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论文题目: Linear sections of determinantal varieties

Linear sections of determinantal varieties

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Abstract

We discuss the nonemptiness of the linear sections of determinantal varieties. More precisely, we search for the minimal dimension of linear subspaces that always have nonempty intersection with the determinantal variety. It turns out that this minimal dimension depends not only on the determinantal variety, but on the base field as well. We first consider the special case where the determinantal variety is the variety defined by the determinant of $n \times n$ square matrices. We prove that, when the field is the real number field, the minimal dimension of linear spaces that always have nonempty intersection with the determinantal variety is the Hurwitz–Radon function $\rho(n)$, whereas the minimal dimension is 1 in the case of the complex number field, and it is n in the case of the rational number field. We then discuss the general case, and we prove that the minimal dimension of linear subspaces that always have nonempty intersection with the determinantal variety is $(m-r)(n-r)$ in the complex number field case. Some partial results are also obtained in the cases of real number field and rational number field.

Keywords: Determinantal varieties, linear sections, Hurwitz–Radon function

Contents

1 Introduction

The determinantal varieties are among the central research objects in algebraic geometry [10]. They are defined as the set of matrices of a given size whose rank is less than or equal to a given number. Although the determinantal varieties can be defined over any fields, most studies focus on the case of an algebraically closed field (e.g. the field of complex numbers $\mathbb{C}$). A thorough presentation of determinantal varieties over $\mathbb{C}$ is given in [3, Chapter II]. The study of the determinantal varieties in non-algebraically closed fields could be harder, as we can see in our work.

In this article, we will consider the following questions concerning the linear sections of determinantal varieties over any field $\Bbbk$. Let us first fix some notations. Let $D_{m,n,r} \subset \mathbb{P}M_{m\times n}(\Bbbk)$ be the subvariety consisting of the projective classes of all $m \times n$ matrices of rank less than or equal to r.

Problem 1.1 *What is the mininal integer $k > 0$, depending on the parameters m, n, r, such that for any linear subspace $L \subset \mathbb{P}M_{m\times n}(\Bbbk)$ of dimension k, the intersetion of L and $D_{m,n,r}$ is nonempty?*

There are two reasons why a complete answer to Problem 1.1 is quite hard. First, the determinantal variety in general, as a geometric object, is very hard to describe by defining equations. Second, we find that the answer to Problem 1.1 depends on the base field $\Bbbk$. We will only discuss, in this article, three fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ that we are most familiar with, and we utilize totally different techniques for these three fields. To summarize,

- When $\Bbbk = \mathbb{R}$, the field of real numbers, the main technique we use to understand Problem 1.1 is a theorem of Adams on the number of linearly independent vector fields on standard spheres, which is a topic in algebraic topology. (c.f. Section 2)
- When $\Bbbk = \mathbb{C}$, the field of complex numbers, the main technique we use is complex projective geometry. (c.f. Section 3.1)
- When $\Bbbk = \mathbb{Q}$, the field of rational numbers, the main technique we use is the theory about algebraic numbers. (c.f. Section 3.2)

Due to the difficulties we described in the previous paragraph, we have only a complete answer to Problem 1.1 when the base field $\Bbbk$ is $\mathbb{C}$ (or more generally, when the base field is algebraically closed).

Theorem 1.2 *The answer to Problem 1.1 when the field $\Bbbk = \mathbb{C}$ is $(m - r)(n - r)$.*

A special case of Problem 1.1 is when $m = n = r + 1$ and it turns out that this case is much more manageable. This is essentially because in this case, the determinantal variety is defined by only 1 equation, the determinant of matrices. Let us reformulate Problem 1.1 in this special case as the following Problem 1.3.

Problem 1.3 *Let D be the hypersurface of $\mathbb{P}M_n(\Bbbk)$ defined by the equation*

$$\det \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \dots & \dots & \dots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} = 0. \tag{1}$$

What is the minimal integer $k > 0$ such that for any linear subspace L of dimension k in $\mathbb{P}M_n(\Bbbk)$, $D \cap L \neq \emptyset$?

For the sake of simplicity of notations, let us denote $\sigma_\Bbbk(n)$ to be the answer of Problem 1.3 over $\Bbbk$. It is relatively easy to see that $\sigma_\Bbbk(n) \leq n$ for any field $\Bbbk$ (Lemma 3.5).

Theorem 1.4 *For $\Bbbk = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$, we can give a complete answer to Problem 1.3 as follows.*
(i) $\sigma_\mathbb{R}(n) = \rho(n)$.
(ii) $\sigma_\mathbb{C}(n) = 1$.
(iii) $\sigma_\mathbb{Q}(n) = n$.

In Theorem 1.4, $\rho(n)$ is the Hurwitz-Radon number that is defined in Section 2.1. Moreover, it is obvious that Theorem 1.4 (ii) is a direct consequence of Theorem 1.2.

Now let us come back to say something more about Problem 1.1. Some partial results for $\Bbbk = \mathbb{Q}$ and $\mathbb{R}$ are obtained in this project.

Theorem 1.5 *Assume $m \geq n$. For any linear subspace $L \subset \mathbb{P}M_{m\times n}(\mathbb{Q})$ of dimension m, we have $L \cap D_{m,n,n-1} \neq \emptyset$. Furthermore, there exists a linear subspace $L \subset \mathbb{P}M_{m\times n}(\mathbb{Q})$ of dimension $m - 1$ such that $L \cap D_{m,n,n-1} = \emptyset$.*

Theorem 1.6 *For any linear subspace $L \subset \mathbb{P}M_{n\times\rho(n)}(\mathbb{R})$ of dimension n, we have $L \cap D_{n,\rho(n),\rho(n)-1} \neq \emptyset$. Furthermore, there exists a linear subspace $L \subset \mathbb{P}M_{n\times\rho(n)}(\mathbb{R})$ of dimension $n-1$ such that $L \cap D_{n,\rho(n),\rho(n)-1} = \emptyset$.*

Let $m \geq n$. Let $\sigma(m,n)$ be the minimal number $k > 0$ such that for any linear subspace $L \subset \mathbb{P}M_{m\times n}(\mathbb{R})$ of dimension k, $L \cap D_{m,n,n-1} \neq \emptyset$. We can calculate $\sigma(m,n)$ for $m \leq 8$. The result is summarized in the following table.

$\sigma(m,n)$	2	3	4	5	6	7	8
2	2	2	4	4	6	6	8
3		1	4	4	4	5	8
4			4	4	4	4	8
5				1	2	3	8
6					2	2	8
7						1	8
8							8

The organization of the article is as follows. In Section 2, we prove Theorem 1.4 (i). In Section 3.1, we prove Theorem 1.2 and Theorem 1.4 (ii). In Section 3.2, we prove Theorem 1.4 (iii) and Theorem 1.5. In Section 3.3, we prove Theorem 1.6 and the table thereafter.

2 Determinantal varieties over $\mathbb{R}$: the special case

In this section, we are going to prove Theorem 1.4 (i), based on works of Adams [1] and Shapiro [9, Chapter 2]. To simplify the notation, let us write $\sigma(n)$ for $\sigma_{\mathbb{R}}(n)$. The strategy of the proof is as follows. First, we reduce Problem 1.3 to the problem of linearly independent vector fields on the standard sphere, and use Adams' theorem (recalled in Theorem 2.4 below) to give the inequality $\sigma(n) \leq \rho(n)$. This idea has already appeared in Adams' original paper [1], and some follow-ups (e.g. [2, 7]) for some problems in linear algebra. Then we are going to explicitly use a construction in [9, Chapter II] to prove the other side of inequality, thus conclude Theorem 1.4 (i).

2.1 Adams' theorem

We present in this subsection the statement of Adams' theorem on the number of linearly independent vector fields on S^n. To start with, we need the definition of the Hurwitz–Radon function.

Definition 2.1 (Hurwitz–Radon function) *The Hurwitz–Radon function is a map $\rho : \mathbb{N} \to \mathbb{N}$ defined as follows. For $n \in \mathbb{N}$, write $n = 2^{4a+b} n_0$, where $a, b, n_0 \in \mathbb{N}$, $b \in \{0, 1, 2, 3\}$ and n_0 is odd. We define $\rho(n) = 8a + 2^b$.*

Remark 2.2 *Despite its weird appearance, the Hurwitz–Radon function is widely used in mathematics. For example,*

- *A composite of quadratic forms of size $[r, n, n]$ exists if and only if $r \leq \rho(n)$. (e.g. [9])*
- *Adams' theorem [1]: the maximal number of linearly independent vector fields on the standard $(n-1)$-sphere S^{n-1} is given by $\rho(n) - 1$.*

The Adams' theorem is a highly nontrivial result that we will employ in our article. To be able to understand its precise meaning, we need to give the definitions of vector fields and linear independence.

Definition 2.3 *Let S^{n-1} be the standard $(n-1)$-sphere located in $\mathbb{R}^n$.*
(i) A vector field *on S^{n-1} is a continuous map*

$$v : S^{n-1} \to \mathbb{R}^n$$

such that for any $x \in S^{n-1}$, the vector $v(x)$ is orthogonal to the vector x.
(ii) A series of vector fields $v_1, \ldots, v_d$ on S^{n-1} are called linearly independent*, if for any $x \in S^{n-1}$, the vectors $v_1(x), \ldots, v_d(x)$ are linearly independent in $\mathbb{R}^n$.*

The statement of Adams' theorem is as follows.

Theorem 2.4 (Adams [1]) *The maximal number of linearly independent vector fields on S^{n-1} is $\rho(n) - 1$, where ρ is the Hurwitz–Radon function.*

Remark 2.5 *If n is odd, then one calculates easily that $\rho(n) - 1 = 0$. In this case, Adams' theorem says that there does not exist any nowhere zero vector fields. This is a classical result in algebraic topology called the "hairy ball theorem".*

In practice, we also utilize another expression of the Hurwitz–Radon function. We will use this expression in Section 2.4.

Lemma 2.6 *Let $n = 2^m n_0$ where n_0 is odd. The Hurwitz–Radon function*

$$\rho(n) = \begin{cases} 2m+1 & \text{if } m \equiv 0 \bmod 4 \\ 2m & \text{if } m \equiv 1 \bmod 4 \\ 2m & \text{if } m \equiv 2 \bmod 4 \\ 2m+2 & \text{if } m \equiv 3 \bmod 4 \end{cases}.$$

Proof If $m \equiv 0 \bmod 4$, then $m = 4a + 0$. Hence, $\rho(n) = 8a + 2^0 = 2m + 1$. If $m \equiv 1 \bmod 4$, then $m = 4a + 1$. Hence, $\rho(n) = 8a + 2^1 = 2m$. If $m \equiv 2 \bmod 4$, then $m = 4a + 2$. Hence, $\rho(n) = 8a + 2^2 = 2m$. If $m \equiv 3 \bmod 4$, then $m = 4a + 3$. Hence, $\rho(n) = 8a + 2^3 = 2m + 2$. □

2.2 The proof of $\sigma(n) \leq \rho(n)$

Proposition 2.7 *There exist $\sigma(n)$ matrices $A_1, \ldots, A_{\sigma(n)}$ of size $n \times n$ satisfying the following property: for any $a_1, \ldots, a_{\sigma(n)} \in \mathbb{R}$, not all being zero, the matrix $a_1 A_1 + \ldots + a_{\sigma(n)} A_{\sigma(n)}$ is invertible. Furthermore, one may assume that $A_1 = I$ is the identity matrix.*

Proof We first construct a subspace $W \subset M_n(\mathbb{R})$ of dimension $\sigma(n)$ such that any nonzero matrix $A \in W$ is invertible. By the minimality of $\sigma(n)$, there exists a subspace $L \subset \mathbb{P}M_n(\mathbb{R})$ with dimension $\sigma(n) - 1$ such that $L \cap D = \emptyset$. Let W be the subspace of $M_n(\mathbb{R})$ whose associate projective subspace is L. Then $\dim W = \dim L + 1 = \sigma(n)$. Moreover, for any nonzero matrix $A \in W$, we may consider the 1-dimensional subspace generated by A, which is an element in L. Since $L \cap D = \emptyset$, whereas D is the set of all 1-dimensional subspaces generated by matrices with nonzero determinant, we get $\det A \neq 0$, i.e. A is invertible.

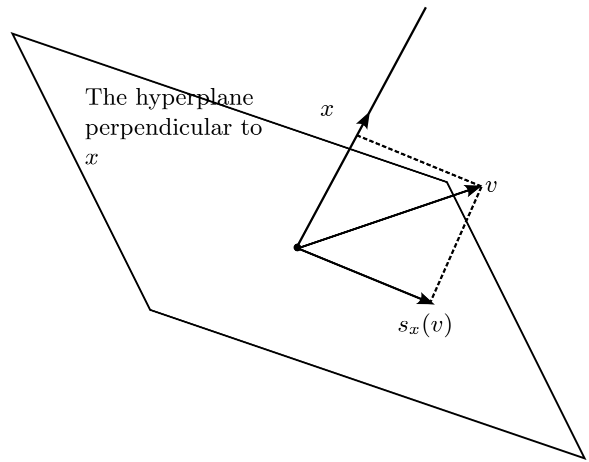
Since W is a vector space whose dimension is $\sigma(n)$, let $\{A_1, \ldots, A_{\sigma(n)}\}$ be a basis of the vector space W. Thus, for any $\lambda_1, \ldots, \lambda_{\sigma(n)}$ that are not zero, $\lambda_1 A_1 + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)} \neq 0$. Since $\lambda_1 A_1 + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)}$ is a nonzero element in W, it is invertible.

Next we want to show that we may assume $A_1 = I$. Let $B_i = A_1^{-1} A_i$. Let us show that $B_1 = I, \ldots, B_{\sigma(n)}$ also satisfy the property as stated in the Proposition 2.7. In fact, for any $\lambda_1, \ldots, \lambda_{\sigma(n)} \in \mathbb{R}$, not all of them being zero, we have

$$\begin{aligned} \lambda_1 B_1 + \ldots + \lambda_{\sigma(n)} B_{\sigma(n)} &= \lambda_1 A_1^{-1} A_1 + \ldots + \lambda_{\sigma(n)} A_1^{-1} A_{\sigma(n)} \\ &= A_1^{-1} (\lambda_1 A_1 + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)}). \end{aligned}$$

Since $\lambda_1 A_1 + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)}$ is invertible, hence $\lambda_1 B_1 + \ldots + \lambda_{\sigma(n)} B_{\sigma(n)}$ is invertible. So we can assume that $A_1 = I$. □

Our next step is to construct $\sigma(n) - 1$ vector fields on S^{n-1} using the matrices we obtained in Proposition 2.7 and then show their linear independence. To this end, we first need the following notation. For any $x \in S^{n-1}$, let $s_x : \mathbb{R}^n \to \mathbb{R}^n$ be the projection mapping to the hyperplane orthogonal to x. Percisely, we can write v in a unique way as $v = \lambda x + s_x(v)$, with $\lambda \in \mathbb{R}$ and $x \cdot s_x(v) = 0$.



Lemma 2.8 *The map $s_x : \mathbb{R}^n \to \mathbb{R}^n$ is linear.*

Proof Since $v = s_x(v)+\lambda x$, we get $v\cdot x = s_x(v)\cdot x+\lambda|x|^2 = \lambda|x|^2$. Since $x \in S^{n-1}$, we have $|x| = 1$ and thus $\lambda = \frac{v\cdot x}{|x|^2} = v\cdot x$. Let $s_x(A) = A-(A\cdot x)x, s_x(B) = B-(B\cdot x)x, s_x(A+B) = A+B-((A+B)\cdot x)x$. Since $A-(A\cdot x)x+B-(B\cdot x)x = A+B-((A+B)\cdot x)x$, we have $s_x(A)+s_x(B) = s_x(A+B)$. Furthermore, $\lambda s_x(A) = \lambda(A-(A\cdot x)x) = \lambda A-(\lambda A\cdot x)x$, and $s_x(\lambda A) = \lambda A-\lambda Ax$, we have $s_x(\lambda A) = \lambda s_x(A)$. Therefore, the map $s_x : \mathbb{R}^n \to \mathbb{R}^n$ is linear. $\square$

Now let $A_1 = I, A_2, \dots, A_{\sigma(n)}$ be the matrices satisfying the property as in Propostion 2.7. For $i = 2, \dots, \sigma(n)$, we can construct a map

$$\begin{array}{cccc} v_i : & S^{n-1} & \to & \mathbb{R}^n \\ & x & \mapsto & s_x(A_i x) \end{array} .$$

Lemma 2.9 *The map $v_i : S^{n-1} \to \mathbb{R}^n$ defined as above is a vector field.*

Proof First, we need to show that $v_i : S^{n-1} \to \mathbb{R}^n$ is continuous. Let $x = (x_1, \dots, x_n)$, $v = (v_1, \dots, v_n)$. Let us find the explicit expression of $s_x(v)$ in terms of their coordinates. Since $v = s_x(v) + \lambda x$, we get $v \cdot x = s_x(v) \cdot x + \lambda|x|^2 = \lambda|x|^2$. Since $x \in S^{n-1}$, we have $|x| = 1$ and thus $\lambda = \frac{v\cdot x}{|x|^2} = v\cdot x = v_1x_1 + \dots + v_nx_n$. Then we can calculate $s_x(v)$ explicitly $s_x(v) = (v_1 - (v_1x_1 + \dots + v_nx_n)x_1, \dots, v_n - (v_1x_1 + \dots + v_nx_n)x_n)$. Therefore, the map $(x, v) \mapsto s_x(v)$ is a continuous map. We know that the map $(x, v) \mapsto s_x(v)$ is a continuous map from $S^{n-1} \times \mathbb{R}^n \to \mathbb{R}^n$. For matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix},$$

the map

$$\begin{array}{cccc} \phi_A : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & \mapsto & Ax = \begin{pmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \dots \\ a_{n1}x_1 + \dots + a_{nn}x_n \end{pmatrix} \end{array}$$

is a continuous map. Therefore, $v_i : S^{n-1} \to \mathbb{R}^n$, being the composition of these two continuous map, is continuous.

Then we need to show that for each $x \in S^{n-1}$, $v_i(x) \perp x$. Since $x \cdot s_x(y) = 0$ for any $x \in S^{n-1}$ and $y \in \mathbb{R}^n$, take $y = A_i x$, then $x \cdot s_x(A_i x) = 0$, i.e., $x \cdot v_i(x) = 0$. So the map

$$\begin{array}{cccc} v_i : & S^{n-1} & \to & \mathbb{R}^n \\ & x & \mapsto & s_x(A_i x) \end{array} .$$

is a vector field of S^{n-1}. $\square$

Proposition 2.10 *The vector fields $v_2, \ldots, v_{\sigma(n)}$ are linearly independent.*

Proof To show that $v_2, \ldots, v_{\sigma(n)}$ are linearly independent, we need to show that $v_2(x), \ldots, v_{\sigma(n)}(x)$ are linearly independent for any $x \in S^{n-1}$. So we need to show that when $\lambda_2 v_2(x) + \ldots + \lambda_{\sigma(n)} v_{\sigma(n)}(x) = 0$, we have $\lambda_2 = \ldots = \lambda_{\sigma(n)} = 0$. We know that

$$\begin{aligned} &\lambda_2 v_2(x) + \ldots + \lambda_{\sigma(n)} v_{\sigma(n)}(x) \\ =&\lambda_2 s_x(A_2(x)) + \ldots + \lambda_{\sigma(n)} s_x(A_{\sigma(n)}(x)) \\ =&s_x(\lambda_2 A_2 x + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)} x) \qquad \text{(by Lemma 2.8)} \end{aligned}$$

Thus, $s_x((\lambda_2 A_2 x + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)})x) = 0$. Therefore, by the definition of s_x, the vector $\lambda_2 A_2 x + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)} x$ is orthogonal to the hyperplane orthogonal to x. So $\lambda_2 A_2 x + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)} x$ is a multiple of x. Thus $\lambda_2 A_2 x + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)} x = \lambda_1 A_1 x$ for some $\lambda_1 \in \mathbb{R}$. Assume by contradiction that $\lambda_2, \ldots, \lambda_{\sigma(n)}$ are not all zero. Then $-\lambda_1 A_1 x + \lambda_2 A_2 x + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)} x = 0$, that is $(-\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)})x = 0$. Let $M = -\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_{\sigma(n)} A_{\sigma(n)}$. Since $\lambda_1, \ldots, \lambda_{\sigma(n)}$ are not all zero, we can see from Proposition 2.7 that M is invertible. Since $Mx = 0$, we have $M^{-1}Mx = 0$, which implies $x = 0$. But $x \in S^{n-1}$ then this is a contradiction. So $\lambda_2, \ldots, \lambda_{\sigma(n)}$ are all zero. Therefore $v_2, \ldots, v_{\sigma(n)}$ are linearly independent vector fields on S^{n-1}. $\square$

Corollary 2.11 *With notation above, we have $\sigma(n) \le \rho(n)$.*

Proof We know that $v_2, \ldots, v_{\sigma(n)}$ are linearly independent vector fields on S^{n-1}. According to Theorem 2.4, the maximum number of the independent fields on S^{n-1} is $\rho(n)-1$. So $\sigma(n)-1 \le \rho(n)-1$, namely, $\sigma(n) \le \rho(n)$. $\square$

2.3 Amicable pairs

The main purpose of this and the following sections is to construct, using three fundamental lemmas suggested in [9], $\rho(n)$ real matrices $A_1, \ldots, A_{\rho(n)}$ satisfying the following Property (*):

(*) For any $\lambda_1, \ldots, \lambda_{\rho(n)} \in \mathbb{R}$, not all being zero, $\lambda_1 A_1 + \ldots + \lambda_{\rho(n)} A_{\rho(n)}$ is invertible.

The existence of such a series of matrices implies that $\sigma(n) \ge \rho(n)$, as will be shown in Section 2.5. Combining with Corollary 2, we get $\sigma(n) = \rho(n)$, and thus we prove Theorem 1.4 (i).

In order to find the matrices $A_1, \ldots, A_{\rho(n)}$ satisfying Property (*), following [9], we define the amicable pairs.

Definition 2.12 *An amicable pair of size (s, t, n) is a pair (S, T) where $S = \{A_1, \ldots, A_s\}$ and $T = \{B_1, \ldots, B_t\}$ consisting of $n \times n$ real matrices satisfying:*

- *A_i is antisymmetric (i.e., ${}^tA_i = -A_i$) and B_j is symmetric (i.e., ${}^tB_j = B_j$).*
- *$A_i^2 = -I$ and $B_j^2 = I$ for any i, j.*
- *The $s + t$ matrices $A_1, \ldots, A_s, B_1, \ldots, B_t$ are pairwise anticommutative.*

The importance of the notion of amicable pair is that the set S contains the matrices that we want.

Proposition 2.13 *Let (S, T) be an amicable pair of size (s, t, n), where $S = \{A_1, \ldots, A_s\}$, then $A_0 = I, A_1, \ldots, A_s$ satisfy Property (*).*

Proof Assume that $\lambda_0, \ldots, \lambda_s$ are not all zero, we want to prove that $\lambda_0 A_0 + \lambda_1 A_1 + \ldots + \lambda_s A_s$ is invertible. Let $A = \lambda_1 A_1 + \ldots + \lambda_s A_s$, then

$$\begin{aligned} A^2 =&(\lambda_1 A_1 + \ldots + \lambda_s A_s)(\lambda_1 A_1 + \ldots + \lambda_s A_s) \\ =&\lambda_1^2 A_1^2 + \lambda_2^2 A_2^2 + \ldots + \lambda_s^2 A_s^2 \\ =&-(\lambda_1^2 + \ldots + \lambda_s^2)I, \end{aligned}$$

where the second equality is due to the anticommutativity: $A_i A_j = -A_j A_i$ for any $i \neq j$, $i, j \in \{1, \ldots, s\}$, and the last equality is due to the second axiom of the amicable pair: $A_i^2 = -I$, for $i \in \{1, \ldots, s\}$. Next, let us show the invertibility of $\lambda_0 A_0 + \lambda_1 A_1 + \ldots + \lambda_s A_s$ by finding its inverse directly. To do this, we calculate

$$\begin{aligned} (\lambda_0 A_0 + A)(\lambda_0 A_0 - A) &= (\lambda_0 A_0)^2 - A^2 \\ &= (\lambda_0 I)^2 + (\lambda_1^2 + \ldots + \lambda_s^2)I \\ &= (\lambda_0^2 + \lambda_1^2 + \ldots + \lambda_s^2)I \end{aligned}$$

Since $\lambda_0, \ldots, \lambda_s$ are not all zero, $\lambda_0^2 + \lambda_1^2 + \ldots + \lambda_s^2 \neq 0$. Hence, $\lambda_0 A_0 + \ldots + \lambda_s A_s$ is invertible, with inverse $\frac{1}{\lambda_0^2 + \lambda_1^2 + \ldots + \lambda_s^2}(\lambda_0 A_0 - A)$. □

2.4 Some constructive lemmas

This part introduces three lemmas associated with amicable pairs. These three lemmas help us enlarge the set S in the amicable pair, thus give us a method to construct matrices satisfying Property (*) in Corollary 2.22. In Appendix B, we implement our Mathematica code for constructing matrices, indicating that there is actually an effective way for us to construct these matrices.

Lemma 2.14 (Construction Lemma) *If there exists an amicable pair of size* (s, t, n)*, then there exists an amicable pair of size* $(s+1, t+1, 2n)$*.*

Proof Let (S, T) be an amicable pair of size (s, t, n), where $S = \{A_1, \ldots, A_s\}$ and $T = \{A_1, \ldots, A_t\}$. For each $1 \leq i \leq s$, define and $2n \times 2n$ matrix A'_i as:

$$A'_i = \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix}.$$

We define and $2n \times 2n$ matrix A'_{s+1} as

$$A'_{s+1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

For each $1 \leq i \leq t$, define and $2n \times 2n$ matrix B'_i as

$$B'_i = \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix}.$$

We define and $2n \times 2n$ matrix B'_{t+1} as

$$B'_{t+1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

To show that (S', T') is an amicable pair of $2n \times 2n$, we need to show that (S', T') satisfy the three axioms of the amicable pair. Since the calculations are tedious, in order to make our presentation conciser, we will leave the explicit calculations in Appendix A.

□

Corollary 2.15 *If $n \in \mathbb{N}$ and $n = 2^m n_0$ with n_0 odd, then there is an amicable pair of size $(m, m+1, n)$.*

Proof We use induction on m. When $m = 0$, it is easy to construct an amicable of size $(0, 1, n)$ with S and T. Let $S = \emptyset$, $T = \{I_n\}$. Then it is clear that (S, T) forms an amicable pair of size $(0, 1, n)$. Now let us assume that we have an amicable pair of size $(k, k+1, 2^k n_0)$. Then by Lemma 2.14, we can get an amicable pair of size $(k+1, k+2, 2^{k+1} n_0)$. Hence the corollary is proved by induction. □

Lemma 2.16 (Shift Lemma) *If we have an amicable pair of size $(s, t+4, n)$, then there exists an amicable pair of size $(s+4, t, n)$.*

Proof Let $S = \{A_1, \dots, A_s\}$, $T = \{B_1, \dots, B_t, C_1, C_2, C_3, C_4\}$. Let $C = C_1C_2C_3C_4$, and $C_i' = C_iC$. Denote $S' = \{A_1, \dots, A_s, C_1', C_2', C_3', C_4'\}$, $T' = \{B_1, \dots, B_t\}$. Next we show that (S', T') is an amicable pair of size $(s+4, t, n)$.

For the first axiom, the assertions that ${}^tA = -A_i$, ${}^tB = -B_i$ come from the fact that (S, T) is an amicable pair. It remains to prove that ${}^tC_i' = C_i'$. In fact, $C_i' = \pm C_jC_kC_l$ where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then

$$\begin{aligned} {}^tC_i' &= \pm {}^tC_l^t C_k^t C_j = \pm C_l C_k C_j \\ &= \pm(-1)^2(-1)^1 C_j C_k C_l \\ &= -(\pm C_j C_k C_l) = -C_i'. \end{aligned}$$

For the second axiom, the assertions $A_i^2 = -I$, $B_i^2 = -I$ come from the fact that (S, T) is an amicable pair. It remains to show that $C_i'^2 = -I$. In fact, $C_i' = \pm C_jC_kC_l$ where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We have $C_i'^2 = C_jC_kC_lC_jC_kC_l$. Since C_i, C_j, C_k are pairwise anticommutative, $C_i'^2 = -C_i^2C_j^2C_k^2 = -I$, thus $C_i'^2 = -I$ is proved.

For the third axiom, we have $A_iB_j = -B_jA_i$ since (S, T) is an amicable pair. It remains to check that $A_iC_j' = -C_j'A_i$ and $B_iC_j' = -C_j'B_i$. Since C_j' is the product of three elements in T different from A_i and B_i, to move A_i (resp. B_i) from the right-hand-side of C_j' to the left-hand-side of C_j', we need to move 3 steps, each generating one -1 by the anticommutativity. Hence, $A_iC_j' = -C_j'A_i$ and $B_iC_j' = -C_j'B_i$.

Thus we have proved (S', T') is an amicable pair of size $(s+4, t, n)$. □

Corollary 2.17 *For $n = 2^m n_0 \in \mathbb{N}$ with n_0 odd, according to the value of m, there is an amicable pair of the size*

$$\begin{array}{cl} (2m, 1, n) & \text{if } m \equiv 0 \bmod 4 \\ (2m-1, 2, n) & \text{if } m \equiv 1 \bmod 4 \\ (2m-2, 3, n) & \text{if } m \equiv 2 \bmod 4 \\ (2m+1, 0, n) & \text{if } m \equiv 3 \bmod 4 \end{array}.$$

Proof By Corollary 2.15, there is an amicable pair of size $(m, m+1, n)$. By Lemma 2.16 above, we can move the elements in T to S in a four-by-four pattern. Hence, if $m \equiv 0 \bmod 4$, we can move m elements from T to S and we get an amicable pair of size $(2m, 1, n)$. If $m \equiv 1 \bmod 4$, we can move $m-1$ elements from T to S and we get an amicable pair of size $(2m-1, 2, n)$. If $m \equiv 2 \bmod 4$, we can move $m-2$ elements from T to S and we get an amicable pair of size $(2m-2, 3, n)$. If $m \equiv 3 \bmod 4$, we can move $m+1$ elements from T to S and we get an amicable pair of size $(2m+1, 0, n)$. □

Lemma 2.18 (Expansion Lemma) *If there exists an amicable pair of size (s, t, n) with $s - t \equiv 2 \bmod 4$, then there is an amicable pair of size $(s+1, t, n)$.*

Proof Let (S, T) be an amicable pair of size (s, t, n) where $s - t \equiv 2 \bmod 4$, where $S = \{A_1, \dots, A_s\}$ and $T = \{B_1, \dots, B_t\}$. Let $Z = A_1 \dots A_sB_1 \dots B_t$ be the product of all these matrices. As long as $(S \cup \{Z\}, T)$ is an amicable pair, Lemma 2.18 is proved.

To prove that $(S \cup \{Z\}, T)$ is an amicable pair, $(S \cup \{Z\}, T)$ must satisfy the three axioms. To this end, we will need the following elementary numerical lemma.

Lemma 2.19 *If $s - t \equiv 2 \bmod 4$, then $s + (s + t - 1) + (s + t - 2) + \ldots + 1$ is an odd number.*

Proof Since $s - t \equiv 2 \bmod 4$, we have $t = s + 2 + 4k$. By the summation formula for the authmetric sequence, we have $s + (s+t-1) + (s+t-2) + \ldots + 1 = s + 1/2(s+t)(s+t-1) = s + 1/2(s+s+2+4k)(s+s+2+4k-1) = s + (s+2k+1)(2s+4k+1)$. By discussing the parity of s we get that $s + (s+2k+1)(2s+4k+1)$ is always odd. □

For the first axiom, we know that ${}^tA_i = -A_i$, ${}^tB_i = -B_i$. Moreover,

$$\begin{aligned}
&{}^t(A_1 \ldots A_s B_1 \ldots B_t) ={}^t B_t \ldots {}^tB_1 {}^tA_s \ldots {}^tA_1\\
=&(-1)^s B_t \ldots B_1 A_s \ldots A_1 \qquad \text{since } A_i \text{ are antisymmetric and } B_i \text{ are symmetric}\\
=&(-1)^s(-1)^{(s+t-1)+(s+t-2)+\ldots+1} A_1 \ldots A_s B_1 \ldots B_t \qquad \text{by pairwise anticommutativity}\\
=&-A_1 \ldots A_s B_1 \ldots B_t \qquad \text{by Lemma 2.19.}
\end{aligned}$$

For the second axiom, we know that $A_i^2 = -I$, $B_i^2 = I$. We have

$$\begin{aligned}
Z^2 =&(A_1 \ldots A_s B_1 \ldots B_t)(A_1 \ldots A_s B_1 \ldots B_t) = A_1 \ldots A_s B_1 \ldots B_t A_1 \ldots A_s B_1\\
=&(-1)^{s+t-1+s+t-2+\ldots+1}(-1)^s(A_1^2 \ldots A_s^2 B_1^2 \ldots B_t^2) \qquad \text{by pairwise anticommutativity}\\
=&(-1)^{s+s+t-1+s+t-2+\ldots+1} I \qquad \text{since } A_i^2 = -I, B_i^2 = I\\
=&-I \qquad \text{by Lemma 2.19.}
\end{aligned}$$

For the third axiom, it suffices to show that $A_iZ = -ZA_i$ and $B_iZ = -ZB_i$. Since Z is the product of all matrices in $S \cup T$, in order to move A_i from the left-hand-side of Z to the right-hand-side of Z, we need to move $s + t$ steps. Among those steps, there are $s + t - 1$ anticommutative steps and 1 commutative step. Hence $A_iZ = (-1)^{s+t-1}ZA_i = -ZA_i$ since $s + t - 1$ is odd, by assumption. □

Corollary 2.20 *For a natural number $n = 2^m n_0$, according to the value of m, there is an amicable pair of the size*

$$\begin{array}{ll}
(2m, 1, n) & \textit{if } m \equiv 0 \bmod 4\\
(2m-1, 2, n) & \textit{if } m \equiv 1 \bmod 4\\
(2m-1, 0, n) & \textit{if } m \equiv 2 \bmod 4\\
(2m+1, 0, n) & \textit{if } m \equiv 3 \bmod 4
\end{array}.$$

Proof Only when $m \equiv 2 \bmod 4$, we can add an element to S in the amicable pair whose size is $(2m-2, t, n)$, which means we need to make sure that $s-t \equiv 2 \bmod 4$, which means $2m-2-t \equiv 2 \bmod 4$, so we have $t = 0$. Then we have an amicable pair $(2m-2, 0, n)$ who satisfies $s - t \equiv 2 \bmod 4$. So we can add an element to S, which means we have $(2m-1, 0, n)$. □

2.5 The proof of $\sigma(n) = \rho(n)$

From Proposition 2.13, Lemmas 2.14, 2.16 and 2.18, we have constructed $\rho(n)$ matrices $A_0, \ldots, A_{\rho(n)-1}$ satisfying Property (*).

We can now finish our argument by using the following useful lemma. Since we will use this lemma constantly in the subsequent of our article, let us state it in the most general form possible.

Lemma 2.21 *Let $\mathbb{k}$ be a fixed base field. Let $D_{m,n,r} \subset \mathbb{P}M_{m\times n}(\mathbb{k})$ be defined as in the Introduction. Then the following two statements are equivalent:*
(i) There exists a linear subspace $L \subset \mathbb{P}M_{m\times n}(\mathbb{k})$ of dimension k such that $L \cap D_{m,n,r} = \emptyset$;
(ii) There exist $k+1$ matrices $A_1, \ldots, A_{k+1}$ of size $m\times n$ such that for any not all zero $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{k}$, the matrix $\lambda_1A_1 + \ldots + \lambda_{k+1}A_{k+1}$ has rank $> r$.

Proof Let us first show (i) $\implies$ (ii). Let $L \subset \mathbb{P}M_{m\times n}(\mathbb{k})$ be a linear subspace of dimension k such that $L \cap D_{m,n,r} = \emptyset$. Let $W \subset M_{m\times n}(\mathbb{k})$ be the vector space associate to L. Then $\dim W = \dim L + 1 = k+1$.

Let $A_1, \ldots, A_{k+1} \in W$ be a basis of W. We verify that the list $A_1, \ldots, A_{k+1}$ satisfies the desired property in (ii). Let $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{k}$ not be all zero. Since the list $A_1, \ldots, A_{k+1}$ is linearly independent, the matrix $\lambda_1 A_1 + \ldots + \lambda_{k+1} A_{k+1}$ is not zero, thus it has an equivalence class, which is an element in the projective space L. Since $L \cap D_{m,n,r} = \emptyset$, then the rank of the matrix $\lambda_1 A_1 + \ldots + \lambda_{k+1} A_{k+1}$ is greater than r, as desired.

Conversely, we show (ii) $\implies$ (i). Let $A_1, \ldots, A_{k+1}$ be $k+1$ matrices of size $m \times n$ such that for any $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{k}$, not all being zero, the matrix $\lambda_1 A_1 + \ldots + \lambda_{k+1} A_{k+1}$ has rank $> r$. Then it is clear that the list $A_1, \ldots, A_{k+1}$ is linearly independent. Let W be the vector space in $M_{m\times n}(\mathbb{k})$ spanned by $A_1, \ldots, A_{k+1}$. Then $\dim W = k+1$. Let L be the projective space associate to W. Then L is of dimension $= k$. It remains to check that $L \cap D_{m,n,r} = \emptyset$. Any element in L can be represented by a matrix of the form $\lambda_1 A_1 + \ldots + \lambda_{k+1} A_{k+1}$ where $\lambda_1, \ldots, \lambda_{k+1}$ are not all zero, since $A_1, \ldots, A_{k+1}$ span W and since L is the projective space associate to W. Therefore, by the property of $A_1, \ldots, A_{k+1}$, we know that the rank of $\lambda_1 A_1 + \ldots + \lambda_{k+1} A_{k+1}$ is greater than r. Hence, this element does not belong to $D_{m,n,r}$. Therefore, $L \cap D_{m,n,r} = \emptyset$, as desired. □

Corollary 2.22 *Let $\mathbb{k}$ be a fixed base field. Let $D \subset \mathbb{P}M_n(\mathbb{k})$ be defined as in Problem 1.3. Then the following two statements are equivalent:*
(i) There exists a linear subspace $L \subset \mathbb{P}M_n(\mathbb{k})$ of dimension k such that $L \cap D = \emptyset$;
(ii) There exist $k+1$ matrices $A_1, \ldots, A_{k+1}$ of size $n \times n$ satisfying Property (*).

We are now ready to prove Theorem 1.4 (i).
Proof of Theorem 1.4 (i). By Corollary 2.11, we have $\sigma(n) \leq \rho(n)$. So it suffices to show that $\sigma(n) \geq \rho(n)$. Since we have found $\rho(n)$ matrices $A_0, \ldots, A_{\rho(n)-1}$ of size $n \times n$ satisfying Property (*), by Corollary 2.22, there exists a linear subspace L of dimension $\rho(n) - 1$ such that $L \cap D = \emptyset$. Thus, by the definition of $\sigma(n)$, we have the inequality $\rho(n) - 1 < \sigma(n)$, i.e. $\rho(n) \leq \sigma(n)$. Thus $\rho(n) = \sigma(n)$. □

3 Determinantal varieties for $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{R}$: the general case

3.1 The complex number case

Lemma 3.1 *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a hypersurface. Let $L \subset \mathbb{P}^N(\mathbb{C})$ be a line. Then $L \cap V \neq \emptyset$.*

Proof Up to projective transformation, we may assume that L is defined by $x_0 = \ldots = x_{N-2} = 0$. Any point $P \in L$ is of the form

$$P = [0 : \cdots : 0 : x_{N-1} : x_N]. \tag{2}$$

Let $V \subset \mathbb{P}^N(\mathbb{C})$ be defined by

$$f(x_0, \ldots, x_N) = 0. \tag{3}$$

Combining (2) and (3), we can find $f(0, 0, \ldots, 0, x_{N-1}, x_N) = 0$. We need to show that the equation always has a nonzero solution. If $f(0, 0, \ldots, 0, x_{N-1}, x_N) \equiv 0$, then any nonzero (x_{N-1}, x_N) is a solution. If not, then $f(0, 0, \ldots, 0, x_{N-1}, x_N)$ is a nonzero homogeneous polynomial. Let us write $f(0, 0, \ldots, 0, x_{N-1}, x_N) = a_0 x_{N-1}^d + a_1 x_{N-1}^{d-1} x_N \ldots + a_d x_N^d$. If $a_0 = 0$, then $(x_{N-1}, x_N) = (1 : 0)$ is a nonzero solution. We now discuss the case when $a_0 \neq 0$. Let $r \in \mathbb{C}$ be a solution of $a_0 x_{N-1}^d + a_1 x_{N-1}^{d-1} + \ldots + a_d = 0$. Such an $r \in \mathbb{C}$ exists since $a_0 x_{N-1}^d + a_1 x_{N-1}^{d-1} + \ldots + a_d$ is a degree d polynomial and we know that any nonconstant polynomial with complex coefficients must have a complex root. Then $(x_{N-1}, x_N) = (r, 1)$ is a solution of $f(0, 0, \ldots, 0, x_{N-1}, x_N) = 0$. □

Corollary 3.2 *Let $D = \{\det = 0\} \subset \mathbb{P}^{n^2-1}(\mathbb{C})$. Any line $L \subset \mathbb{P}^{n^2-1}(\mathbb{C})$ has intersection with D.*

Proof of Theorem 1.4 (ii). For any one dimensional linear subspace L, we have $L \cap D \neq \emptyset$. Take a point $P \notin D$, then $\{P\} \cap D = \emptyset$. Hence there exist 0-dimentional linear subspace L such that $L \cap D = \emptyset$. □

The following classical theorem is a generalization of Lemma 3.1.

Theorem 3.3 ([8] Theorem 1.24) *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a subvariety of codimension n.*
(i) Let $L \subset \mathbb{P}(\mathbb{C})$ be a linear subspace of dimension n. Then $L \cap V \neq \emptyset$.
(ii) There exists a linear subspace L of dimension $n-1$ such that $L \cap V = \emptyset$.

According to [3], we have

Theorem 3.4 ([3] Proposition p. 67) *The codimention of $D_{m,n,r}$ in $\mathbb{P}M_{m\times n}(\mathbb{C})$ is $(m-r)(n-r)$.*

Now we can give an answer to Problem 1.1 in the case of complex number field.
Proof of Theorem 1.2. By Theorem 3.3, the minimal dimension of L that makes sure $L \cap D_{m,n,r} \neq \emptyset$ is equal to the codimension of $D_{m,n,r}$. By Theorem 3.4, the codimension of $D_{m,n,r}$ is $(m-r)(n-r)$. Thus for any linear subspace $L \subset \mathbb{P}M_{m\times n}(\mathbb{C})$ of dimension $(m-r)(n-r)$, the intersection of L and $D_{m,n,r}$ is nonempty, and there exists a linear subspace $L \subset \mathbb{P}M_{m\times n}(\mathbb{C})$ of dimension $(m-r)(n-r)-1$ that does not have intersection with $D_{m,n,r}$. □

3.2 The rational number case

In this section, we show that $\sigma_{\mathbb{Q}}(n) = n$. We have the following observation that holds for any field $\Bbbk$.

Lemma 3.5 *For any field $\Bbbk$, we have $\sigma_{\Bbbk}(n) \leq n$.*

Proof May assume that $\sigma_{\Bbbk}(n) \geq n+1$. By the definition of $\sigma_{\Bbbk}(n)$, there exists some linear subspace $L \subset M_n(\Bbbk)$ of the dimension n satisfies $L \cap D = \emptyset$. Let $L \subset \mathbb{P}M_n(\Bbbk)$ be a linear subspace of dimension n. Let $W \subset M_n(\Bbbk)$ be a vector space whose associated projective space is L. Then $\dim W = \dim L + 1 = n+1$. Let $\{A_1, \ldots, A_{n+1}\}$ be a basis of W, which means $A_1, \ldots, A_{n+1}$ are independent, which means for any $\lambda_1, \ldots, \lambda_{n+1} \in \Bbbk$ not all of them being zero, we have $\lambda_1 A_1 + \ldots + \lambda_{n+1}A_{n+1} \neq 0$. We may consider the 1-dimensional subspace spanned by $\lambda_1 A_1 + \ldots + \lambda_{n+1}A_{n+1}$, which gives an element in L. Since $L \cap D = \emptyset$, we have $\det(\lambda_1 A_1 + \ldots + \lambda_{n+1}A_{n+1}) \neq 0$, i.e. the matrix $\lambda_1 A_1 + \ldots + \lambda_{n+1}A_{n+1}$ is invertible. Let $v \neq 0 \in \Bbbk^n$ be a nonzero vector.

Claim *$A_1 v, \ldots, A_{n+1}v$ are linearly independent vectors on $\Bbbk^n$.*

Proof For any $\lambda_1, \ldots, \lambda_{n+1} \in \Bbbk$, not all of them being zero, we have $\lambda_1 A_1 v + \ldots + \lambda_{n+1}A_{n+1}v = (\lambda_1 A_1 + \ldots + \lambda_{n+1}A_{n+1})v$. Since $\lambda_1 A_1 + \ldots + \lambda_{n+1}A_{n+1}$ is invertible and v is nonzero, we have $\lambda_1 A_1 v + \ldots + \lambda_{n+1}A_{n+1}v \neq 0$, as desired. □
This claim gives us a contradiction since $A_1 v, \ldots, A_{n+1}v$ are $n+1$ linearly independent vectors in an n-dimensional vector space $\Bbbk^n$. □

Now let us consider the case over $\mathbb{Q}$. We recall some preliminaries on algebraic numbers and minimal polynomials. For more details, see [6, V.1].

Definition 3.6 *A complex number $\alpha \in \mathbb{C}$ is called an algebraic number, if α is a root of a nonzero polynomial with rational coefficients.*

Lemma 3.7 *Let α be an algebraic number. Then there exists a unique monic polynomial $\mu_\alpha(X)$ with rational coefficients such that α is a root of μ_α and for any polynomial $f(X)$ with a rational coefficient such that α is a root, μ_α divides f.*

Definition 3.8 *The polynomial μ_α in Lemma 3.7 is called the minimal polynomial of α.*

Proposition 3.9 *Let α be a root of a monic polynomial $f(X)$ with rational coefficients. Then $f(X)$ is the minimal polynomial of α if and only if $f(X)$ is irreducible.*

Now let us go back to prove $\sigma_{\mathbb{Q}}(n) = n$. Actually we just need to prove $\sigma_{\mathbb{Q}}(n) \geq n$. Let us briefly present our strategy here. First, we construct n matrices $A_1, \ldots, A_n \in M_n(\mathbb{Q})$ satisfying Property (*). Through this construction, we can prove that $\sigma_{\mathbb{Q}}(n) \geq n$.

Now let us define n matrices $A_1, \ldots, A_n$ as follows. A_1 is the Identity matrix. For $1 < k \leq n$, A_k is the matrix whose $(i+k, i)$-entries are filled by 1 and whose $(i, i+n-k)$ entries are filled by 2, and all the others are filled by 0.

$$A_1 = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & 0 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & & & & 2 \\ 1 & 0 & 0 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \end{pmatrix},$$

$$\cdots\cdots\cdots$$

$$A_n = \begin{pmatrix} 0 & 2 & & & & \\ 0 & 0 & 2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & 2 \\ 1 & & & & 0 & 0 \end{pmatrix}.$$

Proposition 3.10 *For any $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$, not all being zero, the matrix $\lambda_1 A_1 + \ldots + \lambda_n A_n$ is invertible.*

Proof The expression of $\lambda_1 A_1 + \ldots + \lambda_n A_n$ is explicit. For the sake of convenient, let us denote, for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Q}^n$, $A_\lambda = \lambda_1 A_1 + \ldots + \lambda_n A_n$. Then

$$A_\lambda = \begin{pmatrix} \lambda_1 & 2\lambda_n & 2\lambda_{n-1} & & 2\lambda_3 & 2\lambda_2 \\ \lambda_2 & \lambda_1 & 2\lambda_n & & & 2\lambda_3 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \lambda_2 & \lambda_1 & 2\lambda_n \\ \lambda_n & & & & \lambda_2 & \lambda_1 \end{pmatrix}.$$

We need to prove that $\det A_\lambda = 0$ only when $\lambda = 0$. In order to calculate $\det A_\lambda$, we take $\delta = 2^{\frac{1}{n}}$,

$\mu_i = \delta^{i-1}\lambda_i$, for $i = 1, \dots, n$. Then we have

$$A_\lambda = \begin{pmatrix} 1 & & & & \\ & \delta^{-1} & & & \\ & & \delta^{-2} & & \\ & & & \ddots & \\ & & & & \delta^{-n+1} \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_n & & & \mu_3 & \mu_2 \\ \mu_2 & \mu_1 & \mu_n & & & \mu_3 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ \mu_{n-1} & & & \mu_2 & \mu_1 & \mu_n \\ \mu_n & \mu_{n-1} & & & \mu_2 & \mu_1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & \delta^{1} & & & \\ & & \delta^{2} & & \\ & & & \ddots & \\ & & & & \delta^{n-1} \end{pmatrix}.$$

Since $\det \operatorname{diag}(1, \delta^{-1}, \dots, \delta^{-(n-1)})$ and $\det \operatorname{diag}(1, \delta, \dots, \delta^{n-1})$ are both nonzero, we have $\det A_\lambda = 0$, if and only if

$$\det \begin{pmatrix} \mu_1 & \mu_n & & & \mu_3 & \mu_2 \\ \mu_2 & \mu_1 & \mu_n & & & \mu_3 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ \mu_{n-1} & & & \mu_2 & \mu_1 & \mu_n \\ \mu_n & \mu_{n-1} & & & \mu_2 & \mu_1 \end{pmatrix} = 0$$

But this determinant is relatively easy to calculate, since the matrix is a circulant matrix. We use the following well-known result in the theory of circulant matrices [4, (14.312)].

Lemma 3.11 ([4]) *Let $\epsilon_1, \dots, \epsilon_n \in \mathbb{C}$ be the n-th roots of unity (i.e. $\epsilon_i^n = 1$). Then we have the following formula*

$$\det \begin{pmatrix} \mu_1 & \mu_n & & & \mu_3 & \mu_2 \\ \mu_2 & \mu_1 & \mu_n & & & \mu_3 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ \mu_{n-1} & & & \mu_2 & \mu_1 & \mu_n \\ \mu_n & \mu_{n-1} & & & \mu_2 & \mu_1 \end{pmatrix} = (-1)^{n-1} f(\epsilon_1) f(\epsilon_2) \dots f(\epsilon_n),$$

where $f(X) = \mu_n + \mu_{n-1}X + \dots + \mu_1 X^{n-1}$.

By Lemma 3.14, this determinant can be written as

$$(-1)^{n-1}(\mu_n + \mu_{n-1}\epsilon_1 + \dots + \mu_1\epsilon_1^{n-1})(\mu_n + \mu_{n-1}\epsilon_2 + \dots + \mu_1\epsilon_2^{n-1}) \dots (\mu_n + \mu_{n-1}\epsilon_n + \dots + \mu_1\epsilon_n^{n-1}).$$

Hence, the condition $\det A_\lambda = 0$ implies that for some i, we have $(\mu_n + \mu_{n-1}\epsilon_i + \dots + \mu_1\epsilon_i^{n-1}) = 0$. But we have

$$\begin{aligned} &(\mu_n + \mu_{n-1}\epsilon_i + \dots + \mu_1\epsilon_i^{n-1}) \\ =&(\delta^{n-1}\lambda_n + \delta^{n-2}\epsilon_i\lambda_{n-1} + \dots + \epsilon_i^{n-1}\lambda_1) \\ =&\epsilon_i^{n-1}((\delta\epsilon_i^{-1})^{n-1}\lambda_n + (\delta\epsilon_i^{-1})^{n-2}\lambda_{n-1} + \dots + (\delta\epsilon_i^{-1})^0\lambda_1) \\ =&0. \end{aligned}$$

By Eisenstein's Criterion, it is easy to know that the minimal polynomial of $\delta\epsilon_i^{-1}$ is $x^n - 2$. Hence, $\lambda_1 = \dots = \lambda_n = 0$, since if not, $\delta\epsilon_i^{-1}$ satisfies a nonzero polynominal of degree $\leq n-1$, which is less than n. $\square$

Proof of Theorem 1.4 (ii). This is the same argument as in Corollary 2.22. Let W be a subspace of $M_n(\mathbb{Q})$ spanned by $A_1, \dots, A_n$. Let L be the associate projective space of W. Then $\dim L =$

$\dim W - 1 = n - 1$ and by the Proposition 3.10, we have $L \cap D = \emptyset$. By the definition of $\sigma_{\mathbb{Q}}$, we have $\sigma_{\mathbb{Q}}(n) \geq n$. On the other hand, by Lemma 3.5, $\sigma_{\mathbb{Q}}(n) \leq n$. Therefore, $\sigma_{\mathbb{Q}}(n) = n$. $\square$

Proof of Theorem 1.5. Assume by contradiction that there exists a linear subspace L of dimension m such that $L \cap D_{m,n,n-1} = \emptyset$. By Lemma 2.21, there exist $m+1$ matrices $A_1, \ldots, A_{m+1}$ of size $m \times n$ satisfying that for any $\lambda_1, \ldots \lambda_{m+1} \in \mathbb{Q}$, not all being zero, the matrix $\lambda_1 A_1 + \ldots + \lambda_{m+1} A_{m+1}$ has rank n. Let $v \in \mathbb{Q}^n$ be a nonzero vector.

Lemma 3.12 *The list of vectors $A_1 v, \ldots, A_{m+1} v$ is linearly independent in $\mathbb{Q}^m$.*

Proof For any $\lambda_1, \ldots, \lambda_{m+1} \in \mathbb{Q}$, not all being zero, $\lambda_1 A_1 v + \ldots + \lambda_{m+1} A_{m+1} v = (\lambda_1 A_1 + \ldots + \lambda_{m+1} A_{m+1})v$. Since $\operatorname{rank}(\lambda_1 A_1 + \ldots + \lambda_{m+1} A_{m+1}) = n$, we have $\dim \ker(\lambda_1 A_1 + \ldots + \lambda_{m+1} A_{m+1}) = 0$. Hence, $(\lambda_1 A_1 + \ldots + \lambda_{m+1} A_{m+1})v$ is nonzero, as desired. $\square$

Lemma 3.12 gives us a contradiction since we have $m+1$ linearly independent vectors in an m-dimensional vector space.

It remains to show the last statement. By Proposition 3.10, there exist m matrices $A_1, \ldots, A_m$ of size $m \times m$ such that for any $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}$, not all being zero, $\lambda_1 A_1 + \ldots + \lambda_m A_m$ is invertible. Let A'_i be the $m \times n$ matrix formed by the first n columns of A_i.

Lemma 3.13 *For any $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}$, not all being zero, the matrix $\lambda_1 A'_1 + \ldots + \lambda_m A'_m$ is of rank n.*

Proof Notice that the matrix $\lambda_1 A'_1 + \ldots + \lambda_m A'_m$ is the first n columns of the matrix $\lambda_1 A_1 + \ldots + \lambda_m A_m$. Since $\lambda_1 A_1 + \ldots + \lambda_m A_m$ is invertible by our assumption, its first n columns must be linearly independent. Hence $\operatorname{rank}(\lambda_1 A'_1 + \ldots + \lambda_m A'_m) = n$. $\square$

Therefore, by Lemma 2.21 and Lemma 3.13, there exists a linear subspace L of demension $m-1$ such that $L \cap D_{m,n,n-1} = \emptyset$. $\square$

3.3 The real number case

Theorem 3.14 *For any linear subspace $L \subset \mathbb{P}M_{n \times \rho(n)}(\mathbb{R})$ of dimension n, we have $L \cap D_{n,\rho(n),\rho(n)-1} \neq \emptyset$ and there exists a linear subspace $L \subset \mathbb{P}M_{n \times \rho(n)}(\mathbb{R})$ of dimension $n-1$ such that $L \cap D_{n,\rho(n),\rho(n)-1} = \emptyset$.*

Proof Assume by contradiction that there exists a linear subspace $L \subset \mathbb{P}M_{n \times \rho(n)}(\mathbb{R})$ of dimension n such that $L \cap D_{n,\rho(n),\rho(n)-1} = \emptyset$. By Lemma 2.21, there exist $n+1$ matrices $A_1, \ldots, A_{n+1}$ of size $n \times \rho(n)$ such that for any $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{R}$, not all being zero, the rank of $\lambda_1 A_1 + \ldots + \lambda_{n+1} A_{n+1}$ is $\rho(n)$.

Lemma 3.15 *Let $v \in \mathbb{R}^{\rho(n)}$ be a nonzero vector. The list of vectors $A_1 v, \ldots, A_{n+1} v$ is linearly independent.*

Proof For any $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{R}$, not all of them being zero, we have $\lambda_1 A_1 v + \ldots + \lambda_{n+1} A_{n+1} v = (\lambda_1 A_1 + \ldots + \lambda_{n+1} A_{n+1})v$. Since $\operatorname{rank}(\lambda_1 A_1 + \ldots + \lambda_{n+1} A_{n+1}) = \rho(n)$. We have $\dim \ker(\lambda_1 A_1 + \ldots + \lambda_{n+1} A_{n+1}) = \rho(n) - \operatorname{rank}(\lambda_1 A_1 + \ldots + \lambda_{n+1} A_{n+1}) = 0$. Hence, $(\lambda_1 A_1 + \ldots + \lambda_{n+1} A_{n+1})v \neq 0$, as desired. $\square$

Lemma 3.15 gives us a contradiction, since we get $n+1$ linearly independent vectors in an n-dimensional vector space $\mathbb{R}^n$. This concludes the first statement.

Now let us prove the second statement. Recall that in Sections 2.3 and 2.4, we constructed $\rho(n)$ matrices $A_1, \ldots, A_{\rho(n)}$ of size $n \times n$ satisfying for any $\lambda_1, \ldots, \lambda_{\rho(n)} \in \mathbb{R}$, not all being zero, the matrix $\lambda_1 A_1 + \ldots + \lambda_{\rho(n)} A_{\rho(n)}$ is invertible. Let B_i be the $n \times \rho(n)$ matrix whose j-th column is the i-th column of A_j. Namely, for any $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, \rho(n)\}$, we have

$$B_i^{(j)} = A_j^{(i)},$$

where $B_i^{(j)}$ denotes the j-th column of matrix B_i and similarly for $A_j^{(i)}$. We get n matrices $B_1, \ldots, B_n$ of size $n \times \rho(n)$.

Lemma 3.16 *For any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, not all being zero, the rank of $\lambda_1 B_1 + \ldots + \lambda_n B_n$ is $\rho(n)$.*

Proof Let $B_\lambda = \lambda_1 B_1 + \ldots + \lambda_n B_n$, then $B_\lambda^{(j)} = \lambda_1 B_1^{(j)} + \ldots + \lambda_n B_n^{(j)}$. We need to show that the list of vectors $B_\lambda^{(1)}, \ldots, B_\lambda^{(\rho(n))}$ is linearly independent. For any $\mu_1, \ldots, \mu_{\rho(n)} \in \mathbb{R}$, not all of them being zero, we have

$$\begin{aligned}
\mu_1 B_\lambda^{(1)} + \ldots + \mu_{\rho(n)} B_\lambda^{(\rho(n))} =& \mu_1(\lambda_1 B_1^{(1)} + \ldots + \lambda_n B_n^{(1)}) \\
&+ \ldots \\
&+ \mu_{\rho(n)}(\lambda_1 B_1^{(\rho(n))} + \ldots + \lambda_n B_n^{(\rho(n))}) \\
=& \mu_1(\lambda_1 A_1^{(1)} + \ldots + \lambda_n A_1^{(n)}) \\
&+ \ldots\ldots\ldots \\
&+ \mu_n(\lambda_n A_n^{(1)} + \ldots + \lambda_n A_n^{(n)}) \\
=& \lambda_1(\mu_1 A_1^{(1)} + \mu_2 A_2^{(1)} + \ldots + \mu_{\rho(n)} A_{\rho(n)}^{(1)}) \\
&+ \ldots\ldots\ldots \\
&+ \lambda_n(\mu_1 A_1^{(n)} + \mu_2 A_2^{(n)} + \ldots + \mu_{\rho(n)} A_{\rho(n)}^{(n)}) \\
=& \lambda_1(\mu_1 A_1 + \ldots + \mu_{\rho(n)} A_{\rho(n)})^{(1)} \\
&+ \ldots\ldots\ldots \\
&+ \lambda_n(\mu_1 A_1 + \ldots + \mu_{\rho(n)} A_{\rho(n)})^{(n)} \neq 0.
\end{aligned}$$

The last summation is nonzero because $\mu_1 A_1 + \ldots + \mu_{\rho(n)} A_{\rho(n)}$ is an invertible matrix, and the columns of this matrix is linearly independent. Hence, we have shown that the list of vectors $B_\lambda^{(1)}, \ldots, B_\lambda^{(\rho(n))}$ is linearly independent, as desired. □

By Lemma 2.21, there exists a linear subspace L of dimension $n-1$, such that $L \cap D_{n,\rho(n),\rho(n)-1} = \emptyset$. This terminates the proof of the second statement. □

Definition 3.17 *A map $\phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ is called bilinear if for $v \in \mathbb{R}^s$, the map $u \mapsto \phi(u, v)$ is linear, and for $u \in \mathbb{R}^r$, the map $u \mapsto \phi(u, v)$ is linear.*

Definition 3.18 *Let $\phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ be a bilinear map. This bilinear map is called nonsingular if $\phi(u, v) = 0$ implies either $u = 0$ or $v = 0$.*

Definition 3.19 *Fix $r, s \in \mathbb{N}$. We denote the minimal n as $r\#s$, such that there exists a nonsingular bilinear map $\phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$.*

Assume from now on that $m \geq n$.

Theorem 3.20 *For any $k \in \mathbb{N}$, the following two statements are equivalent:*
(i) There exists a linear subspace $L \subset \mathbb{P}M_{m\times n}(\mathbb{R})$ of dimension $k-1$ such that $L \cap D_{m,n,n-1} = \emptyset$.
(ii) We have $k\#n \leq m$.

Proof First, we prove (i) $\Longrightarrow$(ii). By Lemma 2.21, we have k matrices $A_1, \ldots, A_k$ of size $m \times n$ such that for any $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, not all being zero, we have $\operatorname{rank}(\lambda_1 A_1 + \ldots + \lambda_k A_k) = n$. We define a bilinear map $\phi : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^m$ as follows. Let $u = (u_1, \ldots, u_k) \in \mathbb{R}^k$, $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. Define $\phi(u, v) = u_1 A_1 v + \ldots + u_k A_k v$, and we need to show that $\phi : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^m$ is bilinear and ϕ is nonsingular. It is clear that ϕ is bilinear. To show that ϕ is nonsingular, we need to show that if $u \neq 0$, $v \neq 0$, then $(u_1 A_1 + \ldots + u_k A_k)v \neq 0$. Since $u \neq 0$, we have $\operatorname{rank}(u_1 A_1 + \ldots + u_k A_k) = n$. Thus $\ker(u_1 A_1 + \ldots + u_k A_k) = \{0\}$, which means $(u_1 A_1 + \ldots + u_k A_k)v \neq 0$, since $v \neq 0$. By the definition of $k\#n$, we have $k\#n \leq m$.

Second, we need to prove $(ii) \implies (i)$. By definition of $k\#n$, the fact that $k\#n \leq m$ implies that there exists a nonsingular linear map $\phi : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^m$. Let us construct k matrices $A_1, \ldots, A_k$ of size $m \times n$ as follows: Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^k$ be the coordinate vector whose i-th component is 1 and other components are 0. Then $v \mapsto \phi(e_i, v) : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map which corresponds to an $m \times n$ matrix A_i.

Lemma 3.21 *For any $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ not all being zero, we have* $\operatorname{rank}(\lambda_1 A_1 + \ldots + \lambda_k A_k) = n$.

Proof It suffices to show that $\ker(\lambda_1 A_1 + \ldots + \lambda_k A_k) = \{0\}$. Let $v \in \mathbb{R}^n$ such that $(\lambda_1 A_1 + \ldots + \lambda_k A_k)v = 0$. We have $(\lambda_1 A_1 + \ldots + \lambda_k A_k)v = \lambda_1 \phi(e_1, v) + \ldots + \lambda_k \phi(e_k, v) = \phi((\lambda_1 e_1 + \ldots + \lambda_k e_k), v) = \phi((\lambda_1, \ldots, \lambda_k), v)$. Since ϕ is nonsingular, and $(\lambda_1, \ldots, \lambda_k) \neq 0$, and since $\phi((\lambda_1, \ldots, \lambda_k), v) = 0$, we have $v = 0$. Therefore, $\ker(\lambda_1 A_1 + \ldots + \lambda_k A_k) = \{0\}$. □

By Lemma 2.21 and Lemma 3.21, the statement (i) holds. □

Let $\sigma(m, n)$ be the minimal number $k > 0$ such that for any linear subspace $L \cap D_{m,n,n-1} \neq \emptyset$.

Corollary 3.22 *The number $\sigma(m, n)$ is the minimal integer $k > 0$ such that $k\#n \leq m$.*

Proof By the definition of $\sigma(m, n)$ we know that $\sigma(m, n) - 1$ is the maximal number k such that there exists linear subspace $L \subset \mathbb{P}M_{m\times n}(\mathbb{R})$ of dimension k, such that $L \cap D_{m,n,n-1} = \emptyset$. By Theorem 3.20 the number $\sigma(m, n)$ is the maximal integer k such that $k\#n \leq m$. □

Corollary 3.23 *We can calculate $\sigma(m, n)$ for $m \leq 8$. The result is summarized in the following table.*

$\sigma(m,n)$	*2*	*3*	*4*	*5*	*6*	*7*	*8*
2	*2*	*2*	*4*	*4*	*6*	*6*	*8*
3		*1*	*4*	*4*	*4*	*5*	*8*
4			*4*	*4*	*4*	*4*	*8*
5				*1*	*2*	*3*	*8*
6					*2*	*2*	*8*
7						*1*	*8*
8							*8*

Proof This follows directly from Corollary 3.22 and the following table of $r\#s$ for small r, s that we can find in [5] or [9, Theorem 12.21].

$r\#s$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	2	4	4	6	6	8	8	10
3	3	4	4	4	7	8	8	8	11
4	4	4	4	4	8	8	8	8	12
5	5	6	7	8	8	8	8	8	13
6	6	6	8	8	8	8	8	8	14
7	7	8	8	8	8	8	8	8	15
8	8	8	8	8	8	8	8	8	16
9	9	10	11	12	13	14	15	16	16

□

A Some explicit matrix calculations

In order to make a conciser presentation in the main text, we have omitted the verification of the amicable pair in Lemma 2.14. Let us check it here in this section.

For the first axiom, we have

$$
\begin{aligned}
{}^t A_i' &= \begin{pmatrix} {}^t A_i & 0 \\ 0 & -{}^t A_i \end{pmatrix} = \begin{pmatrix} -A_i & 0 \\ 0 & A_i \end{pmatrix} = -A_i'. \\
{}^t A_{s+1}' &= \begin{pmatrix} 0 & {}^t I \\ -{}^t I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -A_{s+1}' \\
{}^t B_i' &= \begin{pmatrix} {}^t B_i & 0 \\ 0 & -{}^t B_i \end{pmatrix} = \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix} = B_i'. \\
{}^t B_{t+1}' &= \begin{pmatrix} 0 & {}^t I \\ {}^t I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = B_{t+1}'.
\end{aligned}
$$

For the second axiom, we have

$$
\begin{aligned}
A_i'^2 &= \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix}^2 = \begin{pmatrix} -I_n & 0 \\ 0 & -I_n \end{pmatrix} = -I_{2n}. \\
A_{s+1}'^2 &= \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}^2 = \begin{pmatrix} -I_n & 0 \\ 0 & -I_n \end{pmatrix} = -I_{2n} \\
B_i'^2 &= \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix}^2 = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} = I_{2n}. \\
B_{t+1}'^2 &= \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}^2 = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} = I_{2n}.
\end{aligned}
$$

For the third axiom, we do a case-by-case check.

- For $i, j \in \{1, \ldots, s\}$ and $i \neq j$, we have $A_i'A_j' = -A_j'A_i'$. In fact,

$$
\begin{aligned}
A_i'A_j' &= \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix} \begin{pmatrix} A_j & 0 \\ 0 & -A_j \end{pmatrix} = \begin{pmatrix} A_iA_j & 0 \\ 0 & A_iA_j \end{pmatrix} \\
-A_j'A_i' &= \begin{pmatrix} -A_j & 0 \\ 0 & A_j \end{pmatrix} \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix} = \begin{pmatrix} -A_jA_i & 0 \\ 0 & -A_jA_i \end{pmatrix} = \begin{pmatrix} A_iA_j & 0 \\ 0 & A_iA_j \end{pmatrix}.
\end{aligned}
$$

- For $i \in \{1, \ldots, s\}$, we have $A_i'A_{s+1}' = -A_{s+1}'A_i'$. In fact,

$$
\begin{aligned}
A_i'A_{s+1}' &= \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -A_i \\ -A_i & 0 \end{pmatrix} \\
-A_{s+1}'A_i' &= \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix} = \begin{pmatrix} 0 & -A_i \\ -A_i & 0 \end{pmatrix}.
\end{aligned}
$$

- For $i, j \in \{1, \ldots, t\}$ and $i \neq j$, we have $B_i'B_j' = -B_j'B_i'$. In fact,

$$
\begin{aligned}
B_i'B_j' &= \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix} \begin{pmatrix} B_j & 0 \\ 0 & -B_j \end{pmatrix} = \begin{pmatrix} B_iB_j & 0 \\ 0 & B_iB_j \end{pmatrix} \\
-B_j'B_i' &= \begin{pmatrix} -B_j & 0 \\ 0 & B_j \end{pmatrix} \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix} = \begin{pmatrix} -B_jB_i & 0 \\ 0 & -B_jB_i \end{pmatrix} = \begin{pmatrix} B_iB_j & 0 \\ 0 & B_iB_j \end{pmatrix}.
\end{aligned}
$$

- For $i \in \{1, \ldots, t\}$, we have $B'_i B'_{t+1} = -B'_{t+1} B'_i$. In fact,

$$\begin{aligned} B'_i B'_{t+1} &= \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & -B_i \\ -B_i & 0 \end{pmatrix} \\ -B'_{t+1} B'_i &= \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix} = \begin{pmatrix} 0 & -B_i \\ -B_i & 0 \end{pmatrix}. \end{aligned}$$

- For $i \in \{1, \ldots, t\}$, we have $A'_i B'_{t+1} = -B'_{t+1} A'_i$. In fact,

$$\begin{aligned} A'_i B'_{t+1} &= \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_i \\ -A_i & 0 \end{pmatrix} \\ -B'_{t+1} A'_i &= \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix} = \begin{pmatrix} 0 & A_i \\ -A_i & 0 \end{pmatrix}. \end{aligned}$$

- For $i \in \{1, \ldots, s\}$, we have $A'_i B'_{t+1} = -B'_{t+1} A'_i$. In fact,

$$\begin{aligned} A'_i B'_{t+1} &= \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_i \\ -A_i & 0 \end{pmatrix} \\ -B'_{t+1} A'_i &= \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix} = \begin{pmatrix} 0 & A_i \\ -A_i & 0 \end{pmatrix}. \end{aligned}$$

- For $i \in \{1, \ldots, t\}$, we have $A'_{s+1} B'_i = -B'_{t+1} A'_i$. In fact,

$$\begin{aligned} A'_{s+1} B'_i &= \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix} = \begin{pmatrix} 0 & B_i \\ B_i & 0 \end{pmatrix} \\ -B'_{t+1} A'_i &= \begin{pmatrix} B_i & 0 \\ 0 & -B_i \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_i \\ B_i & 0 \end{pmatrix}. \end{aligned}$$

- We have $A'_{s+1} B'_{t+1} = -B'_{t+1} A'_{s+1}$. In fact,

$$\begin{aligned} A'_{s+1} B'_{t+1} &= \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \\ B'_{t+1} A'_{s+1} &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \end{aligned}$$

B Code implementations of three lemmas using Mathematica

In this appendix, we present a code implementation of the three lemmas about the amicable pairs as stated in Section 2.4.

```
(*Define matrices*)
sigma0 = {{1, 0}, {0, 1}};
sigma1 = {{0, 1}, {1, 0}};
sigma2 = {{0, -1}, {1, 0}};
sigma3 = {{1, 0}, {0, -1}};
```

```
(*Construction lemma*)
construction[S_, T_] :=
 Module[{S0 = {}, T0 = {}, i = 1, n = Length[T[[1]]]},
  While[i <= Length[S],
   S0 = Append[S0, KroneckerProduct[sigma3, S[[i]]]];
   i++;
   ];
  S0 = Append[S0, KroneckerProduct[sigma2, IdentityMatrix[n]]];
  i = 1;
  While[i <= Length[T],
   T0 = Append[T0, KroneckerProduct[sigma3, T[[i]]]];
   i++
   ];
  T0 = Append[T0, KroneckerProduct[sigma1, IdentityMatrix[n]]];
  {S0, T0}
  ]

(*Shift lemma*)
shift[S_, T_] := Module[{S1 = S, T1 = T},
   If[Length[T] >= 4,
    B1 = T[[1]];
    B2 = T[[2]];
    B3 = T[[3]];
    B4 = T[[4]];
    S1 =
     Join[S, {B1 . B2 . B3, B1 . B2 . B4, B1 . B3 . B4, B2 . B3 . B4}];
    T1 = Drop[T, 4];
    ];
   {S1, T1}];

(*Expansion lemma*)
expansion[S_, T_] :=
 Module[{n = Length[S[[1]]], S1 = S, T1 = T, i = 1},
  total = IdentityMatrix[n];
  If[Mod[Length[S] - Length[T] - 2, 4] == 0,
   While[i <= Length[S],
    total = total . S[[i]];
    i++];
   i = 1;
   While[i <= Length[T],
    total = total . T[[i]];
    i++];
   S1 = Append[S, total]
   ];
  {S1, T1}
  ]

S = {}; T = {{{1}}};
a = 1; b = 2;
i = 1;
```

```
While[i <= 4 a + b,
  res = construction[S, T];
  S = res[[1]]; T = res[[2]];
  i++];
{MatrixForm /@ S, MatrixForm /@ T}
While[Length[T] >= 4,
 res = shift[S, T];
 S = res[[1]]; T = res[[2]];]

{MatrixForm /@ S, MatrixForm /@ T}
i = 0;
While[i <= Length[T],
  If[Mod[Length[S] - Length[T] + i - 2, 4] == 0,
   T = Drop[T, i];
   res = expansion[S, T];
   S = res[[1]]; T = res[[2]];
   ];
  i++
  ];
{MatrixForm /@ S, MatrixForm /@ T}
```

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$$\begin{cases} xw = yz \\ a_1x + b_1y + c_1z + d_1w = 0 \\ \cdots \\ a_kx + b_ky + c_kz + d_kw = 0 \end{cases}$$

always have a nonzero solution.We try to generalize this problem and conduct research on its broader implications. This investigative pursuit ultimately culminates in the inception of this project.