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论文题目： Using the number of tilings with two colors of square and domino to represent sequence and its algebraic identity

Using the number of tilings with two colors of square and domino to represent sequence and its algebraic identity

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Abstract

In this paper, we redefine sequences such as A002605, A155020, A028860, A063727, and A057087 by using the number of ways to tile a board length n using different colors of squares and dominoes to represent the numbers in the sequence. Moreover, we discuss the algebraic identities of sequence A002605 by conditioning on the location of the last square or last blue tile or last fault and introducing them to other sequences such as A063727 as well as A057087 by changing the coefficients of the formulas.

keywords: Sequence, Tiling, Fault, Breakability

Contents

1 Introduction

The Pell number dates back to 250 BC, when it was first introduced by the Greek mathematician Pythagoras. The Pell number can also be represented by the number of ways to tile a strip of length n using two colors of squares and one color of dominoes only.

Based on the Pell number, let us define g_n to be the number of ways to tile a board of length n with two colors of squares (of width 1) and two colors of dominoes (of width 2). If we look at the different ways to arrange a board of length 2, as shown:

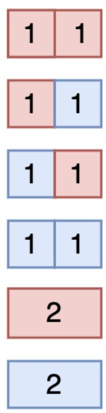

Figure 1: Tilings for a strip of length $n = 2$

We see that there are 6 ways to tile a strip of length 2 using red or blue squares and red or blue dominoes. Hence, $g_2 = 6$.

Here are some more values for g_n :

The first few values for g_n.

n	-1	0	1	2	3	4	5	6	7	8	9
g_n	0	1	2	6	16	44	120	328	896	2448	6688

This is a sequence in the OEIS! It's right here: A002605.

Specifically, for $n \geq 0$,

$$g_n = G_{n+1}.$$

where G_n is the actual numbers in the sequence A002605. For combinatorial convenience, we shall express most of our identities in terms of g_n instead of G_n.

Very quickly, we see the following pattern established:

$$g_n = 2g_{n-1} + 2g_{n-2}$$

To prove this, we first consider all tilings ending with a square. There will be g_{n-1} ways of tiling the rest of the strip. With the two colors of squares available, there will be $2g_{n-1}$ ways to tile the strip. Alternatively, we consider all tilings ending with a domino. There will be g_{n-2} ways of tiling the rest of the strip. With the two colors of dominoes available, there will be $2g_{n-2}$ ways to tile the strip. Together, we get the identity above.

Inspired by the book "Proofs That Really Count" by Art Benjamin and Jennifer Quinn, which introduces a lot of beautiful identities primarily concerning the Fibonacci numbers, we decided to find new theorems concerning the sequence of g_n.

2 New Theorems

2.1 The sequence g_n

Here is our very first theorem. Again, let g_n be defined as the number of ways to tile a strip of length n with two colors of squares (of length 1) and two colors of dominoes (of length 2). For convenience, we will say that the two colors are red and blue. Also, for convenience, we will define g_{-1} to be 0, because of course there is no way to tile a strip of length -1 since length -1 doesn't even exist.

Here is our first theorem about these numbers.

Theorem 1. *For $n \geq 0$, we have*

$$2g_n^2 + g_{n+1}^2 = g_{2n+2}.$$

Proof. How many tilings exist for a $(2n+2)$-board using two colors of squares and two colors of dominoes?

Answer 1: There are g_{2n+2} tilings of a $(2n+2)$-board.

Answer 2: [2] We condition on whether the tiling is breakable at cell $(n+1)$ of the length $(2n+2)$-board. If the tiling is breakable at cell $(n+1)$, there are g_{n+1}^2 ways to tile the strip. If the tiling is unbreakable at cell $(n+1)$, with two

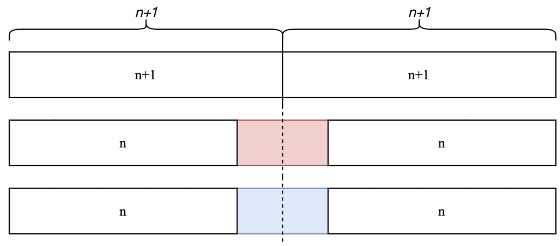

Figure 2: Breakability at cell $(n+1)$

colors of domino for cell $(n+1)$, there are $2g_n^2$ ways to tile the strip. Hence the total number of ways to tile the strip length $(2n+2)$ with two colors of square and two colors of domino would be $2g_n^2 + g_{n+1}^2$.

Since the two answers must be the same, we conclude that

$$2g_n^2 + g_{n+1}^2 = g_{2n+2}.$$

□

To prove this theorem, we can also use the following algebraic method.

Proof. From A002605, we have

$g_n = (1+\sqrt{3})^n \cdot (\frac{1}{2} + \frac{\sqrt{3}}{6}) + (1-\sqrt{3})^n \cdot (\frac{1}{2} + \frac{\sqrt{3}}{6})$.

By using the formula of g_n to represent the left hand side of the theorem we get

$2 \cdot [(1+\sqrt{3})^n \cdot (\frac{1}{2} + \frac{\sqrt{3}}{6}) + (1-\sqrt{3})^n \cdot (\frac{1}{2} + \frac{\sqrt{3}}{6})]^2 + [(1+\sqrt{3})^{n+1} \cdot (\frac{1}{2} + \frac{\sqrt{3}}{6}) + (1-\sqrt{3})^{n+1} \cdot (\frac{1}{2} + \frac{\sqrt{3}}{6})]^2$.

After simplifying the expression, we will get

$(1+\sqrt{3})^{2n} \cdot (6+2\sqrt{3}) \cdot (\frac{1}{2} + \frac{\sqrt{3}}{6})^2 + (1-\sqrt{3})^{2n} \cdot (6-2\sqrt{3}) \cdot (\frac{1}{2} - \frac{\sqrt{3}}{6})^2$

which equals to

$(1+\sqrt{3})^{2n} \cdot (1+\sqrt{3})^2 \cdot (\frac{1}{2} + \frac{\sqrt{3}}{6}) + (1-\sqrt{3})^{2n} \cdot (1-\sqrt{3})^2 \cdot (\frac{1}{2} - \frac{\sqrt{3}}{6})$

and we will soon get

$(1+\sqrt{3})^{2n+2} \cdot (\frac{1}{2} + \frac{\sqrt{3}}{6}) + (1-\sqrt{3})^{2n+2} \cdot (\frac{1}{2} - \frac{\sqrt{3}}{6}) = g_{2n+2}$.

□

Although the sequence g_n's identities can be proved by a myriad of methods, we find the combinatorial approach ultimately satisfying. Through the comparison of the two approaches used to prove Theorem 1, the method of tiling will be applied in the following proofs of sequences' identities for its convenience in calculation.

Theorem 2. *For $n \geq 0$, we have*

$$g_{2n+1} = \sum_{k=0}^{n} g_{2k} \cdot 2^{n-k+1}.$$

Proof. How many tilings of a length $2n+1$ tiling exist?

Answer1: By definition, there are g_{2n+1} such tilings.

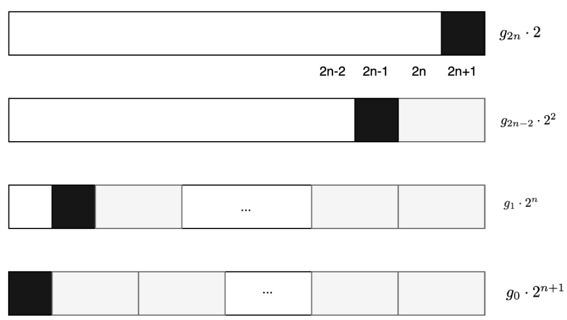

Figure 3: Tile a 2n+1-board with squares and dominoes and condition on the location of the last square

Answer2: [2] Condition on the location of the last square. Since the board has odd length, there must be at least one square and the last square occupies an odd-numbered cell. There are g_{2k} tilings where the last square occupies cell $2k+1$, as illustrated in Figure 3. Considering the colors of square and domino, the number of ways to tile the strip is $g_{2k} \cdot 2^{n-k+1}$. Hence the total number of tilings is

$$\sum_{k=0}^{n} g_{2k} \cdot 2^{n-k+1}.$$

□

Theorem 3. *For $n \geq 1$, we have*

$$\sum_{i=1}^{n} (g_{i-1} + g_{i-2}) \cdot f_{n-i} = g_n - f_n.$$

and we can re-write this in a more pleasing format as the following:

$$\sum_{i=1}^{n} g_i \cdot f_{n-i} = 2(g_n - f_n).$$

Proof. How many tilings exist to tile a length (n) board with at least one blue tile?

Answer 1: There are (g_n) tilings in total of a (n) board. If we exclude the "all red" tilings, then we are excluding f_n such tilings (because f_n gives the number of ways to tile a board with just one color of squares and dominoes). So, this gives ($g_n - f_n$) tilings with at least one blue tile.

Answer 2: [2] Condition on the last blue tile of a n board, which will cover cell i (if a square) and cells $i-1$ and i (if a domino). There are (g_{i-1}) tilings to the left of the blue square, and (g_{i-2}) tilings to the left of the blue domino; since this last blue tile is EITHER a square OR a domino, then we have in all ($g_{i-1} + g_{i-2}$) such tilings. On the right, we have red tiles covering cell ($i+1$) to cell (n), and so there are (f_{n-i}) ways to tile the rest of the board.

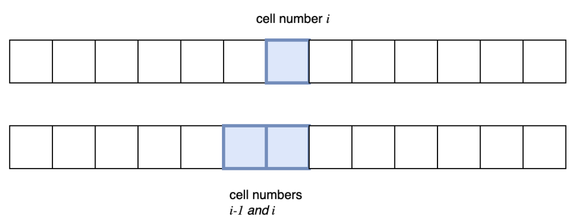

Figure 4: Condition on the last blue tile

Combining the tilings on the left AND the tilings on the right gives us $(g_{i-1} + g_{i-2}) \cdot f_{n-i}$ such tilings.

Summing up all such tilings gives us

$$\sum_{i=1}^{n} (g_{i-1} + g_{i-2}) \cdot f_{n-i}.$$

Since the two answers must be the same, we conclude that

$$\sum_{i=1}^{n} (g_{i-1} + g_{i-2}) \cdot f_{n-i} = g_n - f_n.$$

□

Theorem 4. *For $n \geq 0$,*

$$\sum_{i=0}^{n} g_i^2 \cdot 2^{n+1-i} = g_n \cdot g_{n+1}.$$

Proof. How many tilings of a (n)-board and $(n+1)$-board exist?

Answer 1: By definition, there are $g_n g_{n+1}$ such tilings.

Answer 2: [2] How many tiling pairs have their last fault at cell i, where $0 \leq i \leq n$?

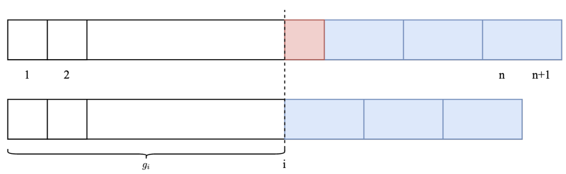

Figure 5: Consider the place of the last fault

There are g_n^2 ways to tile both boards through cell i. To avoid future faults, there is exactly one configuration of squares and dominoes to finish the tiling, as in Figure 5. (Specifically, all tiles after cell i will be dominoes except for a single square placed on cell $i+1$ in the row whose tail length is odd.) Now we have to remember that all these squares and dominoes can be one of two possible colors, and there are $n-i$ dominoes and 1 square in finishing up the tiling. Summing over all possible values of i, and considering the colors of squares and dominoes, gives us

$$\sum_{i=0}^{n} g_i^2 \cdot 2^{n+1-i} = g_n \cdot g_{n+1}.$$

□

Theorem 5. *For $n \geq 1$,*

$$g_{2n} - 2^n = g_1 2^n + g_3 2^{n-1} + g_5 2^{n-2} + \cdots + g_{2n-3} 2^2 + g_{2n-1} 2^1.$$

Proof. How many ways are there to tile a length-$2n$ strip with at least one square?

Answer 1 : There are g_{2n} tilings of a $2n$ board. Excluding the "all domino" tiling gives $g_{2n} - 2^n$ tilings with at least one square.

Answer 2 : Condition on the location of the last square.

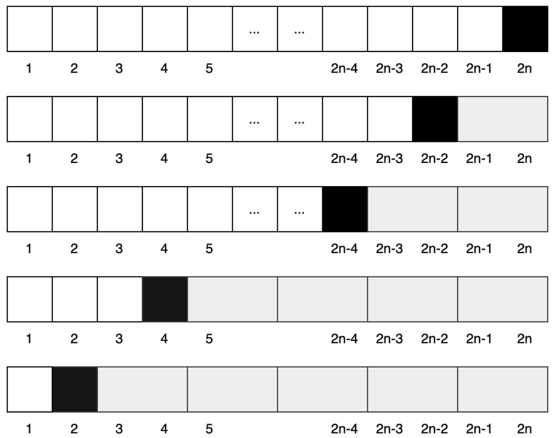

Figure 6: Consider the place of the last square

There are g_i tilings where the last square covers cell g_{n+1}.(Specifically, i must be odd since the total length and the cells left for dominoes must be even.) This is because cells 1 through i can be tiled in g_n ways, cell g_{n+1} must be covered by a square, and cells $n+2$ through $2n$ must be covered by dominoes. Hence the total number of tilings with the restriction of at least one square exists is $g_1 2^n + g_3 2^{n-1} + g_5 2^{n-2} + \cdots + g_{2n-3} 2^2 + g_{2n-1} 2^1$.

Thus, we have

$$g_{2n} - 2^n = g_1 2^n + g_3 2^{n-1} + g_5 2^{n-2} + \cdots + g_{2n-3} 2^2 + g_{2n-1} 2^1.$$

□

Theorem 6. *For $n \geq 0$,*

$$\sum_{i=0}^{n} 2^{n+1-i} \cdot g_i = g_{n+2} - 2^{n+2}.$$

Proof. How many tilings of an n+2-board use at least one domino?

Answer 1: There are g_{n+2} tilings of an $n+2$-board. Excluding the "all square" tiling gives g_{n+2} - 2^{n+2} tilings with at least one domino.

Answer 2: We concider the location of the last domino.

There are g_i tilings where the last domino covers cells $i+1$ and $i+2$. This is because cells 1 through i can be tiled in g_i ways, cells $i+1$ and $i+2$ must be covered by a domino, and cells $i+3$ through $n+2$ must be covered by squares. Hence the total number of tilings with at least one domino is $\sum_{i=0}^{n} 2^{n+1-i} \cdot g_i$.

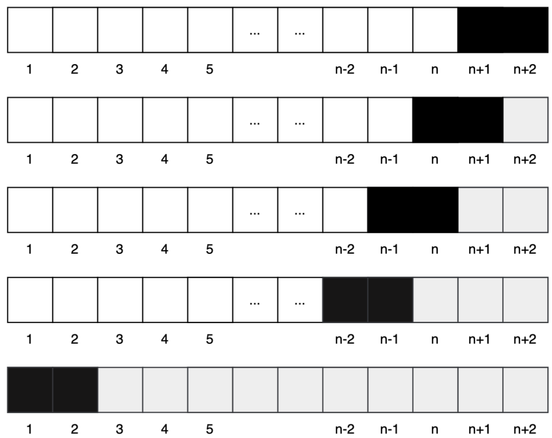

Figure 7: Consider the place of the last domino

Thus, we have

$$\sum_{i=0}^{n} 2^{n+1-i} \cdot g_i = g_{n+2} - 2^{n+2}.$$

□

Theorem 7. *For $n \geq 0$,*

$$\sum_{k \geq 1} 2^n \cdot \binom{n}{k} \cdot g_{k-1} = g_{2n-1}.$$

Proof. How many (2n-1)-tilings exist?

Answer1: By definition, there are g_{2n-1} ways to tile the strip.

Answer2: Condition on the number of squares that appear among the first n tiles. Observe that a $(2n-1)$-board must include at least n tiles, of which at least one is a square. If the first n tiles consist of k squares and $n-k$ dominoes, then these tiles can be arranged $\binom{n}{k}$ ways. Considering the color of tiles as well, there will be $2^n \cdot \binom{n}{k}$ ways to cover cells 1 through $2n-k$. The remaining board has length $k-1$ and can be tiled g_{k-1} ways.

□

Theorem 8. *For $n \geq 0$,*

$$2^n \cdot \binom{n}{0} + 2^{n-1} \cdot \binom{n-1}{1} + 2^{n-2} \cdot \binom{n-2}{2} \cdots = g_n.$$

Proof. We ask, how many tilings of a n board exist?

Answer 1: There are g_n such tilings.

Answer 2: We condition on the number of dominoes. How many n-tilings use exactly i dominoes? For the numbers of tilings to not be zero, $0 \leq i \leq \frac{n}{2}$. Thus, $n - 2i$ squares will be used and in total, $n - i$ tiles. The number of ways to select i of these $n - i$ tiles to be dominoes is $\binom{n-i}{i}$. Taken the number of colors into consideration, there will be $2^{n-i} \cdot \binom{n-i}{i}$ tilings in total. Hence, there are $\sum_{i \geq 0} 2^{n-i} \cdot \binom{n-i}{i}$ n-tilings.

Thus, we have

$$\sum_{i \geq 0} 2^{n-i} \cdot \binom{n-i}{i} = g_n.$$

□

Theorem 9. *For $n \geq 0$,*

$$\sum_{i \geq 0} \sum_{j \geq 0} 2^{n-i} \binom{n-i}{j} \cdot 2^{n-j} \binom{n-j}{i} \cdot 2 = g_{2n+1}.$$

Proof. How many ways are there to tile an 2n+1-board ?

Answer 1: By definition, there are g_{2n+1} ways.

Answer 2: We condition on the number of dominos on each side of the median square. How many tilings contain exactly i dominoes to the left of the median square and exactly j dominoes to the right of the median square?

There are $i + j$ dominoes in total. Consequently, the number of squares is $2n + 1 - 2(i + j)$, with $n - i - j$ squares on each side of the median square. With $n - j$ tiles on the left side of the median square, there are $\binom{n-j}{i}$ ways to choose i dominoes from the $n - j$ tiles. Considering the color of the tiles, there are $2^{n-j}\binom{n-j}{i}$ ways to tile the left side. Similarly, there are $2^{n-i}\binom{n-i}{j}$ ways to tile the right of the median square. Hence, there are $\sum_{i \geq 0} \sum_{j \geq 0} 2^{n-i}\binom{n-i}{j} \cdot 2^{n-j}\binom{n-j}{i} \cdot 2$(considering the color of the median square) tilings altogether.

As i and j vary, we obtain the total number of $(2n + 1)$- tilings as

$$\sum_{i \geq 0} \sum_{j \geq 0} 2^{n-i} \binom{n-i}{j} \cdot 2^{n-j} \binom{n-j}{i} \cdot 2 = g_{2n+1}.$$

□

The next identity is based on the fact that for any $t \geq 0$ a tiling can be broken into segments so that all but the last segment have length t or $t+1$.

Theorem 10. *For $m, p, t \geq 0$*

$$g_{m+(t+1)p} = \sum_{i=0}^{p} \binom{p}{i} g_t^i g_{t-1}^{p-1} g_{m+i}.$$

Proof. How many $(m+(t+1)p)$-tilings exist?

Answer1: $g_{m+(t+1)p}$.

Answer2: For any tiling of length $m+(t+1)p$, we break it into $p+1$ segments of length j_1, j_2, j_3, $\cdots$, j_{p+1}. For $p \geq i \geq 1$, $j_i = t$ unless that would result in breaking a domino in half–in which case we let $j_i = t+1$. Segment $p+1$ consists of the remaining tiles. Count the number of tilings for which i of the first p segments have length t and the other $p-i$ segments have length $t+1$. These p segments have total length $it + (p-i)(t+1) = (t+1)p - i$. Hence $j_{p+1} = m+i$. Since segments of length t can be covered g_t ways and segments of length $t+1$ must end with a domino and can be covered g_{t-1} ways, there are exactly $\binom{p}{i} g_t^i g_{t-1}^{p-1} g_{m+i}$ such tilings.

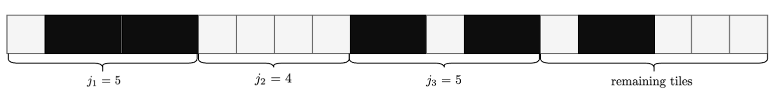

Figure 8: An example of $t = 4$ and $p = 3$

□

Theorem 11. *For $n \geq 0$ and n is even,*

$$\sum_{k=0}^{\frac{n}{2}} 2^{\frac{n-2k+2}{2}} \cdot g_{2k} = \sum_{k=0}^{\frac{n}{2}} 2^{\frac{n-2k+4}{2}} \cdot g_{2k-1} + 2^{1+\frac{n}{2}} + 2 \cdot g_{n-1}.$$

Proof. We question on the number of ways an n+1-board be tiled using squares and dominoes.

Answer 1: Condition on the location of the last square. Since $n+1$ is odd, the last square must exist on an odd cell. For $0 \leq 2k+1 \leq n+1$, the number of ways to tile the rest of a $n+1$-tiling with last square on cell $2k+1$ is $2^{\frac{n-2k+2}{2}}$. For the tilings in front of the last square, there are g_{2k} ways. Altogether, the number of ways to tile n+1-tilings is $\sum_{k=0}^{\frac{n}{2}} 2^{\frac{n-2k+2}{2}} \cdot g_{2k}$.

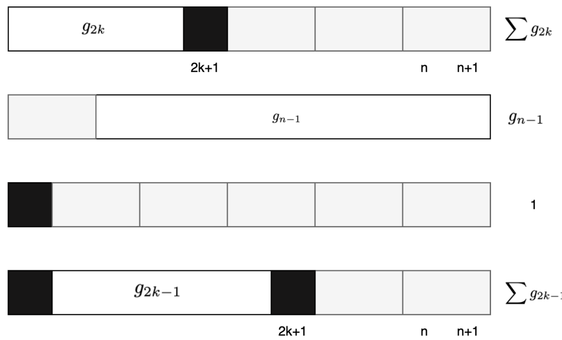

Figure 9: Condition on the location of the last square or the first tile

Answer 2: There are g_{n-1} such tilings that begin with a domino. Considering the color, there are $2 \cdot g_{n-1}$ ways in total. Among those that begin with a square, we condition on the last square. There is one tiling consisting of a single square followed by all dominoes, considering the colors as well, there will be $2^{1+\frac{n}{2}}$ tilings in total. For $3 \leq 2k+1 \leq n+1$, the number of n+1 tilings that begin with a square and whose last square occurs at cell $2k+1$ is

$$\sum_{k=0}^{\frac{n}{2}} 2^{\frac{n-2k+4}{2}}.$$

Altogether, we have

$$\sum_{k=0}^{\frac{n}{2}} 2^{\frac{n-2k+4}{2}} \cdot g_{2k-1} + 2^{1+\frac{n}{2}} + 2 \cdot g_{n-1}.$$

□

The above identities of g_n can fit in the following sequences only with a change of some coefficients.

2.2 The sequence b_n

Let us now introduce a new sequence, which we will call b_n, and we define it to be the number of ways to tile a strip of length n with our two colors of squares and dominos, but with the added restriction that the first tile must be blue.

For example, $b_1 = 1$ because there is just one way to tile a strip of length 1 if we are only able to use a blue tile. Likewise, by direct calculation, $b_2 = 3$ and $b_3 = 8$ and so on.

Here are some values for b_n :

The first few values for b_n.

n	0	1	2	3	4	5	6	7	8	9	10
b_n	1	1	3	8	22	60	164	448	1224	3344	9136

This is a sequence in the OEIS! It's right here: A155020.

Theorem 12. *For $n \geq 3$,*

$$2b_{n-1} + 2b_{n-2} = b_n.$$

Proof. How many tilings of a (n)-board exist when the first tile must be blue?

Answer 1: By definition, there are b_n such tilings.

Answer 2: We consider the last tile. It is either a red or blue square (and if we remove that square, then we have b_{n-1} ways to tile the rest), or it is a red or blue domino (and if we remove that domino, then we have b_{n-2} ways to tile the rest). Summing up these four cases gives us $2b_{n-1} + 2b_{n-2}$ in all.

Comparing the two answers gives us $b_n = 2b_{n-1} + 2b_{n-2}$, as desired.

□

Theorem 13. *For $n \geq 1$,*

$$b_n = \frac{g_n}{2}.$$

Proof. How many tilings of a (n)-board exist when the first tile must be blue?

Answer 1: By definition, there are b_n such tilings.

Answer 2: We consider the tilings of the first two or three cells.

With the restriction that the first cell must be blue, there would be three ways to tile them (starting with a blue square and followed by a square or starting with a blue domino.) and g_{n-2} ways to tile the rest of the strip or two ways to tile them (starting with a blue square followed by a domino.) with g_{n-3} ways to tile the rest of the strip. In total, there are $3g_{n-2} + 2g_{n-3}$ ways to tile the strip. According to the identity:

For $n \geq 2$,

$$2g_{n-1} + 2g_{n-2} = g_n.$$

We have $b_n = 3g_{n-2} + 2g_{n-3} = g_{n-1} + g_{n-2}$. So, we have $b_n = \frac{g_n}{2}$.

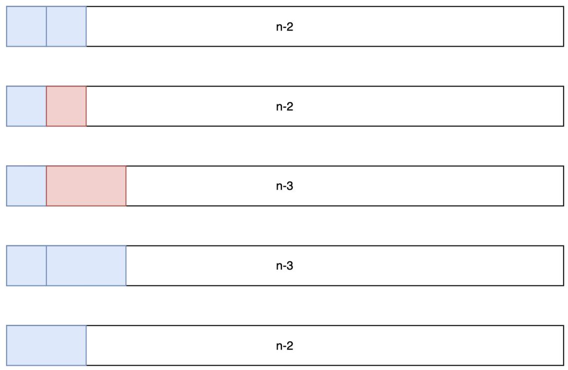

Figure 10: Tilings for a strip of length n with the first cell blue.

□

2.3 The sequence s_n

Let's define s_n to be the number of ways to tile a strip of length n with our two colors of squares and dominoes, but with the added restriction that the first tile must be square.

For example, $s_1 = 2$ because there are two ways to tile a strip of length 1 (with two colors of square.) Likewise, by direct calculation, $s_2 = 4$ and $s_3 = 12$ and so on.

Here are some values for s_n :

The first few values for s_n.

n	0	1	2	3	4	5	6	7	8	9	10
s_n	0	2	4	12	32	88	240	656	1792	4896	13376

This is a sequence in the OEIS! It's right here: A028860.

Theorem 14. *For $n \geq 2$,*

$$2s_{n-1} + 2s_{n-2} = s_n.$$

Proof. How many tilings of a (n)-board exist when the first tile must be a square?

Answer 1: By definition, there are s_n such tilings.

Answer 2: We consider the tilings of the first cell. With the restriction that the first cell must be square, there would be two ways to tile them and g_{n-1} ways to tile the rest of the strip. According to the identity:

For $n \geq 2$,

$$2g_{n-1} + 2g_{n-2} = g_n.$$

We have $2g_{n-1} = 4g_{n-2} + 4g_{n-3}$. According to the identity: $s_n = 2g_{n-1}$

We have $2s_{n-1} + 2s_{n-2} = s_n$.

□

Similarly, the ways of tiling the strip of a (n)-board with the restriction of the first tile must be a domino will also get the sequence of s_n, but with a starting value of s_1=0 and s_2=2.

Theorem 15. *For $n \geq 1$,*

$$s_{n+1} = 2g_n = 4b_n.$$

Proof. How many tilings of a $(n+1)$-board exist when the first tile must be a square?

Answer 1: By definition, there are s_{n+1} such tilings.

Answer 2: We first exclude the first cell of the board and consider the tilings of the rest of the board which is g_n according to definition. Considering the color of the first tile, in total there are $2g_n$ ways to tile the board. According to the identity:

For $n \geq 1$,

$$b_n = \frac{1}{2}g_n$$

.

We have $s_{n+1} = 2g_n = 4b_n$.

□

2.4 The sequence c_n

Let's define c_n to be the number of ways to tile a strip of length n with four colors of squares and two colors of dominoes. For convenience, we will say that the four colors of dominoes are red, blue, green, and brown. The two colors of squares are still red and blue.

For example, $c_1 = 2$ because there are two ways to tile a strip of length 1 (with two colors of square.) Likewise, by direct calculation, $c_2 = 8$ and $c_3 = 24$ and so on.

Here are some values for c_n :

The first few values for c_n.

n	0	1	2	3	4	5	6	7	8	9	10
c_n	1	2	8	24	80	256	832	2688	8704	28160	91136

This is a sequence in the OEIS! It's right here: A063727.

Theorem 16. *For $n \geq 2$,*

$$2c_{n-1} + 4c_{n-2} = c_n.$$

Proof. How many tilings of a (n)-board exist when there are two colors of squares (of width 1) and four colors of dominoes (of width 2) ?

Answer 1: By definition, there are c_n such tilings.

Answer 2: We consider the tilings ending with squares. There will be f_{n-1} ways to tile the rest of the strip. With the two colors of squares available, there will be $2c_{n-1}$ ways to tile the strip. Alternatively, we consider all tilings ending with a domino. There will be c_{n-2} ways to tile the rest of the tiling. With two colors of dominoes available, there will be $4c_{n-2}$ ways to tile the strip. Together, we get $4c_{n-2} + 2c_{n-1}$ ways of tiling.

We have $4c_{n-2} + 2c_{n-1} = c_n$.

□

According to the direct combinatorial approach used in proving theorems in g_n, we can similarly get the following identities of c_n.

For $n \geq 0$, we have

$$4c_n^2 + c_{n+1}^2 = c_{2n+2}.$$

For $n \geq 0$, we have

$$\sum_{i=0}^{n} c_i^2 \cdot 4^{n-i} \cdot 2 = c_n \cdot c_{n+1}.$$

For $n \geq 0$, we have

$$c_{2n} - 4^n = 2 \cdot (c_1 4^{n-1} + c_3 4^{n-2} + c_5 4^{n-3} + \cdots + c_{2n-3} 4^1 + c_{2n-1} 4^0).$$

For $n \geq 0$, we have

$$\sum_{i=0}^{n} 4 \cdot 2^{n-i} \cdot c_i = c_{n+2} - 2^{n+2}.$$

For $n \geq 0$, we have

$$\sum_{i \geq 0} \sum_{j \geq 0} 4^{n-i} \binom{n-i}{j} \cdot 4^{n-j} \binom{n-j}{i} \cdot 2 = c_{2n+1}.$$

2.5 The sequence t_n

Let's define t_n to be the number of ways to tile a strip of length n with four colors of squares and four colors of dominoes. For convenience, we will say that the four colors of dominoes are red, blue, green, and brown, so as the square.

For example, $t_1 = 4$ because there are four ways to tile a strip of length 1 (with four colors of square.) Likewise, by direct calculation, $t_2 = 20$ and $t_3 = 96$ and so on.

Here are some values for t_n :

The first few values for t_n.

n	0	1	2	3	4	5	6	7	8	9
t_n	1	4	20	96	464	2240	10816	52224	252160	1217536

This is a sequence in the OEIS! It's right here: A057087.

Theorem 17. *For $n \geq 2$,*

$$4t_{n-1} + 4t_{n-2} = t_n.$$

Proof. How many tilings of a (n)-board exist when there are four colors of squares (of width 1) and four colors of dominoes (of width 2) ?

Answer 1: By definition, there are t_n such tilings.

Answer 2: We consider the tilings ending with squares. There will be t_{n-1} ways to tile the rest of the strip. With the four colors of squares available, there will be $4t_{n-1}$ ways to tile the strip. Alternatively, we consider all tilings ending with a domino. There will be t_{n-2} ways to tile the rest of the tiling. With four colors of dominoes available, there will be $4t_{n-2}$ ways to tile the strip. Together, we get $4t_{n-1} + 4t_{n-2}$ ways of tiling.

We have $4t_{n-1} + 4t_{n-2} = t_n$.

□

According to the direct combinatorial approach used in proving theorems in g_n, we can similarly get the following identities of t_n.

For $n \geq 0$, we have

$$4t_n^2 + t_{n+1}^2 = t_{2n+2}.$$

For $n \geq 0$, we have

$$\sum_{i=0}^{n} t_i^2 \cdot 4^{n-i} \cdot t = t_n \cdot t_{n+1}.$$

For $n \geq 0$, we have

$$t_{2n} - 4^n = t_1 4^n + t_3 4^{n-1} + t_5 4^{n-2} + \cdots + t_{2n-3} 4^2 + t_{2n-1} 4^1.$$

For $n \geq 0$, we have

$$\sum_{i=0}^{n} 4^{n-i+1} \cdot t_i = t_{n+2} - 4^{n+2}.$$

For $n \geq 0$, we have

$$\sum_{i \geq 0} 4^{n-i} \cdot \binom{n-i}{i} = t_n.$$

For $n \geq 0$, we have

$$\sum_{i \geq 0} \sum_{j \geq 0} 4^{n-i} \binom{n-i}{j} \cdot 4^{n-j} \binom{n-j}{i} \cdot 4 = t_{2n+1}.$$

3 Future Work

[1] There is a possible application of the algebraic identities of the sequences interpreted in this paper which is discovering the number-theoretic aspects of

sequences. Below is one proposal waiting to be proven valid in the future in terms of number theory.

For integers a and b, the greatest common divisor, denoted by $\gcd(a, b)$, is the largest positive number dividing both a and b. It is easy to see that for any integer x.

$$\gcd(a, b) = \gcd(b, a - bx).$$

since any number that divides both a and b must also divide b and (a-bx), and vice versa.

$$\gcd(a, b) = \gcd(b, a - b).$$

We propose,

For $n \geq 1$ and n is odd, $\gcd(G_n, 2G_{n-1}) = 2^{\frac{n-1}{2}}$.

For $n \geq 1$ and n is even, $\gcd(G_n, 2G_{n-1}) = 2^{\frac{n}{2}}$.

4 Conclusion

In conclusion, we see that we can use a few common techniques (taken from Benjamin and Quinn's book) such as:

1. Looking at location of last colored tile,
2. Counting how many tilings have at least one square,
3. Looking at the location of the last fault of two tilings,
4. Considering the breakability of the middle cell,
5. Discussing the number of squares or dominoes in the first n tiles.

These gave us a number of different theorems and equations, thus showing that Benjamin and Quinn's formulas of the Fibonacci sequence can be applied to many other different sequences.

Using those combinatorial approach introduced in Benjamin and Quinn's book can create identities that can be fit into numerous sequences that only need to change a few coefficients depending on the combinatorial definition of the sequence.

In particular, here's one nice formula that I discovered and that I added to the OEIS at A099156.

References

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