**2023 S.T. Yau High School Science Award (Asia)**

**Research Report**

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## Title of Research Report

Desargues' Involution at Action

## Date

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# Desargues' Involution at Action

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## Abstract

Involution is one of the most powerful and fruitful concepts introduced by Desargues and has found numerous novel applications in solving difficult problems in recent years' Mathematical Olympiads. Essentially, an involution is a cross-ratio preserving swap among all the points in a cline or all the lines in a pencil. Typical examples are reflections and inversions restricted to a line. Desargues' Involution Theorem (DIT) and its dual (dDIT) tell us how to build involutions from quadrilaterals and quadrangles. Here we will explore the applications of DIT and dDIT in proving many fundamental theorems in plane geometry in addition to solving challenging problems. In particular, we show that not only the theorems from Pappus, Pascal, Desargues' two-triangle theorem to Butterfly theorem, Ping-Pong Lemma, but also theorems with less projective flavor such as Newton-Gauss line, Jacobi, Poncelet's porism and Protassov's Theorem can arise from DIT and its dual. Furthermore, we find that involution can also improve Casey's solution to Apollonius' problem of finding circles that touch three given circles.

Keywords: Involution; Cross-ratio; Desargues' involution theorem (DIT) and its dual (dDIT); Poncelet's porism; Protassov's theorem; Apollonius' problem

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First, I would like to thank my teachers and my school, Raffles Institution, for their invaluable guidance and for being supportive of me throughout my journey as a student and as a budding researcher. Second, being part of the Singapore International Mathematical Olympiad National Training Team for the past 5 years, I would like to thank Singapore Mathematical Society for their excellent training program that has enabled me to explore more deeply into Mathematical Olympiad topics, in particular, my favourite topic of all: Geometry. Third, I would like to thank my mentor for his continuous encouragement, advice and guidance. Last but not least, I would like to thank my parents for their unending love, for taking good care of me and supporting me and for always being there for me. They are always supportive of my dreams, and are the two greatest pillars of my life, inspiring me to become the best version of myself.

**Commitments on Academic Honesty and Integrity**

We hereby declare that we

1. are fully committed to the principle of honesty, integrity and fair play throughout the competition.
2. actually perform the research work ourselves and thus truly understand the content of the work.
3. observe the common standard of academic integrity adopted by most journals and degree theses.
4. have declared all the assistance and contribution we have received from any personnel, agency, institution, etc. for the research work.
5. undertake to avoid getting in touch with assessment panel members in a way that may lead to direct or indirect conflict of interest.
6. undertake to avoid any interaction with assessment panel members that would undermine the neutrality of the panel member and fairness of the assessment process.
7. observe the safety regulations of the laboratory(ies) where the we conduct the experiment(s), if applicable.
8. observe all rules and regulations of the competition.
9. agree that the decision of YHSA(Asia) is final in all matters related to the competition.

**We understand and agree that failure to honour the above commitments may lead to disqualification from the competition and/or removal of reward, if applicable; that any unethical deeds, if found, will be disclosed to the school principal of team member(s) and relevant parties if deemed necessary; and that the decision of YHSA(Asia) is final and no appeal will be accepted.**

*(Signatures of full team below)*

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Name of team member: Hanzhang Yu

X
Name of supervising teacher: Sixia Yu

Noted and endorsed by

BRIAN LAGMAN ANG
Deputy Principal / Academic Studies (Year 5-6)
Raffles Institution

(signature)

Name of school principal

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## 1. Introduction

The concept of *involution* was first introduced by the French mathematician and architect G. Desargues (1591-1661), one of the founders of projective geometry. Along his journey of laying the foundations of projective geometry, Desargues invented about seventy new terms — most of which were plant related — into the game but, unfortunately, his works were not appreciated during his lifetime and thus almost all of them were forgotten. Involution is the only one that has survived because it provides an extremely powerful tool.

One of Desargues' essential contributions is his Two-Triangle Theorem which states that two triangles are perspective from a point are also perspective from a line. This theorem has become an axiom in the formal system of projective geometry. Compared with his Two-Triangle Theorem, Desargues' involution theorem (DIT) and its dual (dDIT) are regarded to be *"even more remarkable"* by Coxeter [1]. This remarkable theorem specifies how to build involutions from quadrilaterals and quadrangles. The well-known Butterfly Theorem and its extensions are simply a special case of DIT. Another special case is a less well-known theorem called Isogonal Line Lemma. Most recently, DIT has been generalized to more general cases [4].

Starting in 2017, a note by Markbcc168 [2] appears in AoPS with a systematic introduction of DIT in solving problems in various kinds of Mathematical Olympiad competitions. Many difficulty problems can be trivialized by some novel applications of DIT or dDIT. As warmup problems, a few projective theorems such as Pappus and Pascal, are given there as exercises. Most recently, more examples are included, such as the Newton-Gauss line [3]. Along the journey of learning and applying DIT, I found that many more fundamental theorems can be proved by DIT, including Jacobi's Theorem, Poncelet's Porism, and Protassov's Theorem. Moreover, by making use of involution, I can also simplify the constructive solution proposed by Casey to Apollonius' problem of finding circles that touch three given circles.

The rest of the report is organized as follows. In Section 2, we will give the necessary background information on involution and cross-ratio. In Section 3, we will introduce DIT and dDIT with some special cases and different configurations, which are illustrated by detailed diagrams. In Section 4, we will present some typical examples of applying DIT in solving Math Olympiad problems. In Section 5, we will prove some fundamental theorems such as Newton-Gauss line, Jacobi's Theorem, and Poncelet's Porism, and in Section 6, we show how involution can help to improve Casey's solution to Apollonius' problem. In Section 7, we consider some extension of DIT to "invisible" intersections and tangents. Finally, discussions, insights and conclusions are included in Section 8.

## 2. Involution: cross-ratio preserving swap

Simply put, an involution is a swap that preserves cross-ratio. Here, by swapping we mean the exchange of a pair of objects, namely, two points on a line or a circle or two lines in a pencil, which is a bundle of lines passing through a common point. For example, when restricted to some given line, the reflection over a given point on the line is a swap, and the inversion about some point on the line with an arbitrary nonzero power, possibly negative, is another swap.

### 2.1 *Cross-ratio*

A crucial property for a swap to be an involution is that the cross-ratios must be preserved. By cross-ratio [5], we mean a ratio of ratios unique to any four points on a line or a circle of four lines in a bundle, defined as follows: the cross-ratio of four points $A, B, C, D$ on a line (in any order) is given by

$$(AB;CD) := \frac{\frac{AC}{AD}}{\frac{BC}{BD}} = \frac{AC \cdot BD}{AD \cdot BC}. \tag{1}$$

A harmonic division has cross-ratio $(AB;CD) = -1$ with directed lengths. It is clear that under the simultaneous exchanges of any two pairs, the cross-ratio is invariant. For example, $(AB;CD) = (BA;DC)$ and on a line any three points uniquely determines another point via the cross-ratio, e.g., $(AB;CD) = (AB;C'B)$ leads to $C = C'$. Cross-ratio is an extremely powerful tool in solving sophisticated problems and proving fundamental theorems due to the following properties:

1. *Perspectivity* — One nice property of cross-ratio is its invariance under *perspectivity*, which is a projection of a line to another line via a fixed point called the center, say $E$, outside these two lines. Now, if we denote by $A', B', C', D'$ the images of $A, B, C, D$

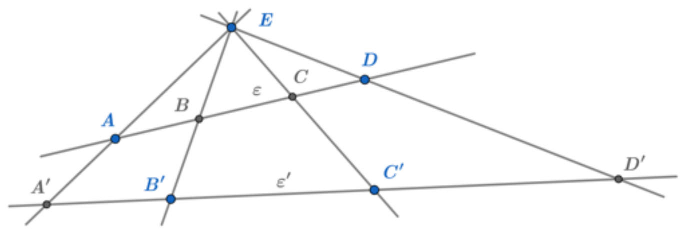


FIG. 1. A perspectivity centered at $E$ from line $\varepsilon$ to line $\varepsilon'$.

under the perspectivity, then $(AB;CD) \stackrel{E}{=} (A'B';C'D')$. This is true because the cross-ratio

$$(AB;CD) = \frac{AC \cdot BD}{AD \cdot BC} = \frac{[ACE] \cdot [BDE]}{[ADE] \cdot [BCE]} = \frac{\sin \angle AEC \cdot \sin \angle BED}{\sin \angle AED \cdot \sin \angle BEC}$$

depends only on the angles spanned by lines $AA', BB', CC', DD'$. As a result, the cross-ratio of four lines in a pencil can be defined by the cross-ratio of the intersections of the pencil with any given line. In a similar manner, the cross-ratio of four points on the circle can be defined by projecting four points on the circle to a given line via a point on the circle, a perspectivity projecting the circle to the given line.

2. *Projectivity* — The composition of a sequence of perspectivities with possibly different centers is called a *projectivity* and it is clear that cross-ratio is preserved. On the other hand, all cross-ratio preserving transformations are projectivities, i.e., they can be realized by a sequence of perspectivities [1].

3. *Collinearity* — Assume that the two line pencils $O(\alpha\beta\gamma\delta)$ and $P(\alpha\beta'\gamma'\delta')$ have the same cross-ratio and common line $\alpha$, then the other corresponding lines intersect at three points $B = \beta \cap \beta'$, $C = \gamma \cap \gamma'$, $D = \delta \cap \delta'$ which are collinear.

   Proof. Let $\beta \cap \beta' = B$ and $\gamma \cap \gamma' = C$ with $A = \alpha \cap BC$ and $D = \delta \cap BC$, $D' = \delta' \cap BC$. By cross-ratio chase, we have

$$(AB;CD) = (AB;CD')$$

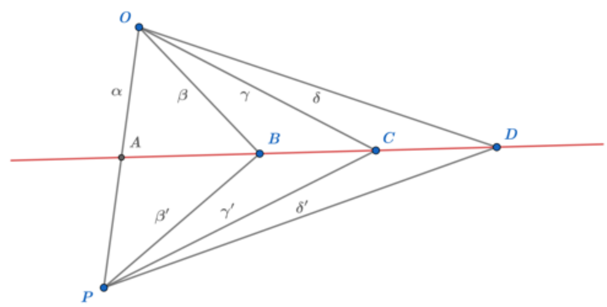


FIG. 2. Proving collinearity by cross-ratio chasing.

from which it follows $D = D'$. $\square$

4. *Concurrence* — Assume that on two lines $\{\alpha, \beta\}$ intersecting at point $O$, there are defined respectively the points $\{A, B, C\}$ and $\{A', B', C'\}$, such that $(OA; BC) = (OA'; B'C')$. Then the lines $\{AA', BB', CC'\}$ pass through a common point or are parallel.

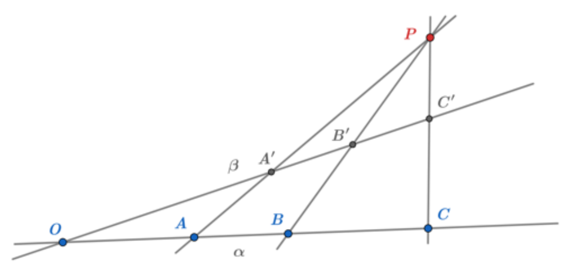


FIG. 3. Proving concurrency by cross-ratio chasing.

Proof. Let $AA' \cap BB' = P$ and $PC \cap \beta = C''$. By cross-ratio chase we have

$$(OA'; B'C'') \stackrel{P}{=} (OA; BC) = (OA'; B'C')$$

meaning that $C' = C''$. $\square$

**2.2** *Properties of involution*

A projectivity can be a transformation of points on the same line or circle or lines in the same pencil. We have an involution when the projectivity is a swap of points on a cline or lines in a pencil. In an involution, two objects swapping to each other form a reciprocal pair. We list below some useful properties of an involution:

1. For an arbitrary projectivity, if there is one reciprocal pair, then it is an involution. Let $\Phi$ be the projectivity and $(AA')$ be a reciprocal pair under $\Phi$, i.e., $A' = \Phi(A)$ and $A = \Phi(A')$. For any given point $C$ and its image $D = \Phi(C)$, we have

$$(AA'; CD) \overset{\Phi}{=} (A'A; D\Phi(D)) = (AA'; \Phi(D)D)$$

As the cross-ratio is preserved and unique, we thus have $C = \Phi(D)$, i.e., a swap.

2. Two reciprocal pairs uniquely determine an involution. We let $\Phi$ and $\Psi$ denote two involutions sharing two identical reciprocal pairs $(AA')$ and $(BB')$, then for any given point $C$ it holds

$$(A'A; B'C') \overset{\Phi}{=} (AA'; BC) \overset{\Psi}{=} (AA'; B'C'')$$

from which it follows $\Phi(C) := C' = C'' =: \Psi(C)$ for all $C$, i.e., $\Phi = \Psi$. That is, to specify an involution, we need only to explicitly show how two different pairs of points are swapped.

3. Inversion restricted on a line is an involution. The inversion around a point $P \in \ell$ with power $c$, i.e., $PA \cdot PA' = c \in \mathbb{R}$ $(\forall A \in \ell)$ induces an involution $A \mapsto A'$ on straight line $\ell$. It is clearly a swap, i.e, $(A')' = A$ and from the inversion distance formula it follows that for any two points $A, B \in \ell$,

$$A'B' = \frac{c}{PA \cdot PB} AB$$

from which the invariance of cross-ratio follows

$$(A'B'; C'D') = \frac{A'C' \cdot B'D'}{B'C' \cdot A'D'} = \frac{c^2 PB \cdot PC}{c^2 PA \cdot PC} \frac{PA \cdot PD}{PB \cdot PD} \frac{AC \cdot BD}{BC \cdot AD} = (AB; CD).$$

4. An involution can have either 2 fixed points or none. Suppose the involution has one fixed point, say $A$, with $A' = A$. Let $(BB')$ be another reciprocal pair with $B \neq B'$ and $C$ be the point such that $(AB; CB') = -1$. Then we have $-1 = (AB'; C'B) = (AB; C'B')$ so that $C = C'$, i.e., $C$ is also a fixed point. If there are 3 or more fixed points then all points are fixed as cross-ratio is preserved, and thus the given involution is the trivial identity map.

5. An involution on a line is either i) a reflection over a fixed point or ii) an inversion of some nonzero, possibly negative, power. That is, for any involution $A' = \Phi(A)$ on a straight line $\ell$ there exists a point $P \in \ell$ such that either i) $PA = PA'$ or ii) $PA \cdot PA' = c \in \mathbb{R}$, for all $A \in \ell$.

   Proof. Let $\ell_\infty$ be the point at infinity on $\ell$. If $P := \Phi(\ell_\infty) \neq \ell_\infty$, since for any two points $A, B \in \ell$ we have $(A, B; \ell_\infty, P) = (A', B'; P, \ell_\infty)$, so we get $PA \cdot PA' = PB \cdot PB'$ ($P = \Phi(\ell_\infty)$ is called the center of involution). If $\Phi(\ell_\infty) = \ell_\infty$ then there exists another fixed point $P \neq \ell_\infty$ such that $(AP; A'\ell_\infty)$ is harmonic for all $A$, i.e., $PA = PA'$, i.e., a reflection over $P$. □

6. An involution on a circle is induced by a perspectivity via some fixed point not lying on the circle.

   Proof. By performing an inversion about any given point $P$ on the circle and for a reciprocal pair $AA'$, we denote by $\tilde{A}, \tilde{A}'$ their images which lie on the line $\ell$, which is the inversion image of the given circle. Let $Q$ be the second intersection of two circles $(\tilde{A}\tilde{A}'P)$ and $(\tilde{B}\tilde{B}'P)$ and $R = PQ \cap \ell$. By Property 5, together with the fact that $PQ$ is the radical axis of these two circles, i.e., $R\tilde{A} \cdot R\tilde{A}' = R\tilde{B} \cdot R\tilde{B}'$, we see that the involution $\tilde{\Phi}$ becomes the inversion about $R$, from which it follows that $RC \cdot R\tilde{C}'$ equals the inversion power, meaning that $R$ also lies on the circle $(\tilde{C}\tilde{C}'P)$, i.e., all circles $(AA'P)$ share the same radical axis. This implies that prior to inversion, they are concurrent. □

7. Any given point on a circle induces an involution on the circle from an involution on a line and vice versa.

8. An involution of a pencil induces an involution on any line not passing through the pencil center. The reciprocal pairs of the involution on the given line are the two intersections of the pair of lines that are reciprocal to each other in the involution of the pencil. Vice versa, an involution on a line gives rise to an involution of the pencil passing any given point outside the given line.

## 3. Desargues' Involution Theorem and its dual

Desargues' involution theorem tells us how to find involutions via quadrangles, quadrilaterals, and circles. Most succinctly, Desargues' Involution Theorem can be formulated via conics: Given four points and a line, the two intersections of the line with any conic passing through those given four points form a reciprocal pair of the same involution on the given line. We note that any five points determine a unique conic, and thus there is a pencil of conics that passes through four given points. However, in order to be applied to Mathematical Olympiad problems, the notion of conics turns out to be unnecessary. Therefore, we focus on the lines and circles that passing through four given points.

### *1. DIT $\ell@(ABCD)$ for cyclic quadrangle*

This is a typical DIT and the most powerful and frequently used configuration with a cyclic quadrilateral $ABCD$ cut by an arbitrary line $\ell$ not passing through any of the vertices of the quadrilateral. We have an involution on the line $\ell$ with the following four reciprocal pairs of points $(\omega \cap \ell), (AB, CD) \cap \ell, (AC, BD) \cap \ell, (AD, BC) \cap \ell$ as indicated by the different colors and shapes in FIG. 4 shown below.

Proof. Consider a cyclic quadrangle $A_1A_2A_3A_4$ with an arbitrary line $\ell$ not passing through any one of those four points. Let $\ell \cap A_iA_j = X_{ij}$ and $\ell \cap \mathcal{C} = \{X, X'\}$. Consider the involution $\Phi$ defined by $(XX')(X_{12}, X_{34})$ on line $\ell$. By cross-ratio chasing, we have

$$(X, X_{13}; X_{12}, X') \stackrel{A_1}{=} (X, A_3; A_2, X') \stackrel{A_4}{=} (X, X_{34}; X_{24}, X') \stackrel{\Phi}{=} (X', X_{12}; X'_{24}, X)$$

which gives $X'_{24} = X_{13}$. Similarly, we have reciprocal pair $(X_{14}, X_{23})$. $\square$

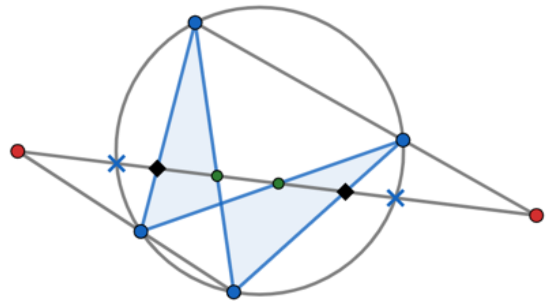

FIG. 4. Reciprocal pairs indicated by different shapes and colors.

We now consider some special cases of this configuration. The first example is the Butterfly Theorem, which can be regarded as a special involution corresponding to reflection. Butterfly Theorem states that if a line $\ell$ intersects the opposite sides, say $AB, CD$, of a complete quadrilateral $ABCDEF$ at two points, say $G, G'$, that are at the same distance to the intersections, say $X, X'$ of the line with the circle, i.e., $GX = G'X'$, then the intersections of the line with the other two pairs of opposite sides are also at the same distance from $X, X'$. By DIT on $\ell @(ABCD)$, we have an involution with reciprocal

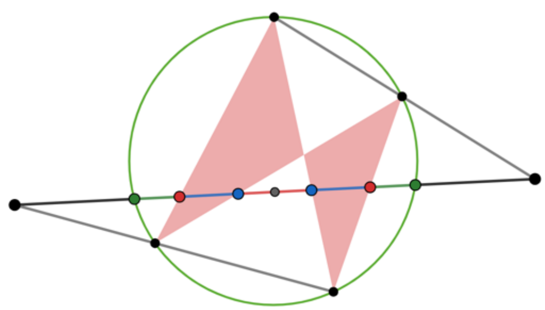

FIG. 5. Butterfly theorem with reflection as involution.

pairs $(GG'), (XX')$, $(AC, BD) \cap \ell, (AD, BC) \cap \ell$. However, as $GX = G'X'$, two pairs $(GG'), (XX')$ determine the involution to be the reflection over the midpoint $M$ of $XX'$, hence $(AC, BD) \cap \ell, (AD, BC) \cap \ell$ are also symmetric with respect to $M$.

The second example is the triangle DIT on $\ell @AABC$ with two vertices of a quadrangle coinciding, in which there are only three reciprocal pairs left and two-point DIT on $\ell@AABB$ with $\ell \cap AB$ being a fixed point (the red point). By Property 4 of Section 2.2, there is another fixed point of the involution which is determined by the harmonic division.

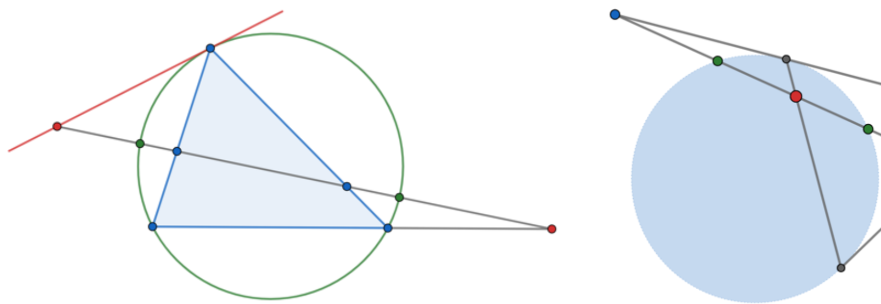

FIG. 6. Triangle and two-point DIT.

Our third special case is due to Pappus. When the quadrangle is not cyclic, namely quadrangular DIT on $\ell@ABCD$: For a given quadrangular $ABCD$ and a line $\ell$ not passing through any of the four given points, we have the following involution on the line $\ell$ with three reciprocal pairs $(AB, CD) \cap \ell, (AC, BD) \cap \ell, (AD, BC) \cap \ell$.

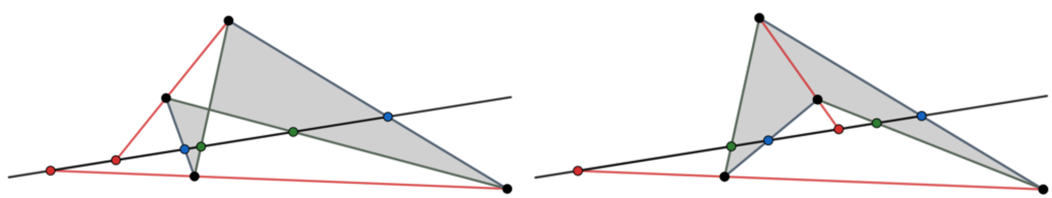

FIG. 7. Reciprocal pairs indicated with the same colors, showing in two different configurations.

### *2. dDIT $P@[ABCD]$ for inscribable quadrilateral*

By duality we mean the conversion of points into lines and vice versa. For example, under duality transformation, a cyclic quadrangle becomes an inscribable quadrilateral, and the range on a line becomes the lines of a pencil. In essence, the corresponding theorem after a duality transformation also holds true: Pascal is the dual of Brianchon, and dDIT is the dual of DIT. For dDIT, we have an involution on the pencil centered at $P$ with reciprocal pairs $P(AC)(BD)(EF)(\omega)$ where $P(\omega)$ denotes two tangents from $P$ to $\omega$. We

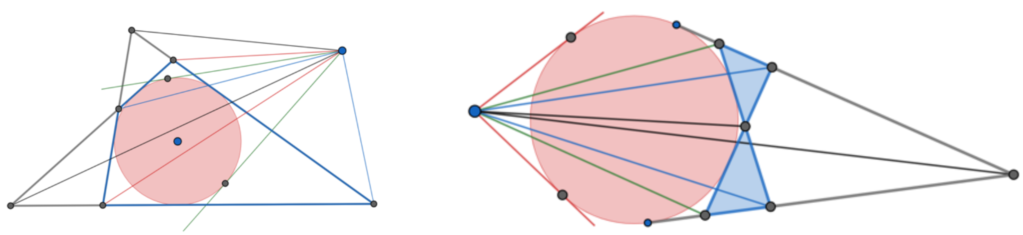

FIG. 8. Reciprocal pairs for $P@[ABCD]$ indicated with the same colors.

consider the following two degenerate cases. First, the incircle (corr. excircle) dDIT on $P@[ABDC]$ with $D$ being the intouch (corr. extouch) point.

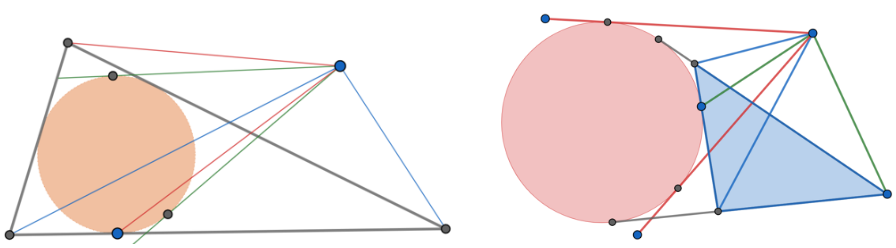

FIG. 9. Incircle (corr. Excircle) dDIT on $P@[ABDC]$: $P(AD)(BC)(\omega)$.

Second, the two-point dDIT on $P@[AABB]$ determines an involution on a pencil centered at $P$ which has a fixed red line $PC$ with $C = AA \cap BB$.

For a general quadrilateral which is not necessarily cyclic, we have dDIT $P@ABCD$: for a given quadrilateral $ABCDEF$ with $E = AC \cap BD$ and $F = AB \cap CD$ and a point $P$ that is not lying on any one of six lines, we have an involution on the pencil

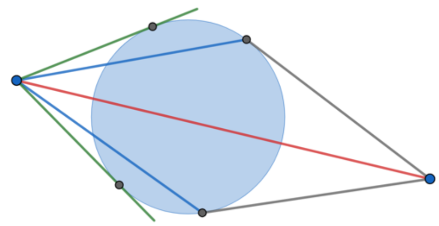

FIG. 10. Two-point dDIT.

passing through $P$, with three reciprocal pairs:

$$P(AB)(CD)(EF).$$

For a given set of four points, there are the following three possible different involutions on the pencil $P$, with reciprocal pairs denoted by lines of the same color:

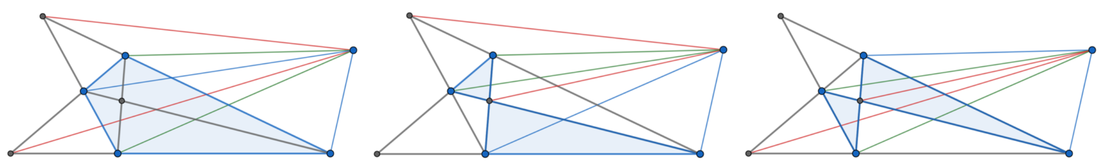

FIG. 11. Three configurations for dDIT on a general quadrilateral.

The special cases for this configuration are in the Isogonal line lemma as well as Parallelogram Isogonality Lemma, where the relevant involution is the reflection over the angle bisector.

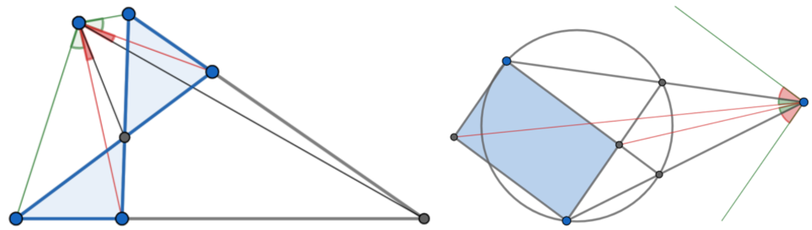

FIG. 12. Isogonal Line Lemma and Parallelogram Isogonality Lemma.

### 4. DIT in Mathematical Olympiads

Desargues' Involution Theorem helps us identify various kinds of involutions in different configurations. There are two ways in which involutions may help us solve problems or prove theorems. The first way is that two reciprocal pairs determine the involution, and thus the rest of the reciprocal pairs will enjoy the same relation. The second way is that as involution preserves cross-ratio, it can be used in cross-ratio chasing, and with the help of Properties 3 and 4 of Section 2.1, we will be able to gain new information on the configuration. Our first example comes from ELMO 2021, which I solved in a mock test by using DIT.

**Problem 0** (ELMO21P1) In $\triangle ABC$, points $P$ and $Q$ lie on sides $AB$ and $AC$, respectively, such that the circumcircle of $\triangle APQ$ is tangent to $BC$ at $D$. Let $E$ lie on side $BC$ such that $BD = EC$. Line $DP$ intersects the circumcircle of $\triangle CDQ$ again at $X$, and line $DQ$ intersects the circumcircle of $\triangle BDP$ again at $Y$. Prove that $D$, $E$, $X$, and $Y$ are concyclic.

Proof. Connect $PY$ and $QX$ and let $Z = BY \cap CX$. By angle chase, $\angle PDQ = \pi - \angle PAQ = \angle PBD + \angle DCQ = \angle PYD + \angle DXQ$, so circle $(BPDY)$ touches circle $(CQDX)$ at $D$. Now, the homothety centred at $D$ sending $(BPDY)$ to $(CQDX)$ sends $P, Y, B$ to $X, Q, C$ respectively, thus $PB \parallel XC$, $YB \parallel QC$ and $PY \parallel XQ$. Thus, $ABZC$ is a parallelogram, so $\angle YDX + \angle YZX = \angle PDQ + \angle PAQ = \pi$, i.e.

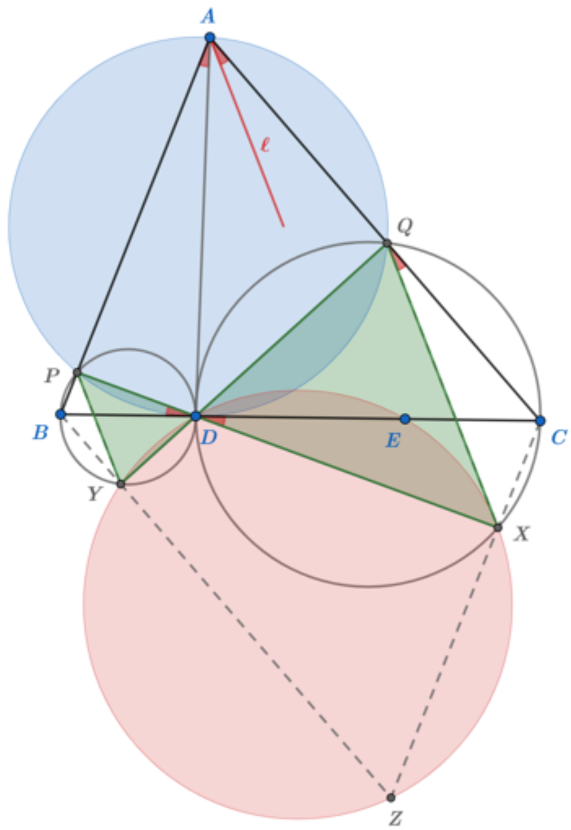


FIG. 13. ELMO 2021 P1.

$(DXZY)$ is cyclic. Denote $(DXZY)$ by $\omega$, and let $\ell$ be the line through $A$ parallel to $PY$ and $QX$. By dDIT on $A@PYQX$, we have the involution

$$(AP, AQ), (AD, \ell), (AX, AY)$$

which is determined by the first two reciprocal pairs to be the reflection over the angle bisector of $PAQ$. Thus, we have $\angle PAY = \angle QAX$. Together with $\angle ABY = \angle ACX$, we get $\triangle ABY \sim \triangle ACX$, so $BY \cdot BZ = BY \cdot AC = CX \cdot BA = CX \cdot CZ$, i.e, $B, C$ have the same power wrt $\omega$. Hence $E'$, the second intersection of $BC$ with $\omega$ satisfying $CE' = BD$, proving $E = E'$ and thus $D, E, X, Y$ are concyclic. $\square$

In this example, we have applied dDIT from the point $A$ to a trapezium $PYQX$

with intersections $PY \cap QX$ at infinity and $D = PX \cap QY$. The involution gives three reciprocal pairs and can be easily identified to be a reflection by two reciprocal pairs and additional information on the third pair helps us solve the problem. In the IMO shortlist alone, there are many problems that are trivialized by DIT and dDIT which include at least the following problems:

05G6, 06G3, 07G3, 08G7, 11G4, 12G2, 12G8, 14G7, 15G7, 19P2, 21P3.

**Problem 1** (ISL08G7) Let $ABCD$ be a convex quadrilateral with $AB \neq BC$. Denote by $\omega_1$ and $\omega_2$ the incircles of triangles $ABC$ and $ADC$. Suppose that there exists a circle $\omega$ inscribed in angle $ABC$, tangent to the extensions of line segments $AD$ and $CD$. Prove that the common external tangents of $\omega_1$ and $\omega_2$ intersect on $\omega$.

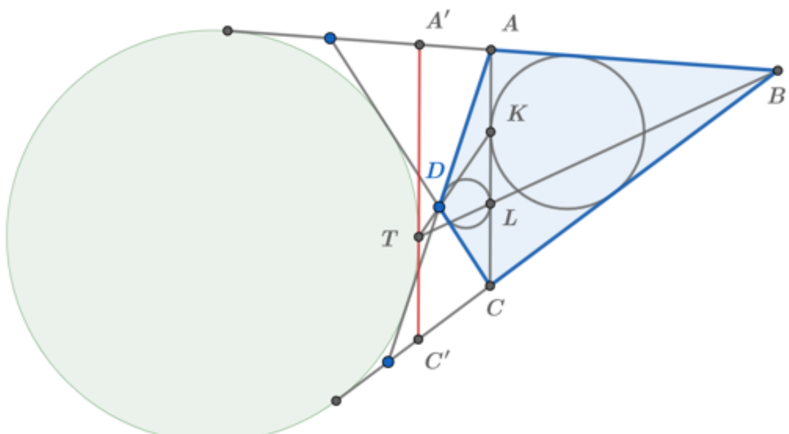


FIG. 14. IMO SL08G7.

Proof. Let $T$ be the touch point of the tangent of $\omega$ that is parallel to $AC$ and $A' = TT \cap AB$, $C' = TT \cap CB$. Now by dDit on $T@[ACBD]$, we get the involution

$$T(\omega)(AC)(BD)$$

which is a reflection when projected onto $AC$ as infinity on $AC$ is mapped into itself. Therefore $AK = CL$. Let $L = TB \cap AC$ and $K = TD \cap AC$. The homothety at

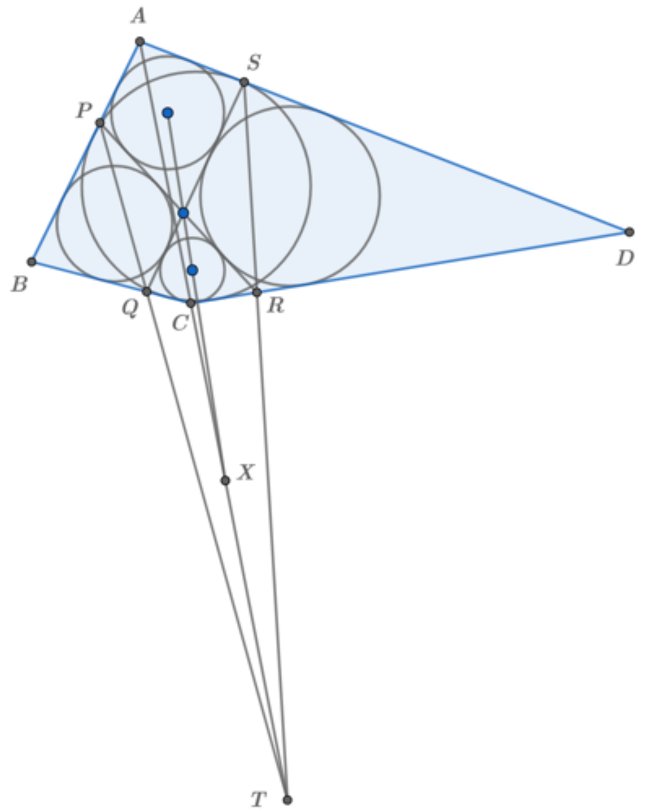


FIG. 15. IMO SL15G7.

$B$ sending $\triangle BAC$ to $\triangle BA'C'$ sends $L$ to $T$, and since $T$ is the extouch point of $\triangle BA'C'$, we have that $L$ is the extouch point of $\triangle ABC$. Similarly, $K$ is the extouch point of $\triangle ADC$. As a result, $K, L$ are also intouch points of $\triangle ABC$ and $\triangle ADC$, respectively, so that $T$ is the exsimilicenter of $\omega_1$ and $\omega_2$. $\square$

**Problem 2** (ISL15G7) Let $ABCD$ be a convex quadrilateral, and let $P, Q, R$, and $S$ be points on the sides $AB, BC, CD$, and $DA$, respectively. Let the line segments $PR$ and $QS$ meet at $O$. Suppose that each of the quadrilaterals $APOS, BQOP, CROQ$, and $DSOR$ has an incircle. Prove that the lines $AC, PQ$, and $RS$ are either concurrent or parallel to each other.

Proof. First, we note that $ABCD$ also has an incircle $\omega$ by the reverse of Pitot's theorem. Let $X$ be the exsimilicenter of $\omega_1 = (APOS)$ and $\omega_2 = (CROQ)$. By Monge's theorem on $\omega_1$, $\omega_2$, and $\omega$, $X$ lies on $AC$. By dDIT on $X@[APOS]$ and dDIT on $X@[CQOR]$, we have involutions

$$X(AO)(PS)(\omega_1), \quad X(CO)(QR)(\omega_2),$$

respectively. Since these two involutions share the pairs $X(AO)(\omega_1)$, they must be identical, which is denoted by $\Phi$. Let $T = PQ \cap RS$. By dDIT on $X@PQRS$, we have the involution

$$X(PR)(QS)(TO)$$

which shares two pairs $X(PR)(QS)$ with $\Phi$. Thus, $X(TO)$ is a reciprocal pair of $\Phi$. But $XO, XAC$ is a pair of $\Phi$, so $T \in AC$. □

**Problem 3** (IMO21P3) Let $D$ be an interior point of the acute triangle $ABC$ with $AB > AC$ so that $\angle DAB = \angle CAD$. The point $E$ on the segment $AC$ satisfies $\angle ADE = \angle BCD$, the point $F$ on the segment $AB$ satisfies $\angle FDA = \angle DBC$, and the point $X$ on the line $AC$ satisfies $CX = BX$. Let $O_1$ and $O_2$ be the circumcenters of the triangles $ADC$ and $EXD$, respectively. Prove that the lines $BC, EF$, and $O_1O_2$ are concurrent.

Proof. Let $S = EF \cap BC$. By introducing the isogonal conjugate $D'$ of $D$ we have by angle chase $(FDD'B)$ and $(EDD'C)$ are cyclic, implying $(ECBF)$ cyclic as $AF \cdot AB = AD \cdot AD' = AE \cdot AC$. Then by dDit on $D@EFBC$ gives the involution

$$D(FC)(EB)(AS)$$

which is the reflection over angle bisector of $\angle EDB$ so that $\angle SDC = \angle ADF = \angle CBD$ and hence $SD$ is tangent to $(DBC)$, giving $SD^2 = SB \cdot SC = SE \cdot SF$ so $SD$ is tangent to $(DEF)$. Hence, $SD$ is tangent to both circles $(DBC)$ and $(DEF)$. Inversion about circle $\omega$ centered at $S$ with radius $SD$ sends $(EF)(BC)$ so $AC$ is

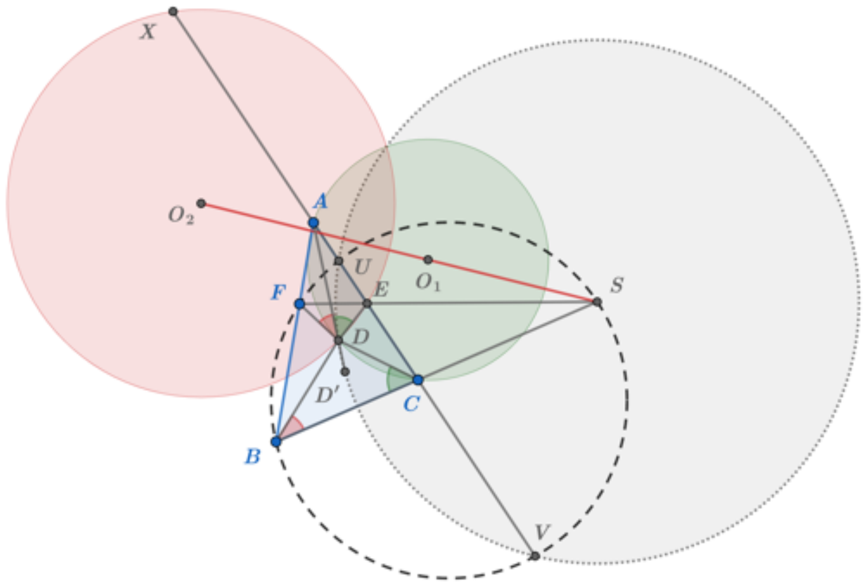


FIG. 16. IMO 21P3.

sent to $(BFS)$. Thus $\{U, V\} = XC \cap (BFS) \in \omega$ and $S$ is the midpoint of arc $UV$ in $(BFS)$. By angle chasing, $\angle XBC = \angle XCB = \angle BVS$ so $XB$ is tangent to $(BFS)$ and thus DIT on $XC@FSBB$ gives the involution

$$(UV)(XE)(AC)$$

on $AC$. Hence, the circles $\omega$, $(ADC)$ and $(DEX)$ are coaxial and thus their centers are collinear. $\square$

**Problem 4** (AoPS) Let $A$ be the external homothetic center of circles $v, g$. Let $B, C$ be points on a line passing through $A$. The tangents through $B$ to $v$ and through $C$ to $g$, intersect at $I, J$ respectively. Also the tangents through $B$ to $v$ and through $C$ to $g$ intersect the common internal tangents of $v, g$ in $M, N, K, L$ respectively. Prove that $MN, KL, IJ$ are concurrent.

Proof. Let $D = ML \cap KN$ and $E = MK \cap NL$ and from Desargues' Two Triangle theorem applied to $\triangle NLJ$ and $\triangle MKI$ it's enough to prove $E \in AC$.

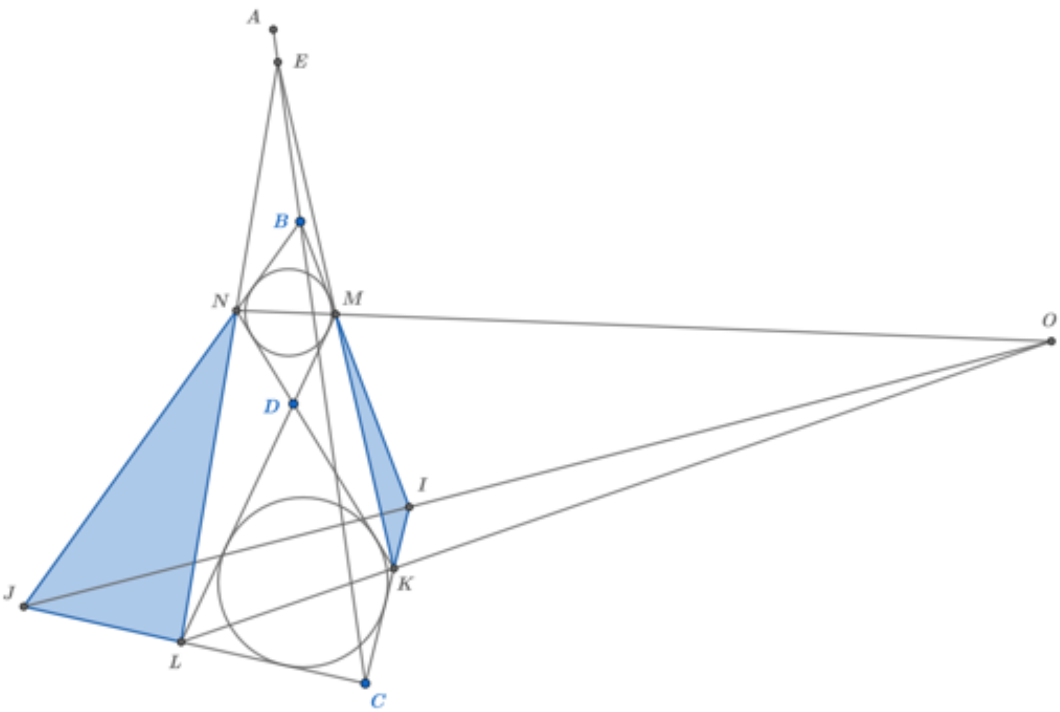


FIG. 17. A problem from AoPS.

The dDITs $A@[BMDN]$ and $A@[KDLC]$ give the same involution

$$A(DC)(v)(MN)(KL)$$

as $A(v) = A(g)$ (circles $v, g$ share the same tangents from $A$). However, dDIT $A@MKNL$ gives also the same involution

$$A(MN)(KL)(DE)$$

and thus $E \in AC$. $\square$

## 5. Some well-known theorems from DIT

### 5.1 *Pappus' theorem*

Given two lines $a, b$ with arbitrary three points on each line $A_k$ and $B_k$. Prove that $C_i = A_jB_k \cap A_kB_j$ with $(i, j, k)$ being cyclic permutations of $(1, 2, 3)$ are concurrent [6].

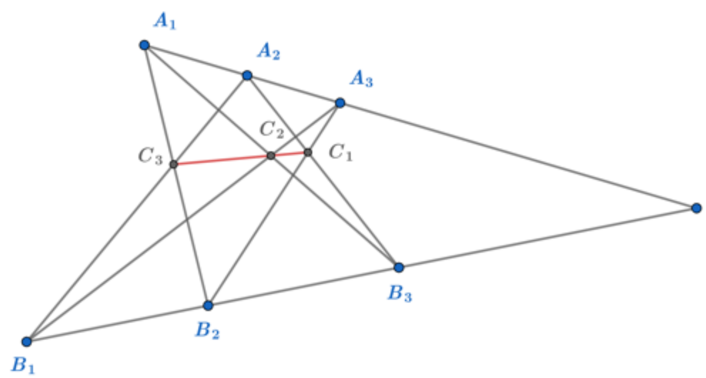


FIG. 18. Pappus' theorem

Proof. Let $O = a \cap b$. By dDIT on $C_1@A_3B_1B_3A_1$, we get the involution

$$C_1(A_3B_3)(A_1B_1)(OC_2)$$

Also, by dDIT on $C_1@A_2B_1B_2A_1$ leads to the involution

$$C_1(A_2B_2)(A_1B_1)(OC_3) = C_1(B_3A_3)(A_1B_1)(OC_3)$$

which must be identical to the first involution since they share two common reciprocal pairs. Thus, $C_1, C_2, C_3$ are collinear. $\square$

**5.2** *Pascal's theorem*

Given 6 points on a circle $A_{1,2,3}$ and $B_{1,2,3}$ and let $C_k = B_iC_j \cap B_jC_i$ with $(i, j, k)$ being cyclic permutations of $(1, 2, 3)$, it hold $C_1 - C_2 - C_3$ collinear [6].

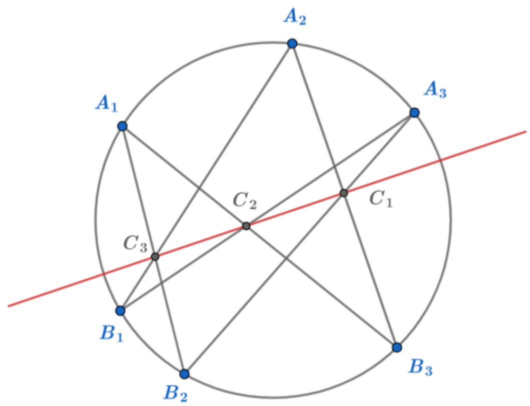


FIG. 19. Pascal's theorem

Proof. Let $\{U, V\} = C_1C_2 \cap (A_1A_2A_3)$. By DIT on $C_1C_2@B_1A_1B_2A_3$, we get the involution

$$(S = B_1A_1 \cap C_1C_2, C_1)(C_2, A_1B_2 \cap C_1C_2)(UV)$$

and by DIT on $C_1C_2@B_1A_1B_3A_2$, we get the involution

$$(S, C_1)(C_2, C_1C_2 \cap A_1B_3)(UV)$$

which is identical to the first one as there are two common reciprocal pairs. Thus $C_1, C_2, C_3$ are concurrent. $\square$

**5.3** *Desargues' 2-triangle theorem*

If two triangles $\triangle A$ and $\triangle B$ are perspective from a line, i.e., $C_k = B_iB_j \cap A_iA_j$ are collinear, with $(i, j, k)$ being cyclic permutations of $(1, 2, 3)$, then they are also perspective from a point, i.e., $A_iB_i$ are concurrent [6].

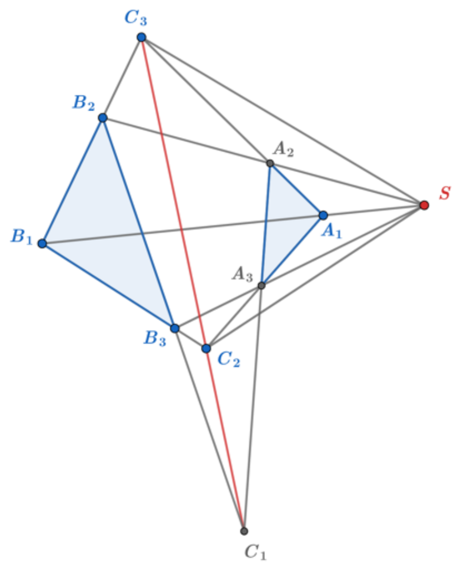


FIG. 20. Desargues' two-triangle theorem

Proof. By DIT on $B_2A_2@A_1C_2B_1C_3$, we get the involution

$$(A_2, U = B_2A_2 \cap B_1C_2)(B_2, V = A_1C_2 \cap A_2B_2)(S = A_2B_2 \cap A_1B_1, W = A_2B_2 \cap C_1C_2)$$

and by DIT on $B_2A_2@C_2A_3C_1B_3$ we get the involution

$$(A_2, U)(B_2, V)(A_2B_2 \cap A_3B_3, W)$$

from which it follows $A_iB_i$ are concurrent. $\square$

### **5.4** *Ping-Pong Lemma*

Consider a cyclic quadrilateral $P_1P_2P_3P_4$ with circumcircle $\Omega$ and a chord $UV$ that cuts the quadrilateral at $A, B, C, D$ such that [7]

$$P_1 \xmapsto{A} P_2 \xmapsto{B} P_3 \xmapsto{C} P_4 \xmapsto{D} P_1.$$

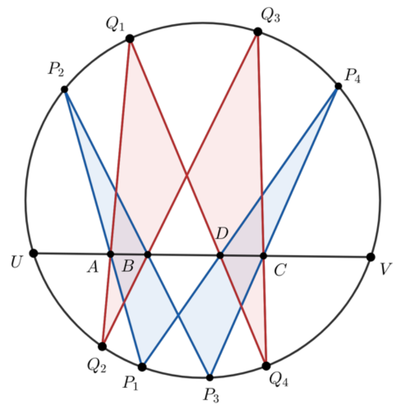


FIG. 21. Ping-pong lemma.

- If we project an arbitrary point $Q_1$ on $\Omega$ via points $A, B, C$, resulting in three points $Q_2, Q_3, Q_4$, i.e., $Q_1 \xmapsto{A} Q_2 \xmapsto{B} Q_3 \xmapsto{C} Q_4$ then we have also $Q_4 \xmapsto{D} Q_1$ i.e., $Q_4, Q_1, D$ collinear.

- If we project an arbitrary point $Q_2$ on $\Omega$ via points $A, D, C$, resulting in three points $Q_1, Q_4, Q_3$, i.e., $Q_2 \xmapsto{A} Q_1 \xmapsto{D} Q_4 \xmapsto{C} Q_3$ then we have also $Q_3 \xmapsto{B} Q_2$ i.e., $Q_3, Q_2, B$ collinear.

Proof. By DIT on $UV@P_1P_2P_3P_4$, we have the involution

$$\Psi = (AC)(BD)(UV)$$

Thus if $A \in Q_1Q_2$ and $B \in Q_2Q_3$, $C \in Q_3Q_4$, letting $D' = UV \cap Q_4Q_1$, we have another involution $(AC)(BD')(UV)$ which equals $\Psi$ because there are two common reciprocal pairs. Thus $D = D'$. $\square$

**5.5** *Newton-Gauss line*

The midpoints of diagonals of a complete quadrilateral are collinear [3].

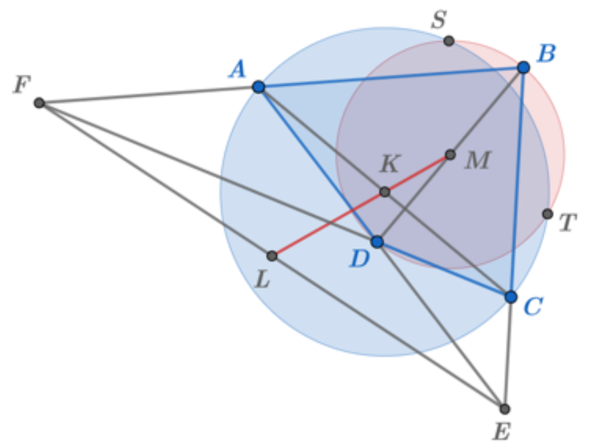


FIG. 22. Newton-Gauss line

Proof. Let two circles with diameters $AC$ and $BD$ (with centers $K$ and $M$ respectively) intersect at $S, T$. By dDIT on $S@ABCD$ and $T@ABCD$, we get the following two involutions

$$S(AC)(BD)(EF), \quad T(AC)(BD)(EF)$$

which are both the rotation of $\frac{\pi}{2}$ as $AC, BD$ are the respective diameters of the circles. Thus $SF \perp SE$ and $TF \perp TE$, meaning that $S, T$ are on the circle with diameter $EF$, i.e., $ST$ is the common radical axis of three circles. $\square$

**5.6** *Jacobi's theorem*

Given triangle $ABC$ and three points $X, Y, Z$ outside of $\triangle ABC$ such that $\angle ZAB = \angle YAC$, $\angle ZBA = \angle XBC$, and $\angle XCB = \angle YCA$, then it holds $AX$, $BY$, and $CZ$ concurrent [6].

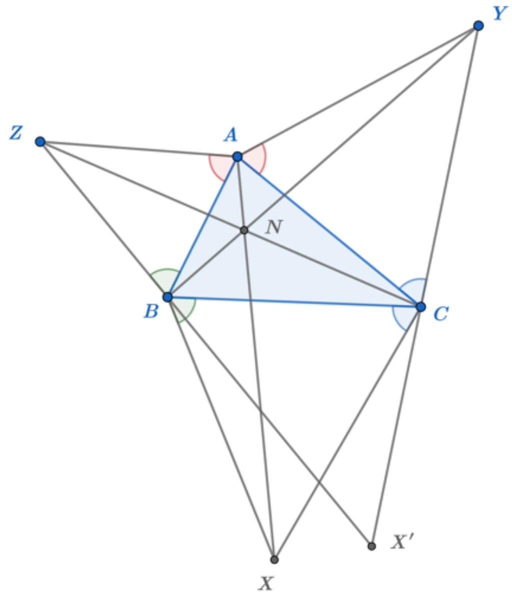


FIG. 23. Jacobi's theorem

Proof. Let $N = BY \cap ZC, X' = ZB \cap YC$. By dDIT on $A@ZBYC$, we get the involution $A(YZ)(BC)(NX')$ which is the reflection over angle bisector of $\angle BAC$. However, $X'$ and $X$ are isogonal conjugates, as a result, $A, N, X$ are collinear. $\square$

**5.7** *Poncelet's Porism*

Given a triangle $ABC$ and its circumcircle and incircle $\omega$. If another triangle $A'B'C'$ with the same circumcircle touches the incircle $\omega$ at two sides, the third side touches also the incircle [8].

Proof. Suppose $A'C'$ and $B'C'$ are tangent to incircle and let the other tangent to the incircle from $B'$ intersect circumcircle at $A''$. We want to show $A' = A''$, which is true by cross-ratio chasing,

$$(AA';BC) = C'(AA';BC) \stackrel{\Psi}{=} C'(DT;CB) = B'(DT;CB) \stackrel{\Phi}{=} B'(AA'';BC)$$

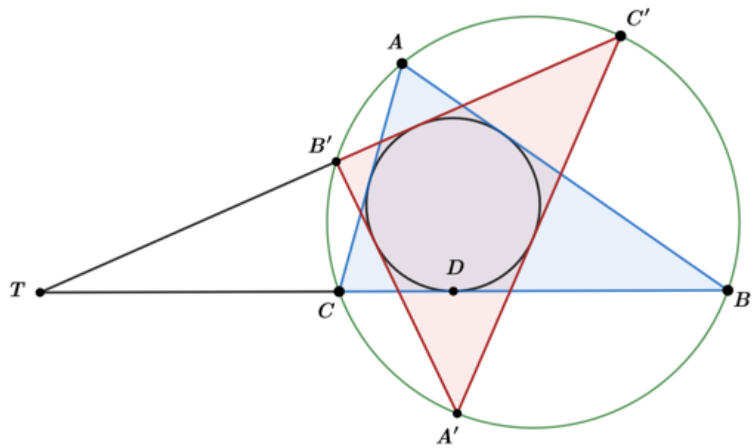


FIG. 24. Poncelet's porism.

where the first involution

$$\Psi = C'(AD)(BC)(A'B')$$

comes from dDIT on $C'@ABDC$ and the second involution

$$\Phi = B'(AD)(BC)(C'A'')$$

comes from dDIT on $B'@ABDC$. $\square$

### 5.8 *Protassov's Theorem*

For a triangle $ABC$ with incircle $\omega$ with incenter $I$, let a circle $\Omega$ passing through $BC$ be arbitrary. Another circle $\Gamma$ touches $AB, AC$ and $\Omega$ at $E, F, T$, respectively. Prove that $TI$ bisects $\angle BTC$ [9].

Proof. Let $E, F$ be the two touch points of $\Gamma$ on $AB, AC$ respectively, and $S = EF \cap BC$. Let $E', F'$ be the intouch points. Let $\Gamma \cap BC = \{U, V\}$ and $R = TI \cap BC$. Redefine $T$ to be the intersection of $\Omega$ and $(SI)$ on the different side of $A$ from $BC$.

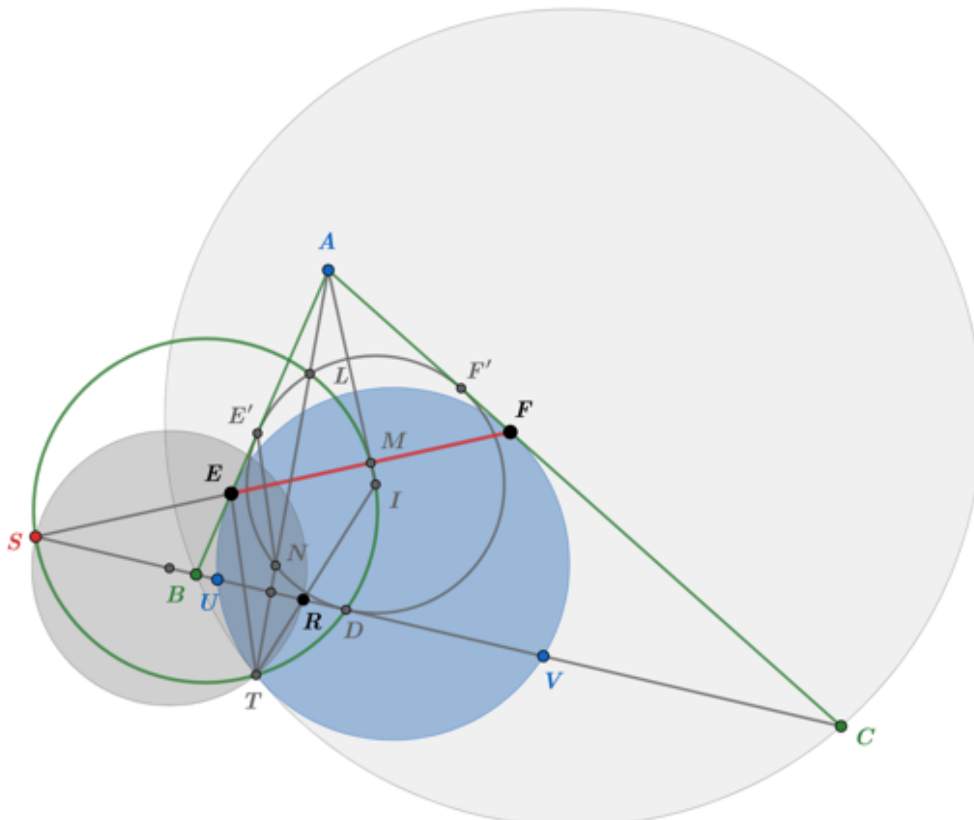


FIG. 25. Protassov's theorem.

We shall prove that $(BTC)$ touches $\Omega$ at $T$. Let $\omega \cap AT = \{L, N\}$. By homothety centered at $A$ between $\omega$ and $\Gamma$, we have $E'N \parallel ET$ so $\angle AE'L = \angle E'NL = \angle ETL$ i.e. $(LE'ET)$ cyclic. As $E', E, M, I$ are concyclic, we also have $L, M, I, T$ are concyclic, and since $S, M, I, T, D$ are concylic by the redefinition of $T$ we have $S, L, M, I, D, T$ are concyclic. Thus, we have $TI$ bisects $\angle LTD = \angle ATD$. By dDit on $T@ABDC$ we have the involution $T(AD)(BC)(\omega)$ which is exactly the reflection over $TI$, so $TI$ bisects $\angle BTC$. By DIT on $BC@EEFF$, we have the involution (pencil attached to $T$) $T(SS)(BC)(UV)$. As $TS \perp TI$ is the external angle bisector of $\angle BTC$ this involution is also the reflection over $TI$ so that $\angle BTU = \angle CTV$, i.e.,

$(BTC)$ is tangent to $\Gamma$. The involution we have here is the reflection over $TI \perp TS$, which is

$$T(BC)(UV)(BC)(AD)(SS)(RR),$$

giving the desired result. $\square$

### 6. Apollonius' Problem

*To construct circles that are tangent to three given circles in a plane by using a straightedge and a compass*, which has been called "the most famous of all" [10] geometry problems. In this section, we will be exploring the proofs of Gergonne and Casey, as well as my own contributions and extensions to the latter.

*Radical center $O$* — Take one point on each given circle and construct the circumcircle $\omega$ of these 3 points. Together with the other 3 intersection points, we have the radical axis of $\omega$ and $\omega_i$. From the pairwise intersection points of these 3 radical axes, we drop perpendiculars to the lines connecting the circumcenters of $\omega_i$ we get the radical center $O$ and construct $(O)$ that is orthogonal to $\omega_i$ with intersections $\{U_i, V_i\}$.

*Monge line $m$* — Let $M_i = U_iV_{i+1} \cap V_iU_{i+1}$ and they are exsimilicenters for $\omega_i, \omega_{i+1}$ as $(U_iV_iU_{i+1}V_{i+1})$ so that the inversion about $M_i$ will send $\omega_i$ to $\omega_{i+1}$. Therefore, by Monge's theorem $M_1, M_2, M_3$ are collinear on some line $m$.

*$m$ as radical axis* — From $W_i = U_iV_i \cap m$ construct tangents to $\omega_i$ with touch points $\{S_i, T_i\}$. Since $OU_i, OV_i$ are tangents to $\omega_i$, $U_iV_i$ is the pole of O with respect to $\omega_i$, hence $W_i$ lies on the polar of $O$ with respect to $\omega_i$. By La Hire's Theorem, $O$ lies on the polar of $W_i$ with respect to $\omega_i$. Since $W_iT_i, W_iS_i$ are tangents to $\omega_i$, $T_iS_i$ is the pole of $W_i$ with respect to $\omega_i$ so $O, T_i, S_i$ are collinear. As $O$ lies on the radical axis of $\omega_i$ and $\omega_{i+1}$, $\{S_i, S_{i+1}\}$ and $\{T_i, T_{i+1}\}$ are antihomologous pairs with respect

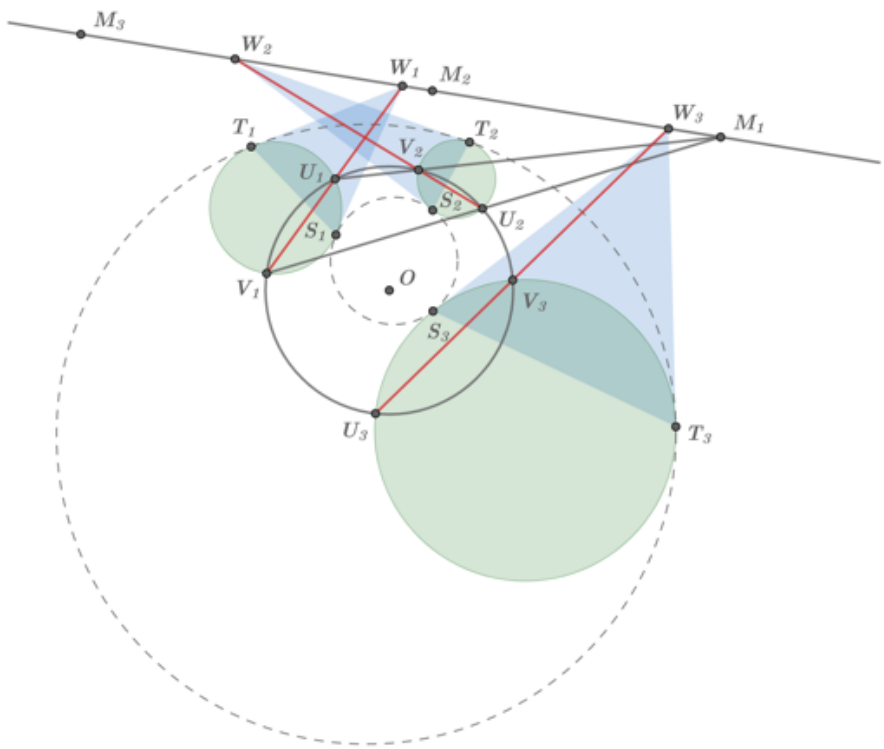


FIG. 26. Gergonne's construction.

to inversion about $M_i$, thus $S_i, S_{i+1}, M_i$ and $T_i, T_{i+1}, M_i$ collinear. As a result, $M_i$ lies on the radical axis of $(O)$ and $\omega_S = (S_1S_2S_3)$ and $\omega_T = (T_1T_2T_3)$.

As $W_i$ is the radical center of $\omega_i, (O), \omega_S, \omega_T$ we have $W_iS_i$ and $W_iT_i$ are also tangent to $\omega_S$ and $\omega_T$, respectively, so that circles $\omega_S$ and $\omega_T$ are tangent to all three given circles $\omega_i$.

In comparison to Gergonne's construction, Casey in his famous book [11] proposed a solution by projectivity: i) find 3 exsimilicenters of each pair of the given 3 circles; ii) for each point $P \in \omega_1$ we recursively find its image under inversion about the exsimilicenter of $\omega_i, \omega_{i+1}$ exchanging $\omega_i$ and $\omega_{i+1}$ with cyclic notation; iii) as a result we get another point $Q \in \omega_1$ and $P \to Q$ (e.g., the red points in FIG. 14.) is a

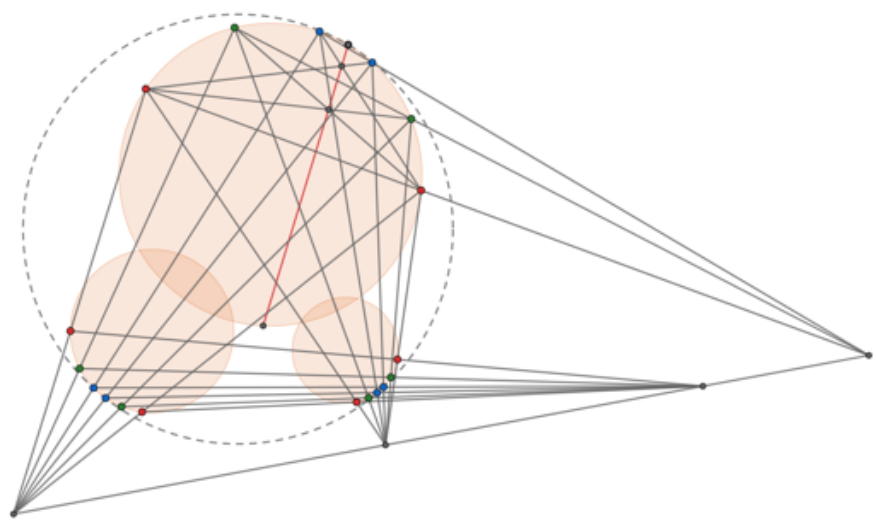

FIG. 27. Casey's construction using projectivity.

projectivity as inversions preserve cross-ratio; iv) by this projectivity we get three images $Q_1, Q_2, Q_3 \in \omega_1$ from 3 general points $P_1, P_2, P_3$ on $\omega_1$; v) these three pairs of points $P_i, Q_i$ determines completely the projectivity on the circle and thus vi) the intersections of the Pascal line, denoted by the red line in FIG. 14., and $\omega_1$ are two fixed points of the projectivity which are exactly the desired tangency points. My contribution is to note that this projectivity is in de facto an involution! Therefore, two points suffice in the determination of its fixed points, i.e., the tangency points.

**Proposition** Given three circles $\omega_i$ as shown above, let $A \in \omega_1$ be arbitrary and $B$ be the image of $A$ under the inversion about exsimilicenter $AB \cap DE$ swapping $\omega_1$ and $\omega_2$ and let $C$ the image of $B$ under the inversion swapping $\omega_2$ and $\omega_3$ and $D$ be the image of $C$ under the inversion swapping $\omega_3$ and $\omega_1$. Then $A \mapsto D$ defines an involution.

Proof. We shall prove first that $A, B, C, D$ are concyclic. Let $AB \cap \omega_1 = A'$, $CD \cap \omega_1 = D'$, and $BC \cap \omega_3 = C'$ and $O_i$ be the centers of $\omega_i$. As $\{B, A'\}$, $\{B, C'\}$, and $\{C, D'\}$ are homologous pairs we have $O_1A' \parallel O_2B \parallel O_3C'$ and $O_1D' \parallel O_3C$.

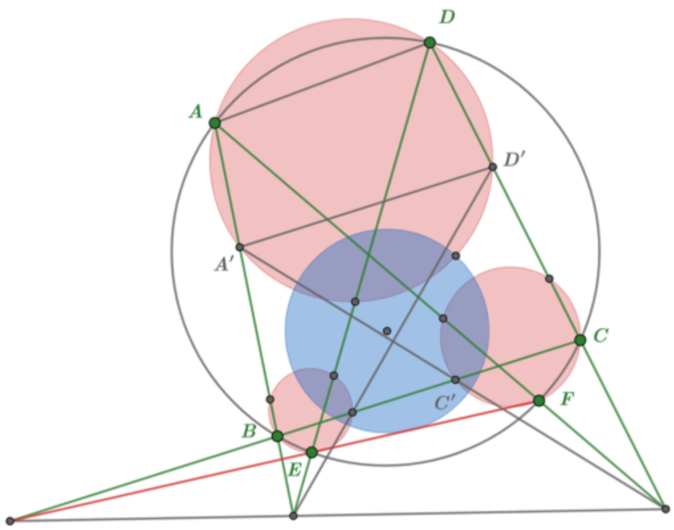


FIG. 28. My construction by involution.

Together with $OA' = OD', OC' = OC$, we get $\triangle O_1A'D' \sim \triangle O_3C'C$. These two similar triangles have two pairs of corresponding sides parallel and thus the last pair must also be parallel i.e. $A'D' \parallel BC$. Now, since $A, A', D', D$ concyclic, by Reim's theorem, we get $A, B, C, D$ are also concyclic. In the same manner, $BCDE$, $CDEF$, and $DEFA'$ are all cyclic, where $A'$ is the image of $F$. As a result, we have cyclic $ABCDEFA'$ and also $A' \in \omega_1$ so that $A = A'$. Thus $A \mapsto D \mapsto A' = A$ is an involution. $\square$

**7. Imaginary realities**

There are certain presumptions in using DIT and dDIT. For example, consider the case of DIT on $\ell@(ABCD)$ for a cyclic quadrangle. If the line $\ell$ does not intersect with the circle then we can only have 3 reciprocal pairs instead of 4. For another instance, the dDIT on $p@[ABCD]$ can have only 3 reciprocal pairs when the point $P$ lies inside the circle, meaning that there are no tangents. Nonetheless, we can assume that there are imaginary intersections or tangents and DIT will still hold, as symmetry is maintained. Most commonly, the involution is a reflection.

### **7.1** *Imaginary Butterfly theorem*

In the following figure, a line $s$ is outside circle $\omega = (O)$ and $OS \perp s$. Let $B, B' \in s$ be two symmetric points wrt $S$, i.e., $BS = B'S$ and two lines $BP$ and $B'Q$ intersect $\omega$ again at $N, M$ respectively. Prove that $C = MN \cap s$ and $C' = PQ \cap s$, $A = NQ \cap s$ and $A' = PM \cap s$ are symmetric wrt $S$.

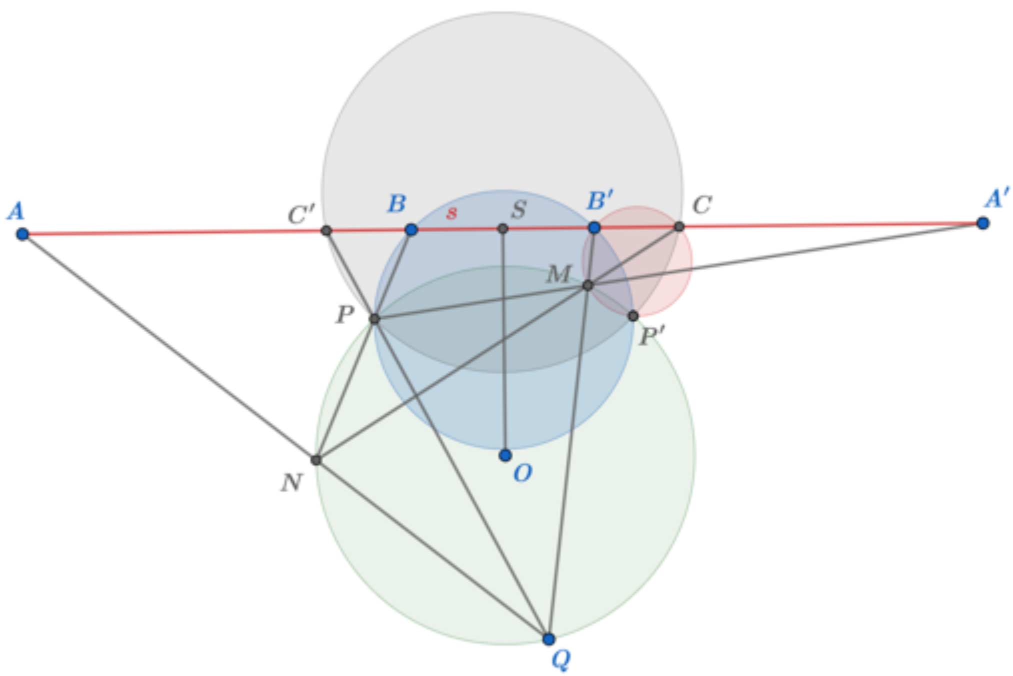


FIG. 29. Imaginary Butterfly theorem.

Proof. It seems like there are two imaginary intersections of line $s$ with circle $\omega$ and these two imaginary points should be symmetric with respect to $S$. Applying DIT to line $s$ and quadrangle $MPNQ$ would lead directly to the conclusion as the involution is a reflection.

Alternatively, the conclusion can also be proved by considering the symmetric point $P' \in \omega$ of $P$. By Miquel Pivot Theorem in $\triangle NBC$, we have $PNP'M$ cyclic and since $PBB'P'$ is also cyclic, $\angle P'MC = \angle NQP' = \angle BPP' = \angle P'B'C$. Hence, $P'MB'C$ is cyclic and by Miquel pivot theorem again in $\triangle QB'C'$, we have $PP'CC'$ cyclic since $PQP'M$ and $P'MB'C$ are cyclic. Thus $C, C'$ are symmetric with respect to $S$. Similarly, $A, A'$ are also symmetric with respect to $S$, and we are done. $\square$

**7.2** *Imaginary Ping-Pong Lemma*

Given three points $A, B, C$ on a line $\ell$ that does not intersect with circle $\Omega$. Each point $P \in \{A, B, C\}$ induces an involution $\tau_P$ from $\Omega$ to itself: $\{X, \tau_A(X)\} = AX \cap \Omega$. Prove that there exists a fourth point $D$ such that $\tau_D \circ \tau_C \circ \tau_B \circ \tau_A$ is identity.

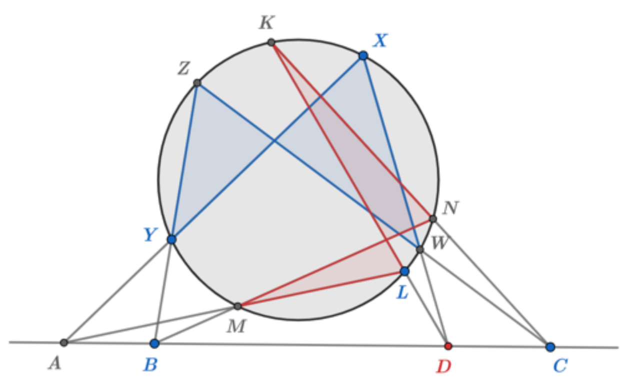


FIG. 30. Imaginary Ping-pong Lemma.

Proof. Let $X$ be an arbitrary point on the circle and $X \xmapsto{A} Y \xmapsto{B} Z \xmapsto{C} W$. It suffices to prove $X, W, D$ collinear. By Pascal's Theorem on $MLXYZN$ we have $A, B, U = XL \cap ZN$ collinear and by Pascal's Theorem on $KLXWZN$ we have $C, U, XW \cap KN$ collinear, i.e., $D, X, W$ collinear. $\square$

### 7.3 *Incircle dDIT with imaginary tangents*

Let the incircle of $ABC$ have center $I$ and touch $BC$ at $D$. Let $X$ be a point so that $XI$ internally bisects $\angle AXD$. Prove that $XI$ also externally bisects $\angle BXC$.

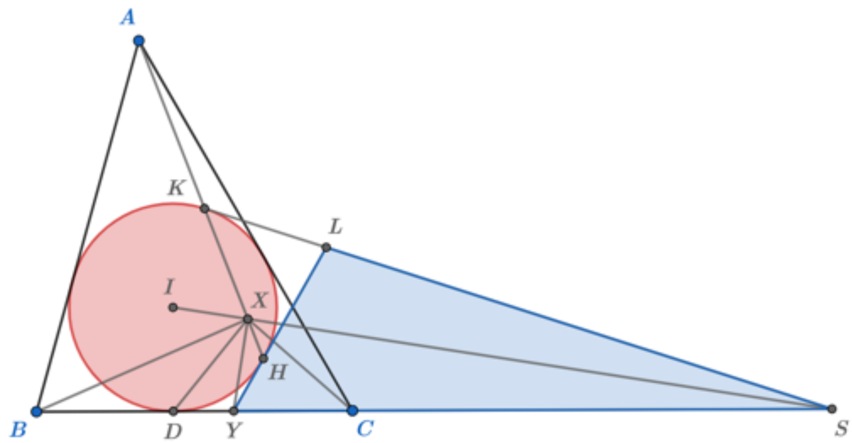


FIG. 31. Imaginary tangents.

Proof. Let $\{K, H\} = AX \cap (I)$ and $S = IX \cap BC$ so that $KS$ also tangent to $(I)$ as $IX$ bisects $\angle AXD$. The tangent at $H$ intersects $BC$ at $Y$ and $KS$ at $L$. By Pole-Polar duality, from $-1 = D(FE; KH)$, where $E, F$ are two touch points on $AB, AC$, we have $(BC; SY) = -1$. Moreover, by Iran Lemma applied to excircle of $\triangle YLS$ we have $YX \perp IS$ so that $IX$ externally bisects $\angle BXC$. $\square$

## 8. Discussions

Being a Math Olympian, I have had the opportunity to come across many wonderful and hard problems to which I have come up with short and quick DIT or dDIT solutions that are able to solve the problem almost instantly. I have compiled the more significant such instances in this Report, each demonstrating a different use of this powerful Theorem. Additionally, I also discovered that DIT and dDIT can be used to prove many other Theorems in Geometry, and be used to further extend their proofs. Furthermore, for all of my solutions presented in this Report, I have managed to completely circumvent the use of conics, demonstrating the fact that the concept of conics, which is commonly associated with the learning and application of DIT, is not necessary at all. Finally, I have included an interesting type of application of this Theorem: the imaginary realities, which is a beautiful concept and extension of Desargues' Involution Theorem.

In conclusion, through this Report, we can see the unlimited power of Desargues' Involution Theorem and its dual in both solving problems and proving other Theorems, and the various extensions of the original version. I believe that there remains much more to be discovered regarding Desargues' Involution Theorem and its dual, and I will continue my research of the yet unknown properties, usages and extensions of this Theorem.

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