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论文题目：On the choice of parameters in MAOR iterative method for solving horizonal linear complementarity problems

On the choice of parameters in MAOR iterative method for solving horizonal linear complementarity problems

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Abstract: The choice of parameters that guarantee the convergence of modulus-based accelerated over-relaxation (abbreviated as **MAOR**) iterative method for solving horizonal linear complementarity problems is analysed. The result we obtained is a generalization of that when the linear complementarity problem is a special case of horizonal linear complementarity problem. Numerical experiments further demonstrate the theoretical analysis.

Keywords: Horizonal linear complementarity problems; Modulus-based accelerated over-relaxation method; Iterative parameters; Convergence.

AMS subject classifications(2010): 65F10

1 Introduction

Given two $n \times n$ real matrices A, B, an n dimentional real column vector q, the horizonal linear complementarity problem (abbreviated as "HLCP(A, B, q)") is to find vectors $z, w \in \mathbb{R}^n$ to satisfy

$$Az - Bw = q, \quad z \geq 0, \quad w \geq 0, \quad z^T w = 0. \tag{1}$$

It is noted that the inequality here is componentwise, z^T is the transpose of vector z. Obviously, when B is the identity matrix, problem (1)will be reduced to the linear complementarity problem, which is denoted as LCP(A, q).

Complementarity problems come from many real problems in scientific computation, such as contact problems, obstacle problems, free boundary problems for journal bearings, the Nash equilibrium point of a bimatrix game, financial pricing problems and so on; details can be found in reference [1]. To solve horizonal linear complementarity problems, the frequently used methods are interior point method [2], reduction to LCP [3], projected splitting method [4] and so on.

In 2010, the author of [5] proposed the modulus-based matrix splitting iteration method to solve linear complementarity problems. The good performance of this method in numerical computation contracted lots of scholars' research, which promotes this type of method to be developed well and have wide applications. For example, the modulus-based type methods to solve linear complementarity problems [6–17]; the modulus-based type methods to solve nonlinear complementarity problems [18–27] and the modulus-based methods to solve implicit complementarity problems [28–33] and so on. In 2019, scholars applied the modulus-based matrix splitting iterative method to solve horizonal linear complementarity problems [34], and obtained the convergence theory when the system matrices A, B are positive definite and H_+-matrices. In practical numerical computations, the modulus-based accelerated over-relaxation iterative method is frequently used, which refers several parameters to chose. Improper iterative parameters may cause the divergence of the iteration sequence. So it is necessary to discuss the range of parameters that guarantee the convergence.

The rest of this paper is organized as follows: the necessary notations, definitions and conclusions, and the description of modulus-based accelerated over-relaxation method are presented in section 2; the choice of parameters which guarantee the convergence of the iterative method is discussed in section 3; in section 4, some numerical examples are given to demonstrate the theoretical analysis; finally, we give some conclusions.

2 The MAOR iterative method

Firstly, the necessary notations, definitions and conclusions will be given, and all of them can be found in [1].

For two given matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m\times n}$, the notation $A \geq B$ ($A > B$) means for arbitrary (i, j), $1 \leq i \leq m, 1 \leq j \leq n$, there holds $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$); specially, if B is the null matrix, then we call A is a nonnegative (positive) matrix. A positive diagonal matrix is a diagonal matrix with positive diagonal entries. Here, we denote $(|a_{ij}|)$ as the absolute value matrix of A, $D_A, -L_A$ and $-U_A$ are the diagonal, strictly upper triangular and strictly lower triangular parts of A, respectively. If A is nonsingular with all the non-diagonal entries being non-positive, and $A^{-1} \geq O$, then we call A is an M-matrix. If $\langle A\rangle = (\langle a\rangle_{ij})$, the comparison matrix of A, is an M-matrix, where

$$\langle a\rangle_{ii} = |a_{ii}|, \quad \langle a\rangle_{ij} = -|a_{ij}|, i, j = 1, \cdots, n, j \neq i,$$

then A is called an H-matrix. Specially, an H-matrix with positive diagonal entries is an H_+-matrix.

For $A = M - N$, if M is nonsingular, we call $A = M - N$ is a splitting of A. Furthermore, if $\langle A\rangle = \langle M\rangle - |N|$, then we call the splitting $A = M - N$ is an H-compatible splitting of A; if $\langle M\rangle - |N|$ is an M-matrix, then $A = M - N$ is an H-splitting of A. Obviously, an H-compatible matrix of an H_+-matrix is an H-splitting, but not vise versa, the counterexample can be found in [23]. For an H-matrix A, we have A is nonsingular, and $|A^{-1}| \leq \langle A\rangle^{-1}$. If $\langle A\rangle - |B|$ is an M-matrix, then $\langle A\rangle$ is also an M-matrix, and there holds $\rho(\langle A\rangle^{-1})|B| < 1$.

Next, we are going to give the modulus-based matrix splitting iteration method for solving horizonal linear complementarity problems. Let $z = \frac{1}{\gamma}(|x| + x), w = \frac{1}{\gamma}\Omega(|x| - x)$ in (1), where Ω is a positive diagonal matrix, γ is a positive constant. Consider the matrix splittings $A = M_A - N_A, B = M_B - N_B$, then (z, w) is the solution of horizonal linear complementarity problem (1) if and only if x satisfies the following fixed-point equation

$$(M_A + M_B\Omega)x = (N_A + N_B\Omega)x + (B\Omega - A)|x| + \gamma q.$$

The modulus-based matrix splitting iteration method proposed in [34] can be described as follows:

Algorithm 2.1 *(1)Chose initial vector $x^{(0)} \in \mathbb{R}^n$, let $k = 0$;*
(2)Find $x^{(k+1)} \in \mathbb{R}^n$ by solving the following system

$$(M_A + M_B\Omega)x^{(k+1)} = (N_A + N_B\Omega)x^{(k)} + (B\Omega - A)|x^{(k)}| + \gamma q, \tag{2}$$

(3)Let $z^{(k+1)} = \frac{1}{\gamma}(|x^{(k+1)}| + x^{(k+1)})$, $w^{(k+1)} = \frac{\Omega}{\gamma}(|x^{(k+1)}| - x^{(k+1)})$;
(4)If $(z^{(k+1)}, w^{(k+1)})$ satisfies the stop criteria, stop the iteration; otherwise, let $k := k + 1$, return to step (2).

Specially, if we chose

$$M_i = \frac{D_i - \beta_i L_i}{\alpha_i}, N_i = \frac{(1 - \alpha_i)D_i + (\alpha_i - \beta_i)L_i + \alpha_i U_i}{\alpha_i},$$

where $\alpha_i \in (0, 2), \beta_i > 0$ are iterative parameters $(i = A, B)$, then we will have the **MAOR** iterative method. The modulus-based succesive over-relaxation (abbreviated as **MSOR**) iterative method, the modulus-based Gauss-Siedel (abbreviated as **MGS**) iterative method and the modulus-based Jacobi (abbreviated as **MJ**) iterative method are the special cases when the quaternion array $(\alpha_A, \alpha_B, \beta_A, \beta_B)$ is chosen as $(\alpha_A, \alpha_B, \alpha_A, \alpha_B)$, $(1, 1, 1, 1)$ and $(1, 1, 0, 0)$, respectively. The existence and uniqueness of the solution of horizonal linear complementarity problem (1) can be found in [34].

When A, B are both H_+-matrices, the following convergence result of Algorithm 2.1 for solving horizonal linear complementarity problems are given in reference [34]:

Theorem 2.1 *Let A, B be H_+-matrices, $A = M_A - N_A$ be an H-splitting of A and $B = M_B - N_B$ be an H-compatible splitting of B. Suppose the positive diagonal matrix Ω satisfies that $M_A + M_B\Omega$ is an H_+-matrix, $\Omega \geq D_A D_B^{-1}$, $|b_{ij}|\omega_{jj} \leq |a_{ij}|, i, j = 1, \cdots, n, i \neq j$, and for $b_{ij} \neq 0$, $\mathrm{sign}(b_{ij}) = \mathrm{sign}(a_{ij})$. Then for arbitrary initial vector $x^{(0)}$, the iterative sequence $\{(z^{(k+1)}, w^{(k+1)})\}_{k=0}^{\infty}$ generated by Algorithm 2.1 will converge to the solution of horizonal linear complementarity problem* (1).

Later, the authors of reference [35] generalize the conditions in Theorem 2.1 and obtained the following conclusion:

Theorem 2.2 *Let A, B be H_+-matrices, $A = M_A - N_A$ be an H-splitting of A, and the splitting $B = M_B - N_B$ satisfies $D_{M_B} > 0$. Suppose the positive diagonal matrix Ω satisfies that $M_A + M_B\Omega$ is an H_+-matrix, and*

$$\langle M_B \rangle \Omega \geq \langle M_A \rangle, \;\; |N_A| \geq |N_B|\Omega.$$

Then for arbitrary initial vector $x^{(0)}$, the iterative sequence $\{(z^{(k+1)}, w^{(k+1)})\}_{k=0}^{\infty}$ generated by Algorithm 2.1 will converge to the solution of horizonal linear complementarity problem (1).

3 Main Results

Firstly, based on the conditions in Theorem 2.1, we derive the choice of the iterative parameters $\alpha_A, \beta_A, \alpha_B, \beta_B$ in the modulus-based accelerated over-relaxation method to guarantee its convergency, which is stated in the following theorem.

Theorem 3.1 *Suppose A, B are H_+-matrices, the positive diagonal matrix Ω satisfies $\Omega \geq D_A D_B^{-1}$, $|b_{ij}|w_{jj} \leq |a_{ij}|, i, j = 1, \cdots, n, i \neq j$, and for $b_{ij} \neq 0$, $\mathrm{sign}(b_{ij}) = \mathrm{sign}(a_{ij})$. When $0 \leq \beta_B \leq \alpha_B \leq 1, \alpha_B \neq 0$ and α_A, β_A satisfy one of the following conditions*

(1)

$$0 \leq \beta_A \leq \alpha_A < \frac{2}{1 + \rho_A}, \;\; \alpha_A \neq 0,$$

(2)

$$2\rho_A \beta_A < \alpha_A < \beta_A < 2 - 2\rho_A \beta_A,$$

where $\rho_A = \rho(D_A^{-1}|C_A|)$, then for arbitrary initial vector $x^{(0)}$, the modulus-based accelerated over-relaxation iterative method for solving horizonal linear complementarity problems will converge.

Proof: From Theorem 2.1, we can see that the conditions $\Omega \geq D_A D_B^{-1}$, $|b_{ij}|\omega_{jj} \leq |a_{ij}|, i, j = 1, \cdots, n, i \neq j$ and $\mathrm{sign}(b_{ij}) = \mathrm{sign}(a_{ij})$ are independent to the splittings of A, B, so we only need to analyse the sufficient conditions for $A = M_A - N_A$ being an H-splitting of A, $B = M_B - N_B$ being an H-compatible splitting of B, and $M_A + M_B\Omega$ being an H_+-matrix, respectively.

Firstly, we will analyse the sufficient conditions for $A = M_A - N_A$ being an H-splitting. From the definition of H-splitting, we only need to find a sufficient condition for $\langle M_A\rangle - |N_A|$ being an M-matrix. Since A is an H_+-matrix, we have $D_A > 0$. From the expressions of M_A, M_B in MAOR iterative method, it can be easily found that

$$\begin{aligned}\langle M_A\rangle - |N_A| &= \frac{D_A - \beta_A|L_A| - |1-\alpha_A|D_A - |\alpha_A - \beta_A||L_A| - \alpha_A|U_A|}{\alpha_A}\\ &= \frac{[1-|1-\alpha_A|]D_A - [\beta_A + |\alpha_A - \beta_A|]|L_A| - \alpha_A|U_A|}{\alpha_A}.\end{aligned}$$

As $0 < \alpha_A < 2$, $|1-\alpha_A| \geq 1-\alpha_A$, there always hold that $1-|1-\alpha_A| > 0$ and $0 < \frac{1-|1-\alpha_A|}{\alpha_A} \leq 1$. In the following, we will discuss from case to case:

(1) When $\alpha_A \geq \beta_A \geq 0, \alpha_A \neq 0$, we have

$$\langle M_A\rangle - |N_A| = \frac{1-|1-\alpha_A|}{\alpha_A}D_A - (|L_A| + |U_A|) = \frac{1-|1-\alpha_A|}{\alpha_A}D_A - |C_A|.$$

From the result in [1], we know that the condition $\rho_A < \frac{1-|1-\alpha_A|}{\alpha_A}$ can make sure that $\langle M_A\rangle - |N_A|$ is an M-matrix. By simple computation, we can see that the validity of inequality $\rho_A < \frac{1-|1-\alpha_A|}{\alpha_A}$ is equivalent to $\rho_A < 1, \rho_A\alpha_A < 1$ and $(1+\rho_A)\alpha_A < 2$ being true simultaneously. It is noticed that when $\rho_A < \frac{1-|1-\alpha_A|}{\alpha_A}$, $\rho_A < 1, \rho_A\alpha_A < 1$ always holds true. Thus a sufficient condition for $\langle M_A\rangle - |N_A|$ being an M-matrix is $(1+\rho_A)\alpha_A < 2$, that is $\alpha_A < \frac{2}{1+\rho_A}$. It follows that, when $0 \leq \beta_A \leq \alpha_A < \frac{2}{1+\rho_A}, \alpha_A \neq 0$, $A = M_A - N_A$ is an H-splitting.

(2) When $0 < \alpha_A < \beta_A$, we have

$$\begin{aligned}\langle M_A\rangle - |N_A| &= \frac{1-|1-\alpha_A|}{\alpha_A}D_A - \frac{(2\beta_A - \alpha_A)|L_A| + \alpha_A|U_A|}{\alpha_A}\\ &> \frac{1-|1-\alpha_A|}{\alpha_A}D_A - \frac{2\beta_A}{\alpha_A}|L_A| - \frac{2\beta_A}{\alpha_A}|U_A|\\ &= \frac{1-|1-\alpha_A|}{\alpha_A}D_A - \frac{2\beta_A}{\alpha_A}|C_A| := K,\end{aligned}$$

From the result in [1], we know that a sufficient condition for $\langle M_A\rangle - |N_A|$ being an M-matrix is that K is an M-matrix, and a sufficient condition for K being an M-matrix is $\rho_A < \frac{1-|1-\alpha_A|}{2\beta_A}$. It is noticed that since $1-|1-\alpha_A| \leq \alpha_A \leq \beta_A$, then for arbitrary $0 < \alpha_A \leq \beta_A$ there always holds $\frac{1-|1-\alpha_A|}{2\beta_A} \leq \frac{1}{2}$. Meanwhile, the validity of inequality $\rho_A < \frac{1-|1-\alpha_A|}{2\beta_A}$ is equivalent to $\rho_A < \frac{1}{2}, 2\rho_A\beta_A < 1$ and $2\rho_A\beta_A < \alpha_A < 2 - 2\rho_A\beta_A$ being true simultaneously. So when $2\rho_A\beta_A < \alpha_A \leq \beta_A < 2 - 2\rho_A\beta_A$, the splitting $A = M_A - N_A$ is an H-splitting.

Secondly, we will analyse the sufficient condition for $B = M_B - N_B$ being an H-compatible splitting. From the definition of H-compatible splitting, we know that the only thing we need to do is to find a sufficient condition for $\langle B\rangle = \langle M_B\rangle - |N_B|$ being true. Since B is an H_+-matrix, we have $\langle B\rangle$ is an M-matrix too, and $D_B > 0$. Then $\langle B\rangle = \langle M_B\rangle - |N_B|$ becomes

$$D_B - |L_A + U_A| = \frac{[1 - |1-\alpha_B|]}{\alpha_B} D_B - \frac{[\beta_B - |\alpha_B - \beta_B|]}{\alpha_B}|L_B| - |U_B|. \tag{3}$$

In order to make (3) true, we only need to make sure that $1-|1-\alpha_B| = \alpha_B$ and $\beta_B + |\alpha_B - \beta_B| = \alpha_B$ hold true simultaneously, which means

$$0 \leq \beta_B \leq \alpha_B \leq 1, \alpha_B \neq 0.$$

Lastly, we will analyse the sufficient condition for $M_A + M_B\Omega$ being an H_+-matrix. Since D_A, D_B, Ω are all positive diagonal matrices and α_A, α_B are all positive, we have that the diagonal part of

$$M_A + M_B\Omega = \frac{D_A - \beta_A L_A}{\alpha_A} + \frac{D_B - \beta_B L_B}{\alpha_B}\Omega,$$

which is $\frac{D_A}{\alpha_A} + \frac{D_B}{\alpha_B}\Omega$, is positive. It is easy to get that

$$\langle M_A + M_B\Omega\rangle = \frac{D_A}{\alpha_A} + \frac{D_B}{\alpha_B}\Omega - |\frac{\beta_A}{\alpha_A}L_A + \frac{\beta_B}{\alpha_B}L_B\Omega|,$$

and matrix $\langle M_A + M_B\Omega\rangle$ is an M-matrix. So in the MAOR iterative method, for arbitrary $\alpha_i \in (0, 2), \beta_i \geq 0, (i = A, B)$, $M_A + M_B\Omega$ is an H_+-matrix.

Combining the above analysis, we can obtain the result of this theorem. ■

Remark 3.1 *We notice that, when matrix B is the identity matrix, the* MAOR *iterative method for solving horizonal linear complementarity problems will be the one for solving linear complementarity problems. At this case, from the result in Theorem 3.1, for arbitrary* $0 < \alpha_B = \beta_B \leq 1$, *when* $0 < \alpha_A = \beta_A < \frac{2}{1+\rho_A}$, *the* MSOR *iterative method converges; we can see that the bounds for* α_A *is bigger than the ones in [5]. It is noticed that, under condition(1), the inequality* $0 < \rho_A < 1$ *naturally holds, then* $1 < \frac{2}{1+\rho_A} < 2$. *Thus, when* $\alpha_B = \beta_B = 1, \alpha_A = \beta_A = 1$, *we can have the convergency of* MGS *iterative method for solving horizonal linear complementarity problems; when* $\alpha_A = \alpha_B = 1, \beta_A = \beta_B = 0$, *we can have the convergency of* MJ *iterative method for solving horizonal linear complementarity problems.*

It is found that in actual implementation the MSOR iterative method will still converge when $\alpha_B > 1$, which inspires us to improve the conditions in Theorem 3.1. From Theorem 2.2, we know that the conditions $A = M_A - N_A$ being an H-splitting and $M_A + M_B\Omega$ being an H_+-matrix are the same as those in Theorem 2.1, and in MSOR iterative method and MJ iterative method, the splitting $B = M_B - N_B$ naturally satisfies $D_{M_B} > 0$. So the only thing we need to do is to find the conditions to guarantee

$$\langle M_B \rangle \Omega \geq \langle M_A \rangle, \ \ |N_A| \geq |N_B|\Omega.$$

Firstly, in the modulus-based Jacobian iterative method, it is easy to find that the condition $\langle M_B \rangle \Omega \geq \langle M_A \rangle$ becomes $\Omega \geq D_A D_B^{-1}$, and $|N_A| \geq |N_B|\Omega$ means $i, j = 1, \cdots, n, i \neq j, |b_{ij}|w_{jj} \leq |a_{ij}|$. All these conditions are the same as those in Theorem 3.1, so MJ iterative method converges.

Secondly, in the MSOR iterative method, since $\alpha_A = \beta_A$ and $\alpha_B = \beta_B$, the constraint on α_A is $0 < \alpha_A < \frac{2}{1+\rho_A}$ from Theorem 3.1. Meanwhile, the condition $\langle M_B \rangle \Omega \geq \langle M_A \rangle, \ \ |N_A| \geq |N_B|\Omega$ is equivalent to

$$\begin{cases} \frac{b_{ii}}{\alpha_B} w_{ii} \geq \frac{a_{ii}}{\alpha_A}, \ \ |b_{ij}|w_{jj} \leq |a_{ij}|, i, j = 1, \cdots, n, i > j, \\ \frac{|1-\alpha_B| b_{ii}}{\alpha_B} w_{ii} \leq \frac{|1-\alpha_A|}{\alpha_A} a_{ii}, \ \ |b_{ij}|w_{jj} \leq |a_{ij}|, i, j = 1, \cdots, n, i < j. \end{cases} \tag{4}$$

Obviously, for all $i, j = 1, \cdots, n, i < j$ there always holds $|b_{ij}|w_{jj} \leq |a_{ij}|$. Then we will discuss by cases:

(i) When $\alpha_B = 1$, then $\frac{|1-\alpha_B| b_{ii}}{\alpha_B} w_{ii} = 0 \leq \frac{|1-\alpha_A|}{\alpha_A} a_{ii}$ is true for arbitrary w_{ii}. In this case, for $0 < \alpha_A < \frac{2}{1+\rho_A}$, if $\Omega \geq \frac{1}{\alpha_A} D_A D_B^{-1}$, the MSOR iterative method will converge.

(ii)When $\alpha_B \neq 1$, from condition (4) we have

$$\frac{\alpha_B}{\alpha_A} a_{ii} b_{ii}^{-1} \leq w_{ii} \leq \frac{\alpha_B |1-\alpha_A|}{\alpha_A |1-\alpha_B|} a_{ii} b_{ii}^{-1}. \tag{5}$$

Then we should find the values of α_A, α_B to make sure that the range of w_{ii} in (5) is non-empty. It is noticed that if $1 > \alpha_A \geq \alpha_B > 0$ or $2 > \alpha_B \geq \alpha_A > 1$, we will have $\Omega = D_A D_B^{-1}$ only when $\alpha_A = \alpha_B$; if $1 > \alpha_B \leq \alpha_A > 0$ or $2 > \alpha_A \geq \alpha_B > 1$ or $2 > \alpha_B > 1 \geq \alpha_A > 0, \alpha_A + \alpha_B \leq 2$ or $2 > \alpha_A \geq 1 > \alpha_B > 0, \alpha_A + \alpha_B \geq 2$, we have $|1 - \alpha_A| \geq |1 - \alpha_B|$, and thus $\frac{\alpha_B}{\alpha_A} D_A D_B^{-1} \leq \Omega \geq \frac{\alpha_B |1-\alpha_A|}{\alpha_A |1-\alpha_B|} D_A D_B^{-1}$.

Summarizing the above analysis, we can have the following theorem.

Theorem 3.2 *Suppose A, B are H_+-matrices, the positive diagonal matrix Ω satisfies $|b_{ij}|w_{jj} \leq |a_{ij}|, i, j = 1, \cdots, n, i \neq j$. Then for arbitrary initial vectors $x^{(0)}$,*

(1)when $\Omega \geq D_A D_B^{-1}$, the modulus-based Jacobian iterative method will converge;

(2)when $\Omega \geq \frac{1}{\alpha_A} D_A D_B^{-1}, \alpha_B = 1, \alpha_A \in (0, \frac{2}{1+\rho_A})$, the modulus-based successive over-relaxation iterative method will converge;

(3)when $\Omega = D_A D_B^{-1}, \alpha_B = \alpha_A \in (0, 1) \bigcup (\frac{2}{1+\rho_A}, 2)$, the modulus-based successive over-relaxation iterative method will converge;

(4)when $\frac{\alpha_B}{\alpha_A}D_AD_B^{-1} \leq \Omega \geq \frac{\alpha_B|1-\alpha_A|}{\alpha_A|1-\alpha_B|}D_AD_B^{-1}$, and one of the following conditions is true,
(4-1)

$$0 < \alpha_A \leq \alpha_B < 1,$$

(4-2)

$$1 < \alpha_B \leq \alpha_A < \frac{2}{1+\rho_A},$$

(4-3)

$$0 < \alpha_B < 1 \leq \alpha_A < \frac{2}{1+\rho_A}, \alpha_A + \alpha_B \geq 2,$$

(4-4)

$$0 < \alpha_A \leq 1 < \alpha_B < 2, \alpha_A + \alpha_B \leq 2,$$

the modulus-based successive over-relaxation iterative method will converge.

4 Numerical Experiment

In this section, we will give some examples to test the above theoretical analysis in section 3.

In all the following numerical experiments, the initial vector is chosen to be $x^{(0)} = 2e \in \mathbb{R}^n$, where e is the column vector with all entries being 1, $\gamma = 1, \Omega = D_AD_B^{-1}$, $q = Az^* - Bw^*$, where $z^* = (0,1,0,1\cdots)^T, w^* = (1,0,1,0,\cdots)^T \in \mathbb{R}^n$. In the MSOR iterative method, the optimal parameters α_A, α_B are chosen as the ones with least iteration steps(which is denoted as 'IT'). The iteration time is denoted as 'CPU', whose unit is second; the stopping criteria is RES $\leq 10^{-6}$, where 'RES' is defined as

$$\text{RES}(z^{(k)}) := \|Az^{(k)} - Bw^{(k)} - q\|_2,$$

or k reaches the maximal number of iteration steps which is 1000 in our paper. All the computations are performed in MATLAB with double machine precision where the CPU is 2.40 GHz and the memory is 4.00 GB.

Let I_n be the n-dimensional identity matrix, consider the following examples.

Example 4.1 [34] *Let m be a given positive integer, $n = m^2$. Chose $A = \hat{A} + \mu I_n, B = \hat{B} + \nu I_n$*

in (1), *where*

$$\hat{A} = \begin{pmatrix} S & -I_m & & & \\ -I_m & S & -I_m & \ddots & \\ & -I_m & S & \ddots & \\ & & & \ddots & -I_m \\ & & & -I_m & S \end{pmatrix}, \hat{B} = \begin{pmatrix} S & & & & \\ & S & & & \\ & & S & & \\ & & & \ddots & \\ & & & & S \end{pmatrix},$$

$S = \mathrm{tridiag}(-1, 4, -1) \in \mathbb{R}^{m \times m}$ *is a tridiagonal matrix.*

Example 4.2 *[34] Let m be a given positive integer, $n = m^2$. Chose $A = \hat{A} + \mu I_n, B = \hat{B} + \nu I_n$ in* (1), *where*

$$\hat{A} = \begin{pmatrix} S & -0.5I_m & & & \\ -1.5I_m & S & -0.5I_m & \ddots & \\ & -1.5I_m & S & \ddots & \\ & & & \ddots & -0.5I_m \\ & & & -1.5I_m & S \end{pmatrix}, \hat{B} = \begin{pmatrix} S & & & & \\ & S & & & \\ & & S & & \\ & & & \ddots & \\ & & & & S \end{pmatrix},$$

$S = \mathrm{tridiag}(-1.5, 4, -0.5) \in \mathbb{R}^{m \times m}$ *is a tridiagonal matrix.*

Example 4.3 *[36] Let m be a given positive integer, $n = m^2$. Chose $A = \hat{A} + \mu I_n, B = \hat{B} + \nu I_n$ in* (1), *where*

$$\hat{A} = \begin{pmatrix} S & -I_m & -I_m & & \\ & S & -I_m & \ddots & \\ & & S & \ddots & -I_m \\ & & & \ddots & -I_m \\ & & & & S \end{pmatrix}, \hat{B} = \begin{pmatrix} S & & & & \\ & S & & & \\ & & S & & \\ & & & \ddots & \\ & & & & S \end{pmatrix},$$

$$S = \begin{pmatrix} 4 & -1 & -1 & \cdots & 0 & 0 \\ 0 & 4 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 4 \end{pmatrix} \in \mathbb{R}^{m \times m},$$

is an upper tridiagonal matrix.

We should note that, since the positive diagonal matrix $\Omega = D_A D_B^{-1}$, the conditions in Theorem 3.1 "$|b_{ij}|w_{jj} \leq |a_{ij}|, i, j = 1, \cdots, n, i \neq j$, and for $b_{ij} \neq 0$ there is $\mathrm{sign}(b_{ij}) = \mathrm{sign}(a_{ij})$" are true for A, B in all the three examples here.

The numerical results when parameters (μ, ν) are $(0, 0)$ and $(0, 4)$ in examples 4.1, 4.2 and 4.3 are listed in Tables 1, 3, 5 and Tables 2, 4, 6, respectively. By computation, it is easy to find the optimal parameters in MSOR iterative methods. In Example 4.1, when $(\mu, \nu) = (0, 0)$, $n \geq 16^2$, the optimal parameters are $(\alpha_A, \alpha_B) = (1, 1.7)$; when $(\mu, \nu) = (0, 4)$, $n \geq 20^2$, the optimal parameters are $(\alpha_A, \alpha_B) = (1, 1.1)$. In Example 4.2, when $(\mu, \nu) = (0, 0)$ and $(\mu, \nu) = (0, 4)$, $n \geq 50^2$时, the optimal parameters (α_A, α_B) are $(0.8, 2.1)$ and $(1.3, 1)$, respectively. In Example 4.3, when $(\mu, \nu) = (0, 0)$, $n \geq 128^2$, the optimal parameters are $(\alpha_A, \alpha_B) = (0.8, 2.1)$; when $(\mu, \nu) = (0, 4)$, $n \geq 50^2$, the optimal parameters are $(\alpha_A, \alpha_B) = (0.5, 1.5)$. Actually, we find that the optimal parameters in MSOR iterative methods for solving all the examples are independent to the sizes of the problems. Thus, in the following numerical results which are solved by MSOR iterative methods, the optimal parameters are chosen as those listed above.

Table 1: Numerical results for Example 4.1 when $\mu = \nu = 0$

m		64	128	256	512	1024
MJ	IT	95	99	102	106	109
	CPU	0.0200	0.0850	0.3220	1.7550	7.0290
	RES	8.8936	8.7364	9.8594	8.7102	9.3671
MGS	IT	59	61	63	65	67
	CPU	0.0110	0.0500	0.1980	1.1250	4.6050
	RES	7.5899	8.7908	9.4246	9.7680	9.9627
MSOR	IT	37	39	40	42	43
	CPU	0.0010	0.0350	0.1350	0.7290	3.1970
	RES	9.3283	7.1322	8.7977	6.0260	6.9788

As we are expected, the MSOR iterative method has the best numerical performance with the optimal parameters, especially when the sizes of problems are large. It is noted that, since the strictly lower triangular parts of A and B in Example 4.3 are zeros, then the numerical results of the MGS iterative method and the MJ iterative method for solving this example are the same, which can also be shown in Tables 5 and 6. Meanwhile, we find that some of the values for the iterative parameters (α_A, α_B) do not satisfy the conditions in the previous theorems, for example, when $\alpha_A > 2$ or $\alpha_B > 2$, the iteration methods can also converge. This means the conditions are sufficient but not necessary for convergency.

Table 2: Numerical results for Example 4.1 when $\mu = 0, \nu = 4$

m		64	128	256	512	1024
MJ	IT	128	133	137	142	146
	CPU	0.0370	0.1360	0.4400	2.3460	10.0290
	RES	9.0080	8.9363	9.9367	9.2581	9.9921
MGS	IT	112	117	121	125	129
	CPU	0.0340	0.1010	0.4100	2.2550	9.2410
	RES	9.8619	8.7882	8.9821	9.0059	8.9457
MSOR	IT	102	106	110	114	118
	CPU	0.0290	0.0860	0.3930	2.1020	8.9950
	RES	9.6586	9.5206	9.0486	8.4547	7.8343

Table 3: Numerical results for Example 4.2 when $\mu = \nu = 0$

m		64	128	256	512	1024
MJ	IT	93	98	102	106	109
	CPU	0.0160	0.0870	0.3130	1.6970	7.6940
	RES	9.9534	9.4476	9.2471	8.4346	9.2158
MGS	IT	41	43	44	46	47
	CPU	0.0100	0.0350	0.1360	0.8240	3.4410
	RES	7.7255	7.1258	9.5827	7.4643	9.3205
MSOR	IT	27	29	29	31	31
	CPU	0.0060	0.0250	0.0830	0.5690	2.4040
	RES	7.6257	4.1188	9.1010	4.1609	8.5184

5 Conclusion

When the systematic matrices A, B are H_+-matrices, we discussed the choice of parameters in modulus-based accelerated over-relaxation method for solving horizonal linear complementarity problems HLCP(A, B, q). Numerical results further demonstrate the theoretical analysis. We must admit that the conditions we obtained are sufficient but not necessary, further research still should be carried out.

Table 4: Numerical results for Example 4.2 when $\mu = 0, \nu = 4$

m		64	128	256	512	1024
MJ	IT	127	132	137	142	146
	CPU	0.0240	0.0930	0.4690	2.4010	9.6400
	RES	9.5224	9.9677	9.7029	9.1461	9.9299
MGS	IT	105	109	113	117	120
	CPU	0.0220	0.0860	0.3890	2.1060	8.6100
	RES	8.3500	8.7889	8.7010	8.3801	9.6259
MSOR	IT	99	103	107	111	115
	CPU	0.0200	0.0810	0.3430	2.0370	8.3850
	RES	8.9971	8.7408	8.1325	7.4450	6.7693

Table 5: Numerical results for Example 4.3 when $\mu = \nu = 0$

m		128	256	512	1024	2048
MJ	IT	120	129	135	140	145
	CPU	0.0780	0.3430	2.1030	8.8880	35.3320
	RES	9.3945	9.3624	9.5031	9.6934	9.2926
MGS	IT	120	129	135	140	145
	CPU	0.1100	0.3120	2.0930	8.7180	35.9480
	RES	9.3945	9.3624	9.5031	9.6934	9.2926
MSOR	IT	103	110	115	120	124
	CPU	0.0620	0.2620	1.7810	7.7870	30.9570
	RES	9.7921	9.5975	9.7330	8.6096	8.6564

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Table 6: Numerical results for Example 4.3 when $\mu = 0, \nu = 4$

m		128	256	512	1024	2048
MJ	IT	216	229	238	247	255
	CPU	0.1870	0.8280	3.9210	16.1280	67.3100
	RES	9.9269	9.3668	9.9368	9.6535	9.8629
MGS	IT	216	229	238	247	255
	CPU	0.1720	0.7130	3.9660	16.3320	69.3720
	RES	9.9269	9.3668	9.9368	9.6535	9.8629
MSOR	IT	202	213	222	230	238
	CPU	0.1410	0.6250	3.6950	15.4770	64.2390
	RES	9.7011	9.6393	9.5099	9.5344	9.2195

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致　谢

一、论文的选题来源和研究背景

水平互补问题的数学模型是: 给定两个$n \times n$的实矩阵A, B和一个n维实列向量q, 我们要寻找列向量$z, w \in \mathbb{R}^n$ 满足

$$Az - Bw = q, \quad z \geq 0, \quad w \geq 0, \quad z^T w = 0.$$

注意, 这里的不等号是分量意义下的不等号, z^T 为向量z 的转置. 当矩阵A 或B 为单位矩阵时, 此水平互补问题便化为线性互补问题. 水平线性互补问题通常被记为HLCP(A, B, q), 它来源于科学计算中的许多实际问题, 如接触问题、障碍问题、径向滑动轴承问题、双矩阵博弈中的纳什均衡点问题和金融定价问题等; 具体可参考文献[1].

关于水平线性互补问题的求解, 常用的方法有内点法[2]、化为线性互补问题[3]、投影分裂法[4]等. 2010年, 文献[5]的作者提出了求解线性互补问题的模基矩阵分裂迭代法. 该方法在数值计算中表现出的优势吸引了众多学者的研究, 这使得模基矩阵分裂迭代法得到了快速发展和广泛应用, 如求解线性互补问题的模系方法[6–17]; 求解非线性互补问题的模系方法[18–27] 以及求解隐互补问题的模系方法[28–33]等. 2019年, 有学者将模基矩阵分裂迭代法应用到了水平线性互补问题[34]的求解中, 给出了当系数矩阵A, B分别为正定矩阵和H_+-矩阵时迭代法的收敛性.

在实际数值计算中, 我们经常选用模基加速超松弛迭代法(MAOR)来进行数值模拟, 其中涉及多个迭代参数的选取. 不恰当的迭代参数的选择会使得迭代序列在规定条件下得不到收敛. 因此, 我们在此论文中将具体讨论保证迭代法收敛的参数的取值范围, 并给出数值算例.

二、每一个队员在论文撰写中承担的工作以及贡献

队员王子睿同学勤奋好学, 思维敏捷, 接受新事物新知识能力很强. 在本项目的研究过程中, 他能积极主动查阅文献, 理解概念, 严格推理, 编写程序, 记录数据并及时整理. 在撰写论文期间, 他主动承担翻译成英文的工作, 论文在指导老师的帮助下不断完善并形成现在的内容. 通过此次学习, 王子睿同学已经具备了一定的科研能力和英语写作能力, 对本论文的形成做出了很大贡献.

三、指导老师与学生的关系, 在论文写作过程中所起的作用, 及指导是否有偿

指导老师1李蕊是嘉兴学院的一名数学专业教师. 听闻王子睿同学对数学学习有及其浓厚的兴趣并具有一定的独立学习能力, 正好嘉兴市在推进未来家“英才计划”项目.为了给在校中学生提供提早感受科研的机会, 经解礁老师推荐, 李蕊老师从自己平时科研中的一些小问题出发, 让王子睿同学参与思考和讨论, 并最终形成此论文.

指导教师2解礁是清华附中嘉兴实验高级中学书记, 执行校长, 他为本项目团队成员参与此项活动提供了信息共享和时间保证, 使学生能有足够的精力推进项目进行. 同时, 解老师对该学生的外文写作技巧进行了指导.

整个项目期间, 师生之间完全出于相互交流和相互学习的目的, 所有指导老师全程均无偿提供指导和帮助.

四、团队成员和指导老师的简历

1. **团队成员简历:** 王子睿, 就读于清华附中嘉兴实验高级中学高二. 自幼痴迷数学, 初中开始自学微积分、线性代数、数学分析、初等数论、复变函数、抽象代数等相关内容. 曾获得2022年全国奥林匹克物理竞赛浙江赛区三等奖. 2023年参加丘成桐中学生数学夏令营获得进一步提升.

2. **指导教师1简历:** 李蕊, 博士, 嘉兴学院副教授. 2015-2018年, 同济大学攻读博士学位; 2005-2007年, 浙江大学攻读理学硕士学位, 方向为黎曼几何; 2001-2005年, 郑州大学攻读数学与应用数学专业. 2007 年至今在嘉兴学院工作, 数学专任教师, 承担《高等代数》、《空间解析几何》、《高等数学》、《概率论与数理统计》、《线性代数》等课程的教学任务, 曾获嘉兴学院优秀教师、教学优秀奖、青年教师讲课十佳、教学设计与技能竞赛一等奖、优秀毕业论文指导老师、优秀班主任、优秀党员等荣誉称号. 主持国家自然科学基金青年项目一项, 参与国家自然科学基金项目2 项、省自然科学基金项目4项, 发表科研论文10 余篇, 均被SCI或EI收录检索, 主要研究方向为数值代数及其应用, 大规模科学计算.

指导教师2简历: 解礁, 清华大学事编教师, 教育管理博士, 中学数学高级教师, 中国数学奥林匹克(CMO)竞赛金牌教练, 北京市海淀区兼职教研员, 北京市紫禁杯优秀班主任, 育人奖获得者, 北京师范大学“未来教师素质大赛”指导教师、评委. 曾执教九届高三毕业班, 多名学生获得高考数学满分150分, 荣获北京市单科状元. 十余篇论文获得全国教育科研类论文评比一、二等奖.