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Methods and Calculations in Domino Tiling Problem

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In this paper, we discuss domino tiling (perfect matching) problem by reviewing the derivation of classic Kasteleyn formula in chapter 2, investigating basic methods in domino tiling problem including linear algebra, Conway's boundary group, several invariants and combinatoric methods in chapter 3 and generalize to solve perfect matching problems especially in Cayley graphs. Our research emphasizes over the relationship between boundary group methods and combinatoric invariants, and the calculation of several examples considering Pfaffian orientation methods. We also conclude that the existence of Pfaffian orientation over Cayley color graphs is negative, but we can still apply the spectra formula([10],[11]) to calculate the parity of determinant.

Key words: domino tiling, combinatorics, algebraic graph theory, Cayley graph;

Contents

1.PREREQUISITE

1.1 Introduction

During our research, we first considered the light-switch problem, which inspired us for all the following topics. The light-switch problem goes like this: there is a light on each vertex of a graph. An operation on one light will change its adjacent lights' state (i.e. if an adjacent light is on at first, it will be off after the operation), while the light operated itself remains unchanged. The goal is to turn all lights (initially on or off) into the same 'off ' state. The problem soon reduces to a linear equation problem in $\mathbb{F}_2$.

The conclusion of [5] is about the parity of determinant of adjacent matrix of the rectangular graph. It also mentioned the formula which holds in the general domino tiling problems:

$$\det A = (\mathrm{Pf}\ A)^2 .$$

While the light-switch problem focuses more on the parity of the determinant, the problem of domino tiling puts its emphasis on the exact value of the determinant, when the Pfaffian can be calculated exactly and is identical to the number of domino tilings.

Then we investigate some basic methods like Kasteleyn Formula and Combinatory methods to get the tiling number respectively over rectangular tiling program and Aztec diamond. We also discuss several invariables like the checkerboard coloring, generalized coloring and height function to judge whether a graph can be tiled or not, especially reviewed their relation with Conway's tiling group. To extend the calculation of Kasteleyn Formula, we investigate the Aztec diamond and other regular graphs. Finally, we return to the parity of determinant to solve the light-switch problem since it is impossible for Cayley color graph to coincide with Pfaffian orientation naturally.

Definition 1.1.1(Domino) A domino is a 1×2 grid.

Definition 1.1.2(Domino tiling) A domino tiling is a kind of covering of a grid that use dominos to place on a 1×2 grid in two ways: horizontal or vertical, like Fig 1.1.2.

Definition 1.1.3(Matching) A matching of a graph C is a matching that each pair of adjacent blocks are represented by an edge and every vertex of the edge is only connected to that edge. Please refer to Fig.1.1.3 for an illustration.

Fig.1.1.1

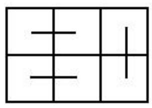


Fig.1.1.2

Fig.1.1.3

Definition 1.1.4(Perfect matching) Every vertex of the graph must be incident to precisely one edge of the matching for the matching to be considered perfect.

If M and N are both odd, it is impossible to tile the graph with 1×2 grids, for the number of squares in an $M\times N$ grid is odd, which cannot be divided by 2.

So, our question should be based on the fact that $M\times N$ should be an even number. To tile a $2\times n$ grid（Fig.1.1.5）, if we wipe out the 1×2 grid on the right, we can get Fig.1.1.6.

$2\times n$

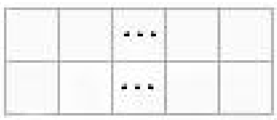


Fig.1.1.5

$2\times(n-1)$

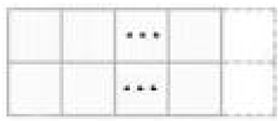


Fig.1.1.6

Furthermore, if we wipe out a 2×2 grid on the right, we will get Fig.1.1.7.

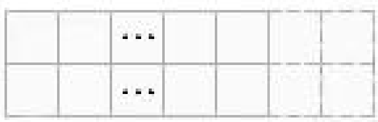


Fig.1.1.7

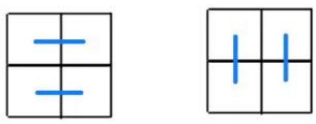


Fig.1.1.8

And for the 2×2 grid, there are two ways to tile it. (Fig.1.1.8). The left picture in Fig.1.1.8 coincides with Fig.1.1.6.So, the number of tilings for a $2\times n$ grid equals the sum of the numbers of tiling for the $2\times(n-1)$ grid and the $2\times(n-2)$ grid. Define a_n $(n\geq 2)$ as the tiling number of the grids.

Conclusion 1.1.5 $a_n = a_{n-1} + a_{n-2}$ $(n\geq 2)$ and $a_1 = 1, a_2 = 2$.

It shows that the numbers of tiling for $2\times n$ grid coincide with Fibonacci sequence.

Definition 1.1.6(Aztec diamond) The Aztec diamond of order n is the union of those lattice squares $[a\,,a+1]\times[b\,,b+1]\subset R$ $(a\,,b\in Z)$ that lie completely inside the tilted square $\{(x\,,y):|\,x\,|+|\,y\,|\leq n+1\}$. (Fig.1.1.9 shows the example of n=3).

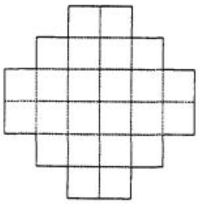


Fig.1.1.9

Definition 1.1.7(Cayley Graph) Let G be a group, and let $S\subseteq G$ be a set of group elements such that the identity element $I\notin S$. The Cayley graph associated with (G,S) can be defined as the directed graph having one vertex associated with each group element and directed edges (g,h) whenever $gh^{-1}\in S$. [7]

The graph $\Gamma(G)$ representing generators $\mathrm{g}_1,\mathrm{g}_2...,\mathrm{g}_\mathrm{n}$ is a directed graph whose vertices are the elements of the group. For each vertex $v\in G$, there will be n outgoing edges, labeled by generators, and n incoming edges. The edge labeled g_i connects v to vg_i.

Let R be relators of the group. Then, $G=\langle g_1,g_2,...,g_n\,|\,R_1=R_2=R_3...=R_k=0\rangle$ (as notation in [8]). For general Cayley graphs we can't equip the Pfaffian orientation naturally (see theorem 3.5.3) , we can still solve the light switch problem. Here is equational definition of the switching problem. (Every entity is in $\mathbb{F}_2=\mathbb{Z}/2\mathbb{Z}$.)

Definition 1.1.8 $\begin{bmatrix} O_1 \\ O_2 \\ \vdots \\ O_n \end{bmatrix} + \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}.$

where $O=\begin{bmatrix} O_1 \\ O_2 \\ O_3 \\ \vdots \\ O_n \end{bmatrix}$ $(O_n \in \mathbb{F}_2)$ shows the initial state of the n lights, $O_i = 0$ means 'off' while $O_i = 1$ means 'on'; $P=\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \end{bmatrix}$ $(P_n \in \mathbb{F}_2)$ shows the parity of the total number of changes in each light; $A=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ $(A_{mn} \in \mathbb{F}_2)$ is the adjacency matrix of the Cayley graph; $F=\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{bmatrix}$ $(F_n \in \mathbb{F}_2)$ shows the final state of the lights. We can assume $F=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

The product of the adjacency matrix and the operands shows the parity of the changes that operate on the lights. By adding the product to the initial state, we can get the final state.

By calculating the adjacency matrix, we can obtain the parity of the determinant. If the value of the determinant is 1, then it is a nonsingular matrix, meaning there is only one solution to get to the final state. If the value is 0, its uniqueness and existence are not obtained.

1.2 Pfaffian of the skew symmetric matrix

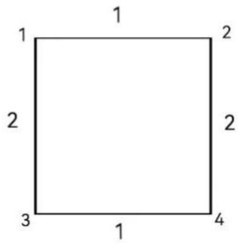
Definition 1.2.1(Pfaffian) The Pfaffian of skew-symmetric matrix A is $\operatorname{Pf} A = \sum_{\tau} \operatorname{sgn}(\tau)\omega(\tau)$, where the sum is taken over the set of all perfect matchings of the graph G. $\omega(\tau)$ is the product of the matrix entities corresponding to the edges of the matching, and $\operatorname{sgn}(\tau)$ is the parity of the matching. [5]

It is known that $\det A = (\operatorname{Pf} A)^2$. The sign $\operatorname{sgn}(\tau)$ is determined as following,

Case 1: If one matching can be transformed into the other by an even permutation, then these two matchings belong to the same class, which is marked by the plus sign $+$.

Case 2: If one matching can be transformed into the other by an odd permutation, then both two matchings belong to another class, which is marked by the minus sign $-$.

Example 1.2.2: Pairs:



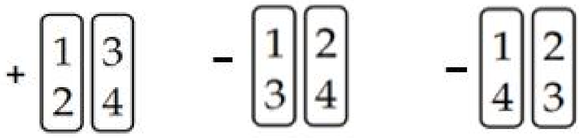


Fig.1.2.2 Fig.1.2.3

The Pfaffian should be the sum of all configurations: $1-4+0=-3$. The determinant of adjacency matrix of Fig.1.2.2 should be:

$$A=\det\begin{bmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ 0 & -2 & -1 & 0 \end{bmatrix}=9$$, which coincides with the formula $\det A=(\mathrm{Pf}\ A)^2$.

Definition 1.2.3(Pfaffian orientation) In graph theory, a Pfaffian orientation of an undirected graph assigns a direction to each edge, so that certain cycles have an odd number of edges in each direction. Given a figure on the square grid and its graph G , there exists an orientation on the edges such that the orientation, combined with the parity of configuration, ensures all matchings of the figure have the same sign. This orientation is referred to as a Pfaffian orientation.

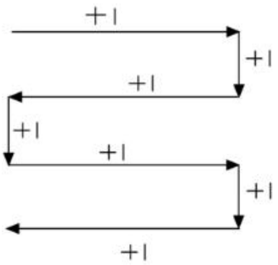
We want to design D such that each non-zero term in the Pfaffian of D correspond to a configuration C , and every configuration C corresponds to a non-zero term of the Pfaffian of D . This way, the number of terms in the Pfaffian of D is the number of domino tiling of the grid $Q_{m,n}$ .To achieve this, we need to find the Pfaffian orientation of the graph. The Pfaffian orientation of domino tiling is:

$$D_{(i,j)(i+1,j)}=z \qquad \text{for } 1 \quad i \quad m-1 \text{ and } 1 \quad j \quad n$$

$$D_{(i,j)(i,j+1)}=(-1)^i z' \qquad \text{for } 1 \quad i \quad m \text{ and } 1 \quad j \quad n-1$$

A more intuitive expression of the Pfaffian orientation is Fig.1.2.4. To create a polygon in the tiling, we need an odd permutation, contributing a factor of (-1) to $sgn(\sigma)$.

Fig.1.2.5 shows how a standard configuration is turned into a polygon through an odd permutation.



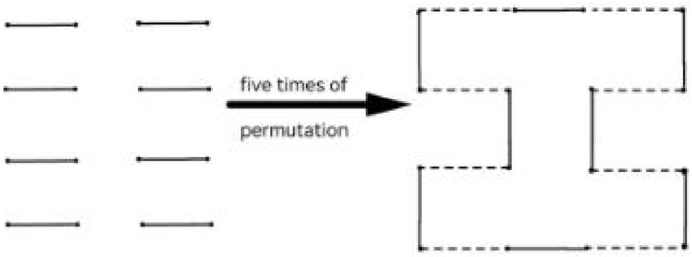


Fig.1.2.4 Fig.1.2.5

And together with the Pfaffian orientation:

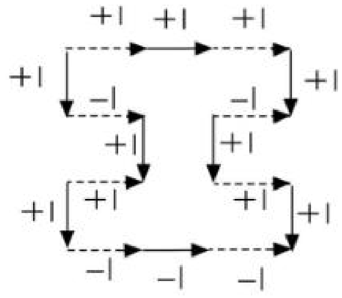


Fig.1.2.6

As the Pfaffian orientation contributes another $(-1)^5$ to the configuration, every configuration has a positive sign. Thus, we can add all the ways of configurations together in order to get the number of tilings.

1.3 Kronecker Product

Let A be an $n \times n$ matrix and let B be an $m \times m$ matrix. The Kronecker product $A \otimes B$ is an $mn \times mn$ matrix that can be written in the form as:

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

Properties 1.3.1

(1) $(A+B)\otimes C = (A \otimes C) + (B \otimes C)$ and $A \otimes (B+C) = (A \otimes B) + (A \otimes C)$.

(2) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

(3) If λ is a scalar, then $(\lambda A) \otimes B = \lambda(A \otimes B) = A \otimes (\lambda B)$.

(4) If A and B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

2. KASTELEYN FORMULA

Take D, the adjacent matrix of the grid which we want to tile. We want to calculate the number of perfect matchings using the methods in [3]. Our goal is to decompose it into the sum of two $mn \times mn$ matrices. Let I_n be the $n \times n$ identity matrix, and

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

Then we can get: $D = z(I_n \otimes Q_m) + z'(Q_n \otimes F_m)$. [3]

We will use an example of a 2×3 grid to prove its correctness.

Let $m=2, n=3$.

1	3	5
2	4	6

Fig.2.1

(The arrangements of numbers satisfy the conditions for the Pfaffian mentioned above.)

$$Q_m=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.\quad Q_n=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}.\quad F_m=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.\quad I_n=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The adjacency matrix of D is:

$$\mathrm{D}=\begin{bmatrix}0 & z & -z' & 0 & 0 & 0\\ -z & 0 & 0 & z' & 0 & 0\\ z' & 0 & 0 & z & -z' & 0\\ 0 & -z' & -z & 0 & 0 & z'\\ 0 & 0 & z' & 0 & 0 & z\\ 0 & 0 & 0 & -z' & -z & 0\end{bmatrix}.$$

According to Properties 1.3.1,

$$z\left(I_n\otimes Q_m\right)+z'\left(Q_n\otimes F_m\right)=\begin{bmatrix}0 & z & -z' & 0 & 0 & 0\\ -z & 0 & 0 & z' & 0 & 0\\ z' & 0 & 0 & z & -z' & 0\\ 0 & -z' & -z & 0 & 0 & z'\\ 0 & 0 & z' & 0 & 0 & z\\ 0 & 0 & 0 & -z' & -z & 0\end{bmatrix}.$$

Thus, $D=z\left(I_n\otimes Q_m\right)+z'\left(Q_n\otimes F_m\right)$.

In order to compute the determinant of D, we should begin by finding the eigenvectors of Q to create a matrix U.

$$\mathrm{U}_{n(l,l')}=(\frac{2}{n+1})^{\frac{1}{2}}i^{l}\sin(\frac{ll'\pi}{n+1}).\qquad \mathrm{U}_{n(l,l')}^{-1}=(\frac{2}{n+1})^{\frac{1}{2}}(-i)^{l'}\sin(\frac{ll'\pi}{n+1}).$$

Let λ be the eigenvalues of Q_n. Fix l', and

$$Q_nU_i=Q_n\left[(\frac{2}{n+1})^{\frac{1}{2}}\begin{pmatrix}i^1\sin l'\frac{1\pi}{n+1}\\ i^2\sin l'\frac{2\pi}{n+1}\\ \vdots\\ i^n\sin l'\frac{n\pi}{n+1}\end{pmatrix}\right],\qquad Q_nU_i=\left[(\frac{2}{n+1})^{\frac{1}{2}}\begin{pmatrix}i^2\sin l'\frac{2\pi}{n+1}\\ i^3\sin l'\frac{3\pi}{n+1}-i\sin l'\frac{\pi}{n+1}\\ \vdots\\ -i^{n-1}\sin l'\frac{(n-1)\pi}{n+1}\end{pmatrix}\right]$$

$$Q_n U_i = \lambda U_i \quad , \quad \lambda_{l'} = \begin{pmatrix} i^2 \sin l' \dfrac{2\pi}{n+1} \\ i^3 \sin l' \dfrac{3\pi}{n+1} - i \sin l' \dfrac{\pi}{n+1} \\ \vdots \\ -i^{n-1} \sin l' \dfrac{(n-1)\pi}{n+1} \end{pmatrix} \div \begin{pmatrix} i^1 \sin l' \dfrac{\pi}{n+1} \\ i^2 \sin l' \dfrac{2\pi}{n+1} \\ \vdots \\ i^n \sin l' \dfrac{n\pi}{n+1} \end{pmatrix} .$$

If $l \in [1,2,\ldots,n]$,

$$\lambda_{l'} = \frac{i^{l+1} \sin \dfrac{(l+1)l'\pi}{n+1} - i^{l-1} \sin \dfrac{(l-1)l'\pi}{n+1}}{i^l \sin \dfrac{ll'\pi}{n+1}} = \frac{-i^{l-1} \times (\sin \dfrac{(l+1)l'\pi}{n+1} + \sin \dfrac{(l-1)l'\pi}{n+1})}{i^l \sin \dfrac{ll'\pi}{n+1}}$$

$$= \frac{-(i^{l-1}) \times 2 \times \sin \dfrac{\frac{2ll'\pi}{n+1}}{2} \times \cos \dfrac{\frac{2l'\pi}{n+1}}{2}}{i^l \sin \dfrac{ll'\pi}{n+1}} = \frac{-i^{l-1} \times 2 \times \cos \dfrac{l'\pi}{n+1}}{i^l} = 2i \cos \frac{l'\pi}{n+1}.$$

Thus, we get the eigenvalue λ_i corresponding to u_i : $\lambda_i = 2i\cos(\dfrac{l'\pi}{n+1})$ (with l' fixed).

U_n is the eigenvalue of Q_n.

The transformation $Q_n' = U_n^{-1} Q_n U_n$ will diagnose Q_n, and the diagonal elements of Q_n' will be the eigenvalues of Q_n.

$$Q_n' = \begin{bmatrix} 2i\cos\dfrac{\pi}{n+1} & 0 & \ldots & 0 \\ 0 & 2i\cos\dfrac{2\pi}{n+1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 2i\cos\dfrac{n\pi}{n+1} \end{bmatrix}.$$

In the same way, $Q_m' = U_m^{-1} Q_m U_m$.

Let e_i be the i th standard basis vector and u_i be the i th column vector of U_m.

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \to \text{the } ith\ position \\ \vdots \\ 0 \end{bmatrix} .$$

So, we can get $U_m e_i = u_i$.

$Q_m U_m e_i = Q_m u_i = \lambda_i u_i$ (λ_i is the eigenvalue corresponding to u_i).

$U_m^{-1}(Q_m U_m e_i) = U_m^{-1}(\lambda_i u_i) = \lambda_i (U_m^{-1} u_i) = \lambda_i e_i$.

Since the entries of $U_{k'}$ are $i^k \sin\dfrac{kk'\pi}{m+1}$ (k is the row number, and we ignore the $\left(\dfrac{2}{m+2}\right)^{\frac{1}{2}}$ coefficient, which has no impact on our calculations.)

The entries of $F_m \times u_{k'}$ takes the form $(-1)^k \sin\dfrac{kk'}{m+1}$.

Let $j = m+1-k'$, thus $k = -j+m+1$.

So, u_j takes the form $u_j = i^k \sin(\dfrac{k(m+1-k')}{m+1}\pi) = i^k \sin(\dfrac{k(m+1-k')}{m+1}\pi) = i^k \sin(k\pi - \dfrac{kk'\pi}{m+1})$.

When k is odd, $i^k \sin(k\pi - \dfrac{kk'\pi}{m+1}) = -i^k \sin(-\dfrac{kk'\pi}{m+1}) = i^k \sin(\dfrac{kk'\pi}{m+1}) = -(-i)^k \sin(\dfrac{kk'\pi}{m+1})$.

When k is even, $i^k \sin(k\pi \quad \dfrac{kk'\pi}{m+1}) = i^k \sin(-\dfrac{kk'\pi}{m+1}) = -i^k \sin(\dfrac{kk'\pi}{m+1}) = (\quad i)^k \sin(\dfrac{kk'\pi}{m+1})$.

So, $i^k \sin(k\pi - \dfrac{kk'\pi}{m+1}) = -(-i)^k \sin(\dfrac{kk'\pi}{m+1}) = u_j$.

Since $F_m u_{k'} = -u_j = -u_{m+1-k'}$, and we've had $-u_{m+1-k'} = -U_m \cdot e_{m+1-k'}$ earlier, so we can get $F_m u_{k'} = -U_m e_{m+1-k'}$.

Then we get $U_m^{-1} \cdot (F_m u_{k'}) = U_m^{-1} \cdot (-U_m e_{m+1-k'})$, thus $u_{k'} = U_m \cdot e_{k'}$.

In this way, $U_m^{-1}(F_m U_m e_{k'}) = -U_m^{-1} \cdot U_m \cdot e_{m+1-k'}$, so $(U_m^{-1} F_m U_m) e_{k'} = -e_{m+1-k'}$.

So, F_m' takes an anti-diagonal form. All the entries of the anti-diagonal of F_m' are (-1) .

$F_m' = U_m^{-1} F_m U_m$.

$$F_m' = \begin{bmatrix} 0 & \cdots\cdots\cdots & 0 & -1 \\ \vdots & & -1 & 0 \\ \vdots & \iddots & & \vdots \\ 0 & -1 & & \vdots \\ -1 & 0 & \cdots\cdots & 0 \end{bmatrix} .$$

Our final goal is to diagonalize D .

Let $V = U_n \otimes U_m$.

Take the transformation $D' = V^{-1} D V$ to diagonalize D .

According to Properties 1.3.1, we get

$V^{-1} D V = (U_n^{-1} \otimes U_m^{-1})[z(I_n \otimes Q_m + z'(Q_n \otimes F_m)](U_n \otimes U_m)$.

$$D' = z(U_n^{-1} \otimes U_m^{-1})(I_n \otimes Q_m)(U_n \otimes U_m) + z'(U_n^{-1} \otimes U_m^{-1})(Q_n \otimes F_m)(U_n \otimes U_m) \quad .$$

So, $D' = z(U_n^{-1} I_n \otimes U_m^{-1} Q_m)(U_n \otimes U_m) + z'(U_n^{-1} Q_n \otimes U_m^{-1} F_m)(U_n \otimes U_m)$,

$$D' = z(U_n^{-1} I_n U_n \otimes U_m^{-1} Q_m U_m) + z'(U_n^{-1} Q_n U_n \otimes U_m^{-1} F_m U_m) \quad .$$

Theorem.2.1 $D' = z(I_n' \otimes Q_m') + z'(Q_n' \otimes F_m')$.

Proof. Let p_k, q_k be

$$p_k = \begin{bmatrix} 0 & 0 & \cdots & -2i\cos\frac{k\pi}{n+1} \\ 0 & -2i\cos\frac{k\pi}{n+1} & & 0 \\ \vdots & \iddots & & \vdots \\ -2i\cos\frac{k\pi}{n+1} & \cdots & 0 & 0 \end{bmatrix}$$, with m rows and m columns.

$$q_k = \begin{bmatrix} 2i\cos\frac{k\pi}{m+1} & 0 & \ldots & 0 \\ 0 & 2i\cos\frac{k\pi}{m+1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 2i\cos\frac{k\pi}{m+1} & \end{bmatrix}$$, with n rows and n columns.

So, $I_n \otimes Q_m' = \begin{bmatrix} q_1 & 0 & \ldots & 0 \\ 0 & q_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & q_m \end{bmatrix}$, $Q_n' \otimes F_m' = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & p_n \end{bmatrix}$.

Then the entries of D' take the form:

$$D'_{(k,l)(k',l')} = 2iz\delta_{k,k'}\delta_{l,l'}\cos\frac{k\pi}{m+1} - 2iz'\delta_{k+k',m+1}\delta_{l,l'}\cos\frac{l\pi}{n+1} \quad ,$$

which is composed of two two-dimensional planes (k,k') and (l,l').

The (k,k') is formed from F_m and $k=k'$. The (l,l') is formed from Q_n and $l=l'$.

The $\delta_{l,j}$ is the Kronecker delta, which is 0 when $i \neq j$ and 1 when $i=j$.

Since D' is an $mn \times mn$ matrix, it can be divided into $n \times n$ parts with $m \times m$ cells.
Now, we can find that D' is formed by $m \times m$ cells along the diagonal where the non-zero elements from a cross shape, with 0 entries everywhere else, as seen left below.
Each swap of two adjacent rows or columns can change the sign of the determinant.
First, we swap the second and last columns, and get the one on the right.

$$\begin{pmatrix} a_1 & 0 & 0 & & 0 & 0 & b_1 \\ 0 & a_2 & 0 & \cdots & 0 & b_2 & 0 \\ 0 & 0 & a_3 & & b_3 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & c_3 & & d_3 & 0 & 0 \\ 0 & c_2 & 0 & \cdots & 0 & d_2 & 0 \\ c_1 & 0 & 0 & & 0 & 0 & d_1 \end{pmatrix} \qquad \begin{pmatrix} a_1 & b_1 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_2 & a_2 \\ 0 & 0 & a_3 & & b_3 & 0 & 0 \\ & \vdots & & & \vdots & & \\ 0 & 0 & c_3 & & d_3 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & d_2 & c_2 \\ c_1 & d_1 & 0 & & 0 & 0 & 0 \end{pmatrix}$$

Next, we need to swap the third and last columns, then the fourth and the last, until the $m-1$ th column becomes the last column. In these steps, we have swapped $m-2$ times and the determinant will not be affected. In this way, we can get a 2×2 cell in the top-left corner and a $(m-2)\times(m-2)$ cross in the bottom-right corner.

Then, we continue to produce 2×2 cells along the diagonal, until finally, D' becomes a cell diagonal matrix.

$$\begin{pmatrix} a_1 & b_1 & 0 & 0 & & 0 & 0 \\ c_1 & d_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & 0 & & 0 & b_2 \\ 0 & 0 & 0 & a_3 & & b_3 & 0 \\ & & \vdots & & & \vdots & \\ 0 & 0 & 0 & c_3 & \cdots & d_3 & 0 \\ 0 & 0 & c_2 & 0 & & 0 & d_2 \end{pmatrix} \qquad \begin{pmatrix} a_1 & b_1 & 0 & 0 & & 0 & 0 \\ c_1 & d_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & b_2 & & 0 & 0 \\ 0 & 0 & c_2 & d_2 & & 0 & 0 \\ & & \vdots & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & a_{\frac{m}{2}} & b_{\frac{m}{2}} \\ 0 & 0 & 0 & 0 & & c_{\frac{m}{2}} & d_{\frac{m}{2}} \end{pmatrix}$$

Conclusion.2.2 The determinant equals the product of all the determinants of each 2×2 block.

Proof.

Let $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = A_1$. $\begin{pmatrix} a_3 & a_4 \\ b_3 & b_4 \end{pmatrix} = A_2$. $\begin{pmatrix} a_{2m-1} & a_{2m} \\ b_{2m-1} & b_{2m} \end{pmatrix} = A_n$.

$$A_1 \cdot \begin{pmatrix} A_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} = A_1 \cdot A_2 \cdot \begin{pmatrix} A_3 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix} = \cdots\cdots = A_1 \cdot A_2 \cdots A_{n-2} \cdot \begin{bmatrix} A_{n-1} & 0 \\ 0 & A_n \end{bmatrix} = A_1 \cdot A_2 \cdots A_n$$

$$= \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{pmatrix}.$$

Conclusion.2.3

$$\det D = \det D' = \prod_{k=1}^{\frac{1}{2}m} \prod_{l=1}^{n} \det \begin{bmatrix} 2iz\cos\frac{k\pi}{m+1} & -2iz'\cos\frac{l\pi}{n+1} \\ -2iz'\cos\frac{l\pi}{n+1} & -2iz\cos\frac{k\pi}{m+1} \end{bmatrix}$$

$$=\prod_{k=1}^{\frac{1}{2}m}\prod_{l=1}^{n}4(z^2\cos^2\frac{k\pi}{m+1}+z'^2\cos^2\frac{l\pi}{n+1}).$$

Definition.2.4 $Z_{m,n}(z,z')=\sum_{h,v}g(h,v)z^h z'^v$ is the generating function in z , z' where the sum ranges over $h,v\geq 0$ and satisfies $2(h+v)=mn$. $g(h,v)$ means the number of tiling with h horizontal dominoes and v vertical dominoes.

Conclusion.2.5 $Z_{m,n}(1,1)$ returns the total number of domino tiling for an $m\times n$ grid.

As we've reached the conclusion that $Z_{m,n}(z,z')=\sqrt{\det D}$,we get:

$$Z_{m,n}(1,1)=\prod_{k=1}^{\frac{1}{2}m}\prod_{l=1}^{n}2\sqrt{\cos^2\frac{k\pi}{m+1}+\cos^2\frac{l\pi}{n+1}}.$$

3.OTHER METHODS AND APPLICATIONS FOR DOMINO TILING

3.1 checkerboard coloring and generalized coloring

Definition 3.1.1(checkerboard coloring) Checkerboard coloring is to color the grids in non-rectangular shapes in black or white, and we need to make sure that every two adjacent grids are not in the same color.

Take the 8×9 grid as an example. Obviously, it can be tiled (Fig.3.1.1). Then we take out the top-left one and the top-right one, and the number of black grids isn't the same as that of white ones, so it cannot be tiled. Another example that can't be tiled has 21 black ones and 15 white ones. (Fig.3.1.3)

Preposition 3.1.2 Any graph with different numbers of black and white cells can't be tiled.

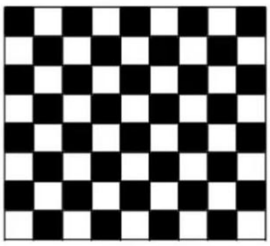


Fig.3.1.1

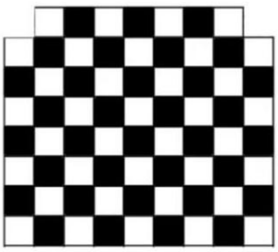


Fig.3.1.2

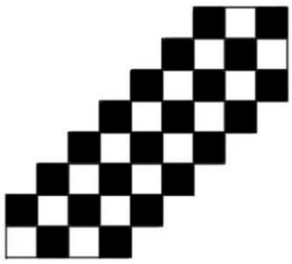


Fig.3.1.3

Preposition 3.1.3 The checkerboard can always be tiled no matter which black block and white cells are removed.

Proof. First, cover the black cell, then all the white cells are an even number of cells away. This means that when we carve out that second white cell, there will always be an even number of cells between the holes along the paths that can be covered with dominoes.

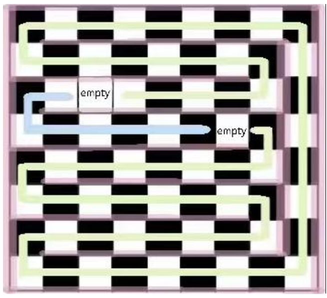


Fig.3.1.4

Consider the square lattice with its associated free group $F=\langle A,U\rangle$ and cycle group $C=[F:F]$. $C=[F:F]$ is an example of the cycle group C consisting of all words W such that $P(W)$ is a close directed path. The group $[C:C]$ consists of all words W such that $P(W)$ is a close directed path in $\mathbb{Z}^2$ with winding number 0 around every cell in $\mathbb{Z}^2$. [7]

Definition 3.1.4 (generalized coloring) A generalized coloring map is any homomorphism $\phi: C \to A$, where A is an abelian group.[7]

Theorem 3.1.5 The group $A_0 = C/[C:C]$ is a direct sum of a countable number of copies of $\mathbb{Z}$, which are in one-to-one correspondence with the cells c_{ij} of the lattice $\mathbb{Z}^2$. The projection map $\pi_{i,j}: C \to \mathbb{Z}$ onto the c_{ij} th $\mathbb{Z}$-summand of A_0 is given by the winding number $w(P(W);c_{ij})$.

Definition 3.1.6 (generalized coloring argument) A generalized coloring argument uses a generalized coloring map ϕ in order to show that R, a simply connected region, cannot be tiled by tiles in a set $\sum$ by showing that the image of the combinatorial boundary $[\partial R]$ under ϕ is not included in the image of the tile group $T(\sum)$ under ϕ .[7] Note that $T(\sum)=N(\langle U^{-1}A^{-2}UA^2, U^{-2}A^{-1}U^2A\rangle)$. According to checkerboard tiling, Fig.3.1.3 cannot be tiled because of the difference in the number of black and white cells. In terms of generalized coloring, $\phi([\partial R])=\{(21,15),(15,21)\}$,while $\phi(T(\sum))=\{(n,n):n\in\mathbb{Z}\}$, so R cannot be tiled by dominoes. Let $B(\sum)$ be the boundary tiling group. $B(\sum)$ is the smallest normal subgroup of C containing $T(\sum)$ and $[C:C]$.

Preposition 3.1.7 $B(\sum)=T(\sum)[C:C]$.

Proof. It is clear that $T(\quad)[C:C] \quad B$. Use the fact that $G_1G_2=\{g_1g_2 : g_1 \quad G_1, g_2 \quad G_2\}$ is a normal subgroup of G whenever G_1 and G_2 are both normal subgroups of G to prove the other inclusion. Since $B(\sum)$ is the smallest normal subgroup of C containing $T(\sum)$ and $[C:C]$, and $T(\sum)[C:C]$ is the normal subgroup of C, so $B(\quad) \quad T(\quad)[C:C]$.

3.2 Height functions and Cayley graph

Let K be the Aztec graph with vertices $\{(a,b)\in Z^2 : |a|+|b|\le n+1\}$ and color it in the way of a black-white checkerboard, with arrows circulating clockwise around white squares and counterclockwise around black squares. (Fig.3.2.1 shows the case $n=3$). Fig.3.2.2 shows one tiling as an example. We set the leftmost point as the origin point and weigh it as 0 .Then, add 1 in the direction of arrow, and we get Fig.3.2.3 which represents a method of tiling. Therefore, we can get a complete and non-contradictory labeling which is a necessary condition for tiling.

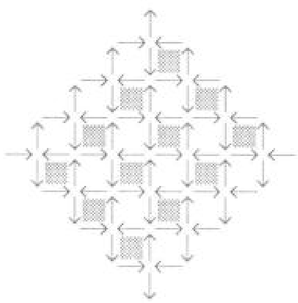


Fig.3.2.1

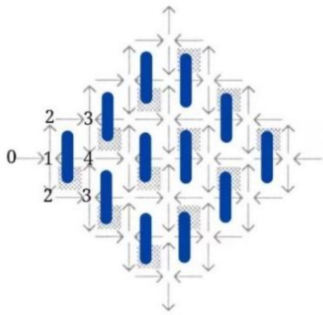


Fig.3.2.2

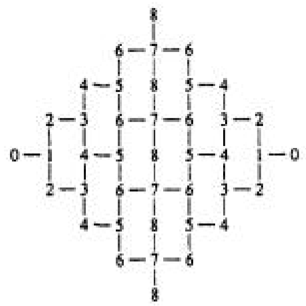


Fig.3.2.3

With the help of height function, we can judge whether a graph can be tiled. If two possible numbers appear on one point, there will be a contradiction, so the graph can't be tiled. (i.e. Fig.3.2.4)

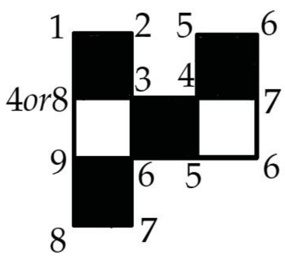


Fig.3.2.4

Take the group $G=\langle x,y \mid xy^2=y^2x,\ yx^2=x^2y\rangle$. Then, we can get the Cayley graph of G (Fig.3.2.5).

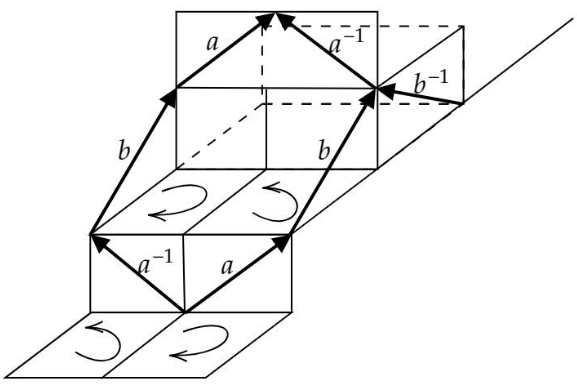


Fig.3.2.5

Fig.3.2.1 is an example of the bottom floor of this Cayley graph. Only the vertices with ingoing and outgoing edges are the vertices belonged to this Cayley graph. As the graph is three-dimensional, the height of the vertices in the Cayley graph corresponds to that of the number labeled by height function.

There are four different directions a, a^{-1}, b, b^{-1} in the Cayley graph, with a, a^{-1} being in the same plane, and b, b^{-1} being in another same plane perpendicular to that of the a and a^{-1} plane. a points right, a^{-1} points left, b points inside and b^{-1} points outside. If the direction of an edge coincides with the circulating direction of the bottom floor, then the edge points upward, and vice versa. The edges collectively form the shape of spirals, with every vertex having its own height.

3.3 Induction over Aztec diamond

Define the order-n Aztec diamond as $A(\mathrm{n})$. $A(2)$ can be tiled and decorated with arrows in opposite directions. (Fig.3.3.1)

Then, move the tiles one step in the direction of the arrows and we get two 2×2 regions in blue. (Fig.3.3.2)

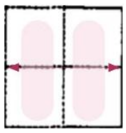


Fig.3.3.1

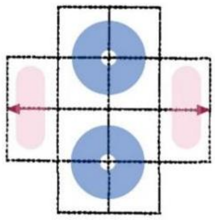


Fig.3.3.2

We fill these 2×2 with pairs of arrowed tiles, either like Fig.3.3.3 or like Fig.3.3.4.

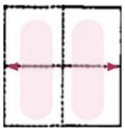


Fig.3.3.3

Fig.3.3.4

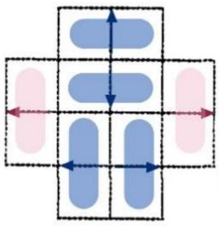


Fig.3.3.5

Then an arrowed tiling of $A(2)$ (Fig.3.3.5) is extended to $A(3)$ by moving all the dominos one step in the direction of the arrows, and we get three 2×2 regions. Again, we can fill every 2×2 pairs with pairs of arrowed dominos in two ways.

To expand to $A(4)$, we can't slide straight way because there are some arrows pointing towards each other. (Fig.3.3.6)

There, remove those yellow tiles, and slide other arrowed dominos.

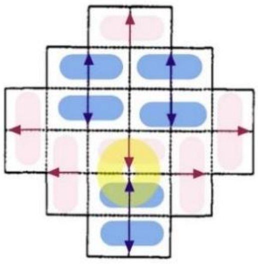


Fig.3.3.6

Fill them in randomly and repeat. In particular, dominos will never end up overlapping after the sliding phase, and at the end of it, the region left to be filled is a collection of 2×2 grids.

It is obvious that the first diamond $A(1)$ has exactly two tilings.

Since all tilings of $A(2)$ come from tilings of $A(1)$, $A(2)$ has 2^{1+2} tilings.

Since $A(3)$ comes from $A(2)$, $A(3)$ has 2^{1+2+3} tilings, and so on. (Fig.3.3.7)

If no arrows are clashing, an order-n Aztec diamond has n 2×2 empty grids.
If two arrows are clashing, an order-n Aztec diamond has $(n+1)$ 2×2 empty grids. But in every 2×2 grid, there are two ways of clashing, so we still can get $\frac{2^{n+1}}{2}=2^n$ ways to tile an order-n Aztec diamond.

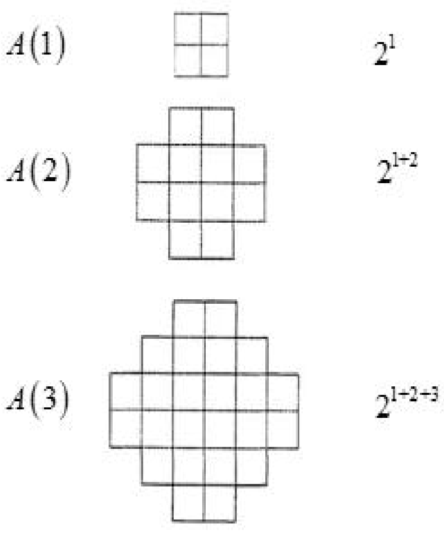


Fig.3.3.7

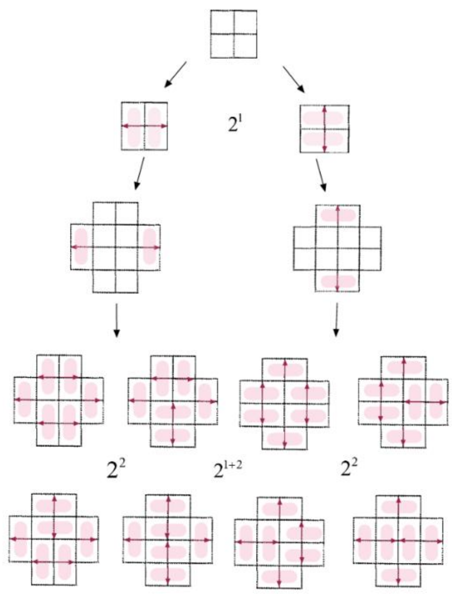


Fig.3.3.8

Conclusion 3.3.1 The number of tiling for an n-order Aztec diamond is $2^{1+2+\dots+n}$.

3.4 Calculations of Pfaffians and ways to find Pfaffian orientation

The 2-order Aztec diamond's Pfaffian orientation can be as follows:

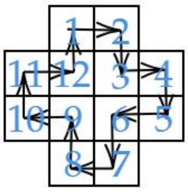


Fig.3.4.1

Number the grids from 1 to 12 in the direction of the Pfaffian orientation.
Suppose the adjacent matrix of the graph is D.

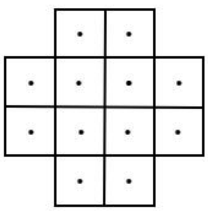
$$D=\begin{bmatrix} 0 & z' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\ -z' & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -z & 0 & z' & 0 & z & 0 & 0 & 0 & 0 & 0 & z' \\ 0 & 0 & -z' & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & 0 & z' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -z & 0 & -z' & 0 & z & 0 & z' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z & 0 & z' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -z' & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -z' & 0 & -z & 0 & z' & 0 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z' & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -z & 0 & z' \\ -z & 0 & -z' & 0 & 0 & 0 & 0 & 0 & -z & 0 & -z' & 0 \end{bmatrix}.$$

Compute the determinant of D, and we get 64. From $(\mathrm{Pf}\ D)^2 = |D|$, we get $\mathrm{Pf}\ D = 8$, which is equal to the number of tilings.

If the graph has a Pfaffian orientation, it is said to be Pfaffian. Every planar graph can find its Pfaffian orientation. An orientation in which each face of a planar graph has an odd number of clockwise-oriented edges is automatically Pfaffian. To find such an orientation, we start with finding an arbitrary orientation of a spanning tree of the graph. The remaining edges, not in this tree, form a spanning tree of the dual graph, and their orientations can be chosen according to a bottom-up traversal of the dual spanning tree in order to ensure that each face of the original graph has an odd number of clockwise edges.

Actually, we can find the Pfaffian orientation as follow:

Step 1. Turn every grid into a vertex. (Figure 3.4.1) Join the vertices, and an adjacent graph will be formed (Figure 3.4.2).



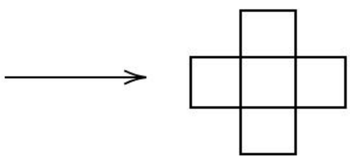


Fig.3.4.1 Fig.3.4.2

Step 2. Turn some of the edges in real lines to dotted lines, so that every vertex is connected to at least one dotted line, and the real lines do not form a loop. (Fig.3.4.3)

Step 3. Find the dual graph (Figure3.4.4), which is formed by turning every grid in the adjacent graph into a vertex.

Step 4. Decide the sequence of the grid of which we will decide its orientation of the edges. We begin by taking a dot outside, independent of the vertices in the dual graph, and draw the connection between those dots according to the positions of the dotted lines shown in Figure3.4.5. It is the spanning tree.

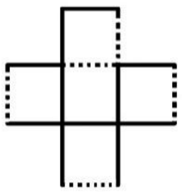


Fig.3.4.3

Fig.3.4.4

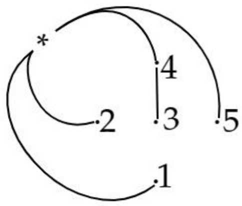


Fig.3.4.5

Step 5. Decide the orientation of each face according to the sequence of numbers. Since each face of the original graph has an odd number of clockwise edges, we can get the following Pfaffian orientation.

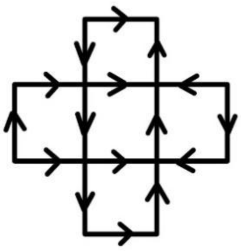


Fig.3.4.6

Here's another example to show how to find the Pfaffian orientation of an octahedron, with a cone cut at every corner. (Fig.3.4.7). Fig.3.4.8 shows its planar structure.

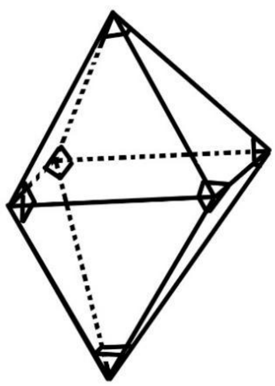


Fig.3.4.7

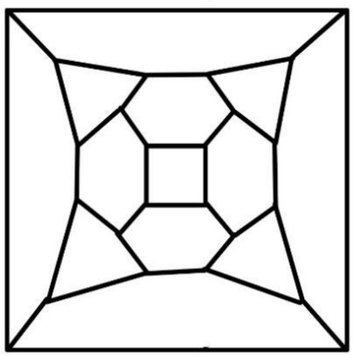


Fig.3.4.8

Then we are able to draw the Pfaffian orientation. It must satisfy the condition that every face has an odd number of clockwise edges.

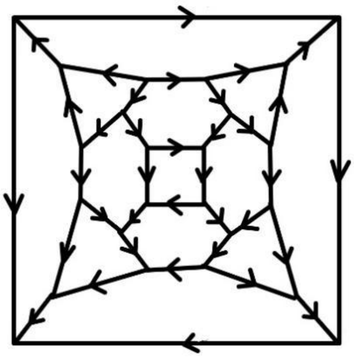


Fig.3.4.9

3.5 More examples of the number of perfect matchings

In this section, to calculate the number of perfect matchings for a spherical shape, we can assign a Pfaffian orientation to them, which means every even circulation should have odd numbers of edges in each direction. It follows by the results of [2] which states that each planar graph has an Pfaffian orientation, and we can get the following theorem easily.

Theorem 3.5.1 The spherical graph with even vertices can be equipped with a Pfaffian orientation.

The following Fig. 3.5.1 is an example of the tiling of a cube.

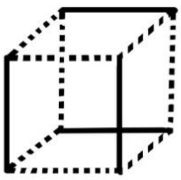
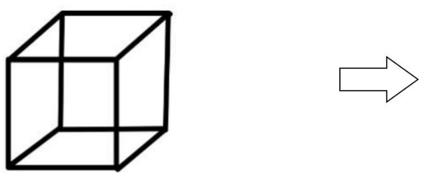


Fig.3.5.1

To figure out the number of tilings for a cube, we need to find out the Pfaffian orientation.

First, turn the three-dimensional cube (Fig.3.5.2) into a planar shape (Fig.3.5.3).



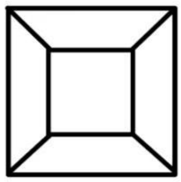
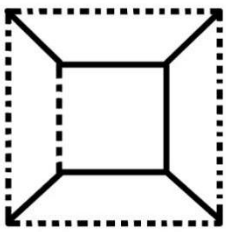


Fig.3.5.2 Fig.3.5.3

Then, turn some of the edges in real lines to dotted lines and it forms a spanning tree. As is shown in Fig.3.5.4.



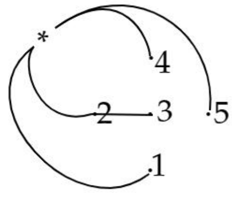


Fig.3.5.4 Fig.3.5.5

Finally, follow the sequence of number and we can get the Pfaffian orientation of the cube. (Fig.3.5.6)

With the help of Pfaffian orientation, we can calculate the value of Pfaffian
Number the grids from 1 to 8 . (Fig.3.5.7)

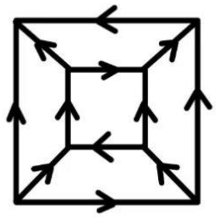


Fig.3.5.6

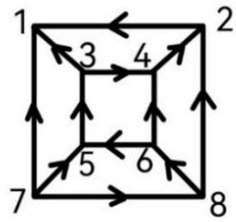


Fig.3.5.7

Suppose the adjacent matrix of the graph is D .

$$D=\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}.$$

$\det D = 81$.

From $(\mathrm{Pf}\ D)^2 = |D|$, we get $\mathrm{Pf}\ D = 9$, which is equal to the number of tiling.

In the same way, we can calculate the number of perfect matchings for a hexagonal prism. (Fig.3.5.8). Fig.3.5.9 is the planar structure of the shape, showing the Pfaffian orientation.

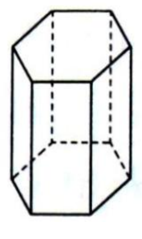


Fig.3.5.8

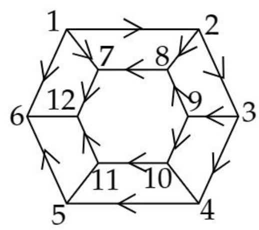


Fig.3.5.9

Suppose the adjacent matrix of the graph is E .

$$E=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Calculation shows $\det E = 400$. From $(\text{Pf } E)^2 = |E|$, we get $\text{Pf } E = 20$, which is equal to the number of tiling.

[9] shows a classification of Cayley graphs of dihedral groups which can be equipped with a Pfaffian orientation. It could be seen as generalization of the above example.

3.6 Perfect matchings of symmetric graph

As we saw in the last section, we want to give a uniform method to calculate the determinant of a graph with symmetric structure. So, we investigate the Pfaffian problem over these graphs. We shall make use of the following results in [10] and [11], where Lovasz concluded that for a graph with a transitive automorphism group， its spectrum can always be reduced to that of a Cayley graph, since its automorphism group has a regular subgroup. Furthermore, Lovasz gave a formula for the spectrum in terms of group characters as demonstrated in [11].

Theorem 3.6.1 The spectrum of the Cayley color graph $X = X(G;\alpha)$ can be arranged as

$$\Lambda=\{\lambda_{ijk}:\text{i}=1,\ldots,\text{h}; \text{ j,k}=1,\ldots,\text{n}_i\}.$$

such that $\lambda_{ij1} = \ldots = \lambda_{ijn_i}$ (this common value will be denoted by λ_{ij}), and

$$\lambda_{i,1}^t + \ldots + \lambda_{i,n_i}^t = \sum_{g_1,\ldots,g_t \in G} \left(\prod_{s=1}^t \alpha(g_s) \right) \chi_i \left(\prod_{s=1}^t g_s \right).$$

Since Cayley graph is a regular graph, the coloring $\alpha : E(X) \to \{-1,0,1\}$ gives a regular orientation while the Pfaffian orientation always comes from some irregular constructions. If there is a Cayley color graph which leads to certain Pfaffian orientations, then we can use the above formula to get the eigenvalues, the determinant and finally the number of tilings. But we answer negatively to this case considering the perfect matching problem where the number of vertices is even.

Theorem 3.6.2 For a Cayley color graph $X = X(G;\alpha)$ with $\alpha : E(X) \to \{-1,0,1\}$ and $|G|$ even, the coloring never leads to a Pfaffian orientation.

Proof. Assume some integers k , $2^k | G$. Then by Sylow lemma, there are some subgroups H with $|H| = 2^k$. So, there should be some elements with even order, which leads to an even circulation with 0 clockwise-oriented or counter-clockwise-oriented edges on the Cayley color graph.

Then the orientation can't be Pfaffian as the Pfaffian orientation needs an odd number of clock-wise oriented edges for each even circulation. So, by theorem 3.6.2, We cannot apply theorem 3.6.1 directly to calculate the number of perfect matchings. While the light-switch problem focuses more on the parity of the determinant, the formula can still be used to solve light-switch problem over Cayley graph. We consider the parity of determinant of Cayley graph for cyclic groups, finite Abelian groups and dihedral groups as examples applying the formula in theorem 3.6.1 and give general conclusions to the light-switch problem over these graphs. The following formula is the brief form of theorem 3.6.1 when the group is Abelian [10].

Theorem 3.6.3 Denoting by $\{\lambda_1,\ldots,\lambda_n\}$ the spectrum of the Cayley color graph $X = X(G;\alpha)$ of the Abelian group G we have:

$$\lambda_i = \sum_{g\in G}\alpha(g)\chi_i(g) \quad i = (1,\ldots,n) .$$

Theorem3.6.4 Cayley graph for cyclic groups $Cay(G,S)$ where $G=\{1, r^1, r^2 \ldots r^{n-1}\}$ and $S=\{r^\theta, r^{-\theta}\}$ have $\det = 0$ (i.e. whose adjacent matrix is singular).

Proof. The cyclic group $G=\{1, r^1, r^2 \ldots r^{n-1}\}$ has *n* one-dimensional representations. Suppose the Cayley graph (G,S), where $S=\{r^\theta, r^{-\theta}\}$. Assume the coloring of the Cayley graph is $\alpha: r^\theta \mapsto 1, r^{-\theta} \mapsto -1$ (i.e. the graph is an oriented one relative to this coloring).

We get

$$\chi_h(r^k) = e^{\frac{2\pi i hk}{n}}, h = 1, 2, \ldots, n .$$

Since $\lambda_h = \sum_{g\in G}\alpha(g)\chi_h(g) = \chi_h(r^\theta) - \chi_h(r^{-\theta}) = e^{\frac{2\pi i h\theta}{n}} - e^{-\frac{2\pi i h\theta}{n}} = 2i\sin(\frac{2\pi h\theta}{n})$, we get

$$\prod_h \lambda_h = (2i)^n \prod_{h=1}^{n} \sin(\frac{2\pi h\theta}{n}) .$$

Especially, when $h = n$, $\sin(\frac{2\pi h\theta}{n}) = 0$, thus $\prod_h \lambda_h = 0$.

Hence $\prod_h \lambda_h \equiv 0 \pmod 2$, which answer negatively to either existence or uniqueness of solution to light-switch problem in this case.

Remark 3.6.5 Actually, for a cyclic group, the parity of the number of 'on' state (resp. 'off' state) is an invariant. For example, an odd number of 'on' lights remain odd after several operations. Thus, if we want to achieve the state of all lights being 'on', an even number of lights should be 'on' at the initial state, which gives an elementary proof to the above result.

Furthermore, we can extend the calculation to all abelian groups.

Theorem 3.6.6 For Cayley graph of finite abelian groups $\Omega = Cay(G,S)$ where $G = \bigoplus_1^T \mathbb{Z}/q_k\mathbb{Z}$

1) $\det\Omega = 0 \pmod 2$ when $q_i \geq 3$ for all $i \in \{1,2,\ldots,T\}$ and $S = \{I_1, I_1^{-1}, I_2, I_2^{-1}, \ldots, I_T, I_T^{-1}\}$, where I_k is the generator of $\mathbb{Z}/q_k\mathbb{Z}$.

2) When $q_i = 2$ for each $i \in \{1,2,\ldots,T\}$ and $S = \{I_1, I_2, \ldots, I_T\}$, where I_k is the generator of each summand $\mathbb{Z}/2\mathbb{Z}$, $\det\Omega \equiv 1 \pmod 2$ when T is odd, and $\det\Omega = 0 \pmod 2$ when T is even.

$\rho_{i_1}^{(1)}, \rho_{i_2}^{(2)} ...... \rho_{i_T}^{(T)}$ $(1 \le i_k \le q_k)$ are the irreducible representations of G, so $\rho_{i_1}^{(1)} \otimes \rho_{i_2}^{(2)} \otimes ...... \otimes \rho_{i_T}^{(T)}$ is an irreducible representation of G. The characters are $\chi_{i_1,i_2......i_T} = \chi_{i_1}^{(1)} \cdot \chi_{i_2}^{(2)} \cdot \chi_{i_3}^{(3)} ...... \chi_{i_T}^{(T)}$.

$$\lambda_{i_1,i_2......i_T} = \sum_{g \in G} \alpha(g) \chi_{i_1,i_2......i_T}(g).$$

$$\chi_{i_k}^{(k)}(I_k{}^m) = e^{\frac{2i\pi i_k m}{q_k}}.$$

Case 1: For non-cubic graphs, $m=1$ or $m=-1$.

Hence, $\det\Omega = \prod_{(i_1,i_2...i_T)} 2i \sum_1^T \sin(\frac{2\pi i_k}{q_k}) = (-1)^{\frac{|G|}{2}} \cdot 2^{|G|} \prod_{(i_1,i_2...i_T)} \sum_1^T \sin(\frac{2\pi i_k}{q_k}) = 0$.

Case 2: For cubes and hypercubes, $m=1$.

$\lambda_{i_1} \equiv T \pmod 2, \lambda_{i_2} \equiv T \pmod 2, ... \lambda_{i_T} \equiv T \pmod 2$, so $\lambda_{i_1} \cdot \lambda_{i_1} ...... \cdot \lambda_{i_T} \equiv T^{|G|} \pmod 2$.

So, the parity of T determines the parity of the determinant.

Remark 3.6.7 Actually, there is an elementary explanation considering the hypercube case.
When T is odd, for a fixed light, we can switch every light adjacent to it to make only this light change its state while any other light remains unchanged. Therefore, for hypercubes in this case, we can change the state of lights one by one to achieve the final state we want.
When T is even, the parity of the number of lights being 'on' or 'off' remains unchanged. If one light is on at the initial state, we will never achieve the state of all lights being 'off'.

Consider any finite abelian group in general: $(\mathbb{Z}/2\mathbb{Z})^{T_1} \oplus \mathbb{Z}/q_1\mathbb{Z} \oplus \mathbb{Z}/q_2\mathbb{Z} \oplus ...... \oplus \mathbb{Z}/q_{T_2}\mathbb{Z}$, where the number of $\mathbb{Z}/2\mathbb{Z}$ summand is T_1.The general finite abelian groups can be reduced to the case of hypercubes $(\mathbb{Z}/2\mathbb{Z})^{T_1}$ case through both spectral calculation similar to case 1 in theorem 3.6.6, or by a combinatoric adjustment method. Both of which can be obtained easily.

Theorem 3.6.8 The determinant of a Cayley graph for dihedral group $\Omega = Cay(D_n, S)$ is even when $n \equiv 0 \pmod 3$ and odd when $n \equiv 1,2 \pmod 3$, i.e.

$$\det\Omega = \begin{cases} 0 \pmod 2 & n \equiv 0 \pmod 3 \\ 1 \pmod 2 & n \equiv 1,2 \pmod 3 \end{cases}.$$

where $D_n = \langle \sigma, H \mid \sigma^n = 1, H^n = 1, \sigma H = H\sigma^{-1} \rangle$ and $S = \{\sigma, \sigma^{-1}, H\}$.

So, we conclude that an n-side prism can find the only solution in the light-switch problem when $n \equiv 1,2 \pmod 3$.

Proof. We set the coloring $\alpha(\sigma) \mapsto 1$, $\alpha(\sigma^{-1}) \mapsto -1$, $\alpha(H) \mapsto 1$, and there are only

representations of 1-dimension and 2-dimension in both the two cases as follows.
Case 1: when n is even,

There are four 1-dimension representations [12], and we denote their characters as $\psi_1, \psi_2, \psi_3, \psi_4$:

	σ^k	$H\sigma^k$
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$

is the character table.

Then we can get :

$$\lambda_{1,1} = \sum_g \alpha(g)\psi_1(g) = \alpha(\sigma)\psi_1(\sigma^1) + \alpha(\sigma^{-1})\psi_1(\sigma^{-1}) + \alpha(H)\psi_1(H\sigma^0) = 1 - 1 + 1 = 1.$$

$$\lambda_{2,1} = \sum_g \alpha(g)\psi_2(g) = 1 - 1 - 1 = -1.$$

$$\lambda_{3,1} = \sum_g \alpha(g)\psi_3(g) = -1 - (-1) + 1 = 1.$$

$$\lambda_{4,1} = \sum_g \alpha(g)\psi_4(g) = -1 - (-1) - 1 = -1.$$

And for 2-dimension representations, the character table is as following:

	σ^k	$H\sigma^k$
$\psi_h(0 < h < \frac{n}{2})$	$2\cos\frac{2\pi hk}{n}$	0

So, we get

$$\lambda_{h,1}^2 + \lambda_{h,2}^2 = \sum_{g_1, g_2 \in G} \left(\prod_{s=1}^{2} \alpha(g_s) \right) \chi_h \left(\prod_{s=1}^{2} g_s \right)$$

$$= 2 \times \left(\alpha(\sigma) \cdot \alpha(\sigma^{-1}) \cdot \psi_h(\sigma^1 \cdot \sigma^{-1}) + \alpha(\sigma) \cdot \alpha(H) \cdot \psi_h(\sigma \cdot H) + \alpha(\sigma^{-1}) \cdot \alpha(H) \cdot \psi_h \ \sigma^{-1} \cdot H) \right)$$

$$+ \alpha(\sigma) \cdot \alpha(\sigma) \cdot \psi_h(\sigma^2) + \alpha(\sigma^{-1}) \cdot \alpha(\sigma^{-1}) \cdot \psi_h(\sigma^{-2}) + \alpha(H) \cdot \alpha(H) \cdot \psi_h(H^2)$$

$$= 2 \times (1 \cdot (-1) \cdot 2 + 0 + 0) + 1 \cdot 1 \cdot \cos\frac{2\pi h \cdot 2}{n} + (-1) \cdot (-1) \cdot \cos\frac{2\pi h \cdot 2}{n} + 1 \cdot 1 \cdot 2 = -2 + 4\cos\frac{4\pi h}{n}.$$

$$\lambda_{h,1}^1 + \lambda_{h,2}^1 = 1 \cdot 2\cos\frac{2\pi h}{n} - 1 \cdot 2\cos\frac{2\pi h \cdot (-1)}{n} = 0.$$

$$\lambda_{h,1}^1 = \sqrt{2\cos\frac{4\pi h}{n} - 1}, \quad \lambda_{h,2}^1 = -\sqrt{2\cos\frac{4\pi h}{n} - 1}.$$

$$\lambda_{h,1}^1 \cdot \lambda_{h,2}^1 = 1 - 2\cos\frac{4\pi h}{n}.$$

$$\det\Omega=\prod_{h=1}^{\frac{n}{2}-1}(1-2\cos\frac{4\pi h}{n})^2=\begin{cases}1 & n\equiv 2\ (\mathrm{mod}\ 12)\\ 9 & n\equiv 4\ (\mathrm{mod}\ 12)\\ 4 & n\equiv 6\ (\mathrm{mod}\ 12)\\ 9 & n\equiv 8\ (\mathrm{mod}\ 12)\\ 1 & n\equiv 10\ (\mathrm{mod}\ 12)\\ 0 & n\equiv 0\ (\mathrm{mod}\ 12)\end{cases}.$$

Case 2: when n is odd.

The character table of 1-dimension representations is:

	σ^k	$H\sigma^k$
ψ_1	1	1
ψ_2	1	-1

And $t=2$, we still have:

	σ^k	$H\sigma^k$
$\psi_h(0<h<\frac{n-1}{2})$	$2\cos\frac{2\pi hk}{n}$	0

In this case, we can get:

$$\lambda_{1,1}=\sum_g\alpha(g)\psi_1(g)=\alpha(\sigma)\psi_1(\sigma^1)+\alpha(\sigma^{-1})\psi_1(\sigma^{-1})+\alpha(H)\psi_1(H\sigma^0)=1-1+1=1.$$

$$\lambda_{2,1}=\sum_g\alpha(g)\psi_2(g)=1-1-1=-1.$$

$$\lambda_{h,1}^2+\lambda_{h,2}^2=\sum_{g_1,g_2\in G}\left(\prod_{s=1}^2\alpha(g_s)\right)\chi_h\left(\prod_{s=1}^2 g_s\right)$$

$$=2\times\left(\alpha(\sigma)\cdot\alpha(\sigma^{-1})\cdot\psi_h(\sigma^1\cdot\sigma^{-1})+\alpha(\sigma)\cdot\alpha(H)\cdot\psi_h(\sigma\cdot H)+\alpha(\sigma^{-1})\cdot\alpha(H)\cdot\psi_h\ \sigma^{-1}\cdot H)\right)$$

$$+\alpha(\sigma)\cdot\alpha(\sigma)\cdot\psi_h(\sigma^2)+\alpha(\sigma^{-1})\cdot\alpha(\sigma^{-1})\cdot\psi_h(\sigma^{-2})+\alpha(H)\cdot\alpha(H)\cdot\psi_h(H^2)$$

$$=2\times(1\cdot(-1)\cdot 2+0+0)+1\cdot 1\cdot\cos\frac{2\pi h\cdot 2}{n}+(-1)\cdot(-1)\cdot\cos\frac{2\pi h\cdot 2}{n}+1\cdot 1\cdot 2=-2+4\cos\frac{4\pi h}{n}.$$

$$\lambda_{h,1}^1+\lambda_{h,2}^1=1\cdot 2\cos\frac{2\pi h}{n}-1\cdot 2\cos\frac{2\pi h\cdot(-1)}{n}=0.$$

$$\lambda_{h,1}^1\cdot\lambda_{h,2}^1=1-2\cos\frac{4\pi h}{n}.$$

Therefore, $\det\Omega=\prod_{h=1}^{\frac{n-1}{2}}(1-2\cos\frac{4\pi h}{n})^2=\begin{cases}1 & n\equiv 1\ (\mathrm{mod}\ 12)\\ 4 & n\equiv 3\ (\mathrm{mod}\ 12)\\ 1 & n\equiv 5\ (\mathrm{mod}\ 12)\\ 1 & n\equiv 7\ (\mathrm{mod}\ 12)\\ 4 & n\equiv 9\ (\mathrm{mod}\ 12)\\ 1 & n\equiv 11\ (\mathrm{mod}\ 12)\end{cases}.$

Above all, $\det\Omega\equiv\begin{cases}0\ (\mathrm{mod}\ 2) & n\equiv 0\ (\mathrm{mod}\ 3)\\ 1\ (\mathrm{mod}\ 2) & n\equiv 1,2\ (\mathrm{mod}\ 3)\end{cases}.$

Remark 3.6.9 Since we get that uniqueness and existence cannot be obtained in Cayley graphs for dihedral group $\Omega = Cay(D_n, S), n \equiv 0 \pmod 3$, we can additionally give an elementary explanation to this conclusion by invariants. In an $n \equiv 0 \pmod 3$ prism, a line between the corresponding top and bottom points is marked with $0,1$. Marked 1 between two points in the same state, while marked 0 in the opposite state. Multiply each mark by a weight in the order of $1, 0, 2, 1, 0, 2, \ldots 1, 0, 2$ alternatively. The weight is an invariant $\bmod 3$. This suggests that we can't change every initial state in $n \equiv 0 \pmod 3$ prisms into the same 'off' state.

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DEDICATION PAGE

Our idea for the research was inspired by the classic light-switch problem. It led us to discuss the main problem of domino tiling, followed with more contents in the field of graph theory and number theory.

Our team consists of three people, student Kunqi Hong, Haoran Bao and the teacher Bowen Chen.

Haoran Bao is responsible for the light-switch problem, Conway's boundary group, Cayley graph and combinatoric methods.

Kunqi Hong is responsible for linear algebra, generalized coloring, Pfaffian orientation methods, and most of the examples' calculation.

Our teacher, Bowen Chen, guided us without any charge. He gave us guidance on theories and how to write a formal thesis.

We are very grateful to Teacher Chen for his trust and guidance during the one-year period of studying. We also appreciate our parents' wholehearted support. In the future, we will further explore the field of Math and continue our enthusiasm for this subject.