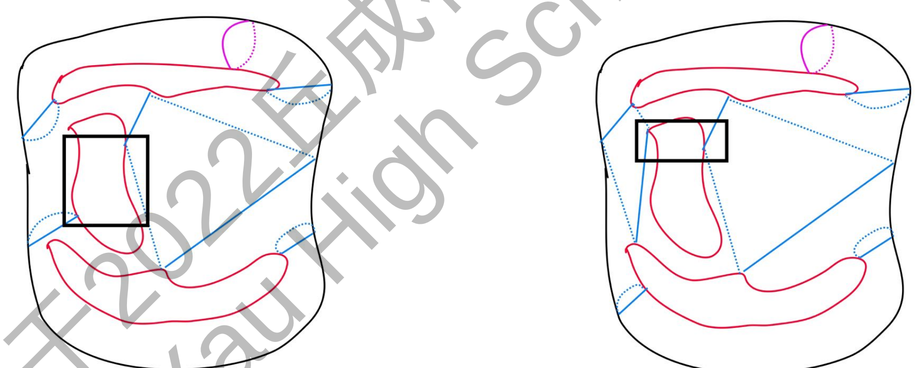
Therefore, we can deduce that

$$U(n,-m,0) \leqslant U(n-1,-m,0)+4m+4 \tag{4.11}$$



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

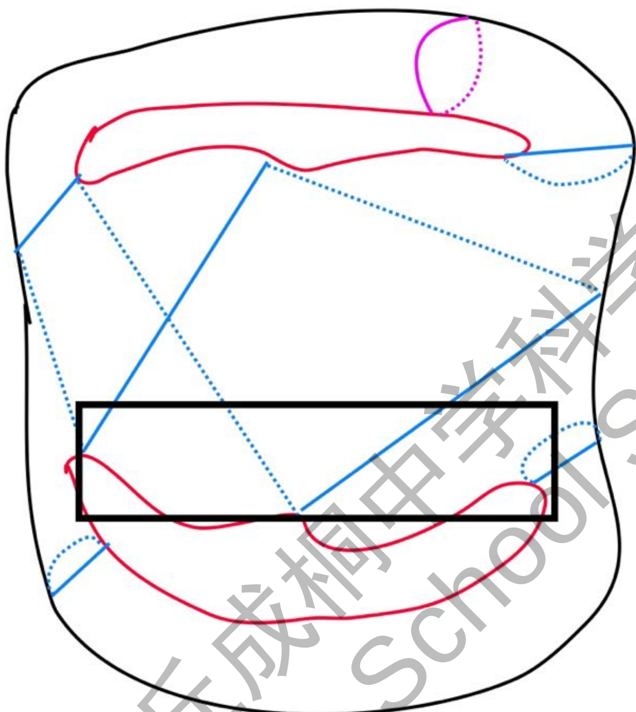
FIGURE 28.



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 29.

By studying $U(1, -m, 0)$, we found out that there also exists a similar deductive inequality that



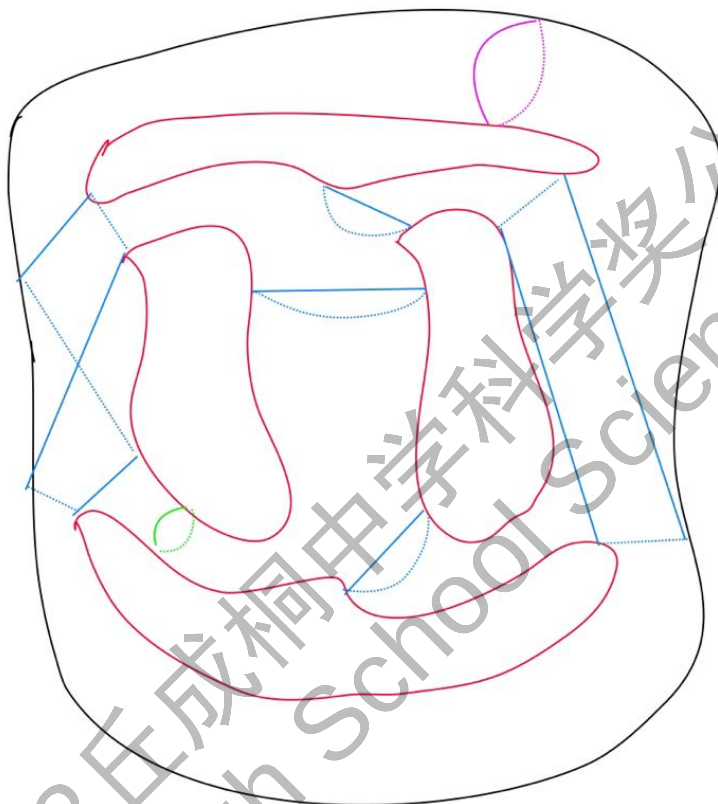
α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 30.

$$U(1,-m,0) \leqslant U(1,-m+1,0)+4 \tag{4.12}$$

Also, we compute that d constructed out of $P(1,0,0)$ is equal to 3
Then, we have that:

$$U(1,-m,0) \leqslant 3+4m \tag{4.13}$$



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

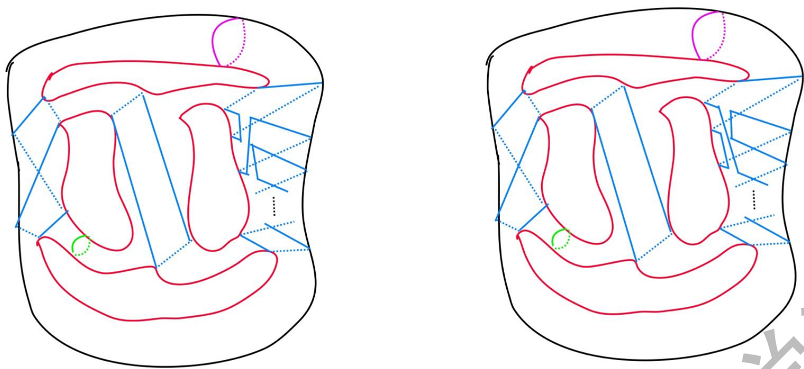
FIGURE 31.

Therefore, based on the above three inequalities, we have

$$U(n, -m, 0) \leqslant 4mn + 4n - 1 \tag{4.14}$$

□

Proposition 4.6. $L(n, -m, 0) = 4mn + 4n - 1$



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 32.

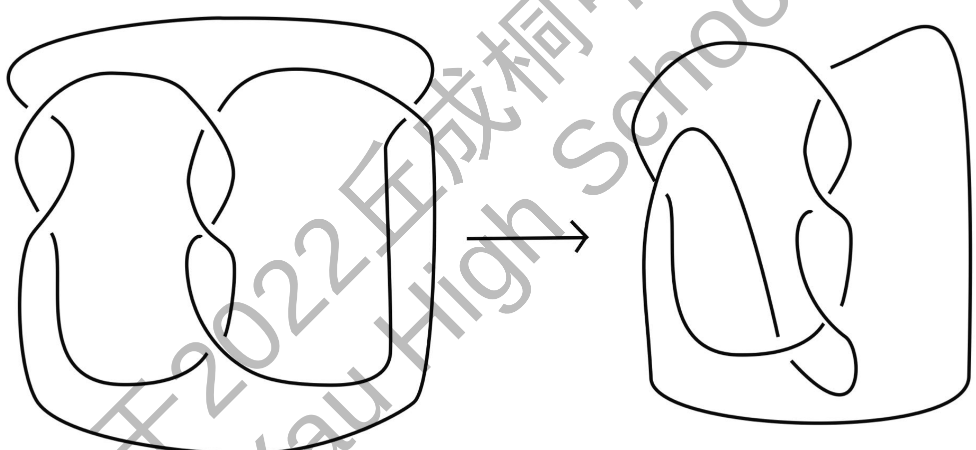


FIGURE 33.

Proof. Giving $P(n,-m,0)$'s knot diagram a direction as in Figure 44, we compute its Alexander polynomial by Theorem2.6. where the K_-, K_+, K_0 are presented in Figure 45.

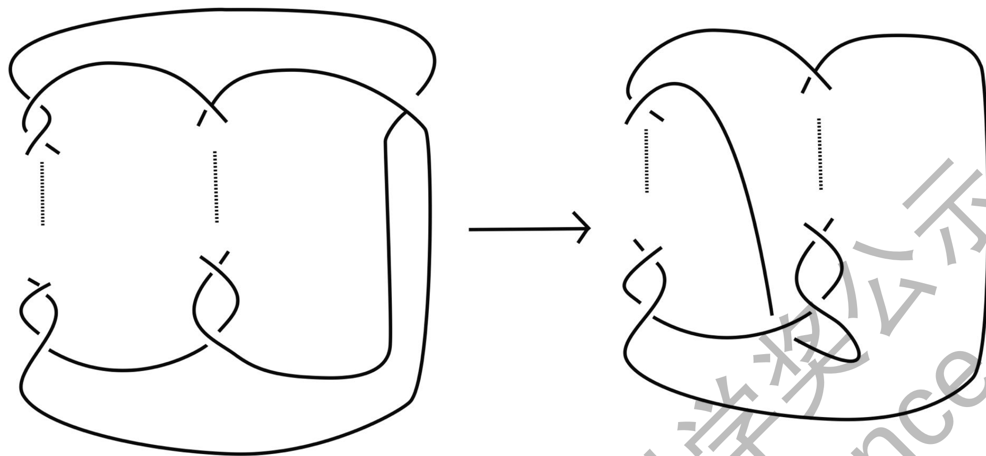


FIGURE 34.

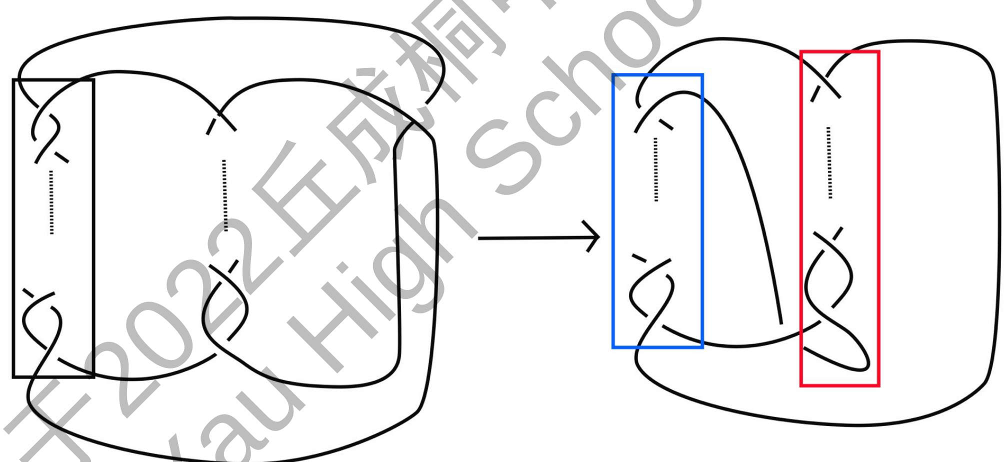


FIGURE 35.

Then, noticing that the K_+ in Figure 45 is equivalent to $P(n-1,-m,0)$ as the left one of the Figure 46, we further study the Alexander polynomial of the left one to find a deduction relationship.

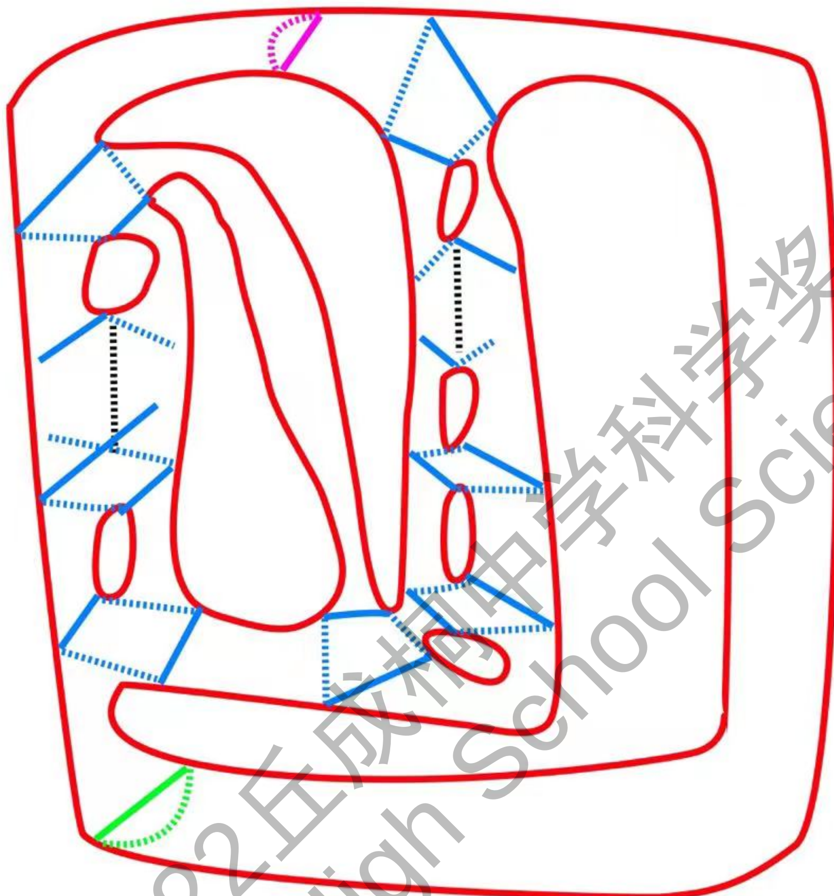
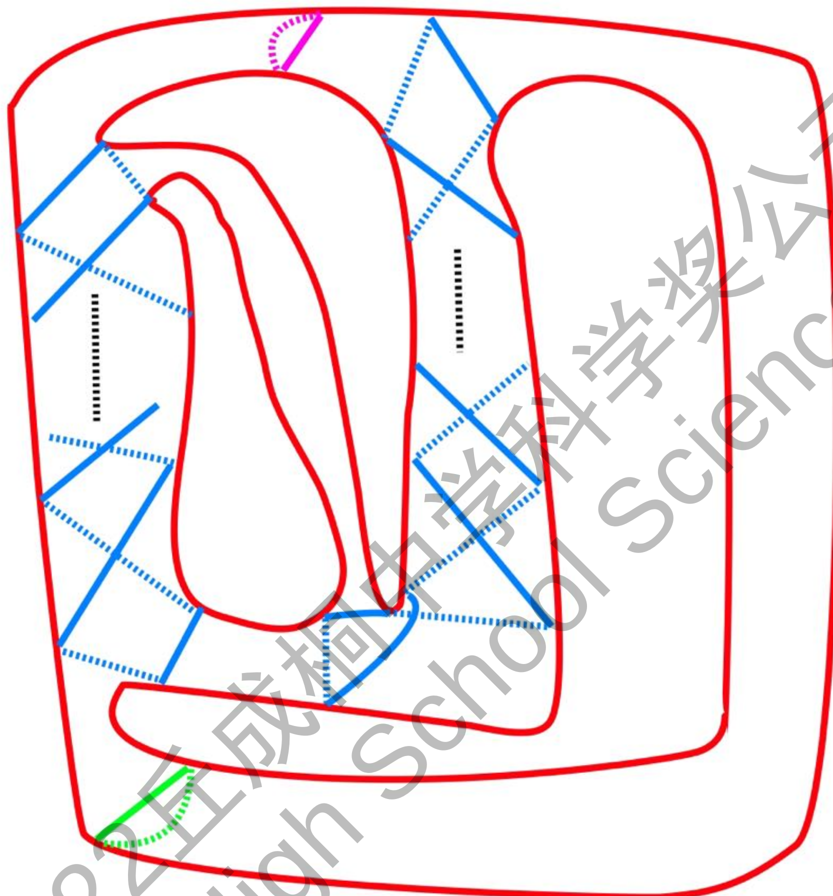


FIGURE 36.

Definition 4.7. $R(c)$ is the Alexander polynomial of the left diagram of Figure 46 where the number of crossing in the blue block in Figure 47 is denoted by c for some $c \in \mathbb{N}$.

Then, by applying the two axioms of Alexander polynomial, we can obtain the following equation by denoting $A = t^{\frac{1}{2}} - t^{\frac{-1}{2}}$.

$$\Delta_K(R(c)) = -A \cdot \Delta_K(R(c-1)) + \Delta_K(R(c-2)) \tag{4.15}$$

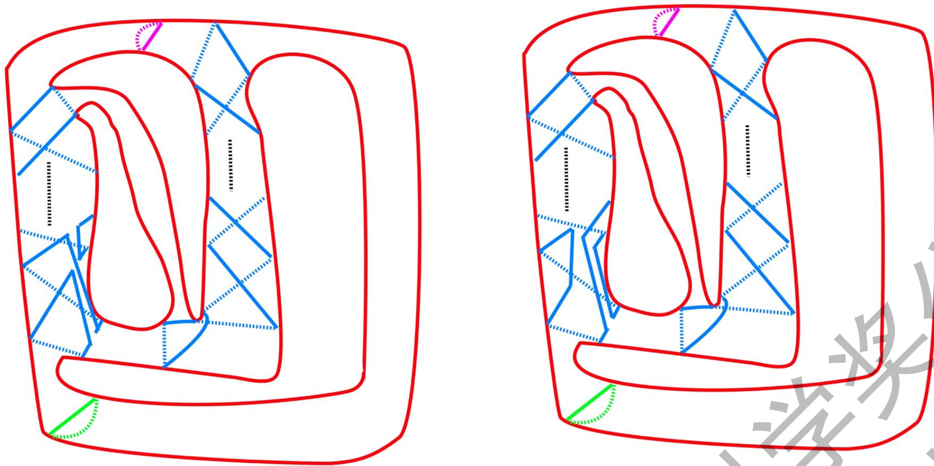


α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 37.

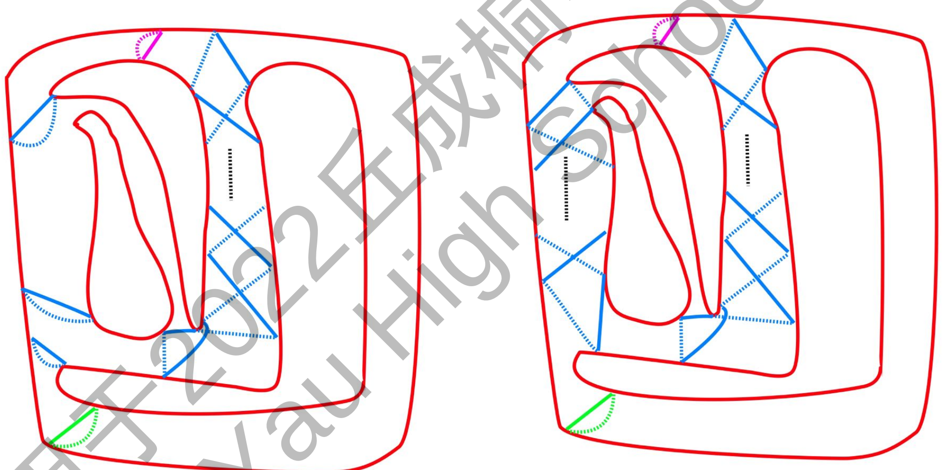
By a simple computation we can tell that $R(1) = -A, R(2) = A^2$. Hence, we can obtain the general term for the above equation by using characteristic root method.

$$R(c) = \frac{(t^{-\frac{1}{2}})^{c+1} - (-t^{\frac{1}{2}})^{c+1}}{t^{\frac{1}{2}} + t^{-\frac{1}{2}}} \tag{4.16}$$



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

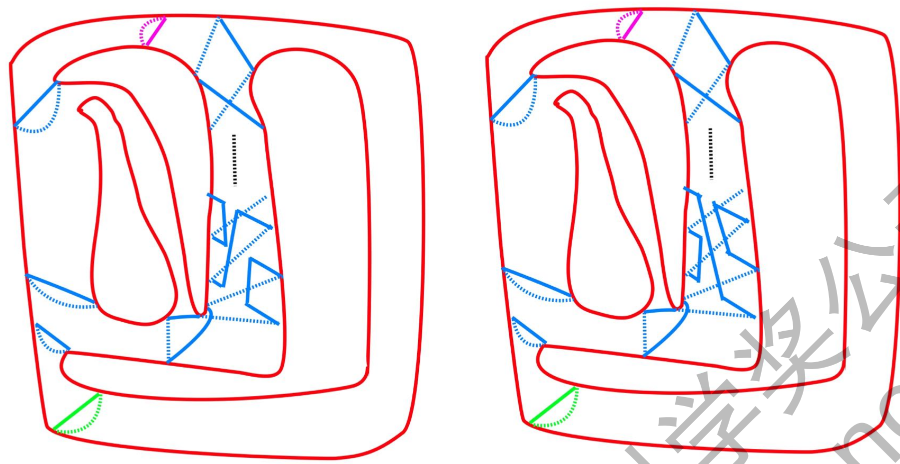
FIGURE 38.



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

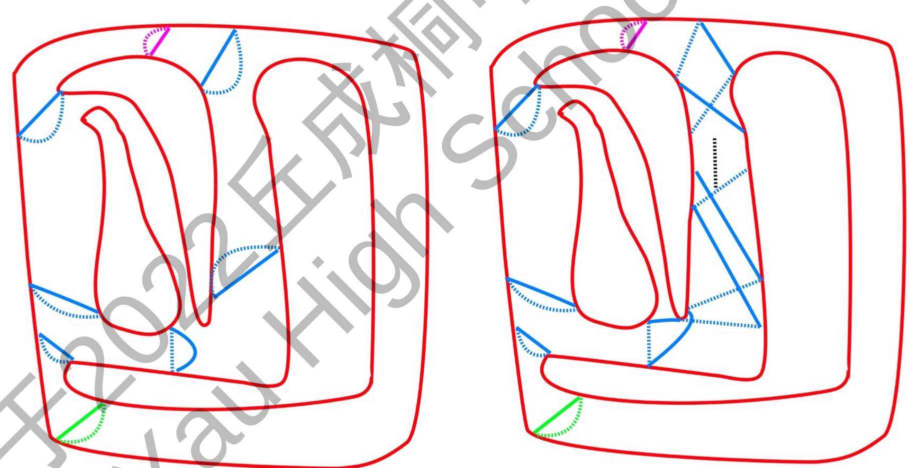
FIGURE 39.

Notice that we only consider $R(c)$ when $c = 2m+1$ for the deduction of the Alexander polynomial for $P(n, -m, 0)$, so when $c = 2m + 1$, namely c is odd and $c + 1$ is even, we have



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

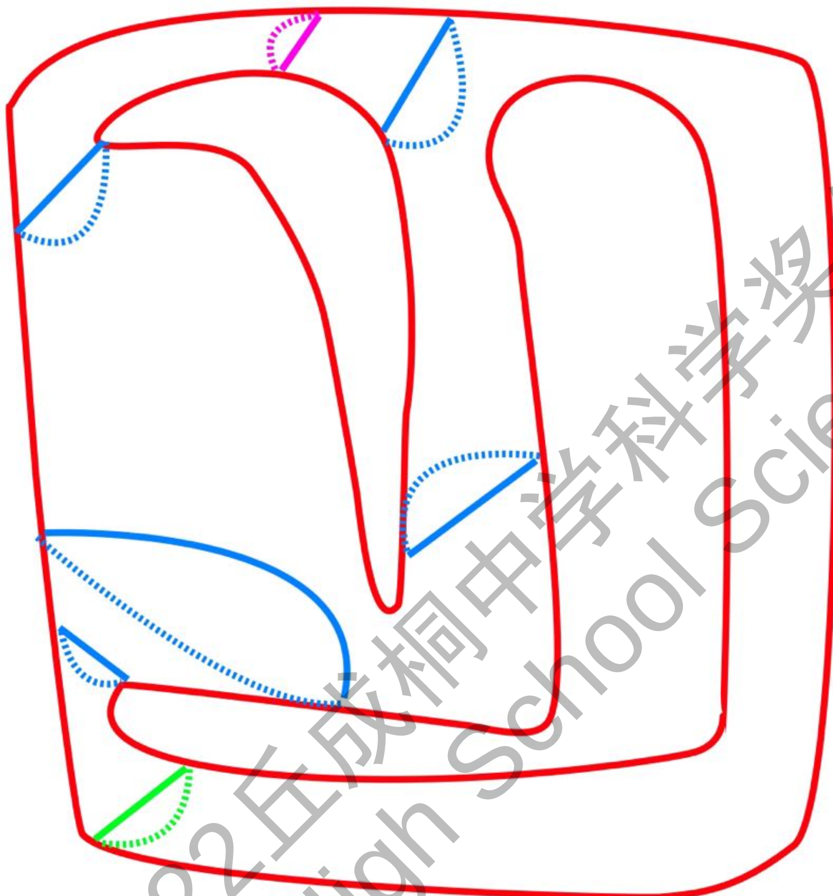
FIGURE 40.



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 41.

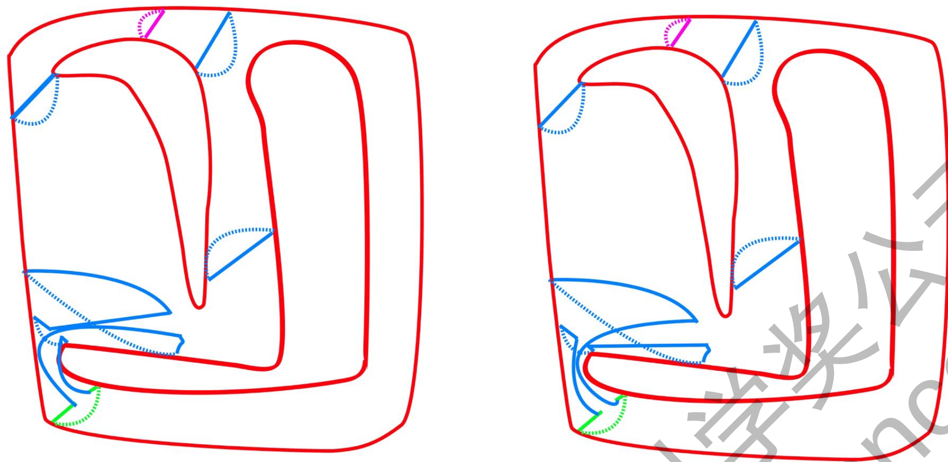
$$R(2m+1) = \frac{(t^{m+1}) + (t^{-m-1})}{t^{\frac{1}{2}} + t^{-\frac{1}{2}}}$$



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

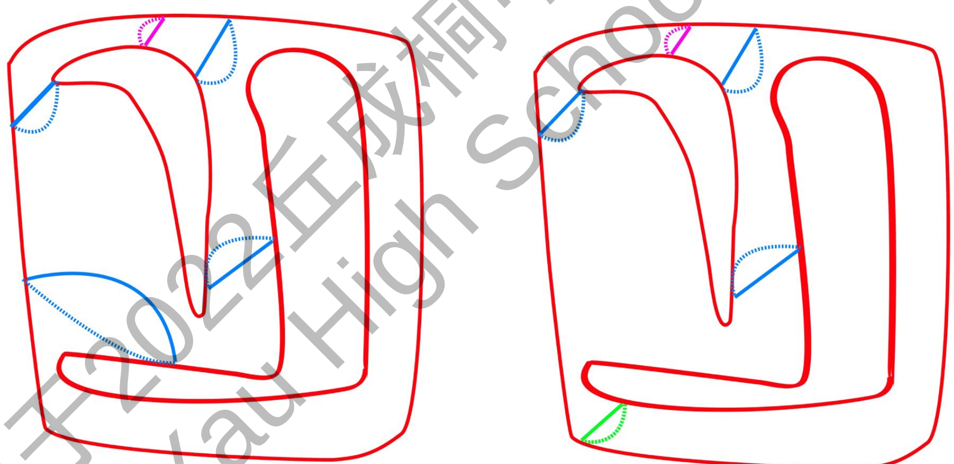
FIGURE 42.

(4.17)



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 43.

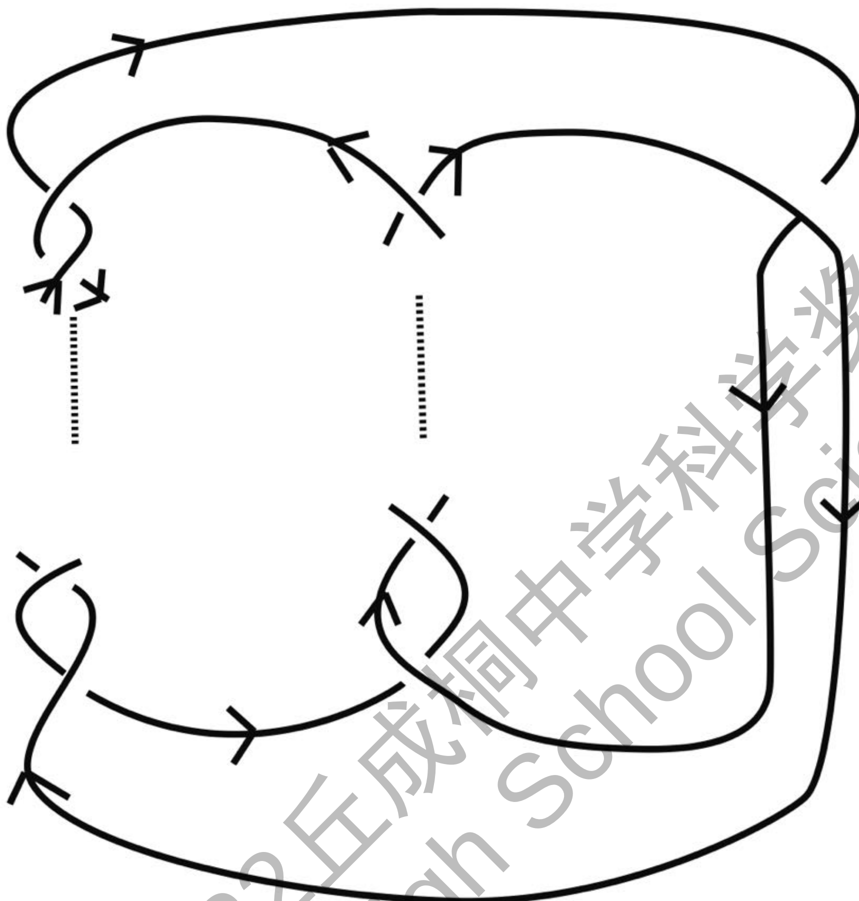


α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 44.

As we have the deduction based on two axioms of Alexander polynomial,

$$\Delta_K(P(n,-m,0) = \Delta_K(P(n-1,-m,0)) - \frac{((t^{m+\frac{3}{2}}) - (t^{-m-\frac{3}{2}})) - ((t^{m+\frac{1}{2}}) - (t^{-m-\frac{1}{2}})))}{t^{\frac{1}{2}} + t^{-\frac{1}{2}}} \tag{4.18}$$



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 45.

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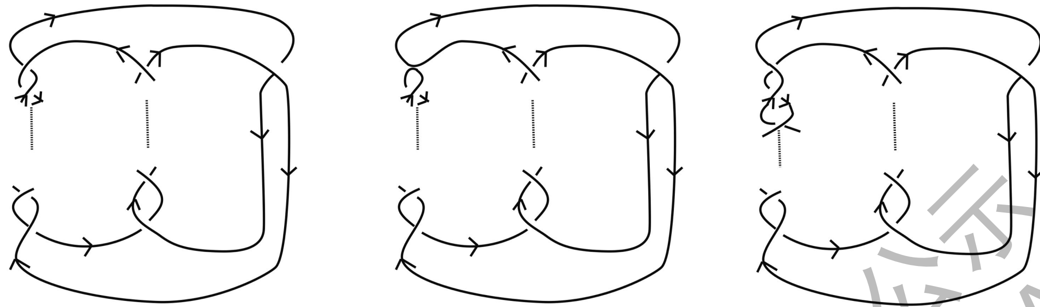
In computing $\Delta_K(P(1,-m,0))$, we find that the $\Delta_K(K_0)$ is $R(2m+1)$ and K_+ is equivalent to torus knot $T(2,2m+1)$ as in Figure



K_- K_0 K_+

FIGURE 46.

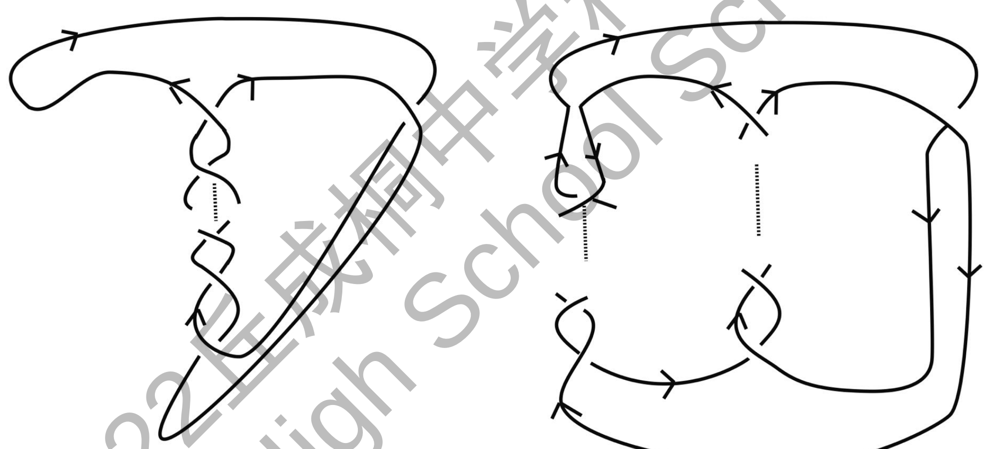


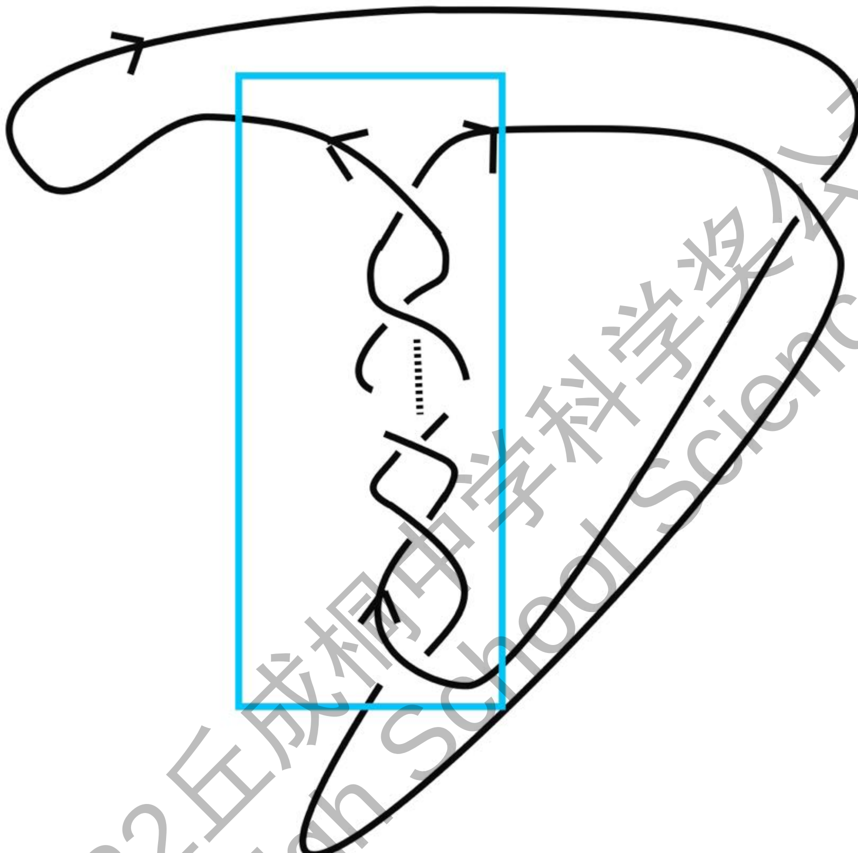
FIGURE 47.

According to [[Rol90]],

$$\Delta_K(T(2,2m+1)) = \frac{t^{2m+2}+1}{t+1} \tag{4.19}$$

As the Alexander polynomial for $P(1,0,0)$ is $t^1 - 1 + t^{-1}$, we have the following equation

$$\Delta_K(P(1,-m,0)) = t^1 - 1 + t^{-1} + \frac{t^{2m+2}+1}{t+1} + \frac{((t^{m+\frac{3}{2}}) - (t^{-m-\frac{3}{2}})) - ((t^{m+\frac{1}{2}}) - (t^{-m-\frac{1}{2}})))}{t^{\frac{1}{2}} + t^{-\frac{1}{2}}} \tag{4.20}$$



α curve (red);meridian (green); second γ curve (blue); last γ curve (purple)

FIGURE 48.

Therefore, based on the above equations, we have:

(4.21)

$$\Delta_K(P(n,-m,0) = t^1 - 1 + t^{-1} + \frac{t^{2m+2}+1}{t+1} + (\frac{((t^{m+\frac{3}{2}}) - (t^{-m-\frac{3}{2}})) - ((t^{m+\frac{1}{2}}) - (t^{-m-\frac{1}{2}}))}{t^{\frac{1}{2}} + t^{-\frac{1}{2}}}) \cdot n$$

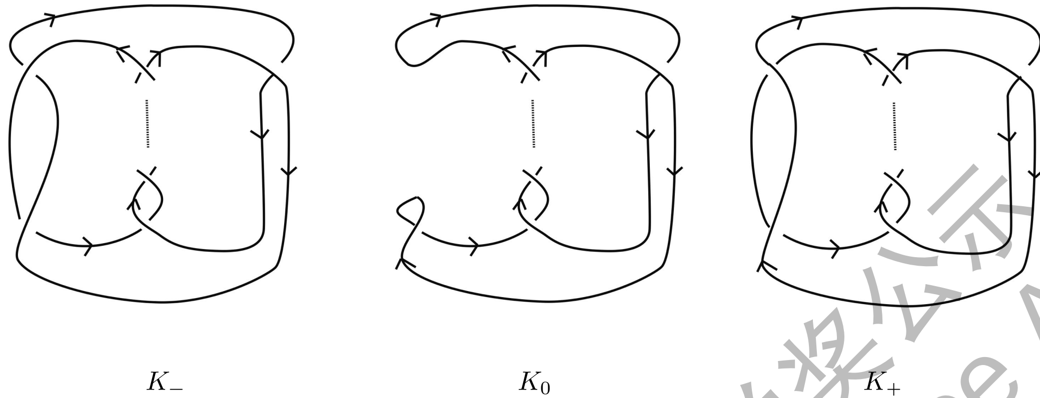


FIGURE 49.

After the reduction, we find that, based on theorem 4.1,

$$\sum_{i\in\mathbb{Z}} |a_i| = 4mn + 4n - 1 \tag{4.22}$$

Hence, we proved that

$$L(n, -m, 0) = 4mn + 4n - 1 \tag{4.23}$$

□

Based on the above proposition, since

$$U(n, -m, 0) \geqslant L(n, -m, 0) \tag{4.24}$$

and

$$U(n, -m, 0) \leqslant 4mn + 4n - 1 = L(n, -m, 0), \tag{4.25}$$

we can tell that

$$U(n, -m, 0) = L(n, -m, 0) = 4mn + 4n - 1 \tag{4.26}$$

Therefore, we proved that

$$\dim KHI(K) = 4mn + 4n - 1 \tag{4.27}$$

□

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