Miyaoka-Yau type inequalities of complete intersection threefolds in products of projective spaces

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Abstract. Geography of projective varieties is one of the fundamental problems in algebraic geometry. There are many researches toward the characteristics of Chern number of some projective spaces, for example Noether's inequalities, the theorem of Chang-Lopez, and the Miyaoka-Yau inequality. In this paper, we compute the Chern numbers of any smooth complete intersection threefold in the product of projective spaces via the standard exact sequences of cotangent bundles. Then we obtain linear Chern number inequalities for $\frac{c_1(X)c_2(X)}{c_1^3(X)}$ and $\frac{c_3(X)}{c_1^3(X)}$ on such threefolds under conditions of $d_{ij} \geq 4$ and $d_{ij} \geq 6$ respectively. They can be considered as a generalization of the Miyaoka-Yau inequality and an improvement of Yau's inequality for such threefolds.

Keywords: Chen class, Miyaoka-Yau inequality, threefold, complete in- tersection.

1. Introduction

One of the fundamental problems in algebraic geometry is to study the geogra- phy of projective varieties, i.e., determining which Chern numbers occur for a complex smooth projective variety M. When M is a minimal surface of general type, we have Noether's inequalities [1]:

$$p_g(M) = h^0(M, \omega_M)$$

$$K_M^2 \geq 2p_g(M) - 4.$$

This implies

$$5c_1^2(M) \geq c_2(M) - 36.$$

While on the other hand, we have the Miyaoka-Yau inequality:

$$c_1^2(M) \leq 3c_2(M).$$

Hence $\frac{c_2(M)}{c_1^2(M)}$ is bounded. When M is a threefold of general type with ample canonical divisor, Yau's famous inequality in [2] says

$$8c_1(M)c_2(M) \leq 3c_1^3(M).$$

Hunt studied the geography of threefolds in [3]. Later, Chang and Lopez proved in [4] that the region described by the Chern ratios $(\frac{c_1^3(M)}{c_1(M)c_2(M)}, \frac{c_3(M)}{c_1(M)c_2(M)})$ of threefolds with ample canonical divisor is bounded. Sheng, Xu and Zhang gave the inequalities of Chern numbers of complete intersection threefolds with ample canonical divisor in [5]:

$$86c_1^3(M) \leq c_3(M) \leq \frac{c_1^3(M)}{6}.$$

The theorem of Chang-Lopez has been generalized to higher dimensional case by Du and Sun in [6].

Theorem 1.1. Let X be a nonsingular projective variety of dimension n over an algebraic closed field κ with any characteristic. Suppose K_X *or* $-K_X$ is ample. If the characteristic of κ is 0 or the characteristic of κ is positive and $O_X(K_X)(O_X(-K_X)$, respectively) is globally generated, then

$$(\frac{c_{2,1^{n-2}}}{c_1^n}, \frac{c_{2,2,1^{n-4}}}{c_1^n}, \cdots, \frac{c_n}{c_1^n}) \in \mathbb{A}^{p(n)}$$

is contained in a convex polyhedron in $A^{p(n)}$ depending on the dimension of X only, where $p(n)$ is the partition number and the elements in the parentheses arranged from small to big in terms of the alphabet order of the lower indices of the numerators.

In this paper, we study the inequalities of Chern numbers of complete intersection threefolds in products of $\mathbb{P}^1$. Throughout this paper, we always let $\pi_i: \underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n+3\text{copies}} \to \mathbb{P}^1$be the i-th projection, and $Q_i = \pi_i^*(P)$, where P is a point of $\mathbb{P}^1$. Take H_i be a general divisor in the linear system $|\sum_{t=1}^{n+3} d_{it}Q_t|$, where d_{it} is a positive integer for $1 \leq i \leq n, 1 \leq t \leq n+3$. By the Bertini theorem, one can assume that H_i is a smooth hypersurface for $i = 1, 2, \cdots, n$, and $X = H_1 \cap H_2 \cap \cdots \cap H_n$ is a smooth threefold.

Our main result is

Theorem1.2. If $d_{ij} \geq 4$ for any $1 \leq i \leq n, 1 \leq j \leq n+3$, then we have $\frac{1}{2} < \frac{c_1(X)c_2(X)}{c_1^3(X)} < \frac{2}{(4n-2)^2} + \frac{2}{4n-2} + 1$. If $d_{ij} \geq 6$ for any $1 \leq i \leq n, 1 \leq j \leq n+3$, then $\frac{c_1(\bar{X})c_2(X)}{c_1^3(X)} - \frac{1}{2} < \frac{c_3(X)}{c_1^3(X)} < \frac{7}{12}$.

In Section 2, In section 2, we recall the basic definitions and properties of Chern classes. In section 3, we will computes the Chern numbers of X. In section 4, we study the upper and lower bounds of $\frac{c_3(X)}{c_1^3(X)}$and$\frac{c_1(X)c_2(X)}{c_1^3(X)}$.

2. Chern classes

In this section, we introduce the definition of Chern classes.

Let M be a smooth projective variety of dimension n. Let $A(M) = \bigoplus_{i=1}^{n} A^i(M)$ be the Chow ring of M. E is a vector bundle on M of rank r. The Chern class $c_i(E)$ is a cycle in $A^i(M)$, here $c_0(E) = 1$. We let $c_t(E) = 1 + c_1(E)t + \cdots + c_r(E)t^r$ be the Chern polynomial of E.

Chern class $c_i(E)$ satisfies the properties below:

(1) If D is a divisor on M and $E \cong \mathcal{O}_M(D)$ is a line bundle, then $c_1(E)=D$.

(2) If $f : M' \to M$ is a morphism of projective varieties, then $c_i(f^*E) = f^*c_i(E)$.

(3) If $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of a vector bundle, then

$$\begin{aligned} c_t(E) &= c_t(E') \cdot c_t(E'') \\ &= (1 + c_1(E')t + \cdots + c_{r'}(E')t^{r'})(1 + c_1(E'')t + \cdots + c_{r''}(E'')t^{r''}) \\ &= c_{r'}(E') \cdot c_{r''}(E'')t^{r'+r''} + \cdots . \end{aligned}$$

Assume that rank $E' = r'$, rank $E'' = r''$, so that rank E = $r' + r''$. As a result, we have $c_{r'+r''}(E) = c_{r'}(E')c_{r''}(E'')$.

(4) Let s be a global section of E. Assume that the zero set $Z(s)$ of s satisfies that dim $Z(s)$ =dim$M - r$, then $c_r(E) = Z(s) \in A^r(M)$.

We call $c_i(M) = c_i(T_M)$ the i-th Chern class of M.

3. Chern numbers of complete intersection three- folds in products of projective spaces

In this section, we compute the Chern numbers of X.

$$M = \underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n+3}$$

then one sees

$$\begin{aligned} c_t(M) = c_t(T_M) &= c_t(\pi_1^* T_{\mathbb{P}^1} \oplus \cdots \oplus \pi_{n+3}^* T_{\mathbb{P}^1}) \\ &= (1 + 2Q_1 t)(1 + 2Q_2 t) \cdots (1 + 2Q_{n+3} t). \end{aligned}$$

From the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{H_1}(-H_1) \longrightarrow \Omega_M|_{H_1} \longrightarrow \Omega_{H_1} \longrightarrow 0,$$

after taking duality, we have

$$0 \longrightarrow T_{H_1} \longrightarrow T_M|_{H_1} \longrightarrow \mathcal{O}_{H_1}(H_1) \longrightarrow 0.$$

Hence we have

$$\begin{aligned} c_t(H_1) &= \frac{c_t(T_M|_{H_1})}{c_t(\mathcal{O}_{H_1}(H_1))} \\ &= \frac{(1+2Q_1 t)(1+2Q_2 t)\cdots(1+2Q_{n+3} t)|_{H_1}}{(1+H_1 t)|_{H_1}}. \end{aligned}$$

From the exact sequence

$$0 \longrightarrow T_{H_1 \cap H_2} \longrightarrow T_{H_1}|_{H_1} \cap H_2 \longrightarrow \mathcal{O}_{H_1 \cap H_2}(H_2) \longrightarrow 0,$$

We obtain

$$c_t(H_1 \cap H_2) = \frac{(1+2Q_1 t)\cdots(1+2Q_{n+3}t)|_{H_1 \cap H_2}}{(1+H_1 t)(1+H_2 t)|_{H_1 \cap H_2}}$$

By repeating the procedure above, it can be obtained that

$$c_t(X) = c_t(H_1 \cap \cdots \cap H_n) = \frac{(1+2Q_1 t)(1+2Q_2 t)\cdots(1+2Q_{n+3}t)|_X}{(1+H_1 t)(1+H_2 t)\cdots(1+H_n t)|_X}$$

It follows that

$$\begin{aligned}&(1 + c_1(X)t + c_2(X)t^2 + c_3(X)t^3)(1 + H_1 t)(1 + H_2 t) \cdots (1 + H_n t)|_X \\ &= (1 + 2Q_1 t)(1 + 2Q_2 t) \cdots (1 + 2Q_{n+3} t)|_X.\end{aligned}$$

By considering the coefficient of t, we can get

$$\begin{aligned}&c_1(X) + H_1|_X + H_2|_X + \cdots + H_n|_X \\ &= 2Q_1|_X + 2Q_2|_X + \cdots + 2Q_{n+3}|_X.\end{aligned}$$

Thus,

$$\begin{aligned}c_1(X) &= (2Q_1 + 2Q_2 + \cdots + 2Q_{n+3} - H_1 - H_2 - \cdots - H_n)|_X \\ &= \textstyle\sum_{i=1}^{n+3} (2 - d_{1i} - d_{2i} - \cdots - d_{ni})Q_i|_X.\end{aligned} \tag{1}$$

As for the coefficient of t^2, we see that

$$\begin{aligned}&c_2(X) + c_1(X)(H_1 + H_2 + \cdots + H_n)|_X + \textstyle\sum_{1 \le i<j \le n} H_i H_j|_X \\ &= 4 \textstyle\sum_{1 \le i<j \le n+3} Q_i Q_j|_X.\end{aligned}$$

Since

$$\begin{aligned}&H_1 + H_2 + \cdots + H_n \\ &= \textstyle\sum_{i=1}^n d_{i1}Q_1 + \sum_{i=1}^n d_{i2}Q_2 + \cdots + \sum_{i=1}^n d_{i,n+3}Q_{n+3},\end{aligned}$$

We obtain

$$\begin{aligned}&c_1(X)(H_1 + H_2 + \cdots + H_n) \\ &= \textstyle\sum_{1 \le i,j \le n+3} (2 - d_{1i} - d_{2i} - \cdots - d_{ni}) \sum_{k=1}^n d_{kj} Q_i Q_j.\end{aligned}$$

Simple computations show that

$$\begin{aligned}H_i H_j &= (d_{i1}Q_1 + d_{i2}Q_2 + \cdots + d_{i,n+3}Q_{n+3})(d_{j1}Q_1 + d_{j2}Q_2 + \cdots + d_{j,n+3}Q_{n+3}) \\ &= \textstyle\sum_{1 \le k,l \le n+3} d_{ik} d_{jl} Q_k Q_l\end{aligned}$$

Hence we have

$$\begin{aligned}c_2(X) = 4 &\textstyle\sum_{1 \le i<j \le n+3} Q_i Q_j|_X - \sum_{1 \le k,l \le n+3} d_{ik} d_{jl} Q_k Q_l|_X - \\ &\textstyle\sum_{1 \le i,j \le n+3} (2 - d_{1i} - d_{2i}|_X - \cdots - d_{ni}) \sum_{k=1}^n d_{kj} Q_i Q_j|_X.\end{aligned} \tag{2}$$

Now considering the coefficient of t^3, we get

$$c_3(X) + c_2(X)\sum_{i=1}^{n} H_{i|}X + c_1(X)\sum_{1\le i<j\le n} H_iH_j|_X + \sum_{1\le i<j<k\le n} H_iH_jH_k|_X = 8(\sum_{1\le i<j<k\le n+3} Q_iQ_jQ_k)|_X.$$

This implies

$$c_3(X) = \sum_{i_1,\cdots,i_n,i,j,k} d_{1i_1}\cdots d_{ni_n}(\frac{4}{3} - \frac{1}{6}\sum_{1\le r,s,t\le n,(r-s)(s-t)(t-r)\ne 0} d_{ri}d_{sj}d_{tk} - \frac{1}{2}(2-\sum_{i=1}^{n} d_{ti})\sum_{1\le s,t\le n,s\ne t} d_{tj}d_{sk} - \left[2 - \frac{1}{2}\sum_{1\le s,t\le n,s\ne t} d_{ti}d_{sj} - (2-\sum_{t=1}^{n} d_{ti})\sum_{t=1}^{n} d_{tj}\right]\sum_{t=1}^{n} d_{tk}) \quad (3)$$

where $i_1,\cdots,i_n,i,j,k$ take all the arrangements of $1,2,\cdots,n+3$.
By (1), (2), (3), we can have

$$c_1^3(X) = \sum_{i_1,\cdots,i_n,i,j,k} d_{1i_1}\cdots d_{ni_n}(2-\sum_{t=1}^{n} d_{ti})(2-\sum_{t=1}^{n} d_{tij})(2-\sum_{t=1}^{n} d_{tk}), \quad (4)$$

$$c_1(X)c_2(X) = \sum_{i_1,\cdots,i_n,i,j,k} d_{1i_1}\cdots d_{ni_n}[2 - \frac{1}{2}\sum_{1\le t,s\le n,t\ne s} d_{ti}d_{sj} - (2-\sum_{t=1}^{n} d_{ti})\sum_{t=1}^{n} d_{tj}](2-\sum_{t=1}^{n} d_{tk}) \quad (5)$$

4. Inequalities of Chern numbers

In this section, we estimate the upper and lower bounds for $\frac{c_1(X)c_2(X)}{c_1^3(X)}$ and $\frac{c_3(X)}{c_1^3(X)}$ respectively. Let

$$A_i = (\sum_{t=1}^{n} d_{ti}) - 2, \quad (6)$$

$$B_{ij} = \sum_{1\le s,t\le n,s\ne t} d_{ti}d_{sj}, \quad (7)$$

$$C_{ijk} = \sum_{1\le r,s,t\le n,(r-s)(s-t)(t-r)\ne 0} d_{ri}d_{sj}d_{tk}, \quad (8)$$

We have

$$\begin{aligned} -c_1^3(X) &= \sum_{i_1,\cdots,i_n,i,j,k} d_{1i_1}\cdots d_{ni_n}A_iA_jA_k, \\ -c_1(X)c_2(X) &= \sum_{i_1,\cdots,i_n,i,j,k} d_{1i_1}\cdots d_{ni_n}\left(2-\frac{1}{2}B_{ij}+A_i(A_j+2)\right)A_k, \\ -c_3(X) &= \sum_{i_1,\cdots,i_n,i,j,k} d_{1i_1}\cdots d_{ni_n}\left[2-\frac{1}{2}B_{ij}+A_i(A_j+2)\right](A_k+2) \\ &\quad -\frac{1}{2}A_iB_{jk} + \frac{1}{6}C_{ijk} - \frac{4}{3}. \end{aligned} \quad (9)$$

4.1. Inequalities of $\frac{c_1(X)c_2(X)}{c_1^3(X)}$

In order to estimate $\frac{c_1(X)c_2(X)}{c_1^3(X)}$, we need to estimate

$$\frac{d_{1i_1}\cdots d_{ni_n}(2-\frac{1}{2}B_{ij}+A_i(A_j+2))A_k}{d_{1i_1}\cdots d_{ni_n}A_iA_jA_k} = \frac{2-\frac{1}{2}B_{ij}+A_i(A_j+2)}{A_iA_j}$$

for any $1 \le i,j \le n+3$ and $i \ne j$.

Lemma 1. If $\mathrm{d}_{ij} \geq 4$ for $1 \leq i \leq n, 1 \leq j \leq n+3, B_{ij} < A_i A_j$.

Proof. If $\mathrm{d}_{ij} \geq 4$, we have

$$\begin{aligned} &\sum_{t=1}^{n} d_{ti} \sum_{t=1}^{n} d_{ij} - 2\sum_{t=1}^{n} d_{ti} - 2\sum_{t=1}^{n} d_{ij} + 4 \\ &= \frac{1}{2}\sum_{t=1}^{n} d_{ti}d_{tj} - 2\sum_{t=1}^{n} d_{ti} + \frac{1}{2}\sum_{t=1}^{n} d_{ti}d_{tj} - 2\sum_{t=1}^{n} d_{ij} + 4 \\ &= \sum_{t=1}^{n} (\frac{1}{2}d_{tj} - 2)d_{ti} + \sum_{t=1}^{n} (\frac{1}{2}d_{ti} - 2)d_{tj} + 4 \geq 4. \end{aligned}$$

Since

$$\begin{aligned} A_i A_j &= (\sum_{t=1}^{n} d_{ti} - 2)(\sum_{t=1}^{n} d_{ij} - 2) \\ &= \sum_{t=1}^{n} d_{ti} \sum_{t=1}^{n} d_{ij} - 2\sum_{t=1}^{n} d_{ti} - 2\sum_{t=1}^{n} d_{ij} + 4 \\ &= B_{ij} + \sum_{t=1}^{n} d_{ti}d_{tj} - 2(A_i + 2) - 2(A_j + 2) + 4 \\ &= B_{ij} - 2A_i - 2A_j - 4 + \sum_{t=1}^{n} d_{ti}d_{tj} \end{aligned}$$

One sees that

$$A_i A_j \geq B_{ij} + 4 > B_{ij}.$$

Lemma 2. When $\mathrm{d}_{ij} \geq 4$ for $1 \leq i \leq n, 1 \leq j \leq n+3$, then we have $\frac{1}{2} < \frac{2-\frac{1}{2}B_{ij}+A_i(A_j+2)}{A_iA_j} < \frac{2}{(4n-2)^2} + \frac{2}{4n-2} + 1$.

Proof. As $\mathrm{d}_{ij} \geq 4$, one sees $A_j \geq 4n-2$, which means $\frac{1}{A_j} \leq \frac{1}{4n-2}$. We can also have $\frac{1}{A_i} > 0$. By Lemma 1, we have

$$\frac{2-\frac{1}{2}B_{ij}+A_i(A_j+2)}{A_iA_j} = \frac{2}{A_iA_j} - \frac{\frac{1}{2}B_{ij}}{A_iA_j} + \frac{2}{A_j} + 1 > 1 - \frac{1}{2} = \frac{1}{2}$$

On the other hand, we have

$$\frac{2}{A_iA_j} - \frac{\frac{1}{2}B_{ij}}{A_iA_j} + \frac{2}{A_j} + 1 < \frac{2}{A_iA_j} + \frac{2}{A_j} + 1 < \frac{2}{(4n-2)^2} + \frac{2}{4n-2} + 1 \tag{10}$$

Theorem 4.1. If $\mathrm{d}_{ij} \geq 4$ for any $1 \leq i, j \leq n+3$, then we have $\frac{1}{2} < \frac{c_1(X)c_2(X)}{c_1^3(X)} < \frac{2}{(4n-2)^2} + \frac{2}{4n-2} + 1$.

Proof. The desired conclusion follows from Lemma 2.

4.2. Inequalities of $\frac{c_3(X)}{c_1^3(X)}$

In order to estimate the range of $\frac{c_3(X)}{c_1^3(X)}$, we need to estimate the range of

$$\frac{\left(2-\frac{1}{2}B_{ij}+A_iA_j+2A_i\right)(A_k+2)-\frac{1}{2}A_iB_{jk}+\frac{1}{6}C_{ijk}-\frac{4}{3}}{A_iA_jA_k}$$

Lemma 3. If $\mathrm{d}_{ij} \geq 6$ for *any* i, j, *then we have* $A_iA_jA_k > C_{ijk}$.

Proof. One sees that

$$\begin{aligned}A_iA_jA_k &= (\textstyle\sum_{t=1}^{n} d_{ti}-2)(\sum_{t=1}^{n} d_{tj}-2)(\sum_{t=1}^{n} d_{tk}-2)\\ &= \textstyle\sum_{t=1}^{n} d_{ti}\sum_{t=1}^{n} d_{tj}\sum_{t=1}^{n} d_{tk}-2\sum_{t=1}^{n} d_{ti}\sum_{t=1}^{n} d_{tj}-2\sum_{t=1}^{n} d_{tj}\sum_{t=1}^{n} d_{tk}\\ &\quad \textstyle -2\sum_{t=1}^{n} d_{ti}\sum_{t=1}^{n} d_{tk}+4\sum_{t=1}^{n}(d_{ti}+d_{tj}+d_{tk})-8,\end{aligned}$$

and

$$\begin{aligned}&\textstyle\sum_{t=1}^{n} d_{ti}\sum_{t=1}^{n} d_{tj}\sum_{t=1}^{n} d_{tk}\\ &= \textstyle\sum_{1\le r,s,t\le n} d_{ri}d_{sj}d_{tk}\\ &= \textstyle C_{ijk}+\sum_{1\le r\ne t\le n} d_{ri}d_{rj}d_{tk}+\sum_{1\le r\ne t\le n} d_{ri}d_{tj}d_{tk}\\ &\quad \textstyle +\sum_{1\le r\ne s\le n} d_{ri}d_{sj}d_{rk}+\sum_{t=1}^{n} d_{ti}d_{tj}d_{tk}.\end{aligned}$$

We can further have that

$$\begin{aligned}&A_iA_jA_k\\ &= \textstyle C_{ijk}+\sum_{1\le r\ne t\le n} d_{ri}d_{rj}d_{tk}-2\sum_{t=1}^{n} d_{tj}\sum_{t=1}^{n} d_{tk}\\ &\quad \textstyle +\sum_{1\le r\ne t\le n} d_{ri}d_{tj}d_{tk}-2\sum_{t=1}^{n} d_{ti}\sum_{t=1}^{n} d_{tk}\\ &\quad \textstyle +\sum_{1\le r\ne s\le n} d_{ri}d_{sj}d_{rk}-2\sum_{t=1}^{n} d_{ti}\sum_{t=1}^{n} d_{tj}+\sum_{t=1}^{n} d_{ti}d_{tj}d_{tk}.\end{aligned}$$

In order to see the relationship between $A_iA_jA_k$ and C_{ijk}, we need to cal- culate the value of

$$\begin{aligned}&\textstyle\sum_{1\le r\ne t\le n} d_{ri}d_{rj}d_{tk}-2\sum_{t=1}^{n} d_{tj}\sum_{t=1}^{n} d_{tk}+\sum_{1\le r\ne t\le n} d_{ri}d_{tj}d_{tk}\\ &\textstyle -2\sum_{t=1}^{n} d_{ti}\sum_{t=1}^{n} d_{tk}+\sum_{1\le r\ne s\le n} d_{ri}d_{sj}d_{rk}-2\sum_{t=1}^{n} d_{ti}\sum_{t=1}^{n} d_{tj}.\end{aligned}$$

One sees that

$$\begin{aligned}&\textstyle\sum_{1\le r\ne t\le n} d_{ri}d_{rj}d_{tk}-2\sum d_{tj}\sum d_{tk}\\ &= \textstyle\sum_{1\le r\ne t\le n} d_{ri}d_{rj}d_{tk}-2\sum_{1\le r,t\le n} d_{rj}d_{tk}\\ &= \textstyle\sum_{1\le r\ne t\le n} d_{ri}d_{rj}d_{tk}-2\sum_{1\le r\ne t\le n} d_{rj}d_{tk}-2\sum_{t=1}^{n} d_{tj}d_{tk}\\ &= \textstyle\sum_{1\le r\ne t\le n} (d_{ri}-2)d_{rj}d_{tk}-2\sum_{t=1}^{n} d_{tj}d_{tk}>-2\sum_{t=1}^{n} d_{tj}d_{tk}.\end{aligned}$$

Similarly, we can obtain that

$$\textstyle\sum_{1\le r\ne t\le n} d_{ri}d_{tj}d_{tk}-2\sum d_{ti}\sum d_{tk}>-2\sum_{t=1}^{n} d_{ti}d_{tk}$$

and

$$\textstyle\sum_{1\le r\ne s\le n} d_{ri}d_{sj}d_{rk}-2\sum d_{ti}\sum d_{tj}>-2\sum_{t=1}^{n} d_{ti}d_{tj}.$$

By (20), (21) and (22), we can have that

$$\begin{aligned}A_iA_jA_k &> \textstyle C_{ijk}-2\sum_{t=1}^{n} d_{tj}d_{tk}-2\sum_{t=1}^{n} d_{ti}d_{tk}-2\sum_{t=1}^{n} d_{ti}d_{tj}+\sum_{t=1}^{n} d_{ti}d_{tj}d_{tk}+\\ &\quad \textstyle 4\sum_{t=1}^{n}(d_{ti}+d_{tj}+d_{tk})-8.\end{aligned}$$

One sees that

$$\sum_{t=1}^{n} d_{ti}d_{tj}d_{tk} - 2\sum_{t=1}^{n} d_{tj}d_{tk} - 2\sum_{t=1}^{n} d_{ti}d_{tk} - 2\sum_{t=1}^{n} d_{ti}d_{tj}$$
$$= \left(\frac{1}{3}\sum_{t=1}^{n} d_{ti}d_{tj}d_{tk} - 2\sum_{t=1}^{n} d_{tj}d_{tk}\right) + \left(\frac{1}{3}\sum_{t=1}^{n} d_{ti}d_{tj}d_{tk} - 2\sum_{t=1}^{n} d_{ti}d_{tk}\right)$$
$$+ \left(\frac{1}{3}\sum_{t=1}^{n} d_{ti}d_{tj}d_{tk} - 2\sum_{t=1}^{n} d_{ti}d_{tj}\right)$$
$$= \sum_{t=1}^{n} \left(\frac{1}{3}d_{ti} - 2\right) d_{tj}d_{tk} + \sum_{t=1}^{n} \left(\frac{1}{3}d_{tj} - 2\right) d_{ti}d_{tk} + \sum_{t=1}^{n} \left(\frac{1}{3}d_{tk} - 2\right) d_{ti}d_{tj}.$$

If $d_{ij} \geq 6$, then we can have that

$$\sum_{t=1}^{n} \left(\frac{1}{3}d_{ti} - 2\right) d_{tj}d_{tk} + \sum_{t=1}^{n} \left(\frac{1}{3}d_{tj} - 2\right) d_{ti}d_{tk} + \sum_{t=1}^{n} \left(\frac{1}{3}d_{tk} - 2\right) d_{ti}d_{tj} \geq 0.$$

This implies that

$$A_iA_jA_k > C_{ijk}.$$

As a result, we have

$$0 < \frac{C_{ijk}}{A_iA_jA_k} < 1.$$

Lemma 4. If $d_{ij} \geq 6$ for any *i*, *j*, then we have

$$\frac{\frac{8}{3} - \frac{1}{2}B_{ij} + A_iA_j + 2A_i - \frac{1}{2}A_iB_{jk} + \frac{1}{6}C_{ijk}}{A_iA_jA_k} > \frac{\frac{1}{2}B_{jk}}{A_jA_k} - \frac{1}{2}.$$

Proof. One sees that

$$\frac{\frac{8}{3} - \frac{1}{2}B_{ij} + A_iA_j + 2A_i - \frac{1}{2}A_iB_{jk} + \frac{1}{6}C_{ijk}}{A_iA_jA_k}$$
$$= \frac{\frac{8}{3} - \frac{1}{2}B_{ij} + A_iA_j + 2A_i + \frac{1}{6}C_{ijk}}{A_iA_jA_k} - \frac{\frac{1}{2}B_{jk}}{A_jA_k}.$$

By Lemma 1, we have

$$B_{ij} < A_iA_j, B_{jk} < A_jA_k.$$

Hence we have

$$\frac{\frac{8}{3} - \frac{1}{2}B_{ij} + A_iA_j + 2A_i - \frac{1}{2}A_iB_{jk} + \frac{1}{6}C_{ijk}}{A_iA_jA_k} >$$
$$\frac{\frac{8}{3} + \frac{1}{2}B_{ij} + 2A_i - \frac{1}{2}A_iB_{jk} + \frac{1}{6}C_{ijk}}{A_iA_jA_k} > 0.$$

This implies

$$\frac{\frac{8}{3} - \frac{1}{2}B_{ij} + A_iA_j + 2A_i - \frac{1}{2}A_iB_{jk} + \frac{1}{6}C_{ijk}}{A_iA_jA_k} > \frac{\frac{1}{2}B_{jk}}{A_jA_k} - \frac{1}{2}. \tag{11}$$

Theorem 4.2. If $d_{ij} \geq 6$ for any *i*, *j*, then we have $\frac{c_1(X)c_2(X)}{c_1^3(X)} - \frac{1}{2} < \frac{c_3(X)}{c_1^3(X)} < \frac{7}{12}$.

Proof. According to Lemma 4, we have that $-c_3(X) > -c_1(X)c_2(X) - \frac{1}{2}c_1^3(X)$, i.e., $\frac{c_3(X)}{c_1^3(X)} > \frac{c_1(X)c_2(X)}{c_1^3(X)} - \frac{1}{2}$.

Now, we consider the upper bound of $\frac{c_3(X)}{c_1^3(X)}$

Because $A_i = \sum_{t=1}^{n} d_{ti} - 2 \geq 6n - 2$, we have

$$\begin{aligned} & \frac{\frac{8}{3}}{A_iA_jA_k} + \frac{1}{A_k} + \frac{2}{A_jA_k} + \frac{\frac{1}{6}C_{ijk}}{A_iA_jA_k} \\ & < \frac{\frac{8}{3}}{(6n-2)^3} + \frac{1}{6n-2} + \frac{2}{(6n-2)^2} + \frac{1}{6} \\ & \leq \frac{8}{3} + \frac{1}{4} + \frac{2}{16} + \frac{1}{6} \\ & = \frac{1}{24} + \frac{1}{4} + \frac{1}{8} + \frac{1}{6} \\ & = \frac{7}{12}. \end{aligned} \tag{12}$$

5. Conclusions

In this paper, we take $M = \underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n+3}$ as an example to calculate the Chern numbers of complete intersection three-folds in products of projective spaces. Thus, in our conclusion, we get its Chern number and the inequalities that it will satisfy:

If $d_{ij} \geq 4$ for any $1 \leq i \leq n, 1 \leq j \leq n+3$, then we have $\frac{1}{2} < \frac{c_1(X)c_2(X)}{c_1^3(X)} < \frac{2}{(4n-2)^2} + \frac{2}{4n-2} + 1$. If $d_{ij} \geq 6$ for any $1 \leq i \leq n, 1 \leq j \leq n+3$, then $\frac{c_1(\bar{X})c_2(X)}{c_1^3(X)} - \frac{1}{2} < \frac{c_3(X)}{c_1^3(X)} < \frac{7}{12}$.

However, those conclusions build up on an important assumption, which is the value of d_{ij}. This means that there is still room for exploration and explanation of those results when applying other values of d_{ij}.

As for the future meaning of research into this field, it may help in the field of physics. For instance, Miyaoka-Yau type inequalities are widely applied to the quantum mechanics and field theory, so we believe researches like this can be applied to more different conditions.

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