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Title: A Study of Non-coercive Problems Arising from the Propagation of Electromagnetic Waves in Metamaterials

A study of non-coercive problems arising from the propagation of electromagnetic waves in metamaterials

Xiaowei Shi

Abstract

In this paper, we are interested in some non-coercive problems arising from the propagation of electromagnetic waves in metamaterials. We apply a method called T-isomorphism, which can be used to transform non-coercive problems into coercive problems, to give the well-posedness results of some Laplace type equations located in different configurations with parameters that change sign.

Keywords: T-isomorphism, non-coercive problems, Laplace type equations

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1 Introduction

Partial Differential Equations (PDEs) represent a category of mathematical equations that involve unknown functions of multiple independent variables and their partial derivatives. PDEs serve as the mathematical foundation for modeling and analyzing various phenomena in multiple disciplines including physics, engineering, biology, and finance.

The variational calculus is a powerful tool for solving PDE problems. By translating PDEs into variational formulations in an appropriate framework, we can use some methods of functional analysis (e.g., the Lax-Milgram theorem) to prove the existence and the uniqueness of solutions of variational formulations. Moreover, we can also adapt the finite element method (FEM) to compute the approximate solution of the variational formulation and also give the corresponding error analysis under some prior conditions.

After reading some papers and books ([3], [4], [5], [8]), we are interested in some non-coercive problems arising in the study of electromagnetic wave propagation in the presence of metals or certain types of metamaterials. Generally speaking, in a material, variations in the electromagnetic field are governed by Maxwell's equations. These involve physical coefficients, dielectric permittivity ε and magnetic permeability μ, which characterize the properties of the medium. For metals, in certain frequency ranges, ε can be considered negative to a first approximation. Similarly, some metamaterials can be represented by negative ε and/or μ. Metamaterials are complex structures made up of resonators that are small in comparison to the wavelength chosen and arranged in such a way as to obtain a material that, at the macroscopic level, exhibits properties of interest for applications. In particular, physicists are conducting major research into the design of doubly negative metamaterials (negative ε and μ) that would exhibit a refractive index n that is itself negative. There are many quite extraordinary applications for these negative metamaterials, ranging from the realization of perfect lenses for observing very small objects, to the manufacture of photon traps.

From a mathematical point of view, the study of Maxwell's equations in media mixing classical positive and negative materials raises original questions. Indeed, it leads us to consider partial differential equations with parameters that change sign. These equations do not fit into the usual frameworks (non-coercive problems), so it is necessary to use some methods to transform these problems into problems that can be solved with the classical approach (Lax-Milgram theorem).

In this paper, we apply a method called T-isomorphism, which can be used to transform non-coercive problems into coercive problems, to study some Laplace type equations with parameters that change sign. More precisely, under some conditions of the contrast of permittivity or permeability, we adapt the T-isomorphism method to give the well-posedness result of the Laplace type equation with the homogeneous Dirichlet boundary condition located in the symmetrical configuration or in the configuration with a corner. Also, we study the Laplace type equation with the non-homogeneous Dirichlet boundary condition located in the symmetrical configuration, and we constructed the T-ismorphism in a different way than before to give the well-posedness result under some conditions of the contrast of permittivity or permeability.

2 Preliminaries

In this paper, Ω is an open set of $\mathbb{R}^N$ (bounded or unbounded), with its boundary denoted $\partial\Omega$. We sometimes assume that Ω is regular and bounded. It suffices to know that a regular open set is roughly an open set whose boundary is a regular hypersurface (a manifold of dimension $N-1$), and this set is locally located on one side of its boundary. We define the exterior normal to the boundary $\partial\Omega$ to be the unit vector $n = (n_i)_{1\leq i\leq N}$ normal to any point on the tangent plane of Ω and pointing to the exterior of Ω. In Ω we note dx the volume measure, or Lebesgue measure of dimension N. In $\partial\Omega$ we note ds the surface measure, or Lebesgue measure of dimension $N-1$ on the submanifold $\partial\Omega$.

2.1 Hilbert Space

Definition 2.1. *Given a real vector space V, a bilinear form b on V is a function of two variables $V \times V \longrightarrow \mathbb{R}$ satisfying the following for any scalar α and any choice of vectors v, w, v_1, v_2, w_1, w_2,*

$$\begin{aligned} b(\alpha v, w) &= b(v, \alpha w) = \alpha b(v, w), \\ b(v_1 + v_2, w) &= b(v_1, w) + b(v_2, w), \\ b(v, w_1 + w_2) &= b(v, w_1) + b(v, w_2). \end{aligned}$$

Definition 2.2. *A real inner product space is a vector space V over $\mathbb{R}$ with a bilinear form*

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$

such that for all vectors $x, y \in V$,

$$\langle x, y\rangle = \langle y, x\rangle,$$
$$\langle x, x\rangle > 0 \text{ for } x \neq 0.$$

$\langle \cdot, \cdot\rangle$ is called the inner product of V.

Definition 2.3. *We say the inner product space V equipped with the inner product $\langle \cdot, \cdot\rangle_V$ is a Hilbert space if it is complete for the associated norm $\| \cdot \|_V$, where*

$$\|v\|_V = \sqrt{\langle v, v\rangle_V}, \forall v \in V.$$

2.2 Lebesgue Space

Definition 2.4. *For $p \in [1, \infty)$, if $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ satisfies*

$$\int_\Omega |u|^p dx < +\infty,$$

then we call $u \in L^p(\Omega)$. Here $L^p(\Omega)$ is a function space equipped with the norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}.$$

We always consider the $L^2(\Omega)$ space and its subspaces in the following subsections. In reality, $L^2(\Omega)$ is also a Hilbert space with the inner product

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega uv dx.$$

2.3 Sobolev Space

Before introducing Sobolev spaces, we define the weak derivative and weak divergence in $L^2(\Omega)$.

Definition 2.5. *Let $v \in L^2(\Omega)$. We call v weakly derivable in $L^2(\Omega)$ if there exist $w_i \in L^2(\Omega)$ for $i \in \{1, \ldots, N\}$, such that for any $\varphi \in C_c^\infty(\Omega)$, we have*

$$\int_\Omega v(x)\frac{\partial \varphi}{\partial x_i}(x)dx = -\int_\Omega w_i(x)\varphi(x)dx.$$

Each w_i is called the i-th weak partial derivative of v and noted henceforth $\frac{\partial v}{\partial x_i}$.

With this definition, we now define the Sobolev space H^m with $m \in \mathbb{N}^+$.

Definition 2.6. *Let Ω be an open domain of $\mathbb{R}^N$. We define the Sobolev space $H^1(\Omega)$ by*

$$H^1(\Omega) = \Big\{ v \in L^2(\Omega) \text{ such that } \forall i \in \{1, \dots, N\},\ \tfrac{\partial v}{\partial x_i} \in L^2(\Omega) \Big\},$$

where $\frac{\partial v}{\partial x_i}$ is the ith weak partial derivative of v. We also define $H^m(\Omega)$ $(m \geq 2)$ by

$$H^m(\Omega) = \big\{ v \in L^2(\Omega) \text{ such that } \forall \alpha \text{ with } |\alpha| \leq m, \partial^\alpha v \in L^2(\Omega) \big\}$$

with

$$\partial^\alpha v(x) = \tfrac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}(x),$$

where $\partial^\alpha v$ is taken in the weak sense. Here $\alpha = (\alpha_1, \dots, \alpha_N)$ is multi-index with $\alpha_i \geq 0$ and $|\alpha| = \sum_{i=1}^N \alpha_i$.

Proposition 2.7. *The Sobolev space $H^m(\Omega)$ is a Hilbert space with the scalar product*

$$\langle u, v \rangle = \int_\Omega \sum_{|\alpha| \leq m} \partial^\alpha u(x) \partial^\alpha v(x) dx \tag{2.1}$$

and the norm

$$\|u\|_{H^m(\Omega)} = \sqrt{\langle u, u \rangle} = \left(\int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha u(x)|^2 dx \right)^{\frac{1}{2}}. \tag{2.2}$$

Proof. We prove the case with $m = 1$ (for $m \geq 2$, the proof is similar). Recall that $L^2(\Omega)$ is a Hilbert space. It's obvious that (2.1) is indeed a scalar product in $H^1(\Omega)$, and it remains to show that $H^1(\Omega)$ is complete for the associated norm. Let $\{u_n\}_{n\geq 1}$ be a Cauchy sequence in $H^1(\Omega)$. By definition of the norm of $H^1(\Omega)$, $\{u_n\}_{n\geq 1}$ as well as $\{\frac{\partial u_n}{\partial x_i}\}_{n\geq 1}$ for $i \in \{1, \dots, N\}$ are Cauchy sequences in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete, there exist limits u and w_i such that u_n converges to u and $\{\frac{\partial u_n}{\partial x_i}\}_{n\geq 1}$ converges to w_i in $L^2(\Omega)$. By definition of the weak derivative of u_n, for any function $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega u_n(x) \frac{\partial \varphi}{\partial x_i}(x) dx = -\int_\Omega \frac{\partial u_n}{\partial x_i}(x) \varphi(x) dx. \tag{2.3}$$

Take the limit $n \to +\infty$ in (2.3) and we obtain

$$\int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x) dx = -\int_\Omega w_i(x) \varphi(x) dx,$$

which proves that u is weakly derivable and that w_i is the i-th weak partial derivative of u. Therefore, u belongs to $H^1(\Omega)$ and $\{u_n\}_{n\geq 1}$ converges to u in $H^1(\Omega)$. $\square$

We now introduce a important subspace of $H^1(\Omega)$, noted as $H_0^1(\Omega)$.

Definition 2.8. *Let Ω be a regular and bounded domain. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ consisting of functions which are null at the boundary $\partial\Omega$.*

Remark 2.9. *In fact, the Sobolev space $H_0^1(\Omega)$ is essentially the completion of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. But with the assumption of Ω and the trace theorems in (2.7) and (2.8), we can prove this statement is equivalent to Definition 2.8.*

We hope to simplify the norms for easier calculation. In order to achieve this, we introduce the definition of equivalent norms.

Definition 2.10. *Two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ defined on X are called equivalent if there exist positive real numbers C and D such that for all $x \in X$*

$$C\|x\|_\alpha \leq \|x\|_\beta \leq D\|x\|_\alpha.$$

We can now apply this definition to the norm of $H_0^1(\Omega)$.

Corollary 2.11. *If Ω is a regular and bounded domain, then the norm of $H_0^1(\Omega)$ can be simplified as*

$$\|v\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla v(x)|^2 dx\right)^{\frac{1}{2}}. \tag{2.4}$$

For $H_0^1(\Omega)$, the norm (2.4) is equivalent to the norm (2.2). To prove this, we introduce the Poincaré inequality.

Proposition 2.12. ***(Poincaré Inequality)*** *Let Ω be a regular and bounded domain. There exists a constant $C > 0$ such that for any function $v \in H_0^1(\Omega)$,*

$$\int_\Omega |v(x)|^2 dx \leq C \int_\Omega |\nabla v(x)|^2 dx. \tag{2.5}$$

Using Poincaré inequality, we can easily prove Corollary 2.11. Before proving Poincaré inequality, we introduce the Rellich–Kondrachov theorem.

Theorem 2.13. ***(Rellich–Kondrachov Theorem)*** *If Ω is a regular and bounded domain, then for any bounded sequence in $H^1(\Omega)$, we can extract a convergent subsequence in $L^2(\Omega)$.*

Then we can use this theorem to prove Poincaré inequality.

Proof. We prove by contradiction. If there is no constant $C > 0$ such that, for any function $v \in H_0^1(\Omega)$,

$$\int_\Omega |v(x)|^2 dx \leq C \int_\Omega |\nabla v(x)|^2 dx.$$

This means that there exists a sequence $v_n \in H_0^1(\Omega)$ such that

$$1 = \int_\Omega |v_n(x)|^2 \, dx > n \int_\Omega |\nabla v_n(x)|^2 \, dx. \tag{2.6}$$

In particular, (2.6) implies that the sequence v_n is bounded in $H_0^1(\Omega)$. Rellich–Kondrachov theorem shows that there exists a sub-sequence $v_{n'}$ that converges in $L^2(\Omega)$. Moreover, (2.6) shows that the sequence $\nabla v_{n'}$ converges to zero in $L^2(\Omega)$ (component by component). Therefore, $v_{n'}$ is a Cauchy sequence in $H_0^1(\Omega)$, which is a Hilbert space, so it converges in $H_0^1(\Omega)$ to a limit v. Since we have

$$\int_\Omega |\nabla v(x)|^2 dx = \lim_{n' \to +\infty} \int_\Omega |\nabla v_{n'}(x)|^2 \, dx \leq \lim_{n' \to +\infty} \frac{1}{n'} = 0,$$

we can deduce that v is a constant in every connected component of Ω. But since v is zero on the boundary $\partial\Omega$, v is identically zero in all Ω. Moreover,

$$\int_\Omega |v(x)|^2 dx = \lim_{n' \to +\infty} \int_\Omega |v_{n'}(x)|^2 \, dx = 1,$$

which is a contradiction with $v = 0$. □

Remark 2.14. *Generally, this proof by contradiction can be used to prove (2.5) for functions in a subspace of $H^1(\Omega)$ which are null at parts of the boundary.*

Because $\partial\Omega$ is a set of zero measure, it's not clear whether one can define the boundary value, or trace of v on the boundary $\partial\Omega$. Fortunately, there is still a way to define the trace $v|_{\partial\Omega}$ of a function in $H^1(\Omega)$. These essential results are demonstrated by the trace theorems.

Theorem 2.15. ***(Trace Theorem** H^1**)** *Let Ω be a regular and bounded domain. We define the trace application γ_0*

$$\begin{aligned} H^1(\Omega) \cap C^1(\bar{\Omega}) &\to L^2(\partial\Omega) \cap C(\overline{\partial\Omega}) \\ v &\to \gamma_0(v) = v|_{\partial\Omega} \,. \end{aligned}$$

This application γ_0 extends by continuity into a continuous linear application from $H^1(\Omega)$ into $L^2(\partial\Omega)$, denoted again as γ_0. In particular, there exists a constant $C > 0$ such that for any function $v \in H^1(\Omega)$, we have

$$\|v\|_{L^2(\partial\Omega)} \leq C \|v\|_{H^1(\Omega)}. \tag{2.7}$$

Theorem 2.16. ***(Trace Theorem*** H^2***)*** *Let Ω be a regular and bounded domain. We define the trace application γ_1*

$$H^2(\Omega) \cap C^1(\bar{\Omega}) \to L^2(\partial\Omega) \cap C(\overline{\partial\Omega})$$
$$v \to \gamma_1(v) = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega}$$

with $\frac{\partial v}{\partial n} = \nabla u \cdot n$. This application γ_1 extends by continuity into a continuous linear application from $H^2(\Omega)$ to $L^2(\Omega)$. In particular, there exists a constant $C > 0$ such that for any function $v \in H^2(\Omega)$, we have

$$\left\| \frac{\partial v}{\partial n} \right\|_{L^2(\partial\Omega)} \leq C \|v\|_{H^2(\Omega)}. \tag{2.8}$$

Using the trace theorems and the density of $C_c^\infty(\bar{\Omega})$ in $H^1(\Omega)$ and $H^2(\Omega)$, we have Green's formulas.

Theorem 2.17. ***(Green's Formula)*** *Let Ω be a regular and bounded domain. If u and v are functions of $H^1(\Omega)$, then we have*

$$\int_\Omega u(x) \frac{\partial v}{\partial x_i}(x) dx = -\int_\Omega v(x) \frac{\partial u}{\partial x_i}(x) dx + \int_{\partial\Omega} u(x) v(x) n_i(x) ds. \tag{2.9}$$

Moreover, if $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have

$$\int_\Omega \Delta u(x) v(x) dx = -\int_\Omega \nabla u(x) \cdot \nabla v(x) dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}(x) v(x) ds. \tag{2.10}$$

For the variational formulations of PDEs, we have a useful tool (Lax-Milgram theorem) to analyze their well-posedness using three assumptions.

2.4 Lax-Milgram Theorem

Most PDEs can be rewritten using (2.9) and (2.10), giving its variational formulation. In fact, the well-posedness of a PDE's variational formulation in a Hilbert space is usually equivalent to the well-posedness of the original PDE.

Remark 2.18. *Different partial differential equations with boundary conditions correspond to different variational formulations. To get the corresponding variational formulation, we roughly take the following steps: we first find a Hilbert space V, multiply both sides of the partial differential equation by $v \in V$ and integrate, then use Green's formula to reduce the differential order of the integral equation by one, and finally obtain the following formulation:*

$$\text{Find } u \in V \text{ such that } a(u, v) = L(v) \text{ for all } v \in V. \tag{2.11}$$

where $a(\cdot,\cdot)$ is a bilinear form on V and $L(\cdot)$ is a linear form on V. The solution of the variational formulation is also called the weak solution of the corresponding partial differential equation.

Theorem 2.19. ***(Lax-Milgram Theorem)*** *Let V be a real Hilbert space,*
1. $a(\cdot,\cdot)$ is a continuous bilinear form on V, i.e., there exists $M > 0$ such that

$$|a(w,v)| \leq M\|w\|_V\|v\|_V \text{ for all } w,v \in V. \tag{2.12}$$

2. $L(\cdot)$ is a continuous linear form on V, i.e., there exists $C > 0$ such that

$$|L(v)| \leq C\|v\|_V \text{ for all } v \in V. \tag{2.13}$$

3. $a(\cdot,\cdot)$ is coercive (or elliptic), i.e., there exists $\alpha > 0$ such that

$$a(v,v) \geq \alpha\|v\|_V^2 \text{ for all } v \in V. \tag{2.14}$$

then the variational formulation (2.11) has a unique solution in V.

3 Analysis of non-coercive problems

Consider $\Omega \subset \mathbb{R}^2$ a bounded domain with Lipschitz boundary $\partial\Omega$. We assume Ω partitioned into two subdomains Ω_1, Ω_2, such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. To model the mixing of positive and negative materials, we introduce the function $\sigma : \Omega \to \mathbb{R}$ such that $\sigma = \sigma_1$ in Ω_1 and $\sigma = \sigma_2$ in Ω_2. Here, σ_1, σ_2 are two constants with $\sigma_1 > 0, \sigma_2 < 0$. We are interested in the following non-coercive problem.

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } -\operatorname{div}(\sigma\nabla u) = f \quad \text{ in } \Omega. \tag{3.1}$$

Here, $f \in L^2(\Omega)$ denotes the source term. We also introduce the following notation

$$\kappa = \frac{\sigma_2}{\sigma_1}.$$

κ will be an important indicator in our analysis of the problem. We will consider the well-posedness of (3.1) in two different domains. For our analysis, we consider the variational formulation corresponding to (3.1) in the following form.

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u,v) = L(v), \quad \forall v \in H_0^1(\Omega) \tag{3.2}$$

where

$$a(u,v) = \sigma_1 \int_{\Omega_1} \nabla u \cdot \nabla v dx + \sigma_2 \int_{\Omega_2} \nabla u \cdot \nabla v dx, \quad L(v) = \int_\Omega f v dx.$$

Before analyzing the problem with the above constants, we resolve the case when both constants σ_1, σ_2 are positive.

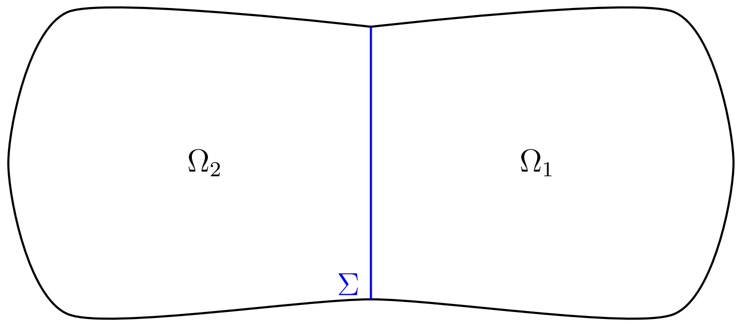

Figure 1: Example of a symmetrical configuration with respect to (Oy) axis

Proposition 3.1. *(3.2) is well-posed if $\sigma_1, \sigma_2 > 0$.*

Proof. We apply the Lax-Milgram theorem. By Cauchy-Schwarz inequality,

1. $|a(w,v)| = \left|\sigma \int_\Omega \nabla w \cdot \nabla v dx\right| \leq \max(\sigma_1,\sigma_2)\|\nabla w\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$
$$= \max(\sigma_1,\sigma_2)\|w\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}, \quad \forall w, v \in H_0^1(\Omega).$$
2. $|L(v)| = \left|\int_\Omega f v dx\right| \leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq C\|v\|_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$
3. $a(v,v) = \sigma \int_\Omega |\nabla v|^2 dx \geq \min(\sigma_1,\sigma_2)\|v\|^2_{H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$

By Theorem 2.19, the variational formulation is well-posed. □

Remark 3.2. *Evidently, $a(\cdot,\cdot)$ is not coercive when $\kappa < 0$. However, the Lax-Milgram theorem is sufficient but not necessary for a PDE to be well-posed. We will try techniques to analyze the well-posedness of (3.2) on different domains.*

3.1 Ill-posedness on symmetric configurations

We consider some simple configurations for which (3.2) is ill-posed. Consider an open set symmetric about the axis (Oy) verifying $\Omega = \{(x,y) \in \Omega | (-x,y) \in \Omega\}$ (see Figure 1). Let us define $\Omega_1 := \{(x,y) \in \Omega | x > 0\}$ and $\Omega_2 := \{(x,y) \in \Omega | x < 0\}$. In this configuration, the interface Σ verifies $\Sigma = \{(x,y) \in \Omega | x = 0\}$. If φ is a function of $L^2(\Omega)$, we denote $\varphi_1 := \varphi|_{\Omega_1}$ and $\varphi_2 := \varphi|_{\Omega_2}$. We now investigate a similar problem on one side of this open set. Let g be an element of $C_c^\infty(\Sigma)$. Then we have the following problem.

$$
\begin{aligned}
\text{Find } v_1 \in\ & H^1(\Omega_1) \text{ such that} \\
-\sigma_1 \nabla v_1 &= 0 \text{ in } \Omega_1, \\
v_1 &= 0 \text{ on } \partial\Omega_1 \setminus \bar{\Sigma}, \\
v_1 &= g \text{ on } \Sigma.
\end{aligned} \tag{3.3}
$$

Lemma 3.3. *The problem (3.3) has an unique solution.*

Proof. Define the domain $\Gamma_1 = \partial\Omega_1 \setminus \bar{\Sigma}$ and the corresponding Hilbert space $H^1_{0,\Gamma_1}(\Omega_1) := \{v \in H^1(\Omega_1) | v = 0 \text{ on } \Gamma_1\}$. Then we have the following variational formulation corresponding to (3.3).

$$
\begin{gathered}
\text{Find } v_1 \in H^1_{0,\Gamma_1}(\Omega_1) \text{ such that} \\
\sigma_1 \int_{\Omega_1} \nabla v_1 \cdot \nabla w dx = \sigma_1 \int_{\Sigma} \frac{\partial g}{\partial n} w dx \quad \forall w \in H^1_{0,\Gamma_1}(\Omega_1).
\end{gathered}
$$

To prove the assumptions of continuous linear and bilinear forms, we reference the technique used in Proposition 3.1. Notice that due to Remark 2.14, (2.6) is applicable for $H^1_{0,\Gamma_1}(\Omega_1)$. It's now evident that there exists constants $C, D > 0$ such that for all $u \in H^1_{0,\Gamma_1}(\Omega_1)$,

$$
C\|u\|_{H^1_0(\Omega_1)} \leq \|u\|_{H^1_{0,\Gamma_1}(\Omega_1)} \leq D\|u\|_{H^1_0(\Omega_1)}.
$$

By Definition 2.10, the norms $\|\cdot\|_{H^1_0(\Omega_1)}$ and $\|\cdot\|_{H^1_{0,\Gamma_1}(\Omega_1)}$ are equivalent. Now it's easy to prove that $a(\cdot,\cdot)$ is coercive. Hence, by Theorem 2.19, problem (3.3) has an unique solution. □

Define the function v_2 on Ω such that $v_2(x, y) = v_1(-x, y)$. Now we consider the following problem.

$$
\begin{aligned}
\text{Find } (u_1, u_2) \in H^1(\Omega_1) \times H^1(\Omega_2) & \text{ such that} \\
-\sigma_1 \nabla u_1 &= 0 \text{ in } \Omega_1, \\
-\sigma_2 \nabla u_2 &= 0 \text{ in } \Omega_2, \\
u_1 - u_2 &= 0 \text{ on } \Sigma := (\bar{\Omega}_1 \cap \bar{\Omega}_2), \\
\sigma_1 \frac{\partial u_1}{\partial n} - \sigma_2 \frac{\partial u_2}{\partial n} &= 0 \text{ on } \Sigma, \\
u_1 &= 0 \text{ on } \partial\Omega \cap \partial\Omega_1, \\
u_2 &= 0 \text{ on } \partial\Omega \cap \partial\Omega_2.
\end{aligned} \tag{3.4}
$$

Corollary 3.4. *The pair (v_1, v_2) is a solution to (3.4) when $\kappa = -1$.*

Proof. Everything is obvious by definition except for $\sigma_1 \frac{\partial u_1}{\partial n} - \sigma_2 \frac{\partial u_2}{\partial n} = 0$ on Σ. By definition of exterior normal, $\frac{\partial u_1}{\partial n} = -\frac{\partial u_2}{\partial n}$. In addition, due to $\sigma_1 = -\sigma_2$, $\sigma_1 \frac{\partial u_1}{\partial n} - \sigma_2 \frac{\partial u_2}{\partial n} = \sigma_1 \frac{\partial u_1}{\partial n} - \sigma_1 \frac{\partial u_1}{\partial n} = 0$. $\square$

Theorem 3.5. *For the geometry considered, (3.2) is not well-posed for $\kappa = -1$.*

Proof. Notice that when $f = 0$, if a pair (u_1, u_2) verifies problem (3.4), then the function u such that $u = u_1$ in Ω_1 and $u = u_2$ in Ω_2 is a solution to (3.2). On the other hand, (3.3) implies that there exists u not always equal to 0 on Ω which satisfies (3.2). Since $u \equiv 0$ is also a solution to (3.2), there is more than one solution to the problem, meaning that (3.2) is not well-posed. Hence our proof is complete. $\square$

3.2 Application of the T-isomorphism method

Obviously, we hope to determine conditions for which (3.2) is well-posed. Since we cannot directly apply the Lax-Milgram theorem, we think of devising techniques to transform the problem into one which can be analyzed using the Lax-Milgram theorem, i.e., to prove the equivalence of its well-posedness with the well-posedness of a more ideal problem. In reality, the non-coerciveness of $a(\cdot, \cdot)$ prevented us from applying the Lax-Milgram theorem, hence it's a suitable starting point for discussion.

Lemma 3.6. *Suppose there exists an isomorphism T of $H_0^1(\Omega)$ such that the bilinear form $(u, v) \mapsto a(u, Tv)$ is coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$. In this case, (3.2) is well-posed.*

Proof. By definition, T is a continuous bijective operator from $H_0^1(\Omega)$ to $H_0^1(\Omega)$. Consider the following variant of (3.2):

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u, Tv) = L(Tv), \quad \forall v \in H_0^1(\Omega). \tag{3.5}$$

If (3.5) is well-posed, since for all $u \in H_0^1(\Omega)$, there always exists $v \in H_0^1(\Omega)$ such that $Tv = u$, (3.2) must also be well-posed. On the other hand, by definition of continuity, for all $u \in H_0^1(\Omega)$ there exists a constant $C > 0$ such that

$$\|Tu\|_{H_0^1(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}.$$

In addition, using the same methods as before, we prove the two assumptions for $a(\cdot, T\cdot)$ and $L(T\cdot)$, which proves that (3.5) is indeed well-posed. $\square$

Remark 3.7. *We wish for the operator T to compensate for the change in sign of σ. Indeed, we want to take $Tu = u$ in Ω_1 and $Tu = -u$ in Ω_2. This leads to*

$$a(u, Tu) = \int_\Omega |\sigma||\nabla u|^2 \geq \min(\sigma_1, |\sigma_2|)\|u\|^2_{H^1_0(\Omega)}.$$

However, this choice of T is not an isomorphism. Hence we cannot deduce the necessary properties for the well-posedness of two problems to be equivalent.

In search for a suitable isomorphism, we introduce the application R such that $(R\varphi)(x, y) = \varphi(-x, y)$ for all $\varphi \in H^1_0(\Omega)$. Recall that $\varphi_1 = \varphi|_{\Omega_1}$. Utilizing R, we introduce the operator T given by

$$T\varphi := \begin{cases} \varphi & \text{in } \Omega_1 \\ -\varphi + 2R\varphi_1 & \text{in } \Omega_2 \end{cases}. \tag{3.6}$$

We now have the following result.

Corollary 3.8. *The operator T is an isomorphism of $H^1_0(\Omega)$.*

Proof. We observe the properties of $(T \circ T)\varphi$,

$$(T \circ T)\varphi : = \begin{cases} T\varphi & \text{in } \Omega_1 \\ -(T\varphi) + 2R(T\varphi)_1 & \text{in } \Omega_2 \end{cases},$$

$$= \begin{cases} \varphi & \text{in } \Omega_1 \\ \varphi & \text{in } \Omega_2 \end{cases}.$$

Hence we see that $T \circ T = \mathbf{Id}$, implying that T is bijective from $H^1_0(\Omega)$ to $H^1_0(\Omega)$. To prove that the weak partial derivative of $T\varphi$ exists on the interface, we take the limit as x approaches 0:

$$\begin{aligned} \lim_{x\to 0^-} T\varphi(x, y) &= -\lim_{x\to 0^-} \varphi(x, y) + 2\lim_{x\to 0^-} R\varphi_1(x, y) \\ &= -\lim_{x\to 0^-} \varphi(x, y) + 2\lim_{x\to 0^+} \varphi_1(x, y). \end{aligned}$$

The desired result immediately follows due to the continuity of φ. In addition, for all $u \in H^1_0(\Omega)$,

$$\begin{aligned} \int_\Omega |\nabla(Tu)|^2 dx &= \int_{\Omega_1} |\nabla u|^2 dx + \int_\Sigma |\nabla u|^2 dx \int_{\Omega_2} |\nabla(-u + 2Ru_1)|^2 dx \\ &\leq C \int_\Omega |\nabla u|^2 dx. \end{aligned}$$

Hence, T is continuous on $H^1_0(\Omega)$, implying that T is an isomorphism of $H^1_0(\Omega)$. $\square$

Theorem 3.9. *(3.2) is well-posed when $-1 < \kappa < 0$.*

Proof. By definition, for all $u \in H_0^1(\Omega)$,

$$\begin{aligned}
a(u, Tu) &= \sigma_1 \int_{\Omega_1} |\nabla u_1|^2 dx + \sigma_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla(Tu_2) dx \\
&= \sigma_1 \int_{\Omega_1} |\nabla u_1|^2 dx + \sigma_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla(-u_2 + 2Ru_1) dx \\
&= \sigma_1 \int_{\Omega_1} |\nabla u_1|^2 dx - \sigma_2 \int_{\Omega_2} |\nabla u_2|^2 dx + 2\sigma_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla(Ru_1) dx.
\end{aligned}$$

Applying the Cauchy-Schwartz inequality backwards, we get

$$\begin{aligned}
a(u, Tu) &\geq \int_{\Omega} |\sigma| |\nabla u|^2 dx - 2|\sigma_2| \|\nabla u_2\|_{L^2(\Omega_2)} \|\nabla(Ru_1)\|_{L^2(\Omega_2)} \\
&\geq \int_{\Omega} |\sigma| |\nabla u|^2 dx - 2|\sigma_2| \|\nabla u_2\|_{L^2(\Omega_2)} \|R\| \|\nabla u_1\|_{L^2(\Omega_1)}.
\end{aligned}$$

Here, $\|R\| = \sup_{u_1 \in H^1_{0,\Sigma}(\Omega_1)} \frac{\|\nabla(Ru_1)\|_{L^2(\Omega_2)}}{\|\nabla u_1\|_{L^2(\Omega_1)}} = 1$. Hence

$$a(u, Tu) \geq \int_{\Omega} |\sigma| |\nabla u|^2 dx - 2|\sigma_2| \|\nabla u_2\|_{L^2(\Omega_2)} \|\nabla u_1\|_{L^2(\Omega_1)}.$$

By application of Young's inequality, for all $\eta > 0$,

$$2|\sigma_2| \|\nabla u_2\|_{L^2(\Omega_2)} \|\nabla u_1\|_{L^2(\Omega_1)} \leq |\sigma_2| \left(\eta \|\nabla u_2\|^2_{L^2(\Omega_2)} + \eta^{-1} \|\nabla u_1\|^2_{L^2(\Omega_1)} \right).$$

Hence,

$$a(u, Tu) \geq \left(\sigma_1 - \eta^{-1}|\sigma_2|\right) \int_{\Omega_1} |\nabla u_1|^2 dx + |\sigma_2| (1 - \eta) \int_{\Omega_2} |\nabla u_2|^2 dx.$$

Notice that $a(u, Tu)$ is coercive if

$$\begin{cases} |\sigma_1| - \eta^{-1}|\sigma_2| > 0 & \Rightarrow \eta > \frac{|\sigma_2|}{\sigma_1} \\ 1 - \eta > 0 & \Rightarrow \eta < 1 \end{cases}.$$

Hence, by Lemma 3.6, (3.2) is well-posed when there exists $\frac{|\sigma_2|}{\sigma_1} < \eta < 1$. In our case, η exists iff $-1 < \frac{\sigma_2}{\sigma_1} = \kappa < 0$, hence proved. □

Now it is known that (3.2) is well-posed when $-1 < \kappa < 0$. Considering the symmetry of Ω, we multiply both sides of the variational formulation (3.2) by -1. Then, using the same technique as above with the isomorphism T, we obtain the following general result.

Corollary 3.10. *Problem (3.2) is well-posed iff $\kappa \neq -1$.*

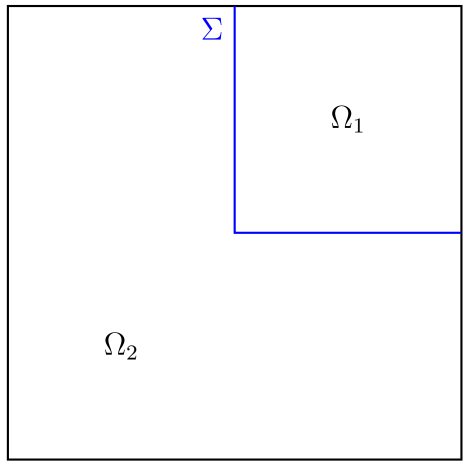

Figure 2: Configuration with a corner

3.3 Analysis on configuration with a corner

We now consider the well-posedness of (3.2) on an open Ω with a corner, inspired by the symmetric configuration introduced in (3.3). Let us define the domains $\Omega = (-1,1)\times(-1,1), \Omega_1 = (0,1)\times(0,1), \Omega_2 = \Omega \setminus \bar{\Omega}_1$ (see Figure 2). Due to the similarities with the symmetric configuration, we take a similar approach. In search for a new isomorphism, we choose the application R such that $(R\varphi)(x,y) = \varphi(|x|,|y|)$ for all $\varphi \in H_0^1(\Omega)$. We introduce the operator T given by

$$T\varphi := \begin{cases} \varphi & \text{in } \Omega_1 \\ -\varphi + 2R\varphi_1 & \text{in } \Omega_2 \end{cases}. \tag{3.7}$$

Corollary 3.11. *The operator T is an isomorphism of $H_0^1(\Omega)$.*

Proof. Referring to Corollary 3.8, we see that

$$(T \circ T)\varphi := \begin{cases} T\varphi & \text{in } \Omega_1 \\ -(T\varphi) + 2R(T\varphi)_1 & \text{in } \Omega_2 \end{cases},$$

$$= \begin{cases} \varphi & \text{in } \Omega_1 \\ \varphi & \text{in } \Omega_2 \end{cases}.$$

Hence we see that $T \circ T = \mathbf{Id}$, implying the desired result. ◻

Theorem 3.12. *(3.2) is well-posed when $\kappa \in (-\infty,-3)\cup(-\frac{1}{3},0)$.*

Proof. Note $\|R\| = \sup_{u_1 \in H^1_{0,\Sigma}(\Omega_1)} \frac{\|\nabla(Ru_1)\|_{L^2(\Omega_2)}}{\|\nabla u_1\|_{L^2(\Omega_1)}} = \sqrt{3}$, then we proceed identically to Theorem 3.9. □

3.4 Configurations with mixed boundary conditions

We consider the following configuration similar to the symmetric configuration introduced previously. Let Ω be the bounded and connected open of $\mathbb{R}^2$ defined by

$$\Omega = \{(x, y) \in \mathbb{R}^2 \text{ such that } -1 < x < 1 \text{ and } 0 < y < 1\}.$$

Correspondingly, we introduce the following domain,

$$\Omega_1 = (-1, 0) \times (0, 1), \Omega_2 = (0, 1) \times (0, 1)$$

so that $\bar{\Omega} = \bar{\Omega}_1 \cap \bar{\Omega}_2$. We will note $\Sigma = \{0\} \times (0, 1)$ the interface and $\Gamma = \{-1\} \times (0, 1)$ the left frontier of the domain. Given two real values $\sigma_1 > 0$ and $\sigma_2 < 0$, let σ be the function defined almost everywhere in Ω by $\sigma(x, y) = \sigma_j$ in Ω_j for $j = 1, 2$. Finally, we define the Hilbert subspace V of $H^1(\Omega)$ as follows,

$$V := \{v \in H^1(\Omega) \text{ such that } v = 0 \text{ on } \Gamma\}.$$

Then we consider the following variational problem.

$$\text{Find } u \in V(\Omega) \text{ such that } a(u, v) = L(v), \quad \forall v \in V(\Omega). \tag{3.8}$$

Lemma 3.13. *For the geometry considered, (3.8) is well-posed when $\frac{\sigma_2}{\sigma_1} > -1$.*

Proof. Referring to the proof of (3.3), it suffices to notice that for $u \in V$, $\|u\|_{V(\Omega)}$ and $\|\nabla u\|_{L^2(\Omega)}$ are equivalent norms, then we proceed to prove the three assumptions of the Lax-Milgram theorem. □

Of course, we wonder if we can apply the isomorphism method directly. We consider a linear operator R from V_2 into V_1 satisfying

$$Ru_2 = u_2 \text{ on } \Sigma, \quad \forall u_2 \in V_2$$

and such that

$$\int_{\Omega_1} |\nabla(Ru_2)|^2 dx \leq C \int_{\Omega_2} |\nabla u_2|^2 dx, \quad \forall u_2 \in V_2,$$

with C a constant independent of u_2.

Corollary 3.14. *The operator R does not exist.*

Proof. Take $u_2 \equiv 1$ in V_2. Then $\nabla(Ru_2) = 0$ which implies $Ru_2 \equiv 1$ on Ω_1, contradicting with the given boundary conditions. □

It's evident that we need a modified approach. For $u_2 \in V_2$, let $m(u_2)$ be its mean

$$m(u_2) = \int_{\Omega_2} u_2 dx$$

and let W_2 be the Hilbert subspace of V_2 consisting of zero-mean functions. With this definition, we have the following property.

Remark 3.15. *For $u_2 \in W_2$, we have the Poincaré inequality (2.6).*

Then, for $u_2 \in V_2$, let R be the operator defined by

$$Ru_2(x, y) = (1 + x)u_2(-x, y), \quad \forall (x, y) \in \Omega_1.$$

Now we have the following result.

Proposition 3.16. *For all $u_2 \in V_2$,*

$$\int_{\Omega_1} |\nabla(Ru_2)|^2 dx \le 2 \int_{\Omega_2} \left(|u_2|^2 + |\nabla u_2|^2\right) dx. \tag{3.9}$$

In addition, there exists a constant $C > 2$ such that

$$\int_{\Omega_1} |\nabla(R(u_2 - m(u_2)))|^2 dx \le C \int_{\Omega_2} |\nabla u_2|^2 dx. \tag{3.10}$$

Proof. For (3.9),

$$\begin{aligned}
\int_{\Omega_1} |\nabla(Ru_2)|^2 dx &= \int_{\Omega_1} |u_2(-x, y) + (1 + x)\nabla u_2(-x, y)|^2 dx \\
&\le \int_{\Omega_1} |u_2(-x, y) + \nabla u_2(-x, y)|^2 dx \\
&\le \int_{\Omega_2} |u_2 + \nabla u_2|^2 dx \\
&\le \int_{\Omega_2} ||u_2| + \nabla |u_2||^2 dx \\
&\le 2 \int_{\Omega_2} \left(|u_2|^2 + |\nabla u_2|^2\right) dx.
\end{aligned}$$

By (3.9),

$$\int_{\Omega_1} |\nabla(R(u_2 - m(u_2)))|^2 dx \le 2 \int_{\Omega_2} \left(|u_2 - m(u_2)|^2 + |\nabla(u_2 - m(u_2))|^2\right) dx.$$

Since $u_2 - m(u_2) \in W_2$, by (2.6) there exists $C > 2$ such that

$$\int_{\Omega_1} |\nabla(R(u_2 - m(u_2)))|^2 dx \le C \int_{\Omega_2} |\nabla(u_2 - m(u_2))|^2 dx$$
$$= C \int_{\Omega_2} |\nabla u_2|^2 dx.$$

□

We now consider constructing another operator T. For $u \in V$, we set

$$Tu = \begin{cases} u_1 - 2R(u_2 - m(u_2)) & \text{in } \Omega_1 \\ -u_2 + 2m(u_2) & \text{in } \Omega_2 \end{cases}. \tag{3.11}$$

Lemma 3.17. *The operator T is an isomorphism of V.*

Proof. As before,

$$\begin{aligned}(T \circ T)u := &\begin{cases} u_1 - 2R(u_2 - m(u_2)) - 2R(-u_2 + 2m(u_2) - m(u_2)) & \text{in } \Omega_1 \\ -(-u_2 + 2m(u_2)) + 2m(-u_2 + 2m(u_2)) & \text{in } \Omega_2 \end{cases} \\ = &\begin{cases} u_1 & \text{in } \Omega_1 \\ u_2 - 2m(u_2) + 2m(u_2) & \text{in } \Omega_2 \end{cases} \\ = &\begin{cases} u_1 & \text{in } \Omega_1 \\ u_2 & \text{in } \Omega_2 \end{cases}.\end{aligned}$$

Hence $T \circ T = \mathbf{Id}$, implying that T is bijective from V to V. To prove that the weak partial derivative of Tu exists on the interface, we take the limit $x \longrightarrow 0$,

$$\begin{aligned}\lim_{x \to 0^-} Tu(x,y) &= \lim_{x \to 0^-} u - 2 \lim_{x \to 0^-} R(u_2(x,y) - m(u_2)) \\ &= \lim_{x \to 0^-} u - 2 \lim_{x \to 0^-} (1+x)(u_2(-x,y) - m(u_2)) \\ &= \lim_{x \to 0^-} u - 2 \lim_{x \to 0^+} u_2(x,y) + 2m(u_2).\end{aligned}$$

The desired result follows immediately by continuity of u. Now we wish to prove that T is continuous on V. By definition,

$$\begin{aligned}\int_\Omega |\nabla(Tu)|^2 dx \le& \int_{\Omega_1} |\nabla u_1 \cdot \nabla(2R(u_2 - m(u_2)))|^2 dx + \int_\Sigma |\nabla(2m(u_2) - u)|^2 dx \\ &+ \int_{\Omega_2} |-\nabla u_2 + \nabla(2m(u_2))|^2 dx \\ \le& \int_{\Omega_1} |\nabla u_1|^2 + 4|\nabla(R(u_2 - m(u_2)))|^2 dx \\ &+ \int_\Sigma |\nabla u|^2 dx + \int_{\Omega_2} |\nabla u_2|^2 dx.\end{aligned}$$

Considering (3.10), there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla(Tu)|^2 dx \le C \int_\Omega |\nabla u|^2 dx.$$

Hence, T is an isomorphism of V. □

Theorem 3.18. *When $\frac{\sigma_2}{\sigma_1} < -C$, where C is the constant in (3.10), the variational problem (3.8) is well-posed.*

Proof. We refer to the proof in Theorem 3.9. By definition, for all $u \in V$,

$$\begin{aligned}a(u, Tu) =& \sigma_1 \int_{\Omega_1} \nabla u_1 \cdot (\nabla u_1 - \nabla(2R(u_2 - m(u_2))))dx \\ &+ \sigma_2 \int_{\Omega_2} \nabla u_2 \cdot (-\nabla u_2 + 2\nabla m(u_2))dx \\ =& \sigma_1 \int_{\Omega_1} \nabla u_1 \cdot (\nabla u_1 - 2\nabla(R(u_2 - m(u_2))))dx - \sigma_2 \int_{\Omega_2} |\nabla u_2|^2 dx \\ =& \sigma_1 \int_{\Omega_1} |\nabla u_1|^2 dx - \sigma_2 \int_{\Omega_2} |\nabla u_2|^2 dx \\ &- 2\sigma_1 \int_{\Omega_1} \nabla u_1 \cdot \nabla(R(u_2 - m(u_2)))dx.\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}a(u, Tu) \ge& \sigma_1 \int_{\Omega_1} |\nabla u_1|^2 dx - \sigma_2 \int_{\Omega_2} |\nabla u_2|^2 dx \\ &- 2\sigma_1 \|\nabla u_1\|_{L^2(\Omega_1)} \|\nabla R(u_2 - m(u_2))\|_{L^2(\Omega_1)}.\end{aligned}$$

By (3.10),

$$\begin{aligned}a(u, Tu) \ge& \sigma_1 \int_{\Omega_1} |\nabla u_1|^2 dx - \sigma_2 \int_{\Omega_2} |\nabla u_2|^2 dx \\ &- 2\sqrt{C}\sigma_1 \|\nabla u_1\|_{L^2(\Omega_1)} \|\nabla u_2\|_{L^2(\Omega_2)}.\end{aligned}$$

Considering Young's inequality,

$$a(u, Tu) \geq (1 - \delta\sqrt{C})\sigma_1 \int_{\Omega_1} u_1 dx - (\sigma_2 + \frac{1}{\delta}\sqrt{C}\sigma_1) \int_{\Omega_2} |\nabla u_2|^2 dx.$$

Notice that $a(u, Tu)$ is coercive if

$$\begin{cases} 1 - \delta\sqrt{C} > 0 & \Rightarrow \delta < \frac{1}{\sqrt{C}} \\ \sigma_2 + \frac{1}{\delta}\sqrt{C}\sigma_1 < 0 & \Rightarrow \delta > -\frac{\sqrt{C}\sigma_1}{\sigma_2}. \end{cases}$$

Hence, by Lemma 3.6, (3.8) is well-posed when there exists $-\frac{\sqrt{C}\sigma_1}{\sigma_2} < \delta < \frac{1}{\sqrt{C}}$. In our case, δ exists iff $\frac{\sigma_2}{\sigma_1} < -C$, hence proved. □

3.5 Further investigation

Having discussed the well-posedness of the problem, there are many questions left unanswered. In this subsection, we investigate some other characteristics of this problem.

Theorem 3.19. *In reality, for an open Ω not necessarily symmetric, if (3.1) is well-posed, then there exists an isomorphism T of $H_0^1(\Omega)$ such that $(u, v) \mapsto a(u, Tv)$ is coercive.*

Proof. Consider the properties of the operator $A : H_0^1(\Omega) \mapsto H_0^1(\Omega)$ defined by means of the Riesz representation theorem such that

$$\int_\Omega \nabla(Au) \cdot \nabla v dx = a(u, v), \quad \forall u, v \in H_0^1(\Omega).$$

Let u be the unique solution to (3.1). By definition,

$$\int_\Omega \nabla(Au) \cdot \nabla v dx = \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega).$$

Hence there exists $L \in H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega \nabla(Au) \cdot \nabla v dx &= \int_\Omega \nabla L \cdot \nabla v dx \\ &\leq \|\nabla L\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &= \|L\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}. \end{aligned}$$

Therefore, $v \mapsto \int_\Omega \nabla L \cdot \nabla v dx$ is a continuous linear form on $H_0^1(\Omega)$. Now, we define the operator $B : H_0^1(\Omega) \to H_0^1(\Omega)$ by

$$B : L \mapsto u_L.$$

Then for all $v \in H_0^1(\Omega)$,

$$\begin{aligned}\int_\Omega \nabla(ABL) \cdot \nabla v dx &= a(BL, v) \\ &= a(u_L, v) \\ &= \int_\Omega \nabla L \cdot \nabla v dx.\end{aligned}$$

Hence, $ABL = L$ implying $AB = \mathbf{Id}$. In addition, for all $v \in H_0^1(\Omega)$,

$$\begin{aligned}a(BAu - u, v) &= a(B(Au), v) - a(u, v) \\ &= \int_\Omega \nabla(Au) \cdot \nabla v dx - a(u, v) \\ &= 0.\end{aligned}$$

Hence $BAu - u = 0$ implying $BA = \mathbf{Id}$. By Banach theorem, it follows that A must be an isomorphism of $H_0^1(\Omega)$, hence proved. $\square$

There are some other naturally arising questions, which were unanswered due to various limitations. One of the highly relevant problems is the analysis of variational problems with weaker requirements for the constants, one of which is introduced below.

Remark 3.20. *We seek a variation of the problem with weaker bounds on the constants σ_1, σ_2. In particular, we wish to explore what property of (3.8) could be proven with the requirement that $\frac{\sigma_2}{\sigma_1} < -1$.*

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4 Acknowledgements

The author expresses his gratitude for Professor Xiaojing Xu for supporting me throughout this process of independent research. The research was conducted as a side project during my participation in the National Talent Program, where I conducted research on mathematical and fluid modelling of smog. Professor Xu introduced me to partial differential equations and their applications in meteorology. In addition, I was introduced to the study of Maxwell's equations with boundary conditions. Professor Xu guided the author in the process of learning background knowledge and introduced formal paper-writing techniques. The author chose to investigate the Laplace type equation with parameters that change sign because of its breadth of applications and large amount of existing research, serving as a good introduction to the theoretical study of partial differential equations.