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Mathematical Modeling of Long-Wave for Interfacial Waves in Two-Layer Fluids Based on the Dirichlet-Neumann Operator

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Abstract

This paper examines the long-wave problem of interfacial waves in a two-layer fluid system. We analyze the linear stability of the two-layer fluid interface wave system, establish its Hamiltonian structure, and extend the Dirichlet-Neumann operator, originally defined for the lower fluid layer, to the upper fluid layer. Using this extended operator, we derive a novel set of nonlinear equations. By applying asymptotic expansions of the Dirichlet-Neumann operator under various approximations and combining them with asymptotic analysis, we derive a series of long-wave model equations for the two-layer fluid interface waves, including the KdV equation, the fifth-order KdV equation, the mKdV equation, and the Benjamin equation. We also performed numerical solutions for these model equations, identifying notable solitary wave solutions, wave-packet solitary waves, and generalized solitary wave solutions.

Keywords: Dirichlet-Neumann operator; asymptotic analysis; nonlinear waves; interfacial waves

Contents

1 Introduction

This paper investigates the mathematical modeling of interfacial waves in a two-layer fluid system. Interfacial waves, which commonly occur at the boundary between two fluid layers with different densities, such as ocean layers of varying salinity and temperature, are significant in both scientific and practical contexts (Phillips and Hasselmann, 1986). These waves, often referred to as internal waves, play a crucial role in ocean dynamics, influencing mixing, nutrient transport, and even climate patterns.

The mathematical modeling and well-posedness of interfacial waves in two-layer fluids have been extensively studied. Early models, like the Korteweg-de Vries (KdV) equation and Benjamin-Ono equation, relied on the assumptions of weak nonlinearity and weak dispersion. Experiments have shown that the KdV equation is widely applicable, particularly for long-wave approximations in interfacial waves (Grue et al., 1999). For nonlinear waves in deep waters, the KdV equation is even superior to the Benjamin-Ono equation (Koop and Butler, 1981). However, the KdV model breaks down when wave amplitudes grow large, and nonlinear effects become more significant, violating the weakly nonlinear assumption of the KdV framework (Helfrich and Melville, 2006). Phenomena such as broad wave platforms and conjugate flows, observed in experiments and oceanographic studies (Benjamin, 1966), are beyond the scope of the KdV equation. The modified KdV (mKdV) equation, introduced by Lee and Beardsley (1974), resolves these issues, effectively modeling such structures. In cases where capillarity becomes important, the Benjamin equation (Benjamin, 1992) predicts wavepacket solitary with decaying oscillatory tails, a phenomenon numerically computed and analyzed by Calvo and Akylas (2003) and predicted theoretically by Grimshaw et al. (1994) using a fifth-order KdV equation. The wavepacket solitary, bifurcate from periodic waves with infinitesimally small amplitudes and are characterized by the nonlinear Schrödinger equation in the small amplitude regimeAkylas (1993).

However, the derivation of these model equations is based on traditional asymptotic analysis methods, which unavoidably involve solving the Laplace equation, making the derivation process cumbersome and computationally expensive. One way to simplify the computation is to avoid directly solving the Laplace equation. Zakharov (1968) made a significant advancement by choosing energy as the Hamiltonian and using wave height and surface potential as canonical variables. This demonstrated that the water wave system in a single fluid layer can be treated as a Hamiltonian system. However, this method still involves solving the Laplace equation and applying boundary conditions. Later, Craig and Sulem (1993) expanded the Dirichlet-Neumann operator using a Taylor series and reformulated the kinematic and dynamic boundary conditions in terms of the canonical variables introduced by Zakharov (1968). This approach avoids solving the Laplace equation directly, thereby reducing computational effort. It is important to note that the above studies are all based on a single fluid layer. Whether these methods can be extended to two-layer fluids and used to derive model equations for interfacial waves remains an open question.

This paper addresses this gap by demonstrating that the two-layer interfacial wave system retains a Hamiltonian structure. Moreover, we extend the Dirichlet-Neumann operator, traditionally applied to the lower fluid layer, to the upper fluid, enabling the derivation of new nonlinear equations. Using expansions of the Dirichlet-Neumann operator in conjunction with asymptotic analysis, we derive a series of long-wave model equations for interfacial waves, including the KdV, fifth-order KdV, mKdV, and Benjamin equations, and present numerical solutions to these models.

The structure of the paper is as follows. Section 2 outlines the problem formulation, discusses dispersion relations, and proves that the two-layer interfacial wave system is Hamiltonian. We also extend the Dirichlet-Neumann operator to the upper fluid layer, along with its Taylor expansion. Section 3 derives a series of nonlinear long-wave model equations, while Section 4 provides numerical results for the derived models. Finally, Section 5 concludes with a discussion of the research findings.

2 Mathematical Formulation

2.1 Governing equations

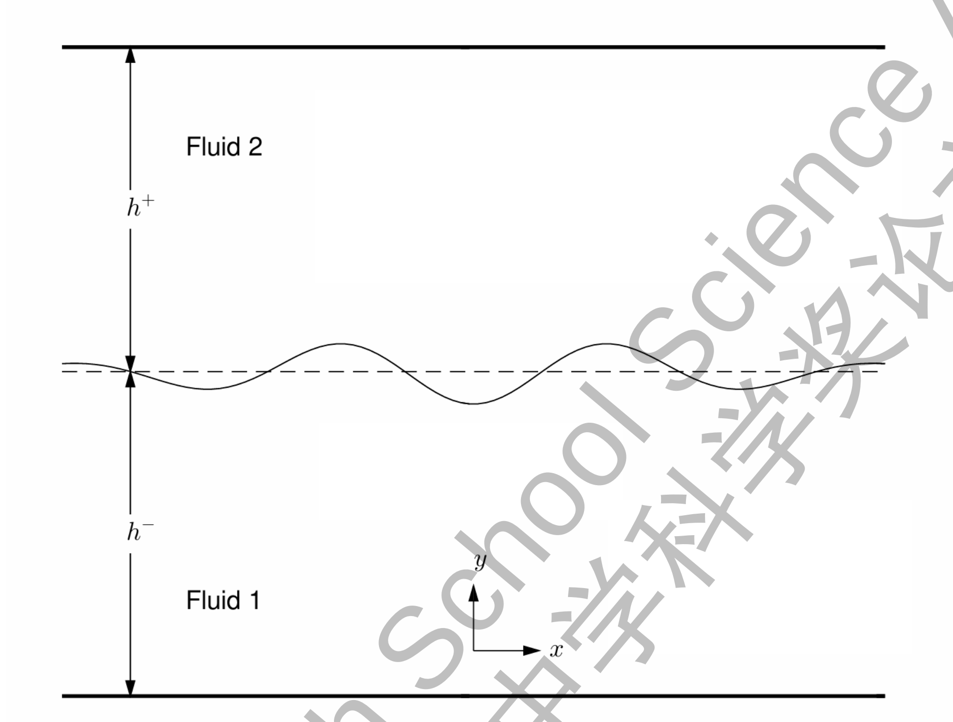

Figure 1: **Diagram of the Two-Layer Fluid Interface Wave Model**

Consider two mutually incompressible, ideal, inviscid fluids as shown in Figure (1). The fluids are bounded by solid walls above and below. When the fluids are at rest, the thicknesses and densities of the upper and lower layers are denoted as $h^{\pm}$ and $\rho^{\pm}$, where the superscripts $+$ and $-$ refer to the upper and lower layers, respectively. We establish a Cartesian coordinate system with the y-direction aligned with the opposite direction of gravity. The interface between the two fluid layers when at rest is located at $y = 0$, and the x-direction is horizontal. We examine the irrotational flow of the fluids; indeed, for ideal, inviscid, and incompressible fluids, irrotationality is preserved as long as it is initially present, in accordance with Helmholtz's theorem (Kundu et al., 2016). Consequently, the flow is potential, and the potential functions $\phi^{\pm}$ for the upper and lower layers of fluid satisfy Laplace's equation:

$$\phi^+_{xx} + \phi^+_{yy} = 0, \quad \eta < y < h^+, \tag{1}$$

$$\phi^-_{xx} + \phi^-_{yy} = 0, \quad -h^- < y < \eta, \tag{2}$$

where, $\eta = \eta(x,t)$ represents the shape of the interface between the two fluid layers. At the interface between the two fluid layers, the kinematic and dynamic boundary conditions are satisfied as follows:

$$\eta_t = \phi_y^+ - \eta_x \phi_x^+, \quad y = \eta(x,t), \tag{3}$$

$$\eta_t = \phi_y^- - \eta_x \phi_x^-, \quad y = \eta(x,t), \tag{4}$$

$$\rho^- \left[\phi_t^- + \frac{1}{2}\left|\nabla\phi^-\right|^2 + g\eta\right] - \rho^+ \left[\phi_t^+ + \frac{1}{2}\left|\nabla\phi^+\right|^2 + g\eta\right] - \frac{\sigma\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}} = 0, \quad y = \eta(x,t), \tag{5}$$

where, $\nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is gradient operator, σ denotes the surface tension coefficient. At the fixed wall boundaries of the upper and lower layers, the no-penetration boundary condition is satisfied:

$$\frac{\partial \phi^\pm}{\partial y} = 0, \quad y = \pm h^\pm. \tag{6}$$

2.2 Linear stability analysis

We first investigate the linear theoretical solution to equations (1)-(6). Without loss of generality, we consider a linear solution of the form for η:

$$\eta = \hat{\eta} e^{\mathrm{i}(kx-\omega t)}, \tag{7}$$

Thus, using the method of separation of variables, it is straightforward to solve equations (1), (2), and (6):

$$\phi^\pm = \hat{\phi}^\pm \cosh\left(|k|\left(y \mp h^\pm\right)\right) \mathrm{e}^{\mathrm{i}(kx-\omega t)}. \tag{8}$$

Substituting equations (7) and (8) into the linearized equations (3), (4), and (5), and setting $y = 0$, we obtain:

$$\begin{aligned} &-\mathrm{i}\omega\hat{\eta} = -|k|\hat{\phi}^+ \sinh\left(|k|h^+\right), \\ &-\mathrm{i}\omega\hat{\eta} = |k|\hat{\phi}^- \sinh\left(|k|h^-\right), \\ &\mathrm{i}\omega\left[\rho^+\hat{\phi}^+ \cosh\left(|k|h^+\right) - \rho^-\hat{\phi}^- \cosh\left(|k|h^-\right)\right] + \left(\rho^- - \rho^+\right) g\hat{\eta} + \sigma k^2 \hat{\eta} = 0, \end{aligned} \tag{9}$$

From the above three equations, it is straightforward to derive the dispersion relation:

$$\omega^2 = \frac{|k|\left(\left(\rho^- - \rho^+\right) g + \sigma k^2\right)}{\rho^- \coth\left(|k|h^-\right) + \rho^+ \coth\left(|k|h^+\right)}. \tag{10}$$

According to linear stability theory, the system is linearly stable when $\omega^2 > 0$, while $\omega^2 < 0$ indicates linear instability. From the dispersion relation, it is evident that surface tension contributes to the stability of the system. When considering gravitational effects, if $\rho^- > \rho^+$, where the density of the lower fluid is greater than that of the upper fluid, gravity acts to stabilize the system. Conversely, if $\rho^- < \rho^+$, with the lower fluid being less dense than the upper fluid, gravity destabilizes the system, leading to the Rayleigh-Taylor instability. In this latter case, a critical wavenumber k_c exists:

$$k_c = \sqrt{\frac{(\rho^+ - \rho^-)\, g}{\sigma}}, \tag{11}$$

For wavenumbers $k > k_c$, linear waves are stable, whereas for $k < k_c$, they are unstable. Thus, in the long-wave limit, where $k \to 0$, the system is stable only if $\rho^- > \rho^+$. In the short-wave limit, where $k \to \infty$, the system remains stable regardless of whether $\rho^- > \rho^+$ or $\rho^- < \rho^+$.

2.3 Hamilton structure

In this section, we will show that, given equations (1), (2), and (6), the equations (3), (4), and (5) can be expressed as a Hamiltonian system. Zakharov (1968) first demonstrated that deep-water gravity waves are equivalent to a Hamiltonian system by utilizing energy as the Hamiltonian and surface wave height η and surface potential function $\xi(x,t) = \phi\,(x, \eta(x,t), t)$ as canonical variables. Inspired by Zakharov (1968), we will define the surface potential functions for the upper and lower fluid layers as $\xi^{\pm} = \phi^{\pm}\,(x, \eta(x,t), t)$ and note that:

$$\frac{\partial \xi^{\pm}}{\partial t} = \left(\frac{\partial \phi^{\pm}}{\partial t} + \frac{\partial \phi^{\pm}}{\partial y} \eta_t \right) \Big|_{y=\eta(x,t)}. \tag{12}$$

Combining equations (3) and (4), we obtain:

$$\frac{\partial \phi^{\pm}}{\partial t} \Big|_{y=\eta(x,t)} = \xi_t^{\pm} - \phi_y^{\pm} \left(\phi_y^{\pm} - \eta_x \phi_x^{\pm} \right) \big|_{y=\eta(x,t)}. \tag{13}$$

Substituting equation (13) into equation (5), we obtain:

$$\begin{aligned} &\rho^- \left[\xi_t^- + \frac{1}{2} \left(\phi_x^-\right)^2 - \frac{1}{2} \left(\phi_y^-\right)^2 + \phi_y^- \phi_x^- \eta_x + g\eta \right] \\ &- \rho^+ \left[\xi_t^+ + \frac{1}{2} \left(\phi_x^+\right)^2 - \frac{1}{2} \left(\phi_y^+\right)^2 + \phi_y^+ \phi_x^+ \eta_x + g\eta \right] - \frac{\sigma \eta_{xx}}{\left(1 + \eta_x^2\right)^{3/2}} = 0, \quad y = \eta(x,t). \end{aligned} \tag{14}$$

The total energy is chosen as the Hamiltonian:

$$\begin{aligned} \mathcal{H} = &\frac{\rho^-}{2} \int_{\mathbb{R}} \int_{-h^-}^{\eta} |\nabla \phi^-|^2 \mathrm{d}x\mathrm{d}y + \frac{\rho^+}{2} \int_{\mathbb{R}} \int_{\eta}^{h^+} |\nabla \phi^+|^2 \mathrm{d}y\mathrm{d}x \\ &+ \frac{g\,(\rho^- - \rho^+)}{2} \int_{\mathbb{R}} \eta^2 \mathrm{d}x + \sigma \int_{\mathbb{R}} \left(\sqrt{1 + \eta_x^2} - 1 \right) \mathrm{d}x. \end{aligned} \tag{15}$$

In equation (15), the first two terms represent the kinetic energy, the third term denotes the gravitational potential energy, and the fourth term signifies the surface tension potential energy. Applying Green's theorem, the kinetic energy component of the Hamiltonian can be transformed into:

$$
\begin{aligned}
E_k = \frac{\rho^-}{2}\int_{\mathbb{R}}\int_{-h^-}^{\eta} |\nabla\phi^-|^2 \mathrm{d}x\mathrm{d}y + \frac{\rho^+}{2}\int_{\mathbb{R}}\int_{\eta}^{h^+} |\nabla\phi^+|^2 \mathrm{d}y\mathrm{d}x \\
= \frac{\rho^-}{2}\int_l \xi^- \frac{\partial\phi^-}{\partial \boldsymbol{n}}\mathrm{d}l - \frac{\rho^+}{2}\int_l \xi^+ \frac{\partial\phi^+}{\partial \boldsymbol{n}}\mathrm{d}l, \\
= \frac{\rho^-}{2}\int_{\mathbb{R}} \xi^- \frac{\partial\phi^-}{\partial \boldsymbol{n}}\sqrt{1+\eta_x^2}\mathrm{d}x - \frac{\rho^+}{2}\int_{\mathbb{R}} \xi^+ \frac{\partial\phi^+}{\partial \boldsymbol{n}}\sqrt{1+\eta_x^2}\mathrm{d}x
\end{aligned}
\tag{16}
$$

where, $\boldsymbol{n} = (-\eta_x, 1)/\sqrt{1+\eta_x^2}$ represents the outward normal direction to the curve in the lower fluid region, and dl denotes the differential line element along the curve. The normal derivative can be expressed using the Green's function for the boundary value problem of the Laplace equation:

$$
\frac{\partial\phi^\pm(l)}{\partial \boldsymbol{n}} = \int G(l, l_1)\xi^\pm(l_1)dl_1, \tag{17}
$$

where, l and l_1 represent points on the interface $y = \eta(x,t)$. The Green's function is symmetric, i.e., $G(l,l_1) = G(l_1,l)$. Consequently, the variation of the kinetic energy contains four terms:

$$
\begin{aligned}
\delta E_k = \frac{\rho^-}{2}\int_l \delta\xi^-(l)\frac{\partial\phi^-(l)}{\partial \boldsymbol{n}}\mathrm{d}l + \frac{\rho^-}{2}\int_l \xi^-(l)\frac{\partial\delta\phi^-(l)}{\partial \boldsymbol{n}}\mathrm{d}l \\
-\frac{\rho^+}{2}\int_l \delta\xi^+(l)\frac{\partial\phi^+(l)}{\partial \boldsymbol{n}}\mathrm{d}l - \frac{\rho^+}{2}\int_l \xi^+(l)\frac{\partial\delta\phi^+(l)}{\partial \boldsymbol{n}}\mathrm{d}l.
\end{aligned}
\tag{18}
$$

Exploiting the symmetry of the Green's function, Equation (18) can be simplified to:

$$
\begin{aligned}
\delta E_k &= \rho^-\int_l \delta\xi^-(l)\frac{\partial\phi^-(l)}{\partial \boldsymbol{n}}\mathrm{d}l - \rho^+\int_l \delta\xi^+(l)\frac{\partial\phi^+(l)}{\partial \boldsymbol{n}}\mathrm{d}l \\
&= \rho^-\int_{\mathbb{R}} \delta\xi(x)^-\frac{\partial\phi^-}{\partial \boldsymbol{n}}\sqrt{1+\eta_x^2}\mathrm{d}x - \rho^+\int_{\mathbb{R}} \delta\xi(x)^+\frac{\partial\phi^+}{\partial \boldsymbol{n}}\sqrt{1+\eta_x^2}\mathrm{d}x \\
&= \int_{\mathbb{R}} \delta\left(\rho^-\xi^-(x) - \rho^+\xi^+(x)\right)\frac{\partial\phi^-}{\partial \boldsymbol{n}}\sqrt{1+\eta_x^2}\mathrm{d}x \\
&= \int_{\mathbb{R}} \delta\left(\rho^-\xi^-(x) - \rho^+\xi^+(x)\right)\frac{\partial\phi^+}{\partial \boldsymbol{n}}\sqrt{1+\eta_x^2}\mathrm{d}x.
\end{aligned}
\tag{19}
$$

Inspired by (19), new canonical variables η and $\xi = \rho^-\xi^- - \rho^+\xi^+$ are introduced. Based on Equation (15) and Equation (19), the variation of the Hamiltonian with respect to the canonical variable ξ can be obtained:

$$
\frac{\delta\mathcal{H}}{\delta\xi} = \frac{\delta E_k}{\delta\xi} = \frac{\partial\phi^\pm}{\partial \boldsymbol{n}}\sqrt{1+\eta_x^2} = \phi_y^\pm - \eta_x\phi_x^\pm, \quad y = \eta. \tag{20}
$$

Thus, by combining Equations (3), (4), and Equation (20), we immediately obtain:

$$\eta_t = \frac{\delta \mathcal{H}}{\delta \xi}. \tag{21}$$

Next, the variation of the Hamiltonian with respect to η, while keeping ξ constant, is considered. The variation of the potential energy with respect to η is:

$$\begin{aligned} \frac{\delta E_p}{\delta \eta} &= \frac{\delta}{\delta \eta}\left[\frac{g\left(\rho^- - \rho^+\right)}{2}\int_{\mathbb{R}} \eta^2 \mathrm{d}x + \sigma \int_{\mathbb{R}} \left(\sqrt{1+\eta_x^2} - 1\right) \mathrm{d}x\right] \\ &= g\left(\rho^- - \rho^+\right)\eta - \frac{\sigma \eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}. \end{aligned} \tag{22}$$

The variation of the kinetic energy with respect to η is:

$$\begin{aligned} \delta E_k = & \frac{\rho^-}{2}\int_{\mathbb{R}} |\nabla \phi^-|^2 \delta\eta \mathrm{d}x + \rho^- \int_{\mathbb{R}}\int_{h^-}^{\eta} \nabla\phi^- \cdot \nabla\delta\phi \mathrm{d}y\mathrm{d}x \\ & - \frac{\rho^+}{2}\int_{\mathbb{R}} |\nabla \phi^+|^2 \delta\eta \mathrm{d}x + \rho^+ \int_{\mathbb{R}}\int_{\eta}^{h^+} \nabla\phi^+ \cdot \nabla\delta\phi \mathrm{d}y\mathrm{d}x \\ = & \frac{\rho^-}{2}\int_{\mathbb{R}} |\nabla \phi^-|^2 \delta\eta \mathrm{d}x - \frac{\rho^+}{2}\int_{\mathbb{R}} |\nabla \phi^+|^2 \delta\eta \mathrm{d}x \\ & + \rho^- \int_{\mathbb{R}} \left(-\phi_y^- + \eta_x \phi_x^-\right) \phi_y^-\big|_{y=\eta} \delta\eta \mathrm{d}x - \rho^+ \int_{\mathbb{R}} \left(-\phi_y^+ + \eta_x \phi_x^+\right) \phi_y^+\big|_{y=\eta} \delta\eta \mathrm{d}x. \end{aligned} \tag{23}$$

Thus, by combining equations (14), (22), and (23), we can obtain:

$$\xi_t = -\frac{\delta}{\delta\eta}\left(E_k + E_p\right) = -\frac{\delta \mathcal{H}}{\delta \eta}. \tag{24}$$

Therefore, we have demonstrated that the interfacial wave system of a two-layer fluid is a Hamiltonian system:

$$\begin{aligned} \eta_t &= \frac{\delta \mathcal{H}}{\delta \xi}, \\ \xi_t &= -\frac{\delta \mathcal{H}}{\delta \eta}. \end{aligned} \tag{25}$$

The earliest proof was given by Benjamin and Bridges (1997), but they did not consider capillary forces. Here, we take into account the effect of capillary forces and prove that the interfacial wave system with capillary forces is still a Hamiltonian system.

2.4 Dirichlet-Neumann operator

In this section, the well-known Dirichlet-Neumann (DtN) operator, initially introduced and expanded by Craig and Sulem (1993), will be discussed. The DtN operator transforms Dirichlet boundary conditions into Neumann boundary conditions, thereby eliminating the need to solve the

Laplace equation directly. This approach significantly reduces the computational effort involved in deriving model equations. It is important to note that the DtN operator introduced by Craig and Sulem (1993) is specific to the lower fluid layer, and an extension to the upper fluid layer is necessary.

First, the DtN operator for the lower fluid layer will be described. This operator is applied to the potential function $\phi^-(x, y)$ that satisfies the following boundary value problem:

$$\begin{cases} \phi^-_{xx} + \phi^-_{yy} = 0, & -h^- < y < \eta, \\ \phi^-_y = 0, & y = -h^-, \\ \phi^- = \xi^-, & y = \eta. \end{cases} \tag{26}$$

At this point, the Dirichlet-Neumann (DtN) operator for the lower fluid layer is defined as:

$$G^-\xi^- = \left(\phi^-_y - \eta_x \phi^-_x\right)\big|_{y=\eta(x,t)} = \frac{\partial \phi^-}{\partial \boldsymbol{n}} \sqrt{1+\eta_x^2}. \tag{27}$$

Similarly, the Dirichlet-Neumann (DtN) operator for the upper fluid layer is defined for the function $\phi^+(x, y)$ satisfying the following boundary value problem:

$$\begin{cases} \phi^+_{xx} + \phi^+_{yy} = 0, & \eta < y < h^+, \\ \phi^+_y = 0, & y = h^+, \\ \phi^+ = \xi^+, & y = \eta. \end{cases} \tag{28}$$

Thus the DtN operator for the upper fluid is defined as:

$$G^+\xi^+ = \left(\eta_x \phi^+_x - \phi^+_y\right)\big|_{y=\eta(x,t)} = -\frac{\partial \phi^+}{\partial \boldsymbol{n}} \sqrt{1+\eta_x^2}. \tag{29}$$

By introducing the Dirichlet-Neumann (DtN) operators for the upper and lower fluids, as defined in equations (27) and (29), the kinematic boundary conditions (3) and (4) can be rewritten as:

$$\eta_t = -G^+\xi^+ = G^-\xi^-. \tag{30}$$

Note,

$$\xi^\pm_x = \phi^\pm_x + \phi^\pm_y \eta_x. \tag{31}$$

By combining the above with (3) and (4), we obtain:

$$\phi^\pm_x = \frac{\xi^\pm_x - \eta_t \eta_x}{1+\eta_x^2}, \tag{32}$$

$$\phi^\pm_y = \frac{\eta_t + \eta_x \xi^\pm_x}{1+\eta_x^2}, \tag{33}$$

and

$$
\begin{aligned}
&\rho^{-}\left[\xi_t^{-}+\frac{1}{2}\left(\phi_x^{-}\right)^2-\frac{1}{2}\left(\phi_y^{-}\right)^2+\phi_y^{-}\phi_x^{-}\eta_x\right] \\
&-\rho^{+}\left[\xi_t^{+}+\frac{1}{2}\left(\phi_x^{+}\right)^2-\frac{1}{2}\left(\phi_y^{+}\right)^2+\phi_y^{+}\phi_x^{+}\eta_x\right] \\
&=\rho^{-}\left[\xi_t^{-}+\frac{1}{2}\left(\frac{\xi_x^{-}-\eta_t\eta_x}{1+\eta_x^2}\right)^2-\frac{1}{2}\left(\frac{\eta_t+\eta_x\xi_x^{-}}{1+\eta_x^2}\right)^2+\frac{\xi_x^{-}-\eta_t\eta_x}{1+\eta_x^2}\cdot\frac{\eta_t+\eta_x\xi_x^{-}}{1+\eta_x^2}\cdot\eta_x\right] \\
&-\rho^{+}\left[\xi_t^{+}+\frac{1}{2}\left(\frac{\xi_x^{+}-\eta_t\eta_x}{1+\eta_x^2}\right)^2-\frac{1}{2}\left(\frac{\eta_t+\eta_x\xi_x^{+}}{1+\eta_x^2}\right)^2+\frac{\xi_x^{+}-\eta_t\eta_x}{1+\eta_x^2}\cdot\frac{\eta_t+\eta_x\xi_x^{+}}{1+\eta_x^2}\cdot\eta_x\right] \\
&=\rho^{-}\left[\xi_t^{-}+\frac{1}{2}\left(\xi_x^{-}\right)^2-\frac{\left(\eta_t+\eta_x\xi_x^{-}\right)^2}{2\left(1+\eta_x^2\right)}\right]-\rho^{+}\left[\xi_t^{+}+\frac{1}{2}\left(\xi_x^{+}\right)^2-\frac{\left(\eta_t+\eta_x\xi_x^{+}\right)^2}{2\left(1+\eta_x^2\right)}\right].
\end{aligned}
\tag{34}
$$

Therefore, from equations (14) and (34), the new dynamic boundary conditions can be derived as:

$$
\begin{aligned}
\rho^{-}\left[\xi_t^{-}+\frac{1}{2}\left(\xi_x^{-}\right)^2-\frac{\left(\eta_t+\eta_x\xi_x^{-}\right)^2}{2\left(1+\eta_x^2\right)}\right]-\rho^{+}\left[\xi_t^{+}+\frac{1}{2}\left(\xi_x^{+}\right)^2-\frac{\left(\eta_t+\eta_x\xi_x^{+}\right)^2}{2\left(1+\eta_x^2\right)}\right]& \\
+\left(\rho^{-}-\rho^{+}\right)\eta-\frac{\sigma\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}=0.&
\end{aligned}
\tag{35}
$$

In the previous section, we introduced a pair of new canonical variables:

$$
\begin{cases}
\eta, \\
\xi=\rho^{-}\xi^{-}-\rho^{+}\xi^{+}.
\end{cases}
\tag{36}
$$

It is important to note that ξ^{-}, ξ^{+}, and ξ are not completely independent; ξ^{-} and ξ^{+} can both be expressed in terms of ξ. By applying the operators G^{+} and G^{-} to equation (36) and using the relationship (30), we can derive:

$$
G^{+}\xi=\left(\rho^{-}G^{+}+\rho^{+}G^{-}\right)\xi^{-}\ \Rightarrow\ \xi^{-}=\left(\rho^{-}G^{+}+\rho^{+}G^{-}\right)^{-1}G^{+}\xi, \tag{37}
$$

$$
G^{-}\xi=-\left(\rho^{-}G^{+}+\rho^{+}G^{-}\right)\xi^{+}\ \Rightarrow\ \xi^{+}=-\left(\rho^{-}G^{+}+\rho^{+}G^{-}\right)^{-1}G^{-}\xi, \tag{38}
$$

where, the operator raised to the power of -1 represents the inverse of the operator. The kinematic boundary conditions (30) can be rewritten as

$$
\eta_t=G^{-}\left(\rho^{-}G^{+}+\rho^{+}G^{-}\right)^{-1}G^{+}\xi. \tag{39}
$$

We now summarize the equations that have been rewritten using the DtN operator(Lannes, 2013):

$$
\eta_t=G^{-}\left(\rho^{-}G^{+}+\rho^{+}G^{-}\right)^{-1}G^{+}\xi, \tag{40}
$$

$$
\begin{aligned}
\rho^{-}\left[\xi_t^{-}+\frac{1}{2}\left(\xi_x^{-}\right)^2-\frac{\left(\eta_t+\eta_x\xi_x^{-}\right)^2}{2\left(1+\eta_x^2\right)}\right]-\rho^{+}\left[\xi_t^{+}+\frac{1}{2}\left(\xi_x^{+}\right)^2-\frac{\left(\eta_t+\eta_x\xi_x^{+}\right)^2}{2\left(1+\eta_x^2\right)}\right]& \\
+\left(\rho^{-}-\rho^{+}\right)g\eta-\frac{\sigma\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}}=0,& \qquad (41)
\end{aligned}
$$

$$
\xi^{-}=\left(\rho^{-}G^{+}+\rho^{+}G^{-}\right)^{-1}G^{+}\xi, \tag{42}
$$

$$
\xi^{+}=-\left(\rho^{-}G^{+}+\rho^{+}G^{-}\right)^{-1}G^{-}\xi. \tag{43}
$$

For the DtN operator, if η is less than a certain value, the DtN operator is analytic, which also means we can perform a Taylor expansion of the DtN operator (Craig and Sulem, 1993). Following the method of Craig and Sulem (1993), we first expand the operator G^{-}. Consider a solution to equation (26):

$$
\phi_k^{-}=\cosh\left(k\left(y+h^{-}\right)\right)\mathrm{e}^{\mathrm{i}kx}. \tag{44}
$$

It is clear that this solution (44) is a harmonic function that satisfies the boundary condition $\phi_y^{-}\left(x,y=-h^{-}\right)=0$. Substituting equation (44) into equation (27) yields:

$$
G^{-}\xi_k^{-}=G^{-}\phi_k^{-}\left(x,y=\eta\right)=\left(\frac{\partial\phi_k^{-}}{\partial y}-\eta_x\frac{\partial\phi_k^{-}}{\partial x}\right)\bigg|_{y=\eta(x,t)}. \tag{45}
$$

To determine the specific expression for the expansion $G^{-}(\eta)=\sum_{n=0}^{\infty}G_n^{-}(\eta)$, we perform a Taylor expansion of $\cosh(k(\eta+h^{-}))$ and $\sinh(k(\eta+h^{-}))$ around $\eta=0$. After some calculations, we obtain:

$$
\begin{aligned}
\left(\sum_{l=0}^{\infty}G_l^{-}\left(\eta\right)\right)\left(\sum_{j\ even}\frac{1}{j!}\left(k\eta\right)^j\cosh\left(kh^{-}\right)\mathrm{e}^{\mathrm{i}kx}+\sum_{j\ odd}\frac{1}{j!}\left(k\eta\right)^j\sinh\left(kh^{-}\right)\mathrm{e}^{\mathrm{i}kx}\right)& \\
=\sum_{j\ even}\frac{1}{j!}\left(k\eta\right)^j\left(k\sinh\left(kh^{-}\right)-\mathrm{i}k\eta_x\cosh\left(kh^{-}\right)\right)\mathrm{e}^{\mathrm{i}kx}& \qquad (46)\\
+\sum_{j\ odd}\frac{1}{j!}\left(k\eta\right)^j\left(k\cosh\left(kh^{-}\right)-\mathrm{i}k\eta_x\sinh\left(kh^{-}\right)\right)\mathrm{e}^{\mathrm{i}kx}.&
\end{aligned}
$$

Next, by comparing terms of equal order in η on both sides of the equation, we can derive the specific expression for the expansion $G^{-}(\eta)$. For $j=0$, we have:

$$
G^{-}\left(0\right)\mathrm{e}^{\mathrm{i}kx}=k\tanh\left(kh\right)\mathrm{e}^{\mathrm{i}kx}. \tag{47}
$$

For a general function $\zeta(x)$, according to Fourier analysis, we have

$$
G^{-}\left(0\right)\zeta(x)=D\tanh\left(h^{-}D\right)\zeta(x), \tag{48}
$$

where, $D=-\mathrm{i}\partial_x$, it is important to note that the DtN operator is actually a pseudodifferential operator. Understanding this operator should be done in Fourier space. For example, for equation (48), this should be interpreted as follows:

$$G^{-}(0)\,\zeta(x)=D\tanh\left(h^{-}D\right)\zeta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}k\tanh\left(kh^{-}\right)\hat{\zeta}\mathrm{e}^{\mathrm{i}kx}dk, \tag{49}$$

where, $\hat{\zeta}$ denotes the Fourier transform of ζ, and the operator itself can be represented in Fourier space. Specifically, the Fourier transform of $G^{-}(0)$ is given by:

$$\widehat{G}^{-}(0)=k\tanh\left(kh^{-}\right). \tag{50}$$

For higher-order expansions of the operator G^{-}, they can be derived from equation (46). For $j>0$ and even values of j, the expansion is:

$$\begin{aligned}G_j^{-}(\eta)=&\frac{1}{j!}\left(\eta^j D^{j+1}\tanh(hD)-\mathrm{i}\left(\eta^j\right)_x D^j\tanh\left(h^{-}D\right)\right)\\&-\sum_{l<j\ and\ l\ even}G_l^{-}(\eta)\frac{1}{(j-l)!}\eta^{j-l}D^{j-l}\\&-\sum_{l<j\ and\ l\ odd}G_l^{-}(\eta)\frac{1}{(j-l)!}\eta^{j-l}D^{j-l}\tanh\left(h^{-}D\right),\end{aligned} \tag{51}$$

For $j>0$ and odd,

$$\begin{aligned}G_j^{-}(\eta)=&\frac{1}{j!}\left(\eta^j D^{j+1}-\mathrm{i}\left(\eta^j\right)_x D^j\right)\\&-\sum_{l<j\ and\ l\ odd}G_l^{-}(\eta)\frac{1}{(j-l)!}\eta^{j-l}D^{j-l}\\&-\sum_{l<j\ and\ l\ even}G_l^{-}(\eta)\frac{1}{(j-l)!}\eta^{j-l}D^{j-l}\tanh\left(h^{-}D\right).\end{aligned} \tag{52}$$

In deriving equations (51) and (52), the Cauchy product formula was employed:

$$\left(\sum_{n=0}^{\infty}a_n\right)\cdot\left(\sum_{n=0}^{\infty}b_n\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_k b_{n-k}. \tag{53}$$

In this study, only the first three terms of the DtN operator are considered:

$$\begin{aligned}G_0^{-}=&D\tanh\left(h^{-}D\right),\\G_1^{-}=&D\eta D-D\tanh\left(h^{-}D\right)\eta\tanh\left(h^{-}D\right)D,\\G_2^{-}=&-\frac{1}{2}D^2\eta^2\tanh\left(h^{-}D\right)D-\frac{1}{2}D\tanh\left(h^{-}D\right)\eta^2D^2\\&+D\tanh\left(h^{-}D\right)\eta D\tanh\left(h^{-}D\right)\eta D\tanh\left(h^{-}D\right).\end{aligned} \tag{54}$$

Additionally, it is important to note that the composition of multiple operators is performed from right to left. For example:

$$D\eta D f = (-\mathrm{i}\partial_x)\,\eta\,(-\mathrm{i}\partial_x)\,f = -\partial_x \eta \partial_x f = (\eta f_x)_x\,. \tag{55}$$

A similar approach will be used to expand the operator G^+Craig et al. (2010). Consider a solution to equation (28):

$$\phi_k^+ = \cosh\left(k\left(y - h^+\right)\right)\mathrm{e}^{\mathrm{i}kx}, \tag{56}$$

It is evident that this solution satisfies the boundary condition $\phi_y^+\left(x, y = h^+\right) = 0$, making it a harmonic function. Substituting equation (56) into equation (29) gives:

$$G^+\xi_k^+ = G^+\phi_k^+\left(x, y = \eta\right) = \left.\left(\eta_x \frac{\partial \phi_k^+}{\partial x} - \frac{\partial \phi_k^+}{\partial y}\right)\right|_{y=\eta(x,t)}. \tag{57}$$

To derive the specific expression for the expansion $G^+(\eta) = \sum_{n=0}^{\infty} G_n^+(\eta)$, a Taylor expansion of $\cosh(k(\eta - h^+))$ and $\sinh(k(\eta - h^+))$ around $\eta = 0$ is required. After some calculations, the result is:

$$\begin{aligned}
&\left(\sum_{l=0}^{\infty} G_l^+(\eta)\right)\left(\sum_{j\ even} \frac{1}{j!}(k\eta)^j \cosh\left(kh^+\right)\mathrm{e}^{\mathrm{i}kx} - \sum_{j\ odd} \frac{1}{j!}(k\eta)^j \sinh\left(kh^+\right)\mathrm{e}^{\mathrm{i}kx}\right)\\
&\quad= \sum_{j\ even} \frac{1}{j!}(k\eta)^j\left(k\sinh\left(kh^+\right) + \mathrm{i}k\eta_x\cosh\left(kh^+\right)\right)\mathrm{e}^{\mathrm{i}kx}\\
&\quad- \sum_{j\ odd} \frac{1}{j!}(k\eta)^j\left(k\cosh\left(kh^+\right) + \mathrm{i}k\eta_x\sinh\left(kh^+\right)\right)\mathrm{e}^{\mathrm{i}kx}.
\end{aligned} \tag{58}$$

Next, by comparing terms of the same order in η on both sides of the equation, the specific expansion expression for $G^+(\eta)$ can be obtained. For $j = 0$:

$$G^+(0)\,\mathrm{e}^{\mathrm{i}kx} = k\tanh\left(kh^+\right)\mathrm{e}^{\mathrm{i}kx}, \tag{59}$$

For the higher-order expansion of the operator G^+, it can be derived from equation (58). When $j > 0$ and j is even:

$$\begin{aligned}
G_j^+(\eta) =& \frac{1}{j!}\left(\eta^j D^{j+1}\tanh\left(h^+D\right) - \mathrm{i}\left(\eta^j\right)_x D^j \tanh\left(h^+D\right)\right)\\
&- \sum_{l<j\ and\ l\ even} G_l^-(\eta)\,\frac{1}{(j-l)!}\eta^{j-l}D^{j-l}\\
&+ \sum_{l<j\ and\ l\ odd} G_l^-(\eta)\,\frac{1}{(j-l)!}\eta^{j-l}D^{j-l}\tanh\left(h^+D\right).
\end{aligned} \tag{60}$$

When $j > 0$ and j is odd:

$$\begin{aligned} G_j^+(\eta) = &- \frac{1}{j!}\left(\eta^j D^{j+1} - \mathrm{i}\left(\eta^j\right)_x D^j\right) \\ &- \sum_{l<j \text{ and } l \text{ odd}} G_l^+(\eta) \frac{1}{(j-l)!}\eta^{j-l} D^{j-l} \\ &+ \sum_{l<j \text{ and } l \text{ even}} G_l^+(\eta) \frac{1}{(j-l)!}\eta^{j-l} D^{j-l} \tanh\left(h^+ D\right). \end{aligned} \tag{61}$$

In our study, only the first three terms of the DtN operator expansion are considered:

$$\begin{aligned} G_0^+ =& D\tanh\left(h^+D\right), \\ G_1^+ =& -D\eta D + D\tanh\left(h^+D\right)\eta\tanh\left(h^+D\right)D, \\ G_2^+ =& -\frac{1}{2}D^2\eta^2\tanh\left(h^+D\right)D - \frac{1}{2}D\tanh\left(h^+D\right)\eta^2 D^2 \\ &+ D\tanh\left(h^+D\right)\eta D\tanh\left(h^+D\right)\eta D\tanh\left(h^+D\right). \end{aligned} \tag{62}$$

Additionally, from equations (51), (52), (60), and (61), it is observed that:

$$G^+\left(\eta; h^+\right) = \sum_{j=0}^{\infty} G_j^+\left(\eta; h^+\right) = \sum_{j=0}^{\infty} (-1)^j G_j^-\left(\eta; h^+\right), \tag{63}$$

Thus, the extension of the DtN operator method from a single-layer fluid to a two-layer fluid has been successfully accomplished, in line with the approach of Craig and Sulem (1993).The expansion formula of the DtN operator for the two-layer fluid system was first provided by Craig et al. (2010). To streamline our discussion, we now proceed with the non-dimensionalization of equations (40) and (41). Without loss of generality, we select h^-, $\sqrt{gh^-}$, and $\sqrt{g(h^-)^3}$ as the characteristic length, characteristic velocity, and characteristic potential function, respectively. We introduce the following dimensionless quantities:

$$h = \frac{h^+}{h^-}, \rho = \frac{\rho^+}{\rho^-}, B = \frac{\sigma}{\rho^- g\left(h^-\right)^2}, \tag{64}$$

the three dimensionless quantities introduced are the ratio of the depths of the two fluid layers, the ratio of the densities of the two fluid layers, and the Bond number, which represents the ratio of capillary forces to gravitational forces. Consequently, the dimensionless kinematic and dynamic boundary conditions can be expressed as follows(Lannes, 2013):

$$\eta_t = G^-\left(G^+ + \rho G^-\right)^{-1} G^+\xi, \tag{65}$$

$$\xi_t + \frac{1}{2}\left[\left(\xi_x^-\right)^2 - \frac{\left(\eta_t + \eta_x\xi_x^-\right)^2}{\left(1+\eta_x^2\right)}\right] - \frac{\rho}{2}\left[\left(\xi_x^+\right)^2 - \frac{\left(\eta_t + \eta_x\xi_x^+\right)^2}{\left(1+\eta_x^2\right)}\right] + (1-\rho)\eta - \frac{B\eta_{xx}}{\left(1+\eta_x^2\right)^{3/2}} = 0, \tag{66}$$

$$\xi^- = \left(G^+ + \rho G^-\right)^{-1} G^+ \xi, \tag{67}$$

$$\xi^+ = -\left(G^+ + \rho G^-\right)^{-1} G^- \xi. \tag{68}$$

$$\xi = \xi^- - \rho \xi^+. \tag{69}$$

3 Nonlinear Long-Wave Model

In this section, the Dirichlet-Neumann operator expansion method will be employed to derive a series of nonlinear long-wave models. The derivation assumes a long-wave approximation where the characteristic wavelength is substantially larger than the depth of the lower fluid layer. This assumption is quantified by the parameter $\mu = h^-/\lambda \ll 1$, where h^- is the depth of the lower fluid layer and λ is the characteristic wavelength.

3.1 Shallow(lower layer)-shallow(upper layer) model

Introducing the long wave parameter $\mu = 1/\lambda \ll 1$ and assuming that the depth ratio $h = O(1)$, these two assumptions imply that the depth of the two fluid layers is of the same order of magnitude, while the wavelength is much larger than the depth of the fluid layers. We first consider the classical Boussinesq scaling (Johnson):

$$\partial_x = O\left(\mu\right), \partial_t = O\left(\mu\right), \eta = O\left(\mu^2\right), \xi = O\left(\mu\right), \rho = O\left(1\right), B = O\left(1\right). \tag{70}$$

Thus, by asymptotically expanding the operator $G^{\pm}$ in terms of the small parameter μ, and omitting the detailed mathematical derivations, we obtain:

$$\begin{aligned}
G_0^- &= -\partial_{xx} - \frac{1}{3}\partial_{xxxx} + O\left(\mu^6\right), \\
G_1^- &= -\partial_x \eta \partial_x - O\left(\mu^6\right), \\
G_0^+ &= -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} + O\left(\mu^6\right), \\
G_1^+ &= \partial_x \eta \partial_x - O\left(\mu^6\right), \\
G^- &= -\partial_{xx} - \frac{1}{3}\partial_{xxxx} - \partial_x \eta \partial_x + O\left(\mu^6\right), \\
G^+ &= -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} + \partial_x \eta \partial_x + O\left(\mu^6\right), \\
G^+ + \rho G^- &= -\left(h+\rho\right)\partial_{xx}\left(1 + \frac{h^3+\rho}{3\left(h+\rho\right)}\partial_{xx} - \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + O\left(\mu^4\right)\right), \\
\left(G^+ + \rho G^-\right)^{-1} &= -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3\left(h+\rho\right)^2} - \frac{1-\rho}{\left(h+\rho\right)^2}\partial_x^{-1}\eta\partial_x^{-1} + O\left(\mu^2\right).
\end{aligned} \tag{71}$$

By substituting equation (71) into (67) and (68), we obtain:

$$\begin{aligned}
\xi^- &= \frac{h}{h+\rho}\xi + O\left(\mu^2\right), \\
\xi^+ &= -\frac{1}{h+\rho}\xi + O\left(\mu^2\right).
\end{aligned} \tag{72}$$

By combining equations (65), (66), (71), and (72), and neglecting higher-order terms, we obtain:

$$\eta_t + \frac{h}{h+\rho}\xi_{xx} + \frac{h^2(1+h\rho)}{3(h+\rho)^2}\xi_{xxxx} + \frac{h^2-\rho}{(h+\rho)^2}\partial_x(\eta\xi_x) = 0, \tag{73}$$

$$\xi_t + (1-\rho)\eta - B\eta_{xx} + \frac{h^2-\rho}{2(h+\rho)^2}\xi_x^2 = 0. \tag{74}$$

By combining equations (73) and (74), and eliminating η, we obtain:

$$\begin{aligned}\xi_{tt} - \frac{h(1-\rho)}{h+\rho}\xi_{xx} + \frac{h}{h+\rho}\left[B - \frac{(h-h\rho)(1+h\rho)}{3(h+\rho)}\right]\xi_{xxxx} + \frac{h^2-\rho}{2(1+\rho)^2}\partial_t\xi_x^2 \\ + \frac{h^2-\rho}{(h+\rho)^2}\partial_x(\xi_t\xi_x) = 0.\end{aligned} \tag{75}$$

We first examine the coefficients of the linear term ξ_{xx} and the quartic term ξ_{xxxx}:

$$c^2 := \frac{h(1-\rho)}{h+\rho}, \quad \alpha := \frac{h}{h+\rho}\left[B - \frac{(h-h\rho)(1+h\rho)}{3(h+\rho)}\right], \tag{76}$$

By performing a Taylor expansion of the dispersion relation (10) under the specified scaling, we find that the first two terms of the expansion align with those in the above expression. Subsequently, introducing the coordinate transformation $X = x - ct$, the new variable $\tau = \mu^3 t$, and setting $H = \xi_X$, we obtain the well-known Korteweg-de Vries (KdV) equation from equation (75):

$$H_\tau - \frac{\alpha}{2c}H_{XXX} + \frac{3(h^2-\rho)}{2(h+\rho)}HH_X = 0. \tag{77}$$

From equation (73), it can also be deduced that:

$$\eta = \frac{h}{c(h+\rho)}H + O(\mu^4). \tag{78}$$

Substituting equation (78) into (77) yields:

$$\eta_\tau - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_X = 0. \tag{79}$$

Thus, the KdV model for wave height in the case of shallow water-shallow water has been derived, which is known for its solitary wave solution:

$$\eta = A\mathrm{sech}^2\left\{\sqrt{\frac{\alpha h(h+\rho)}{36c^2(\rho-h^2)}A}\left[X + \frac{\alpha}{6c}A\tau\right]\right\}, \tag{80}$$

At this stage, the solitary wave height demonstrates exponential decay. According to equation (80), it can be observed that if $\alpha(\rho - h^2) > 0$, then $A > 0$, producing an upward convex solitary wave. Conversely, if $\alpha(\rho - h^2) < 0$, then $A < 0$, resulting in a downward concave solitary wave (Ramollo, 1996).

It is important to note that the coefficient α in the third-order dispersion term of equations (77) and (79) can potentially be much smaller than 1. In such scenarios, the asymptotic expansion method may no longer be valid. Therefore, we need to choose new scaling relations:

$$\partial_x = O(\mu), \partial_t = O(\mu), \eta = O\left(\mu^4\right), \xi = O\left(\mu^3\right), \rho = O(1), \alpha = O\left(\mu^2\right), \tag{81}$$

Through a derivation similar to that of the KdV equation, we obtain the fifth-order KdV equation (Ramollo, 1996; Craig et al., 2005):

$$\eta_\tau - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)}\eta\eta_X + \frac{\beta}{2c}\eta_{XXXXX} = 0, \tag{82}$$

where,

$$\beta = \frac{h}{h+\rho}\left[\frac{2h^3\left(1+h^3\rho\right)}{15}c^2 - \frac{\rho\left(1-h^2\right)^2}{9\left(h+\rho\right)}c^2 - \frac{h\left(1+h\rho\right)}{3\left(h+1\right)^2}B\right], \tag{83}$$

The dispersion relation for the fifth-order KdV equation can be derived as follows:

$$c_p = \frac{\alpha}{2c}k^2 + \frac{\beta}{2c}k^4. \tag{84}$$

Note that when $\alpha\beta < 0$, the phase speed c_p exhibits a global minimum at $k = \sqrt{-\alpha/2\beta}$. This indicates the presence of wave-packet solitary waves near this wavenumber (Vanden-Broeck, 2010). These solitary wave solutions bifurcate from a periodic wave solution with an infinitesimally small amplitude (Grimshaw et al., 1994). For small amplitudes, such solitary waves can be interpreted as specific envelope solitary wave solutions to the nonlinear Schrödinger equation (Akylas, 1993).

In addition to the fact that the coefficient α of the third-order dispersion term in equations (77) and (79) might be much smaller than 1, the coefficient of the nonlinear term $\frac{3c(h^2-\rho)}{2(h^2+h\rho)}$ can also be significantly less than 1. In such cases, the original asymptotic expansion method and scaling used for deriving the KdV equation are no longer valid. Therefore, a new scaling relationship must be adopted (Ramollo, 1996).

$$\partial_x = O(\mu), \partial_t = O(\mu), \eta = O(\mu), \xi = O(1), \rho = O(1), \frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)} = O(\mu). \tag{85}$$

Unlike the derivation of the fifth-order KdV equation, here we need to re-expand the Dirichlet-

Neumann operator asymptotically:

$$
\begin{aligned}
&G^{-}=-\partial_{xx}-\frac{1}{3}\partial_{xxxx}-\partial_x\eta\partial_x+O\left(\mu^5\right),\\
&G^{+}=-h\partial_{xx}-\frac{1}{3}h^3\partial_{xxxx}+\partial_x\eta\partial_x+O\left(\mu^5\right),\\
&G^{+}+\rho G^{-}=-\left(h+\rho\right)\partial_{xx}\left(1+\frac{h^3+\rho}{3\left(h+\rho\right)}\partial_{xx}-\frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x+O\left(\mu^3\right)\right),\\
&\left(G^{+}+\rho G^{-}\right)^{-1}=-\frac{\partial_{xx}^{-1}}{h+\rho}+\frac{h^3+\rho}{3\left(h+\rho\right)^2}-\frac{1-\rho}{\left(h+\rho\right)^2}\partial_x^{-1}\eta\partial_x^{-1}-\frac{\left(1-\rho\right)^2}{\left(h+\rho\right)^3}\partial_x^{-1}\eta^2\partial_x^{-1}+O\left(\mu\right).
\end{aligned}
\tag{86}
$$

Substitute equation (86) into (67) and (68) to obtain:

$$
\begin{aligned}
\xi^{-}&=\frac{h}{h+\rho}\xi-\frac{\left(1+h\right)\rho}{\left(h+\rho\right)^2}\eta\xi_x+O\left(\mu^3\right),\\
\xi^{+}&=\frac{1}{h+\rho}\xi-\frac{\left(1+h\right)}{\left(h+\rho\right)^2}\eta\xi_x+O\left(\mu^3\right),
\end{aligned}
\tag{87}
$$

By combining equations (65), (66), (86), and (87), and neglecting higher-order terms, we obtain:

$$
\eta_t+\frac{h}{h+\rho}\xi_{xx}+\frac{h^2\left(1+h\rho\right)}{3\left(h+\rho\right)^2}\xi_{xxxx}+\frac{h^2-\rho}{\left(h+\rho\right)^2}\partial_x\left(\eta\xi_x\right)-\frac{\left(1+h\right)^2\rho}{\left(h+\rho\right)^3}\partial_x\left(\eta^2\xi_x\right)=0,
\tag{88}
$$

$$
\xi_t+\left(1-\rho\right)\eta-B\eta_{xx}+\frac{h^2-\rho}{2\left(h+\rho\right)^2}\xi_x^2+\frac{\left(1+h\right)^2\rho}{\left(h+\rho\right)^3}\eta\xi_x^2=0.
\tag{89}
$$

Following a similar derivation method to that used for the KdV equation, we obtain the well-known modified Korteweg-de Vries (mKdV) equation:

$$
\eta_\tau-\frac{\alpha}{2c}\eta_{XXX}+\frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)}\eta\eta_X-\frac{3c\rho\left(1+h\right)^2}{h\left(h+\rho\right)^2}\eta^2\eta_X=0,
\tag{90}
$$

The mKdV equation also admits solitary wave solutions (Ivanov et al., 2022):

$$
\eta=\frac{A}{1+B\cosh\left(\left(x-vt\right)/C\right)},
\tag{91}
$$

where, A, B, and C depend on the coefficients of the mKdV equation and the wave speed v.

3.2 Shallow(lower layer)-deep(upper layer) model

We now assume that the upper water layer is relatively deep, with its depth comparable to the wavelength. This implies $h = h^+/h^- \gg 1$ and $h \sim O(\lambda)$. Consistent with the previous discussion, we introduce the small parameter $\mu = 1/\lambda$ and set $h = O\left(1/\mu\right)$. We also adopt the following scaling relationships:

$$
\partial_x=O\left(\mu\right),\partial_t=O\left(\mu\right),\eta=O\left(\mu\right),\xi=O\left(1\right),\rho=O\left(1\right),B=O\left(1/\mu\right),
\tag{92}
$$

The scaling relationships above effectively assume strong surface tension, with $B \sim O(1/\mu)$. Under these new scaling conditions, we can derive the asymptotic expansions for the DtN operators $G^{\pm}$:

$$
\begin{aligned}
G_0^- &= -\partial_{xx} - \frac{1}{3}\partial_{xxxx} + O\left(\mu^6\right), \\
G_1^- &= -\partial_x \eta \partial_x - O\left(\mu^5\right), \\
G_0^+ &= D \tanh\left(hD\right), \\
G_1^+ &= -D\eta D + G_0^+ \eta G_0^+ = \partial_x \eta \partial_x + G_0^+ \eta G_0^+, \\
G^- &= -\partial_{xx} - \frac{1}{3}\partial_{xxxx} - \partial_x \eta \partial_x + O\left(\mu^5\right), \\
G^+ &= G_0^+ + \partial_x \eta \partial_x + G_0^+ \eta G_0^+ + O\left(\mu^5\right), \\
G^+ + \rho G^- &= G_0^+ + \partial_x \eta \partial_x + G_0^+ \eta G_0^+ + \rho\left(-\partial_{xx} - \frac{1}{3}\partial_{xxxx} - \partial_x \eta \partial_x\right) + O\left(\mu^5\right), \\
\left(G^+ + \rho G^-\right)^{-1} &= \left(G_0^+\right)^{-1} + \rho\left(G_0^+\right)^{-1}\partial_{xx}\left(G_0^+\right)^{-1} + O\left(\mu\right).
\end{aligned}
\tag{93}
$$

It is important to note that G_0^+ is a non-local pseudodifferential operator, with its Fourier transform given by $\widehat{G_0^+} = k \tanh(kh)$. Consequently, the Fourier transform of its inverse operator is $\widehat{\left(G_0^+\right)^{-1}} = \coth(kh)/k$. By combining equations (65), (66), and (93), and neglecting higher-order small quantities, we obtain:

$$
\eta_t + \xi_{xx} + \partial_x\left(\eta \xi_x\right) - \rho \mathcal{K}\left[\xi_{xx}\right] = 0, \tag{94}
$$

$$
\xi_t + (1-\rho)\,\eta + \frac{1}{2}\xi_x^2 - B\eta_{xx} = 0, \tag{95}
$$

where, $\mathcal{K}$ is a pseudodifferential operator, with its Fourier transform given by $\widehat{\mathcal{K}} = k \coth(kh)$. By combining equations (94) and (95), and eliminating η while neglecting higher-order terms, we obtain:

$$
\xi_{tt} - c^2 \xi_{xx} + \partial_x\left(\xi_t \xi_x\right) + \frac{1}{2}\partial_t\left(\xi_x^2\right) + B\xi_{xxxx} + \rho c^2 \mathcal{K}\left[\xi_{xx}\right] = 0, \tag{96}
$$

where, $c^2 = 1 - \rho$. By introducing the new variables $X = x - ct$ and $\tau = \mu t$, the above equation can be simplified to:

$$
H_\tau + \frac{3}{2} H H_X - \frac{B}{2c} H_{XXX} - \frac{\rho c}{2}\mathcal{K}\left[H_X\right] = 0, \tag{97}
$$

where, $H = \xi_X$. This equation is known as the Benjamin equation (Benjamin, 1992). Additionally, from equation (94), it follows that $\eta = H/c + O(\mu)$. Thus, we have:

$$
\eta_\tau + \frac{3c}{2}\eta \eta_X - \frac{B}{2c}\eta_{XXX} - \frac{\rho c}{2}\mathcal{K}\left[\eta_X\right] = 0, \tag{98}
$$

From equation (98), we observe that, compared to the traditional KdV equation, this equation has an additional term $\rho c \mathcal{K}[\eta_X]/2$.

We can also introduce new scaling relationships:

$$\partial_x = O(\mu), \partial_t = O(\mu), \eta = O(1), \xi = O(1/\mu), \rho = O(1), B = O(1/\mu). \tag{99}$$

The scaling relationship (99) implies that the wave height is comparable to the depth of the lower fluid layer. Similarly, we can obtain the asymptotic expansion of the DtN operators $G^{\pm}$:

$$\begin{aligned}
&G^- = -\partial_{xx} - \partial_x\eta\partial_x + O\left(\mu^4\right), \\
&G^+ = |D| + \partial_x\eta\partial_x + |D|\eta|D| + O\left(\mu^3\right), \\
&G^+ + \rho G^- = |D| + \partial_x\eta\partial_x + |D|\eta|D| + \rho\left(-\partial_{xx} - \partial_x\eta\partial_x\right) + O\left(\mu^3\right), \\
&\left(G^+ + \rho G^-\right)^{-1} = |D|^{-1} - \eta - \rho - (1-\rho)\,|D|^{-1}\partial_x\eta\partial_x|D|^{-1} + O(\mu),
\end{aligned} \tag{100}$$

where, $|D| = (-\partial_{xx})^{1/2}$. By combining equations (65)-(68) with (100), we can obtain:

$$\begin{aligned}
&\xi^- = \xi - \rho|D|\xi + \rho|D|^{-1}\partial_x\left(\eta\xi_x\right) + O(\mu), \\
&\xi^+ = -|D|\xi + |D|^{-1}\partial_x\left(\eta\xi_x\right) + O(\mu),
\end{aligned} \tag{101}$$

Similarly, by introducing $H = \xi_x$, we obtain a strongly nonlinear models:

$$\eta_t + H_x - \rho|D|H_x + \partial_x(\eta H) - \rho\partial_x\left(|D|\eta H + \eta|D|H + \eta|D|\eta H\right) = 0, \tag{102}$$

$$H_t + (1-\rho)\,\eta_x + \eta\eta_x - B\eta_{xxx} - \rho\partial_x\left(H|D|H\right) - \rho\partial_x\left(H|D|\eta H\right) = 0. \tag{103}$$

3.3 Deep(lower layer)-shallow(upper layer) model

Next, following the methods of Barannyk et al. (2012, 2015), a long-wave model is constructed for deep water (bottom layer) over shallow water (top layer). This model introduces a new small parameter $\mu^2 = h/\lambda$ and new scaling relationships:

$$\partial_x = O(1), \partial_t = O(\mu), \eta = O\left(\mu^2\right), \xi = O(\mu), \rho = O(1), B = O(1), h = O\left(\mu^2\right). \tag{104}$$

At this point, the asymptotic expansion of the DtN operator $G^{\pm}$ is given by:

$$\begin{aligned}
&G_0^- = D\tanh(D), \\
&G_1^- = D\eta D + G_0^-\eta G_0^- = -\partial_x\eta\partial_x - G_0^-\eta G_0^-, \\
&G_0^+ = -h\partial_{xx} + O\left(\mu^4\right), \\
&G_1^+ = \partial_x\eta\partial_x - O\left(\mu^6\right), \\
&G^- = G_0^- - \partial_x\eta\partial_x - G_0^-\eta G_0^- + O\left(\mu^4\right), \\
&G^+ = -h\partial_{xx} + \partial_x\eta\partial_x + O\left(\mu^4\right), \\
&G^+ + \rho G^- = -h\partial_{xx} + \partial_x\eta\partial_x + \rho\left(G_0^- - \partial_x\eta\partial_x - G_0^-\eta G_0^-\right) + O\left(\mu^4\right), \\
&\left(G^+ + \rho G^-\right)^{-1} = \frac{1}{\rho}\left(G_0^-\right)^{-1} + O\left(\mu^2\right).
\end{aligned} \tag{105}$$

From equations (65) and (66), we can obtain:

$$\eta_t + \frac{h}{\rho}\xi_{xx} - \frac{1}{\rho}\partial_x\left(\eta\xi_x\right) = 0, \tag{106}$$

$$\xi_t + \left(1-\rho\right)\eta + \frac{1}{2\rho}\xi_x^2 - B\eta_{xx} = 0. \tag{107}$$

Introducing the variable substitution:

$$\eta = h\left(1-\varLambda\right), \xi_x = -\rho\sqrt{h}U, t = \tau/\sqrt{h}, \tag{108}$$

we can obtain:

$$\varLambda_\tau + \partial_x\left(U\varLambda\right) = 0, \tag{109}$$

$$\rho\left(U_\tau + UU_x\right) - B\varLambda_{xxx} + \left(1-\rho\right)\varLambda_x = 0. \tag{110}$$

4 Numerical Computation

In this chapter, spectral methods (Trefethen) will be employed alongside the iterative schemes of Ablowitz et al. (2006) to numerically solve the model equations derived earlier. The numerical approach will be introduced, noting that the model equations are essentially nonlinear ordinary differential equations, which can be expressed in the general form:

$$\eta_t + \mathcal{L}_1[\eta] + \mathcal{N}[\eta] = 0, \tag{111}$$

where, $\mathcal{L}_1[\eta]$ represents the linear part of the equation, while $\mathcal{N}[\eta]$ denotes the nonlinear part. To obtain traveling wave solutions, the transformation $X = x - ct$ is introduced to eliminate the linear part in equation (111), yielding:

$$-c\eta_X + \mathcal{L}_1[\eta] + \mathcal{N}[\eta] = 0, \tag{112}$$

By choosing the new linear operator $\mathcal{L}[\eta] := -c\eta_X + \mathcal{L}_1[\eta]$, equation (112) can be rewritten as:

$$\mathcal{L}[\eta] + \mathcal{N}[\eta] = 0. \tag{113}$$

Applying the Fourier transform to equation (113) yields:

$$\widehat{\eta} = -\frac{\widehat{\mathcal{N}[\eta]}}{\widehat{\mathcal{L}}} = \mathcal{P}[\widehat{\eta}]. \tag{114}$$

Next, following the method provided by Ablowitz et al. (2006), the numerical solution of equation (114) can be obtained using the following iterative scheme:

$$\widehat{\eta}_{n+1} = \left(\frac{\int |\hat{\eta}_n|^2 \, dk}{\int \hat{\eta}_n^* \mathcal{P}[\hat{\eta}_n] \, dk}\right)^m \mathcal{P}[\hat{\eta}_n], \tag{115}$$

m is a tunable parameter, and the initial guess for the iteration can be chosen as $\eta = A\mathrm{sech}^2(X)$, where $|A|$ is typically chosen to be relatively small.

4.1 Numerical solution of the KdV equation

In the previous sections, we have derived the form of the KdV equation:

$$\eta_\tau - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_X = 0. \tag{116}$$

Applying the transformation $X = X - V\tau$ to the KdV equation yields:

$$-V\eta_X - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_X = 0. \tag{117}$$

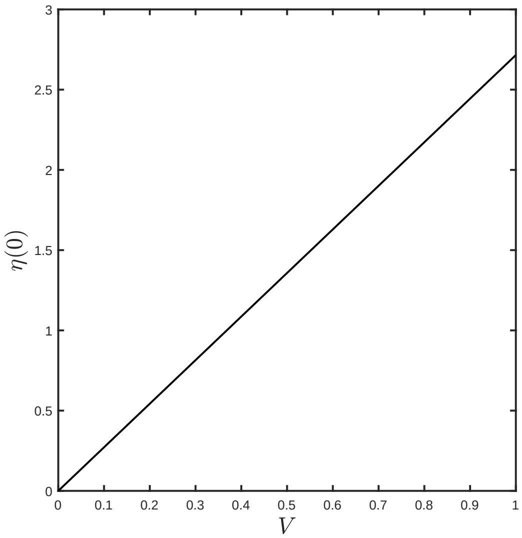

Figure 2: **The speed-amplitude bifurcation curve for solitary wave solutions of the KdV equation when $h = 2$, $\rho = 0.2$, and $B = 0.1$.**

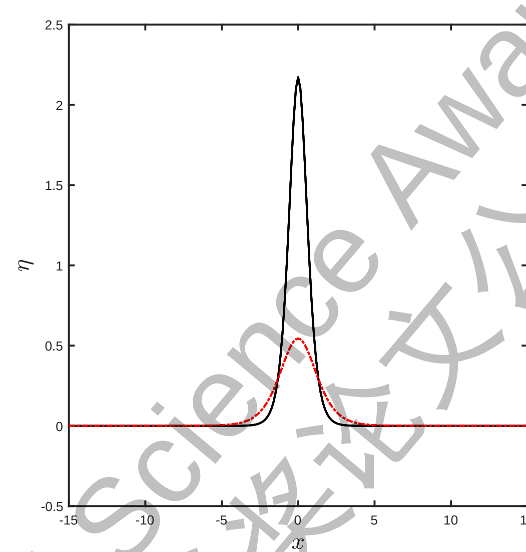

Figure 3: **The solitary wave solutions of the KdV equation for $h = 2$, $\rho = 0.2$, and $B = 0.1$: The black solid line and the red dashed line represent the wave profiles at $V = 0.8$ and $V = 0.2$, respectively.**

Corresponding to the linear operator:

$$\mathcal{L} = -V\partial_X - \frac{\alpha}{2c}\partial_{XXX}, \tag{118}$$

Its Fourier transform:

$$\widehat{\mathcal{L}} = -\mathrm{i}Vk + \mathrm{i}\frac{\alpha}{2c}k^3. \tag{119}$$

From equation (114), it can be obtained that:

$$\widehat{\eta} = -\frac{1}{-\mathrm{i}Vk + \mathrm{i}\frac{\alpha}{2c}k^3}\mathcal{F}\left[\frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)}\eta\eta_X\right] = \mathcal{P}_{KdV}\left[\hat{\eta}\right]. \tag{120}$$

The iterative scheme for the numerical solution of the KdV equation is:

$$\widehat{\eta}_{n+1} = \left(\frac{\int\left|\hat{\eta}_n\right|^2\,dk}{\int\hat{\eta}_n^*\mathcal{P}_{KdV}\left[\hat{\eta}_n\right]\,dk}\right)^m\mathcal{P}_{KdV}\left[\hat{\eta}_n\right], \tag{121}$$

We computed the solitary wave solutions of the KdV equation for $h = 2$, $\rho = 0.2$, and $B = 0.1$, and plotted the bifurcation curve of wave speed versus wave amplitude for KdV solitary waves, as shown in Figure (2). The numerical results reveal that the wave amplitude $\eta(0)$ of the KdV solitary waves is proportional to the speed V, with the wave profile representing a typical single-peaked solitary wave. Figure (3) illustrates the wave profiles for $V = 0.8$ and $V = 0.2$, demonstrating that the wave amplitude for $V = 0.8$ is significantly greater than that for $V = 0.2$.

4.2 Numerical solution of the fifth-order KdV equation

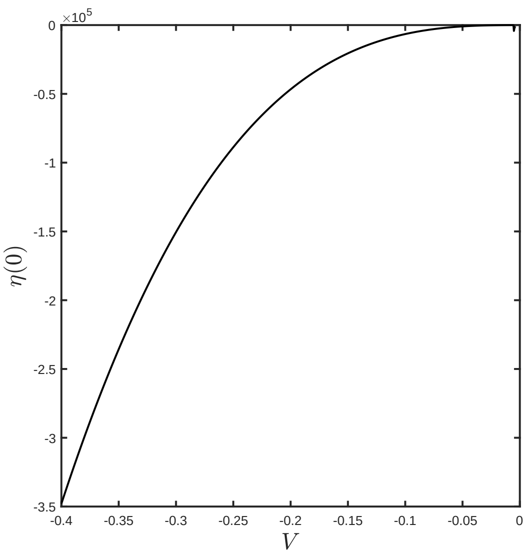

Figure 4: **The speed-amplitude bifurcation curve for the wavepacket solitary wave solution of the fifth-order KdV equation with $h = 2, \rho = 0.8, B = 0.1$.**

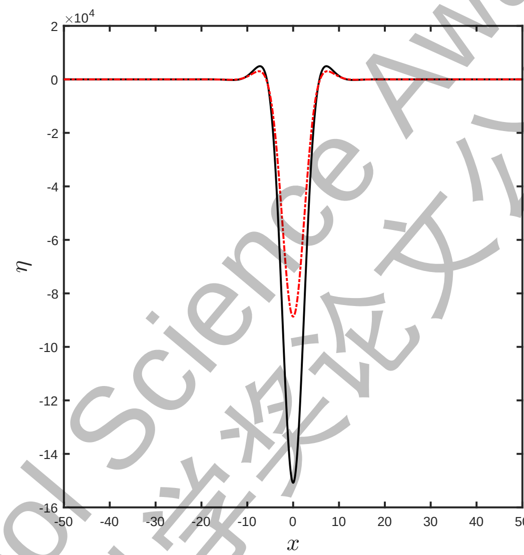

Figure 5: **The wavepacket solitary solutions of the fifth-order KdV equation for $h = 2, \rho = 0.2, B = 0.1$: The black solid line and red dashed line represent the wave profiles at $V = -0.3$ and $V = -0.25$, respectively.**

The form of the fifth-order KdV equation derived in the previous section is

$$\eta_\tau - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)}\eta\eta_X + \frac{\beta}{2c}\eta_{XXXXX} = 0. \tag{122}$$

By applying the transformation $X = X - V\tau$ to the fifth-order KdV equation, we obtain:

$$-V\eta_X - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)}\eta\eta_X + \frac{\beta}{2c}\eta_{XXXXX} = 0. \tag{123}$$

By performing a Fourier transform on the equation, we obtain:

$$\widehat{\eta} = -\frac{1}{-\mathrm{i}Vk + \mathrm{i}\frac{\alpha}{2c}k^3 + \mathrm{i}\frac{\beta}{2c}k^5}\mathcal{F}\left[\frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)}\eta\eta_X\right] = \mathcal{P}_{5thKdV}\left[\hat{\eta}\right]. \tag{124}$$

Therefore, the iterative scheme for the numerical solution of the 5th KdV equation is:

$$\widehat{\eta}_{n+1} = \left(\frac{\int \left|\hat{\eta}_n\right|^2 dk}{\int \hat{\eta}_n^* \mathcal{P}_{5thKdV}\left[\hat{\eta}_n\right] dk}\right)^m \mathcal{P}_{5thKdV}\left[\hat{\eta}_n\right], \tag{125}$$

We computed the solitary wave solutions of the fifth-order KdV equation for $h = 2$, $\rho = 0.8$, and $B = 0.1$, and plotted the velocity-amplitude bifurcation curve for these solitary waves, as shown in Figure (4). The numerical results indicate that the wave amplitude $\eta(0)$ of the solitary wave is positively correlated with the velocity V. However, unlike the solitary wave solutions of the KdV equation, the solitary wave solutions of the fifth-order KdV equation are no longer single-peaked but instead exhibit multi-peaked wave packets. Figure (5) illustrates the wave profiles for $V = -0.3$ and $V = -0.25$, showing that the solitary wave solutions of the fifth-order KdV equation manifest as multi-peaked wave packets, with the wave amplitude for $V = -0.3$ being significantly greater than that for $V = -0.25$. Similar multi-peaked wave packet solutions were first observed in gravity-capillary waves (Vanden-Broeck and Dias, 1992).

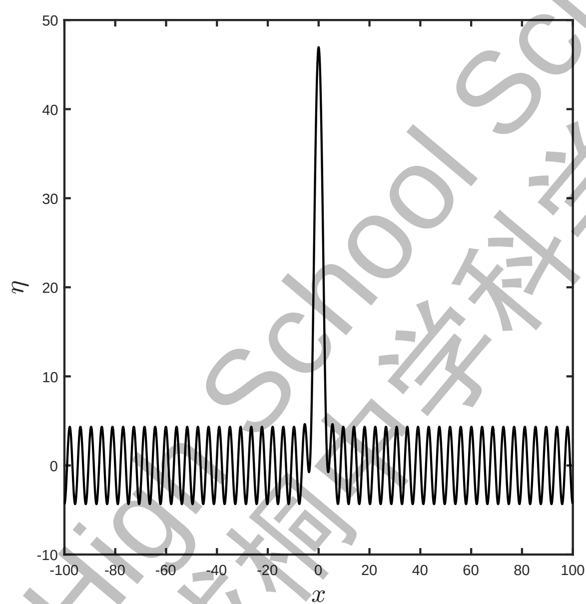

Figure 6: **The generalized solitary wave solution of the 5th KdV equation for $h = 2$, $\rho = 0.8$, $B = 0.1$, and $V = 5$.**

Additionally, we discovered a particularly intriguing numerical solution for the fifth-order KdV equation with parameters $h = 2$, $\rho = 0.8$, and $B = 0.1$. As illustrated in Figure (6), for a wave speed $V = 5$, the numerical solution exhibits both a solitary wave-like pulse and a non-decaying periodic wave train. This solution represents a superposition of a solitary wave and a periodic wave, known as a generalized solitary wave solution. The existence of generalized solitary waves was first proved by Beale (1991) for gravity-capillary waves, and Champneys et al. (2002) computed these solutions for gravity-capillary waves. Our numerical findings suggest that generalized solitary waves may also occur in interfacial waves between two fluids.

4.3 Numerical solution of the mKdV equation

The mKdV equation that we derived has the following form:

$$\eta_\tau - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)}\eta\eta_X - \frac{3c\rho\left(1+h\right)^2}{h\left(h+\rho\right)^2}\eta^2\eta_X = 0. \tag{126}$$

Applying the transformation $X = X - V\tau$ to the mKdV equation yields:

$$-V\eta_X - \frac{\alpha}{2c}\eta_{XXX} + \frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)}\eta\eta_X - \frac{3c\rho\left(1+h\right)^2}{h\left(h+\rho\right)^2}\eta^2\eta_X = 0. \tag{127}$$

Applying the Fourier transform to the equation yields:

$$\widehat{\eta} = -\frac{1}{-\mathrm{i}Vk + \mathrm{i}\frac{\alpha}{2c}k^3}\mathcal{F}\left[\frac{3c\left(h^2-\rho\right)}{2h\left(h+\rho\right)}\eta\eta_X - \frac{3c\rho\left(1+h\right)^2}{h\left(h+\rho\right)^2}\eta^2\eta_X\right] = \mathcal{P}_{5thKdV}\left[\hat{\eta}\right]. \tag{128}$$

Thus, the iterative scheme for the numerical computation of the mKdV equation is:

$$\widehat{\eta}_{n+1} = \left(\frac{\int\left|\hat{\eta}_n\right|^2 dk}{\int \hat{\eta}_n^* \mathcal{P}_{5thKdV}\left[\hat{\eta}_n\right] dk}\right)^m \mathcal{P}_{5thKdV}\left[\hat{\eta}_n\right]. \tag{129}$$

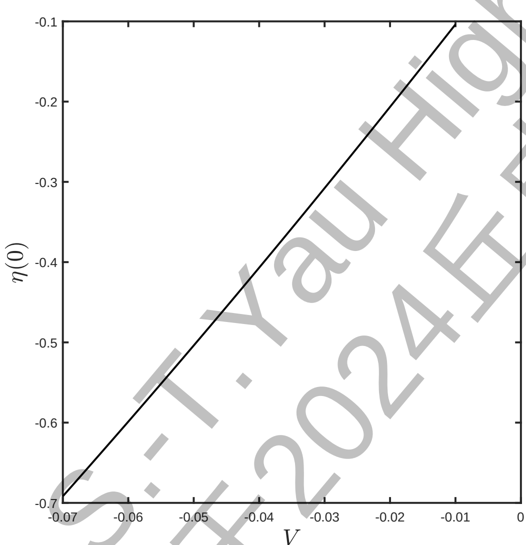

Figure 7: **The speed-amplitude bifurcation curve for the solitary wave solutions of the fifth-order KdV equation with parameters $h = 1$, $\rho = 0.5$, and $B = 2$.**

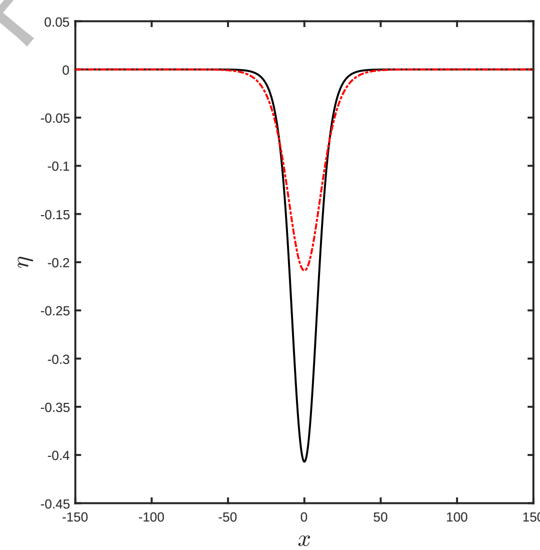

Figure 8: **The solitary wave solutions of the mKdV equation for parameters $h = 2$, $\rho = 0.2$, and $B = 0.1$: the black solid line and the red dashed line represent the wave profiles for $V = -0.04$ and $V = -0.02$, respectively.**

The bifurcation curve of wave speed V versus wave amplitude $\eta(0)$ for solitary wave solutions of the mKdV equation with parameters $h = 1$, $\rho = 0.5$, and $B = 2$ is shown in Figure (7). Numerical results indicate that the wave amplitude $\eta(0)$ of the mKdV solitary waves is positively correlated with the wave speed V, and the wave profile is a typical single-peaked solitary wave. Figure (8) illustrates the wave profiles for $V = -0.02$ and $V = -0.04$, where it is evident that the wave height for $V = -0.04$ is significantly greater than that for $V = -0.02$.

4.4 Numerical solution of the Benjamin equation

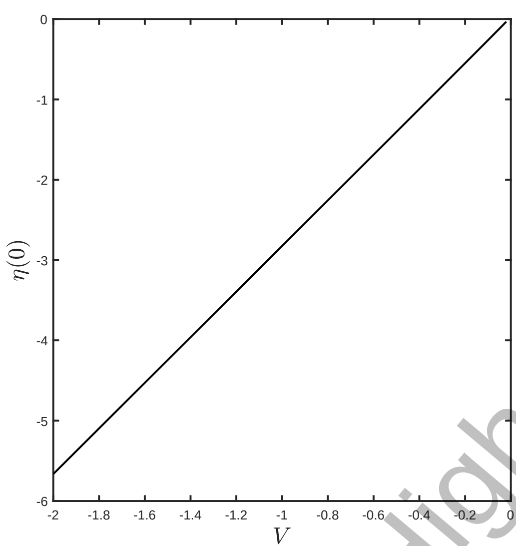

Figure 9: **The bifurcation curve of wave speed-amplitude for solitary wave solutions of the Benjamin equation with parameters $h = 100$, $\rho = 0.5$, and $\widetilde{B} = 1$.**

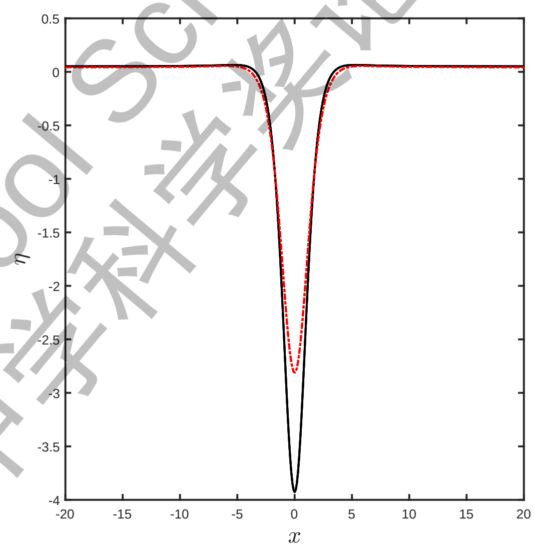

Figure 10: **Solitary wave solutions of the Benjamin equation with parameters $h = 100$, $\rho = 0.5$, and $\widetilde{B} = 1$: The black solid line and the red dashed line represent wave profiles at $V = -1.41$ and $V = -1.01$, respectively.**

The form of the Benjamin equation that we have derived is:

$$\eta_\tau + \frac{3c}{2}\eta\eta_X - \frac{B}{2c}\eta_{XXX} - \frac{\rho c}{2}\mathcal{K}\left[\eta_X\right] = 0. \tag{130}$$

Applying the transformation $X = X - V\tau$ to the Benjamin equation yields:

$$-V\eta_X + \frac{3c}{2}\eta\eta_X - \frac{\widetilde{B}}{2c}\eta_{XXX} - \frac{\rho c}{2}\mathcal{K}\left[\eta_X\right] = 0. \tag{131}$$

Taking the Fourier transform of the equation yields:

$$\widehat{\eta} = -\frac{1}{-\mathrm{i}Vk + \mathrm{i}\frac{B}{2c}k^3 - \mathrm{i}\frac{\rho c}{2}k^2\coth(kh)}\mathcal{F}\left[\frac{3c}{2}\eta\eta_X\right] = \mathcal{P}_{Benjamin}\left[\hat{\eta}\right]. \tag{132}$$

The iterative scheme for the numerical solution of the Benjamin equation is:

$$\widehat{\eta}_{n+1} = \left(\frac{\int |\hat{\eta}_n|^2 \, dk}{\int \hat{\eta}_n^* \mathcal{P}_{Benjamin} [\hat{\eta}_n] \, dk} \right)^m \mathcal{P}_{Benjmain} [\hat{\eta}_n] \, . \tag{133}$$

Finally, the bifurcation curve of wave speed versus wave amplitude for solitary wave solutions of the Benjamin equation with parameters $h = 100$, $\rho = 0.5$, and $\widetilde{B} = 1$ is presented in Figure (9). Numerical results show that the wave amplitude $\eta(0)$ of the solitary waves described by the Benjamin equation is positively correlated with the wave speed V. Figure (10) displays the wave profiles for $V = -1.41$ and $V = -1.01$, clearly illustrating that the wave amplitude for $V = -1.41$ is significantly greater than that for $V = -1.01$.

5 Conclusions

In this study, we systematically investigated the mathematical modeling and numerical computation of nonlinear interface waves in two-layer fluid systems. We began by developing a mathematical model for the dynamics of interface waves in two-layer fluids and analyzed their linear stability using the regular perturbation method. Our findings indicate that the system remains linearly stable when the density of the lower fluid exceeds that of the upper fluid. Conversely, Rayleigh-Taylor instability arises when the lower fluid density is less than the upper fluid density. We then extended the method introduced by Zakharov (1968) and Benjamin and Bridges (1997) to demonstrate that the two-layer fluid interface wave system with capillary possesses a Hamiltonian structure. Following this, Craig et al. (2010) extended the Dirichlet-Neumann operator, originally applicable to the lower fluid only, to include the upper fluid. Reformulating the original equations using this extended Dirichlet-Neumann operator, we derived a new set of nonlinear equations that circumvent the direct solution of the Laplace equation, further simplifying the problem by making the equations independent of y. Applying long-wave assumptions, we expanded the Dirichlet-Neumann operator and used asymptotic analysis to derive model equations for various scenarios, including the KdV equation, the fifth-order KdV equation, the mKdV equation, the Benjamin equation, and other strong nonlinear models. Our research demonstrates that the Dirichlet-Neumann operator expansion method is more efficient and computationally less intensive compared to traditional methods for deriving model equations for two-layer fluid interface waves. We numerically solved the KdV equation, the fifth-order KdV equation, the mKdV equation, and the Benjamin equation using spectral methods combined with the iterative scheme proposed by Ablowitz et al. (2006). Our numerical results revealed a range of solutions, including traditional single-peaked solitary waves, wave packet-type solitary waves, and notably, generalized solitary waves in the fifth-order KdV equation.

In the future, the study of interface waves in two-layer fluids is expected to remain a prominent area of research. Building on the findings of this paper, several key directions for future research are proposed:

1. Investigation of Short-Wave Problems Using the Dirichlet-Neumann Operator Method: While this study focuses on long-wave phenomena, short-wave problems also play a significant role in two-layer fluid interface waves. These problems involve phenomena such as three-wave resonance, long-short wave interactions, and Benjamin-Feir instability, which are governed by the three-wave resonance equation, long-short wave interaction equations, and the nonlinear Schrödinger equation. Future research should explore how to adapt the Dirichlet-Neumann operator method to address these short-wave scenarios.

2. Modeling and Numerical Computation of Three-Dimensional Interface Waves: The cur-

rent study is limited to two-dimensional problems. Three-dimensional interface waves introduce additional complexities and may be described by equations such as the Kadomtsev-Petviashvili (KP) equation, the Benjamin-Ono equation, and the Davey-Stewartson equations. Further research is needed to extend the Dirichlet-Neumann operator method to three-dimensional contexts and develop appropriate numerical techniques.

3. Exploration of Nonlinear Stability and Time-Dependent Dynamics: This paper primarily examines traveling wave solutions for specific model equations. However, aspects such as nonlinear stability and time-dependent behavior of these waves have not been fully explored. Future studies should address the stability of these waves, including both linear and nonlinear stability analyses, and investigate the time evolution of interface waves to gain a more comprehensive understanding of their dynamics.

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A Numerical Computation Code

Listing 1: MATLAB Code

```
  %% KdV
% y_t + a*y_xxx + b*y*y_x = 0
clear, clc
N = 512*4; L = 150; % half wave length
dx = 2*L/N; x = -L:dx:L-dx; x = x';
k = pi/L*[0:N/2-1 0 -N/2+1:-1]';
h = 2; r = .2; B = .1;
c = sqrt(h*(1-r)/(h+r)); alpha = h/(h+r)*(B-(h-h*r)*(1+h*r)/(3*(h+r)));
a = -alpha/(2*c); b = 3*c*(h^2-r)/(2*h*(h+r));
bif = []; X = []; Y = [];
Vf = 0.0004; Vs = 1; dV = (Vf-Vs)/100;
% V = -alpha/6c*A alpha(r-h^2)
for V=Vs:dV:Vf-dV
y = 0.01*exp(-x.^2);
yhat = fft(y);
err = 1; count = 0;
while (err>1e-9)
    count = count + 1
    yhat_new = real(b/2*fft(y.^2)./(V+a*k.^2)); %yhat_new(1) = 0;
    Q = (yhat'*yhat)/(yhat'*yhat_new);
    yhat_new = Q^2*yhat_new;
    err = abs(1-Q)
    yhat = yhat_new;
    y = real(ifft(yhat));
end
plot(x,y, 'k','LineWidth',1), drawnow
% H_w = y(N/2+1) - y(1)
bif = [bif; V, y(N/2+1) - y(1)];
    X = [X x]; Y = [Y y];
end
%% 5thKdV
clear; clc;
% INITIAL SETTING
N = 1024*2; L = 100; % half wave length
dx = 2*L/N; x = -L:dx:L-dx; x = x';
k = pi/L*[0:N/2-1 0 -N/2+1:-1]';
% parameters
h = 2; r = .8; B = .1;
c = sqrt(h*(1-r)/(h+r));
b1 = 2*h^3*(1+h^3*r)*c^2/15;
```

```
b2 = -r*(1-h^2)^2/(h+r)/9*c^2;
b3 = -h*(1+h*r)/(3*(1+h)^2)*B;
alpha = h/(h+r)*(B-(h-h*r)*(1+h*r)/(3*(h+r)));
beta = h*(b1+b2+b3)/(h+r);
% y_t + a*y_xxx + b*y*y_x + a1*y_xxxxx = 0
a = -alpha/2/c;
b = 3*c*(h^2-r)/(2*h*(h+r));
a1 = beta/2/c;
kc = linspace(0,.2,1024);
figure(1)
cp = -a*kc.^2+a1*kc.^4;
plot(kc,cp,'k')
%%
bif = []; X = []; Y = [];
Vf = -.4; Vs = -0.004; dV = (Vf-Vs)/1000;
for V=Vs:dV:Vf
err = 1; count = 0;
% INITIAL DATA
y = -0.01*exp(-x.^2/10);
yhat = fft(y);
% c = 1;
while (err>1e-7) && (err<1e+3)
    count = count + 1
    yhat_new = real(b/2*fft(y.^2)./(V + a*k.^2-a1*k.^4));%yhat_new(1) = 0;
    Q = (yhat'*yhat)/(yhat'*yhat_new);
    yhat_new = yhat_new*Q^2;
    err = norm(yhat_new - yhat)
    yhat = yhat_new;
end
y = real(ifft(yhat));
%y = [y(N/2+1:end); y(1:N/2)];
figure(2)
plot(x,y,'k')
% plot(x(1:8:end),y(1:8:end), 'ko', 'markersize', 5)
% H_w = y(N/2+1) - y(1)
bif = [bif; V, y(N/2+1) - y(1)];
    X = [X x]; Y = [Y y];

end
%% mKdV
clear; clc;
N = 512*8; L = 150; % half wave length
dx = 2*L/N; x = -L:dx:L-dx; x = x';
```

```
k = pi/L*[0:N/2-1 0 -N/2+1:-1]';
y = 0.01*exp(-x.^2);
yhat = fft(y);
h = 1; r = .5; B = 2;
c = sqrt(h*(1-r)/(h+r)); alpha = h/(h+r)*(B-(h-h*r)*(1+h*r)/(3*(h+r)));
V = -.0106;
Vs = -0.01;  Vf = -0.07; dV=(Vf-Vs)/100;
% y_t + a*y_xxx + b*y*y_x + b1*y^2*y_x
a = -alpha/(2*c);
b = 3*c*(h^2-r)/(2*h*(h+r));
b1 = -3*c*r*(1+h)^2/(h*(h+r)^2);
% a = -.1;
% b = -1;
% b1 = -2;
bif = []; X = []; Y = [];
for V = Vs:dV:Vf
y = 0.01*exp(-x.^2);
yhat = fft(y);
err = 1; count = 0;
while (err>1e-9)
    count = count + 1
    yhat_new = (b/2*fft(y.^2) - b1/3*k.^2.*fft(y.^3))./(V+a*k.^2);
    Q = (yhat'*yhat)/(yhat'*yhat_new);
    yhat_new = Q^2*yhat_new;
    err = norm(yhat_new - yhat)
    yhat = yhat_new;
    y = real(ifft(yhat));
end
plot(x,y, 'k')
H_w = y(N/2+1) - y(1), drawnow
    bif = [bif; V, y(N/2+1) - y(1)];
    X = [X x]; Y = [Y y];
end
%% Benjamin
clear, clc
N = 512*4; L = 100; % half wave length
dx = 2*L/N; x = -L:dx:L-dx; x = x';
k = pi/L*[0:N/2-1 0 -N/2+1:-1]';
% y_t + a*y_xxx + b*y*y_x + d*K[y_x] = 0
r = 0.5; Bb = 1; h = 100;
c = sqrt(1-r);
a = -Bb/2/c;
b = 3*c/2;
```

```
d = -r*c/2;
% a = -1; b = 1; d=1;
V = -.1;
bif = []; X = []; Y = [];
Vf = -2; Vs = -0.02; dV = (Vf-Vs)/100;
for V=Vs:dV:Vf
    err = 1; count = 0;
    y = 0.01*exp(-x.^2);
    yhat = fft(y);
    % V = -0.12;
    % plot(x,y, 'k','LineWidth',1), drawnow
    while (err>1e-9)
        count = count + 1;
        yhat_new = real((b/2*fft(y.^2))./(V+a*k.^2-d*k.*coth(k*h)));
        yhat_new(1) = 0;
        yhat_new(N/2+1) = 0;
        Q = (yhat'*yhat)/(yhat'*yhat_new);
        yhat_new = Q^2*yhat_new;
        err = abs(1-Q);
        yhat = yhat_new;
        y = real(ifft(yhat));
    end
    hold on
    plot(x,y,'LineWidth',1), drawnow
    H_w = y(N/2+1) - y(1)
    bif = [bif; V, y(N/2+1) - y(1)];
    X = [X x]; Y = [Y y];
end
```

B The Process of Deriving Mathematical Formulation

Linear Stability Analysis 去掉所有非线性项

A

$\phi^+_{xx} + \phi^+_{yy} = 0$, $0 < y < h^+$

$\phi^-_{xx} + \phi^-_{yy} = 0$, $-h^- < y < \eta$

$\eta_t - \phi^+_y = 0$, $y = 0$ ①

$\eta_t - \phi^-_y = 0$, $y = 0$ ②

$\rho^-(\phi^-_t + g\eta) - \rho^+(\phi^+_t + g\eta) - \sigma\eta_{xx} = 0$, $y = 0$ ③

$\dfrac{\partial \phi^\pm_y}{\partial y} = 0$, $y = \pm h^\pm$

$\eta = \hat{\eta} e^{i(kx-wt)}$

$\phi^\pm = \hat{\phi}^\pm \cosh(|k|(y \mp h^\pm)) e^{i(kx-wt)}$

①: $\eta_t = \phi^+_y$, $y = 0$

$\eta_t = -iw\hat{\eta} e^{i(kx-wt)}$

$\phi^+_y\big|_{y=0} = |k|\hat{\phi}^+ \sinh(|k|(y-h^+)) e^{i(kx-wt)}\big|_{y=0}$

$= -|k|\hat{\phi}^+ \sinh(|k|h^+) e^{i(kx-wt)}$

$\boxed{-iw\hat{\eta} = -|k|\hat{\phi}^+ \sinh(|k|h^+)}$

② $\eta_t - \phi_y^- = 0 \ , \ y=0$

$$\eta_t = -i\omega\hat{\eta}e^{i(kx-\omega t)}$$

$$\phi_y^-\big|_{y=0} = |k|\hat{\phi}^- \sinh(|k|(y+h^-))e^{i(kx-\omega t)}\Big|_{y=0}$$

$$= |k|\hat{\phi}^- \sinh(|k|h^-)e^{i(kx-\omega t)}$$

$$\Rightarrow \boxed{-i\omega\hat{\eta} = |k|\hat{\phi}^- \sinh(|k|h^-)}$$

③ $\rho^-(\phi_t^- + g\eta) - \rho^+(\phi_t^+ + g\eta) - \sigma\eta_{xx} = 0 \ , \ y=0$

$$\phi_t^-\big|_{y=0} = -i\omega\cosh(|k|(y+h^-))e^{i(kx-\omega t)}\Big|_{y=0} = -i\omega\cosh(|k|h^-)e^{i(kx-\omega t)}$$

$$\phi_t^+\big|_{y=0} = -i\omega\cosh(|k|(y-h^+))e^{i(kx-\omega t)}\Big|_{y=0} = -i\omega\cosh(|k|h^+)e^{i(kx-\omega t)} \neq$$

$$-\sigma\eta_{xx} = -\sigma(ik)^2\hat{\eta}e^{i(kx-\omega t)} = \sigma k^2\hat{\eta}e^{i(kx-\omega t)}$$

$$\left\{-i\omega\rho^-\cosh(|k|h^-)\hat{\phi}^- + \rho^- g\hat{\eta} - \rho^+[-i\omega\cosh(|k|h^+)\hat{\phi}^+ + g\hat{\eta}] + \sigma k^2\hat{\eta}\right\}e^{i(kx-\omega t)} = 0$$

$$\boxed{i\omega[\rho^+\cosh(|k|h^+)\hat{\phi}^+ - \rho^-\cosh(|k|h^-)\hat{\phi}^-] + (\rho^- - \rho^+)g\hat{\eta} + \sigma k^2\hat{\eta} = 0}$$

$$-i\omega\hat{\eta} = -|k|\hat{\phi}^+ \sinh(|k|h^+) \Rightarrow \hat{\phi}^+ = \frac{i\omega}{|k|}\frac{1}{\sinh(|k|h^+)}\hat{\eta}$$

$$-i\omega\hat{\eta} = |k|\hat{\phi}^- \sinh(|k|h^-) \Rightarrow \hat{\phi}^- = -\frac{i\omega}{|k|}\frac{1}{\sinh(|k|h^-)}\hat{\eta}$$

$$i\omega\left[\rho^+\hat{\phi}^+\cosh(|k|h^+) - \rho^-\hat{\phi}^-\cosh(|k|h^-)\right] + (\rho^- - \rho^+)g\hat{\eta} + \sigma k^2\hat{\eta} = 0$$

$$i\omega\rho^+\frac{i\omega}{|k|}\frac{\cosh(|k|h^+)}{\sinh(|k|h^+)}\hat{\eta} - i\omega\rho^-\left(-\frac{i\omega}{|k|}\right)\frac{\cosh(|k|h^-)}{\sinh(|k|h^-)}\hat{\eta} + (\rho^- - \rho^+)g\hat{\eta} + \sigma k^2\hat{\eta} = 0$$

$$-\omega^2\rho^+\frac{\coth(|k|h^+)}{|k|}\hat{\eta} - \omega^2\rho^-\frac{\coth(|k|h^-)}{|k|}\hat{\eta} + (\rho^- - \rho^+)g\hat{\eta} + \sigma k^2\hat{\eta} = 0$$

$$-\omega^2\left[\rho^+\coth(|k|h^+) + \rho^-\coth(|k|h^-)\right]\cancel{\hat{\eta}} + (\rho^- - \rho^+)g|k|\cancel{\hat{\eta}} + \sigma k^2|k|\cancel{\hat{\eta}} = 0$$

$$\Rightarrow \omega^2 = \frac{|k|\left[(\rho^- - \rho^+)g + \sigma k^2\right]}{\rho^+\coth(|k|h^+) + \rho^-\coth(|k|h^-)}$$

$(\rho^- - \rho^+)g + \sigma k^2$

若 $\rho^- - \rho^+ > 0 \Rightarrow (\rho^- - \rho^+)g + \sigma k^2 > 0$

若 $\rho^- - \rho^+ < 0$ $\quad (\rho^- - \rho^+)g + \sigma k^2 < 0$

$$\boxed{\begin{matrix}\omega^2 > 0 \text{ 稳定} \\ \omega^2 < 0 \text{ 不稳定}\end{matrix}}$$

$\rho^- > \rho^+$ 时 一定稳定

$\sigma k^2 < (\rho^+ - \rho^-)g$

$k^2 < \frac{(\rho^+ - \rho^-)g}{\sigma}$

$\rho^- < \rho^+$ 时，$k < k_c = \sqrt{\frac{(\rho^+ - \rho^-)g}{\sigma}}$ 不稳定

$k < \sqrt{\frac{(\rho^+ - \rho^-)g}{\sigma}}$

$\Downarrow$

Rayleigh-Taylor (RT) instability

$\phi^+_{xx} + \phi^+_{yy} = 0, \ \eta < y < h^+$ ①

$\phi^-_{xx} + \phi^-_{yy} = 0, \ -h^- < y < \eta$ ②

$\eta_t = \phi^+_y - \eta_x \phi^+_x, \ y = \eta(x,t)$ ③

$\eta_t = \phi^-_y - \eta_x \phi^-_x, \ y = \eta(x,t)$ ④

B $\rho^-\left[\phi^-_t + \frac{1}{2}|\nabla\phi^-|^2 + g\eta\right] - \rho^+\left[\phi^+_t + \frac{1}{2}|\nabla\phi^+|^2 + g\eta\right] - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0, \ y = \eta(x,t).$ $\quad \nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ ⑤

$\frac{\partial\phi^\pm}{\partial y} = 0, \ y = \pm h^\pm$ ⑥

定义表面势函数：$\xi^\pm(x,t) = \phi^\pm(x,\eta(x,t),t)$

$$\frac{\partial\xi^\pm}{\partial t} = \left(\frac{\partial\phi^\pm}{\partial t} + \frac{\partial\phi^\pm}{\partial y}\eta_t\right)\Bigg|_{y=\eta(x,t)}$$

注意到 $\eta_t = \phi^\pm_y - \eta_x\phi^\pm_x$ at $y = \eta(x,t)$

$$\Rightarrow \frac{\partial\xi^\pm}{\partial t} = \frac{\partial\phi^\pm}{\partial t}\Bigg|_{y=\eta(x,t)} + \frac{\partial\phi^\pm}{\partial y}\left(\phi^\pm_y - \eta_x\phi^\pm_x\right)\Bigg|_{y=\eta(x,t)}$$

$$\Rightarrow \frac{\partial\phi^\pm}{\partial t}\Bigg|_{y=\eta(x,t)} = \xi^\pm_t - \phi^\pm_y\left(\phi^\pm_y - \eta_x\phi^\pm_x\right)\Big|_{y=\eta(x,t)}$$

Var-1

$$\rho^-\left[\phi_t^- + \frac{1}{2}|\nabla\phi^-|^2 + g\eta\right] - \rho^+\left[\phi_t^+ + \frac{1}{2}|\nabla\phi^+|^2 + g\eta\right] - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0,\quad y=\eta(x,t) \qquad \nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

$$|\nabla\phi^\pm|^2 = \left(\frac{\partial\phi^\pm}{\partial x}, \frac{\partial\phi^\pm}{\partial y}\right)\cdot\left(\frac{\partial\phi^\pm}{\partial x}, \frac{\partial\phi^\pm}{\partial y}\right) = (\phi_x^\pm)^2 + (\phi_y^\pm)^2$$

$$\rho^-\left[\xi_t^- - \phi_y^-(\phi_y^- - \eta_x\phi_x^-) + \frac{1}{2}(\phi_x^-)^2 + \frac{1}{2}(\phi_y^-)^2 + g\eta\right] \cancel{+} - \rho^+\left[\xi_t^+ - \phi_y^+(\phi_y^+ - \eta_x\phi_x^+) + \frac{1}{2}(\phi_x^+)^2 + \frac{1}{2}(\phi_y^+)^2 + g\eta\right] - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$\rho^-\left[\xi_t^- - \phi_y^-(\phi_y^- - \eta_x\phi_x^-) + \frac{1}{2}|\nabla\phi^-|^2 + g\eta\right] - \rho^+\left[\xi_t^+ - \phi_y^+(\phi_y^+ - \eta_x\phi_x^+) + \frac{1}{2}|\nabla\phi^+|^2 + g\eta\right] - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

哈密顿

$$H = \frac{\rho^-}{2}\int_{\mathbb{R}}\int_{-h}^{\eta} |\nabla\phi^-|^2\,dx\,dy + \frac{\rho^+}{2}\int_{\mathbb{R}}\int_{\eta}^{h^+} |\nabla\phi^+|^2\,dy\,dx + \frac{g(\rho^- - \rho^+)}{2}\int_{\mathbb{R}} \eta^2\,dx + \sigma\int_{\mathbb{R}}\left(\sqrt{1+\eta_x^2} - 1\right)dx$$

$$E_K = \frac{\rho^-}{2}\int_{\mathbb{R}}\int_{-h}^{\eta} |\nabla\phi^-|^2\,dx\,dy + \frac{\rho^+}{2}\int_{\mathbb{R}}\int_{\eta}^{h^+} |\nabla\phi^+|^2\,dy\,dx$$

$$E_K = \frac{\rho^-}{2}\int_{\mathbb{R}}\int_{-h}^{\eta} \nabla\phi^-\cdot\nabla\phi^-\,dx\,dy + \frac{\rho^+}{2}\int_{\mathbb{R}}\int_{\eta}^{h^+} \nabla\phi^+\cdot\nabla\phi^+\,dy\,dx$$

1/av-2

Var-3

$$E_k = \frac{\rho^-}{2}\int_l (\vec{n}\phi^- \cdot \nabla\phi^-)_{y=\eta}\, dl - \frac{\rho^+}{2}\int_l (\vec{n}\phi^+ \cdot \nabla\phi^+)\, dl$$

$$E_k = \frac{\rho^-}{2}\int_l \left(\phi^- \cdot \frac{\partial\phi^-}{\partial\vec{n}}\right)_{y=\eta} dl - \frac{\rho^+}{2}\int_l \left(\phi^+ \frac{\partial\phi^+}{\partial\vec{n}}\right)_{y=\eta} dl$$

dl, dx, $dy = \eta_x\, dx$

$$\boxed{\frac{\partial\phi}{\partial\vec{s}} = \underset{(单位)}{\vec{s}} \cdot \nabla\phi}$$

$$E_k = \frac{\rho^-}{2}\int_{\mathbb{R}} \xi^- \frac{\partial\phi^-}{\partial\vec{n}}\sqrt{1+\eta_x^2}\, dx - \frac{\rho^+}{2}\int_{\mathbb{R}} \xi^+ \frac{\partial\phi^+}{\partial\vec{n}}\sqrt{1+\eta_x^2}\, dx \qquad \vec{n} = \frac{(-\eta_x, 1)}{\sqrt{1+\eta_x^2}}$$

$$\boxed{dl = \sqrt{1+\eta_x^2}\, dx}$$

$$\delta E_k = \underline{\frac{\rho^-}{2}\int_{\mathbb{R}} \delta\xi^- \frac{\partial\phi^-}{\partial\vec{n}}\sqrt{1+\eta_x^2}\, dx} + \underline{\frac{\rho^-}{2}\int_{\mathbb{R}} \xi^- \frac{\partial\delta\phi^-}{\partial\vec{n}}\sqrt{1+\eta_x^2}\, dx}$$
$$- \frac{\rho^+}{2}\int_{\mathbb{R}} \delta\xi^+ \frac{\partial\phi^+}{\partial\vec{n}}\sqrt{1+\eta_x^2}\, dx - \frac{\rho^+}{2}\int_{\mathbb{R}} \xi^+ \frac{\partial\delta\phi^+}{\partial\vec{n}}\sqrt{1+\eta_x^2}\, dx$$

$$\boxed{\begin{aligned} L &= L_1 \cdot L_2 \\ \delta L &= \delta L_1 \cdot L_2 + L_1 \delta L_2 \end{aligned}}$$

$$\delta E_k = \rho^-\int_{\mathbb{R}} \delta\xi^- \frac{\partial\phi^-}{\partial\vec{n}}\sqrt{1+\eta_x^2}\, dx - \rho^+\int_{\mathbb{R}} \delta\xi^+ \frac{\partial\phi^+}{\partial\vec{n}}\sqrt{1+\eta_x^2}\, dx$$

$$\begin{cases} \frac{\partial\phi^-}{\partial\vec{n}}\sqrt{1+\eta_x^2} = -\eta_x\phi_x^- + \phi_y^- = \eta_t \\ \qquad \| \\ \frac{\partial\phi^+}{\partial\vec{n}}\sqrt{1+\eta_x^2} = -\eta_x\phi_x^+ + \phi_y^+ = \eta_t \end{cases}$$

$$= \int_{\mathbb{R}} \delta(\rho^-\xi^- - \rho^+\xi^+)\boxed{\frac{\partial\phi^-}{\partial\vec{n}}}\sqrt{1+\eta_x^2}\, dx$$

$$\|$$

$$= \int_{\mathbb{R}} \delta(\rho^-\xi^- - \rho^+\xi^+)\boxed{\frac{\partial\phi^+}{\partial\vec{n}}}\sqrt{1+\eta_x^2}\, dx$$

Var-4

$$E_p = \frac{g(\rho^- - \rho^+)}{2}\int_{\mathbb{R}} \eta^2 dx + \sigma \int_{\mathbb{R}} \left(\sqrt{1+\eta_x^2} - 1\right) dx$$

对 ξ 求变分

$$\delta \xi = \delta(\rho^- \xi^- - \rho^+ \xi^+)$$

$$\delta E_p = \int_{\mathbb{R}} 0 \cdot \delta(\rho^- \xi^- - \rho^+ \xi^+) dx$$

$$\Rightarrow \delta H = \delta E_k + \delta E_p = \int_{\mathbb{R}} \delta(\rho^- \xi^- - \rho^+ \xi^+) \frac{\partial \phi^-}{\partial \vec{n}} \sqrt{1+\eta_x^2}\, dx + 0$$

$$\boxed{令\ \xi = \rho^- \xi^- - \rho^+ \xi^+}$$

$$= \int_{\mathbb{R}} \delta(\rho^- \xi^- - \rho^+ \xi^+) \frac{\partial \phi^+}{\partial \vec{n}} \sqrt{1+\eta_x^2}\, dx + 0$$

$$= \int_{\mathbb{R}} \delta \xi \frac{\partial \phi^-}{\partial \vec{n}} \sqrt{1+\eta_x^2}\, dx = \int_{\mathbb{R}} \delta \xi \frac{\partial \phi^+}{\partial \vec{n}} \sqrt{1+\eta_x^2}\, dx$$

$$\boxed{\begin{aligned} \frac{\delta H}{\partial \xi} &= \frac{\partial \phi^-}{\partial \vec{n}} \sqrt{1+\eta_x^2} = -\eta_x \phi_x^- + \phi_y^- = \eta_t \\ &= \frac{\partial \phi^+}{\partial \vec{n}} \sqrt{1+\eta_x^2} = -\eta_x \phi_x^+ + \phi_y^+ = \eta_t \end{aligned}}$$

at $y = \eta(x,t)$

$$\frac{\partial \phi^-}{\partial \vec{n}} = \vec{n} \cdot \nabla \phi^- = \frac{1}{\sqrt{1+\eta_x^2}} (-\eta_x, 1)(\phi_x^-, \phi_y^-)$$

$$= \frac{1}{\sqrt{1+\eta_x^2}} (-\eta_x \phi_x^-, \phi_y^-)$$

$$\boxed{\eta_t = \frac{\delta H}{\delta \xi}}$$

Var-5

$$\rho^-\left\{\xi_t^- - \phi_y^-(\phi_y^- - \eta_x\phi_x^-) + \frac{1}{2}|\nabla\phi^-|^2 + g\eta\right\} - \rho^+\left\{\xi_t^+ - \phi_y^+(\phi_y^+ - \eta_x\phi_x^+) + \frac{1}{2}|\nabla\phi^+|^2 + g\eta\right\} - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$(\rho^-\xi_t^- - \rho^+\xi_t^+) = \rho^-\left\{\phi_y^-(\phi_y^- - \eta_x\phi_x^-) - \frac{1}{2}|\nabla\phi^-|^2 + g\eta\right\} + \rho^+\left\{-\phi_y^+(\phi_y^+ - \eta_x\phi_x^+) + \frac{1}{2}|\nabla\phi^+|^2 + g\eta\right\} + \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}$$

$$\xi = \rho^-\xi^- - \rho^+\xi^+$$

$$\xi_t = \rho^-\left\{\phi_y^-(\phi_y^- - \eta_x\phi_x^-) - \frac{1}{2}|\nabla\phi^-|^2\right\} - \rho^- g\eta + \rho^+\left\{-\phi_y^+(\phi_y^+ - \eta_x\phi_x^+) + \frac{1}{2}|\nabla\phi^+|^2\right\} + \rho^+ g\eta + \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}$$

~~$\xi_t = \rho^- \{ 0$~~

$$\xi_t = -\rho^-\left\{(-\phi_y^- + \eta_x\phi_x^-)\phi_y^- + \frac{1}{2}|\nabla\phi^-|^2\right\} + \rho^+\left\{(-\phi_y^+ + \eta_x\phi_x^+)\phi_y^+ + \frac{1}{2}|\nabla\phi^+|^2\right\} - (\rho^- - \rho^+)g\eta + \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}$$

$$E_k = \frac{\rho^-}{2}\int_{\mathbb{R}}\int_{-h^-}^{\eta}|\nabla\phi^-|^2\,dx\,dy + \frac{\rho^+}{2}\int_{\mathbb{R}}\int_{\eta}^{h^+}|\nabla\phi^+|^2\,dx\,dy$$

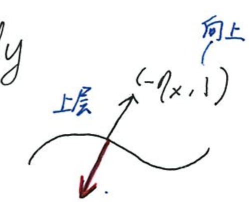
对 η 变分

$$\delta E_k = \frac{\rho^-}{2}\int_{\mathbb{R}}|\nabla\phi^-|^2\delta\eta\,dx + \rho^-\int_{\mathbb{R}}\int_{h^-}^{\eta}\nabla\phi^-\cdot\nabla\delta\phi\,dx\,dy$$
$$-\frac{\rho^+}{2}\int_{\mathbb{R}}|\nabla\phi^+|^2\delta\eta\,dx + \rho^+\int_{\mathbb{R}}\int_{\eta}^{h^+}\nabla\phi^+\cdot\nabla\delta\phi^+\,dx\,dy$$

$$\delta\int_{\mathbb{R}}\int_{-h}^{\eta}|\nabla\phi^-|^2\,dy\,dx$$
$$= \int_{\mathbb{R}}(|\nabla\phi^-|^2)_{y=\eta}\,\delta\eta\,dx + \int_{\mathbb{R}}\int_{-h}^{\eta}\frac{\delta|\nabla\phi^-|^2}{\partial\eta}\delta\eta\,dy\,dx$$
$$= \int_{\mathbb{R}}(|\nabla\phi^-|^2)_{y=\eta}\,\delta\eta\,dx$$
$$+ \int_{\mathbb{R}}\int_{-h}^{\eta} 2\nabla\phi^-\cdot\nabla\frac{\partial\phi^-}{\partial\eta}\delta\eta\,dy\,dx$$
$$= \int_{\mathbb{R}}(|\nabla\phi^-|^2)_{y=\eta}\,\delta\eta\,dx$$
$$+ \int_{\mathbb{R}}\int_{-h}^{\eta} 2\nabla\phi^-\cdot\nabla\delta\phi^-\,dy\,dx$$

Var-6

$$\delta E_k = \frac{\rho^-}{2}\int_{\mathbb{R}} |\nabla\phi^-|^2 \delta\eta\, dx - \frac{\rho^+}{2}\int_{\mathbb{R}} |\nabla\phi^+|^2 \delta\eta\, dx + \rho^-\int_{\mathbb{R}}\int_{h^-}^{\eta} \nabla\phi^-\cdot\nabla\delta\phi^-\, dx dy + \rho^+\int_{\mathbb{R}}\int_{\eta}^{h^+} \nabla\phi^+\cdot\nabla\delta\phi^+\, dx dy$$



$$\delta E_k = \frac{\rho^-}{2}\int_{\mathbb{R}} |\nabla\phi^-|^2 \delta\eta\, dx - \frac{\rho^+}{2}\int_{\mathbb{R}} |\nabla\phi^+|^2 \delta\eta\, dx + \rho^-\int_l \nabla\phi^-\cdot\vec{n}\,\delta\phi^-\, dl \mp \rho^+\int_l \nabla\phi^+\cdot\vec{n}\,\delta\phi^+\, dl$$

$$\boxed{\vec{n} = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1)}$$

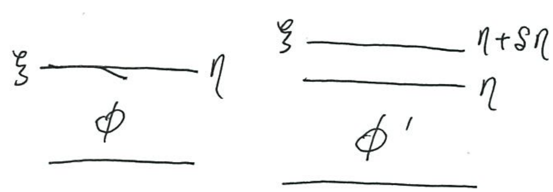
$\cancel{\nabla\phi^-\cdot\vec{n}}$

$$\boxed{\begin{aligned}\nabla\phi^-\cdot\vec{n} &= (\phi_x^-, \phi_y^-)\frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1) = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x\phi_x^- + \phi_y^-)\\ \nabla\phi^+\cdot\vec{n} &= (\phi_x^+, \phi_y^+)\frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1) = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x\phi_x^+ + \phi_y^+)\end{aligned}}$$

$$\delta E_k = \frac{\rho^-}{2}\int_{\mathbb{R}} |\nabla\phi^-|^2 \delta\eta\, dx - \frac{\rho^+}{2}\int_{\mathbb{R}} |\nabla\phi^+|^2 \delta\eta\, dx + \rho^-\int_{\mathbb{R}} \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x\phi_x^- + \phi_y^-)\underline{\sqrt{1+\eta_x^2}}\,\delta\phi^-\,\underline{dx} - \rho^+\int_{\mathbb{R}} \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x\phi_x^+ + \phi_y^+)\underline{\sqrt{1+\eta_x^2}}\,\delta\phi^+ \underline{d}$$

dl

$$\delta E_k = \frac{\rho^-}{2}\int_{\mathbb{R}} |\nabla\phi^-|^2 \delta\eta\, dx - \frac{\rho^+}{2}\int_{\mathbb{R}} |\nabla\phi^+|^2 \delta\eta\, dx + \rho^-\int_{\mathbb{R}} (-\eta_x\phi_x^- + \phi_y^-)\,\delta\phi^-\, dx - \rho^+\int_{\mathbb{R}} (-\eta_x\phi_x^+ + \phi_y^+)\,\delta\phi^+\, dx$$



$$\delta\phi(x, y=\eta, t)$$
$$\parallel$$
$$\phi'(x, y=\eta, t) - \phi(x, y=\eta, t)$$
$$= \phi'(x, y=\eta, t) - \phi'(x, y=\eta+\delta\eta, t)$$
$$= -\frac{\partial\phi'}{\partial y}\delta\eta\Big|_{y=\eta} = -\frac{\partial\phi}{\partial y}\delta\eta\Big|_{y=\eta} \quad (\delta\eta\to 0)$$
$$\delta\phi = -\frac{\partial\phi}{\partial y}\delta\eta\Big|_{y=\eta}$$

Var-7

$$\delta E_k = \frac{\rho^-}{2}\int_{\mathbb{R}} |\nabla\phi^-|^2 \delta\eta\, dx - \frac{\rho^+}{2}\int_{\mathbb{R}} |\nabla\phi^+|^2 \delta\eta\, dx + \rho^-\int_{\mathbb{R}} (-\eta_x\phi_x^- + \phi_y^-)\left(-\frac{\partial\phi^-}{\partial y}\delta\eta\right)dx - \rho^+\int_{\mathbb{R}} (-\eta_x\phi_x^+ + \phi_y^+)\left(-\frac{\partial\phi^+}{\partial y}\delta\eta\right)dx$$

$$\delta E_k = \frac{\rho^-}{2}\int_{\mathbb{R}} |\nabla\phi^-|^2 \delta\eta\, dx - \frac{\rho^+}{2}\int_{\mathbb{R}} |\nabla\phi^+|^2 \delta\eta\, dx + \rho^-\int_{\mathbb{R}} (-\phi_y^- + \eta_x\phi_x^-)\frac{\partial\phi^-}{\partial y}\delta\eta\, dx - \rho^+\int_{\mathbb{R}} (-\phi_y^+ + \eta_x\phi_x^+)\frac{\partial\phi^+}{\partial y}\delta\eta\, dx$$

$$\frac{\delta E_k}{\delta\eta} = \frac{\rho^-}{2}|\nabla\phi^-|^2 - \frac{\rho^+}{2}|\nabla\phi^+|^2 + \rho^-(-\phi_y^- + \eta_x\phi_x^-)\frac{\partial\phi^-}{\partial y} - \rho^+(-\phi_y^+ + \eta_x\phi_x^+)\frac{\partial\phi^+}{\partial y}$$

$$E_p = \frac{1}{2}g(\rho^- - \rho^+)\int_{\mathbb{R}} \eta^2 dx + \sigma\int_{\mathbb{R}} \left(\sqrt{1+\eta_x^2} - 1\right)dx$$

$$\frac{\delta E_p}{\delta\eta} = g(\rho^- - \rho^+)\eta - \frac{\sigma\eta_{xx}}{\sqrt{1+\eta_x^2}}$$

$$\frac{\delta H}{\delta\eta} = \frac{\delta(E_k + E_p)}{\delta\eta} = \frac{\rho^-}{2}|\nabla\phi^-|^2 + \frac{\rho^+}{2}|\nabla\phi^+|^2 + \rho^-(-\phi_y^- + \eta_x\phi_x^-)\phi_y^- - \rho^+(-\phi_y^+ + \eta_x\phi_y^+)\phi_y^+$$
$$+ g(\rho^- - \rho^+)\eta - \frac{\sigma\eta_{xx}}{\sqrt{1+\eta_x^2}} = -\xi_t$$

$$\Rightarrow \boxed{\xi_t = -\frac{\delta H}{\delta\eta}}$$

DtN-1

$$\begin{cases} \phi^-_{xx} + \phi^-_{yy} = 0, & -h^- < y < \eta \\ \phi^-_y = 0, & y = -h^- \\ \phi^- = \xi^-, & y = \eta \end{cases}$$

C

下层DtN算子的定义 $\boxed{G^-\xi^- = (\phi_y^- - \eta_x \phi_x^-)\big|_{y=\eta(x,t)} = \frac{\partial \phi^-}{\partial \vec{n}}\sqrt{1+\eta_x^2}}$

$$\vec{n} = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1) \begin{cases} \frac{\partial \phi^-}{\partial \vec{n}} = \vec{n}\cdot\nabla\phi^- = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1)(\phi_x^-, \phi_y^-) = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x\phi_x^- + \phi_y^-) \\ \frac{\partial \phi^+}{\partial \vec{n}} = \vec{n}\cdot\nabla\phi^+ = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x, 1)(\phi_x^+, \phi_y^+) = \frac{1}{\sqrt{1+\eta_x^2}}(-\eta_x\phi_x^+ + \phi_y^+) \end{cases}$$

$$\begin{cases} \phi^+_{xx} + \phi^+_{yy} = 0, & \eta < y < h^+ \\ \phi^+_y = 0, & y = h^+ \\ \phi^+ = \xi^+, & y = \eta \end{cases}$$

上层DtN算子的定义 $\boxed{G^-\xi^+ = \cancel{\phi_y^+}(\eta_x \phi_x^+ - \phi_y^+)\big|_{y=\eta(x,t)} = -\frac{\partial \phi^+}{\partial \vec{n}}\sqrt{1+\eta_x^2}}$

运动学边界条件：
$$\boxed{\begin{aligned} \eta_t &= (\phi_y^- - \eta_x\phi_x^-)\big|_{y=\eta(x,t)} = G^-\xi^- \\ \eta_t &= (\phi_y^+ - \eta_x\phi_x^+)\big|_{y=\eta(x,t)} = -G^+\xi^+ \end{aligned}} \Rightarrow \boxed{\eta_t = G^-\xi^- = -G^+\xi^+}$$

令 $\xi = \rho^-\xi^- - \rho^+\xi^+ \Rightarrow G^+\xi = \rho^- G^+\xi^- - \rho^+ G^+\xi^+ = \rho^- G^+\xi^- + \rho^+ G^-\xi^-$

$$G^+\xi = (\rho^- G^+ + \rho^+ G^-)\xi^- \Rightarrow \boxed{\xi^- = (\rho^- G^+ + \rho^+ G^-)^{-1} G^+\xi}$$

$$\eta_t = G^-\xi^- = -G^+\xi^+$$

令 $\xi = \rho^-\xi^- - \rho^+\xi^+$

$$G^+\xi = G^+(\rho^-\xi^- - \rho^+\xi^+) = \rho^-G^+\xi^- - \rho^+G^+\xi^+ = \rho^-G^+\xi^- + \rho^+G^-\xi^-$$

$$G^+\xi = (\rho^-G^+ + \rho^+G^-)\xi^-$$

$$\Rightarrow \boxed{\cancel{\xi^+ =}\ \xi^- = (\rho^-G^+ + \rho^+G^-)^{-1}G^+\xi}$$

$$\Rightarrow \boxed{\eta_t = G^-\xi^- = G^-(\rho^-G^+ + \rho^+G^-)^{-1}G^+\xi} \longrightarrow \text{运动学边界条件}$$

$$G^-\xi = G^-(\rho^-\xi^- - \rho^+\xi^+) = \rho^-G^-\xi^- - \rho^+G^-\xi^+ = -\rho^-G^+\xi^+ - \rho^+G^-\xi^+ = -(\rho^-G^+ + \rho^+G^-)\xi^+$$

$$\Rightarrow G^-\xi = -(\rho^-G^+ + \rho^+G^-)\xi^+ \quad \Rightarrow \quad \boxed{\xi^+ = -(\rho^-G^+ + \rho^+G^-)^{-1}G^-\xi}$$

D+N-3

$$\rho^-\left\{\xi_t^- - \phi_y^-(\phi_y^- - \eta_x\phi_x^-) + \frac{1}{2}(\phi_x^-)^2 + \frac{1}{2}(\phi_y^-)^2 + g\eta\right\} - \rho^+\left\{\xi_t^+ - \phi_y^+(\phi_y^+ - \eta_x\phi_x^+) + \frac{1}{2}(\phi_x^+)^2 + \frac{1}{2}(\phi_y^+)^2 + g\eta\right\} - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$\rho^-\left\{\xi_t^- - (\phi_y^-)^2 + \eta_x\phi_x^-\phi_y^- + \frac{1}{2}(\phi_x^-)^2 + \frac{1}{2}(\phi_y^-)^2 + g\eta\right\} - \rho^+\left\{\xi_t^+ - (\phi_y^+)^2 + \eta_x\phi_x^+\phi_y^+ + \frac{1}{2}(\phi_x^+)^2 + \frac{1}{2}(\phi_y^+)^2 + g\eta\right\} - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$\rho^-\left\{\xi_t^- + \frac{1}{2}(\phi_x^-)^2 - \frac{1}{2}(\phi_y^-)^2 + \phi_x^-\phi_y^-\eta_x + g\eta\right\} - \rho^+\left\{\xi_t^+ + \frac{1}{2}(\phi_x^+)^2 - \frac{1}{2}(\phi_y^+)^2 + \phi_x^+\phi_y^+\eta_x + g\eta\right\} - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

Note: $\eta_t = \phi_y^\pm - \eta_x\phi_x^\pm$ $\quad \xi^\pm = \phi^\pm(x, y=\eta(x,t), t)$ $\quad \xi_x^\pm = \phi_x^\pm + \phi_y^\pm\eta_x$

$$\Rightarrow \begin{cases} \phi_x^\pm + \eta_x\phi_y^\pm = \xi_x^\pm & ① \\ -\eta_x\phi_x^\pm + \phi_y^\pm = \eta_t & ② \end{cases}$$

$\eta_x ① + ② \Rightarrow \eta_x\phi_x^\pm + \eta_x^2\phi_y^\pm - \eta_x\phi_x^\pm + \phi_y^\pm = \eta_x\xi_x^\pm + \eta_t$

$$\Rightarrow (1+\eta_x^2)\phi_y^\pm = \eta_t + \eta_x\xi_x^\pm \Rightarrow \phi_y^\pm = \frac{\eta_t + \eta_x\xi_x^\pm}{1+\eta_x^2}$$

$① - \eta_x ② \Rightarrow \phi_x^\pm + \cancel{\eta_x\phi_y^\pm} + \eta_x^2\phi_x^\pm - \cancel{\eta_x\phi_y^\pm} = \xi_x^\pm - \eta_t\eta_x$

$$(1+\eta_x^2)\phi_x^\pm = \xi_x^\pm - \eta_t\eta_x$$

$$\boxed{\phi_x^\pm = \frac{\xi_x^\pm - \eta_t\eta_x}{1+\eta_x^2} \qquad \phi_y^\pm = \frac{\eta_t + \eta_x\xi_x^\pm}{1+\eta_x^2}}$$

$$\rho^-\left[\xi_t^- + \frac{1}{2}(\phi_x^-)^2 - \frac{1}{2}(\phi_y^-)^2 + \phi_y^-\phi_x^-\eta_x\right] - \rho^+\left[\xi_t^+ + \frac{1}{2}(\phi_x^+)^2 - \frac{1}{2}(\phi_y^+)^2 + \phi_y^+\phi_x^+\eta_x\right]$$

$$= \rho^-\left[\xi_t^- + \frac{1}{2}\left(\frac{\xi_x^- - \eta_t\eta_x}{1+\eta_x^2}\right)^2 - \frac{1}{2}\left(\frac{\eta_t+\eta_x\xi_x^-}{1+\eta_x^2}\right)^2 + \frac{\xi_x^- - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\eta_t+\eta_x\xi_x^-}{1+\eta_x^2}\cdot\eta_x\right]$$

$$-\rho^+\left[\xi_t^+ + \frac{1}{2}\left(\frac{\xi_x^+ - \eta_t\eta_x}{1+\eta_x^2}\right)^2 - \frac{1}{2}\left(\frac{\eta_t+\eta_x\xi_x^+}{1+\eta_x^2}\right)^2 + \frac{\xi_x^+ - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\eta_t+\eta_x\xi_x^+}{1+\eta_x^2}\cdot\eta_x\right]$$

$$= \rho^-\left[\xi_t^- + \frac{1}{2}\left(\frac{\xi_x^- - \eta_t\eta_x}{1+\eta_x^2}\right)^2 - \frac{1}{2}\left(\frac{\eta_t+\eta_x\xi_x^-}{1+\eta_x^2}\right)^2 + \frac{1}{2}\,\frac{\xi_x^- - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\eta_t+\eta_x\xi_x^-}{1+\eta_x^2}\cdot\eta_x + \frac{1}{2}\,\frac{\xi_x^- - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\eta_t+\eta_x\xi_x^-}{1+\eta_x^2}\cdot\eta_x\right]$$

$$-\rho^+\left[\xi_t^+ + \frac{1}{2}\left(\frac{\xi_x^+ - \eta_t\eta_x}{1+\eta_x^2}\right)^2 - \frac{1}{2}\left(\frac{\eta_t+\eta_x\xi_x^+}{1+\eta_x^2}\right)^2 + \frac{1}{2}\,\frac{\xi_x^+ - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\eta_t+\eta_x\xi_x^+}{1+\eta_x^2}\cdot\eta_x + \frac{1}{2}\,\frac{\xi_x^+ - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\eta_t+\eta_x\xi_x^+}{1+\eta_x^2}\cdot\eta_x\right]$$

$$= \rho^-\left[\xi_t^- + \frac{1}{2}\left(\frac{\xi_x^- - \eta_t\eta_x}{1+\eta_x^2}\right)\left(\frac{\xi_x^- \cancel{-\eta_t\eta_x} \cancel{+\eta_t\eta_x} + \eta_x^2\xi_x^-}{1+\eta_x^2}\right) - \frac{1}{2}\left(\frac{\eta_t+\eta_x\xi_x^-}{1+\eta_x^2}\right)\left(\frac{\eta_t + \cancel{\eta_x\xi_x^-} - \cancel{\xi_x^-\eta_x} + \eta_t\eta_x^2}{1+\eta_x^2}\right)\right]$$

$$-\rho^+\left[\xi_t^+ + \frac{1}{2}\left(\frac{\xi_x^+ - \eta_t\eta_x}{1+\eta_x^2}\right)\left(\frac{\xi_x^+ \cancel{-\eta_t\eta_x} \cancel{+\eta_t\eta_x} + \eta_x^2\xi_x^-}{1+\eta_x^2}\right) - \frac{1}{2}\left(\frac{\eta_t+\eta_x\xi_x^-}{1+\eta_x^2}\right)\left(\frac{\eta_t + \cancel{\eta_x\xi_x^+} - \cancel{\xi_x^+\eta_x} + \eta_t\eta_x^2}{1+\eta_x^2}\right)\right]$$

DtN-4

DtN-5

$$= \rho^-\left[\xi_t^- + \frac{1}{2}\frac{\xi_x^- - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\xi_x^- + \eta_x^2\xi_x^-}{1+\eta_x^2} - \frac{1}{2}\frac{\eta_t+\eta_x\xi_x^-}{1+\eta_x^2}\,\frac{\eta_t+\eta_t\eta_x^2}{1+\eta_x^2}\right]$$

$$- \rho^+\left[\xi_t^+ + \frac{1}{2}\frac{\xi_x^+ - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\xi_x^+ + \eta_x^2\xi_x^+}{1+\eta_x^2} - \frac{1}{2}\frac{\eta_t+\eta_x\xi_x^+}{1+\eta_x^2}\,\frac{\eta_t+\eta_t\eta_x^2}{1+\eta_x^2}\right]$$

$$= \rho^-\left[\xi_t^- + \frac{1}{2}\frac{\xi_x^- - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\cancel{(1+\eta_x^2)}\,\xi_x^-}{\cancel{1+\eta_x^2}} - \frac{1}{2}\frac{\eta_t+\eta_x\xi_x^-}{1+\eta_x^2}\,\frac{\eta_t\cancel{(1+\eta_x^2)}}{\cancel{1+\eta_x^2}}\right]$$

$$- \rho^+\left[\xi_t^+ + \frac{1}{2}\frac{\xi_x^+ - \eta_t\eta_x}{1+\eta_x^2}\,\frac{\cancel{(1+\eta_x^2)}\,\xi_x^+}{\cancel{1+\eta_x^2}} - \frac{1}{2}\frac{\eta_t+\eta_x\xi_x^+}{1+\eta_x^2}\,\frac{\eta_t\cancel{(1+\eta_x^2)}}{\cancel{1+\eta_x^2}}\right]$$

$$= \rho^-\left[\xi_t^- + \frac{1}{2}\frac{(\xi_x^- - \eta_t\eta_x)\xi_x^-}{1+\eta_x^2} - \frac{1}{2}\frac{(\eta_t+\eta_x\xi_x^-)\eta_t}{1+\eta_x^2}\right] - \rho^+\left[\xi_t^+ + \frac{1}{2}\frac{(\xi_x^+ - \eta_t\eta_x)\xi_x^+}{1+\eta_x^2} - \frac{1}{2}\frac{(\eta_t+\eta_x\xi_x^+)\eta_t}{1+\eta_x^2}\right]$$

$$= \rho^-\left[\xi_t^- + \frac{1}{2(1+\eta_x^2)}\left((\xi_x^-)^2 - \eta_t\eta_x\xi_x^- - (\eta_t)^2 - \eta_x\eta_t\xi_x^-\right)\right] - \rho^+\left[\xi_t^+ + \frac{1}{2(1+\eta_x^2)}\left((\xi_x^+)^2 - \eta_t\eta_x\xi_x^+ - (\eta_t)^2 - \eta_x\eta_t\xi_x^+\right)\right]$$

$$= \rho^-\left[\xi_t^- + \frac{1}{2(1+\eta_x^2)}\left((\xi_x^-)^2 - (\eta_t)^2 - 2\eta_t\eta_x\xi_x^- - (\eta_x\xi_x^-)^2 + (\eta_x\xi_x^-)^2\right)\right]$$

$$- \rho^+\left[\xi_t^+ + \frac{1}{2(1+\eta_x^2)}\left((\xi_x^+)^2 - (\eta_t)^2 - 2\eta_t\eta_x\xi_x^+ - (\eta_x\xi_x^+)^2 + (\eta_x\xi_x^+)^2\right)\right]$$

$$= \rho^- \left[\xi_t^- + \frac{1}{2(1+\eta_x^2)} \left((\xi_x^-)^2 (1+\eta_x^2) - (\eta_t + \eta_x \xi_x^-)^2 \right) \right] - \rho^+ \left[\xi_t^+ + \frac{1}{2(1+\eta_x^2)} \left((\xi_x^+)^2 (1+\eta_x^2) - (\eta_t + \eta_x \xi_x^+)^2 \right) \right]$$

$$= \rho^- \left[\xi_t^- + \frac{(\xi_x^-)^2}{2} - \frac{(\eta_t + \eta_x \xi_x^-)^2}{2(1+\eta_x^2)} \right] - \rho^+ \left[\xi_t^+ + \frac{(\xi_x^+)^2}{2} - \frac{(\eta_t + \eta_x \xi_x^+)^2}{2(1+\eta_x^2)} \right]$$

$$\Rightarrow \rho^- \left[\xi_t^- + \frac{1}{2}(\xi_x^-)^2 - \frac{(\eta_t + \eta_x \xi_x^-)^2}{2(1+\eta_x^2)} \right] + \rho^- g\eta - \rho^+ \left[\xi_t^+ + \frac{1}{2}(\xi_x^+)^2 - \frac{(\eta_t + \eta_x \xi_x^+)^2}{2(1+\eta_x^2)} \right] - \rho^+ g\eta - \frac{\sigma \eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$\Rightarrow \rho^- \left[\xi_t^- + \frac{1}{2}(\xi_x^-)^2 - \frac{(\eta_t + \eta_x \xi_x^-)^2}{2(1+\eta_x^2)} \right] - \rho^+ \left[\xi_t^+ + \frac{1}{2}(\xi_x^+)^2 - \frac{(\eta_t + \eta_x \xi_x^+)^2}{2(1+\eta_x^2)} \right] + (\rho^- - \rho^+) g\eta - \frac{\sigma \eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$\Rightarrow \quad \xi = \rho^- \xi^- - \rho^+ \xi^+$$

$$\boxed{\xi_t + \frac{1}{2}\rho^- \left[(\xi_x^-)^2 - \frac{(\eta_t + \eta_x \xi_x^-)^2}{(1+\eta_x)^2} \right] - \frac{1}{2}\rho^+ \left[(\xi_x^+)^2 - \frac{(\eta_t + \eta_x \xi_x^+)^2}{(1+\eta_x^2)} \right] + (\rho^- - \rho^+) g\eta - \frac{\sigma \eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0}$$

DHB-7

总结：$\eta_t = G^-(\rho^- G^+ + \rho^+ G^-)^{-1} G^+ \xi$

$$\xi_t + \frac{1}{2}\rho^-\left[(\xi_x^-)^2 - \frac{(\eta_t + \eta_x \xi_x^-)^2}{(1+\eta_x^2)}\right] - \frac{1}{2}\rho^+\left[(\xi_x^+)^2 - \frac{(\eta_t+\eta_x\xi_x^+)^2}{(1+\eta_x^2)}\right] + (\rho^- - \rho^+)g\eta - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$\xi^- = (\rho^- G^+ + \rho^+ G^-)^{-1} G^+ \xi$$

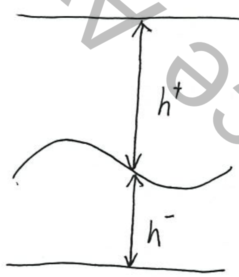
$$\xi^+ = -(\rho^- G^+ + \rho^+ G^-)^{-1} G^- \xi$$

$$\xi = \rho^- \xi^- - \rho^+ \xi^+$$

D

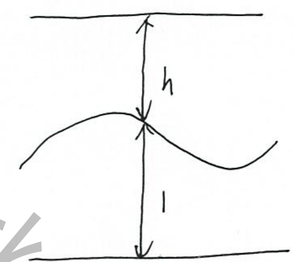
无量纲：长度 h^-，速度 $\sqrt{gh^-}$，势函数 $\sqrt{g(h^-)^3}$，时间 $\frac{h^-}{\sqrt{gh^-}} = \sqrt{\frac{h^-}{g}}$

Dim
~~DIN~~-1



$(h^+)^* = \frac{h^+}{h^-} = h$

$(h^-)^* = \frac{h^-}{h^-} = 1$



$(h^+)^* = h$

$(h^-)^* = 1$

① $\xi = \rho^- \xi^- - \rho^+ \xi^+$

$\xi = \xi^* \rho^- \sqrt{g(h^-)^3}$ $\quad \xi^- = \sqrt{g(h^-)^3}\,(\xi^-)^*$ $\quad \xi^+ = \sqrt{g(h^-)^3}\,(\xi^+)^*$

$\Rightarrow \rho^- \sqrt{g(h^-)^3}\,\xi^* = \rho^- \sqrt{g(h^-)^3}\,(\xi^-)^* - \rho^+ \sqrt{g(h^-)^3}\,(\xi^+)^*$

$\xi^* = (\xi^-)^* - \frac{\rho^+}{\rho^-}(\xi^+)^* = \xi^- - \rho\,\xi^+$

为方便，省略

Dim-2

$$\xi^- = (\rho^- G^+ + \rho^+ G^-)^{-1} G^+ \xi$$

$$\xi^- = \sqrt{g(h^-)^3}\,(\xi^-)^*$$

$$\xi = \rho^- \sqrt{g(h^-)^3}\,(\xi)^*$$

$$(G^+) = \frac{1}{h^-}(G^+)^*(G^-) = \frac{1}{h^-}(G^-)^*$$

$$\sqrt{g\cancel{(h^-)^3}}\,(\xi^-)^* = \left(\rho^- \cancel{\frac{1}{h^-}}(G^+)^* + \rho^+ \cancel{\frac{1}{h^-}}(G^-)^*\right)^{-1} \cancel{\frac{1}{h^-}}(G^+)^* \rho^- \sqrt{g\cancel{(h^-)^3}}\,(\xi)^*$$

$$(\xi^-)^* = \left((G^+)^* + \frac{\rho^+}{\rho^-}(G^-)^*\right)^{-1}(G^+)^*(\xi)^*$$

$$(\xi^-)^* = \left((G^+)^* + \rho(G^-)^*\right)^{-1}(G^+)^*(\xi)^*$$

$$\eta_t = G^-(\rho^- G^+ + \rho^+ G^-)^{-1} G^+ \xi$$

$$\eta_t = \frac{h^-}{\sqrt{\frac{h^-}{g}}}\eta^*_{t^*} = \sqrt{gh^-}\,\eta^*_{t^*}$$

$$\sqrt{gh^-}\,\eta^*_{t^*} = \frac{1}{h^-}(G^-)^*\left(\rho^- \frac{1}{h^-}(G^+)^* + \rho^+ \frac{1}{h^-}(G^-)^*\right)^{-1} \frac{1}{h^-}(G^+)^* \rho^- \sqrt{g(h^-)^3}\,(\xi^*)$$

$$\eta^*_{t^*} = (G^-)^*\left((G^+)^* + \frac{\rho^+}{\rho^-}(G^-)^*\right)^{-1}(G^+)^* \xi^*$$

Dim-3

$$\xi_t + \frac{1}{2}\rho^-\left[(\xi_x^-)^2 - \frac{(\eta_t+\eta_x\xi_x^-)^2}{(1+\eta_x^2)}\right] - \frac{1}{2}\rho^+\left[(\xi_x^+)^2 - \frac{(\eta_t+\eta_x\xi_x^+)^2}{1+\eta_x^2}\right] + (\rho^- - \rho^+)\eta g - \frac{\sigma\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$\rho^-(\sqrt{gh^-})^2\xi^*_{t^*} + \frac{1}{2}\rho^-(\sqrt{gh^-})^2\left[(\xi^{-*}_{x^*})^2 - \frac{(\eta^*_{t^*}+\eta^*_{t^*}\xi^{-*}_{x^*})^2}{(1+\eta^{*\,2}_{x^*})}\right]$$

$$-\frac{1}{2}\rho^+(\sqrt{gh^-})^2\left[(\xi^{+*}_{x^*})^2 - \frac{(\eta^*_{t^*}+\eta^*_{t^*}\xi^{+*}_{x^*})^2}{1+\eta^{*\,2}_{x^*}}\right] + (\rho^- - \rho^+)h^-\eta^* g - \frac{\sigma\frac{1}{h^-}\eta^*_{x^*x^*}}{(1+\eta^{*\,2}_{x^*})^{\frac{3}{2}}} = 0$$

$$\xi^*_{t^*} + \frac{1}{2}\left[(\xi^{-*}_{x^*})^2 - \frac{(\eta^*_{t^*}+\eta^*_{t^*}\xi^{-*}_{x^*})^2}{1+\eta^*_{x^*}}\right] - \frac{1}{2}\left[(\xi^{+*}_{x^*})^2 - \frac{(\eta^*_{t^*}+\eta^*_{t^*}\xi^{+*}_{x^*})}{1+\eta^{*\,2}_{x^*}}\right] - \left(1-\frac{\rho^+}{\rho^-}\right)\eta$$

$$-\frac{\sigma}{\rho^- g h^{-2}}\frac{\eta^*_{x^*x^*}}{(1+\eta^{*\,2}_{x^*})^{\frac{3}{2}}} = 0$$

$$\rho = \frac{\rho^+}{\rho^-} \qquad B = \frac{\sigma}{\rho^- g h^{-2}}$$

$G_0^- = D\tanh(h^- D)$ $\quad G_0^- = D\tanh(h^- D)$

$$D^* = h^- D \implies D = \frac{1}{h^-} D^*$$

$$(G_0^-)^* = \frac{1}{h^-} D^* \tanh\left(h^- \frac{1}{h^-} D^*\right) = \frac{1}{h^-} D^* \tanh(D^*)$$

$$G_0^+ = D\tanh(h^+ D)$$

$$(G_0^+)^* = \frac{1}{h^-} D \tanh\left(h^+ \frac{1}{h^-} D^*\right) = \frac{1}{h^-} D^* \tanh(hD^*)$$

Dim-4

DtN算子

$$(\star)\begin{cases}\phi^-_{xx}+\phi^-_{yy}=0, & -h^-<y<\eta\\ \phi^-_y=0, & y=-h^-\\ \phi^-=\xi^-, & y=\eta\end{cases}$$

$\phi_k^- = \cosh(k(y+h^-))e^{ikx}$ 是 $(\star)$的一个解

$$\boxed{\text{DtN定义: } G^-\xi_k^- = G^-\phi_k^-(x,y=\eta) = \left(\frac{\partial\phi_k^-}{\partial y} - \eta_x\frac{\partial\phi_k^-}{\partial x}\right)\Big|_{y=\eta}}$$

$$\underbrace{G^-\phi_k^-(x,y=\eta)}_{\text{左边}} = \underbrace{\left(\frac{\partial\phi_k^-}{\partial y} - \eta_x\frac{\partial\phi_k^-}{\partial x}\right)}_{\text{右边}}\Big|_{y=\eta}$$

$G^-(\eta) = \sum_{n=0}^{\infty} G_n^-(\eta)$，$n$为$\eta$的次数

左边：$G^-\phi_k^-(x,y=\eta) = G^-\cosh(k\eta+kh^-)e^{ikx} = G^-[\cosh(k\eta)\cosh(kh^-)+\sinh(k\eta)\sinh(kh^-)]e^{ikx}$

泰勒展开：$\left(\sum_{l=0}^{\infty} G_l^-(\eta)\right)\left(\sum_{j\,\text{even}} \frac{1}{j!}(k\eta)^j\cosh(kh^-)e^{ikx} + \sum_{j\,\text{odd}}\frac{1}{j!}(k\eta)^j\sinh(kh^-)e^{ikx}\right)$

右边：$\left(\frac{\partial\phi_k^-}{\partial y} - \eta_x\frac{\partial\phi_k^-}{\partial x}\right)_{y=\eta} = \left[\frac{\partial}{\partial y}\cosh(k(y+h^-))e^{ikx} - \eta_x\frac{\partial}{\partial x}\cosh(k(y+h^-))e^{ikx}\right]_{y=\eta}$

右 $= \left[k\sinh(k(y+h^-))e^{ikx} - ik\cosh(k(y+h^-))\eta_x e^{ikx}\right]_{y=\eta}$

右 $= \left[k\sinh(k\eta+kh^-)e^{ikx} - ik\cosh(k\eta+kh^-)\eta_x e^{ikx}\right]$

右 $= [(k\sinh(k\eta)\cosh(kh^-)e^{ikx} + k\cosh(k\eta)\sinh(kh^-)e^{ikx}) - ik\cosh(k\eta)\cosh(kh^-)\eta_x e^{ikx} - ik\sinh(k\eta)\sinh(k$
$\eta_x e^{ikx}]$

For $j=0$，我们让 $\eta=0$

$$G_{(0)}^{-}\cosh(kh^{-})e^{ikx}=k\sinh(kh^{-})e^{ikx} \Rightarrow G_{(0)}^{-}e^{ikx}=k\tanh(kh^{-})e^{ikx}$$

$$\boxed{\text{定义 Fourier 变换对} \begin{cases} \zeta(x)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}\hat{\zeta}e^{ikx}dk \\ \hat{\zeta}(x)=\displaystyle\int_{-\infty}^{+\infty}\zeta(x)e^{-ikx}dx \end{cases}}$$

$$G_{(0)}^{-}\zeta(x)=\frac{1}{2\pi}G_{(0)}^{-}\int_{-\infty}^{+\infty}\hat{\zeta}e^{ikx}dk=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\zeta}_{(k)}G_{(0)}^{-}e^{ikx}dk$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\zeta}_{(k)}k\tanh(kh^{-})e^{ikx}dk$$

于是 $\widehat{G_{(0)}^{-}}=k\tanh(kh^{-})$ 我们让 $\boxed{D=-i\partial_x}$，$\hat{D}=-i\widehat{\partial_x}=-i(ik)=k$

$$\Rightarrow G_{(0)}^{-}=D\tanh(Dh^{-}) \qquad \boxed{G_0^{-}=D\tanh(h^{-}D)}$$

$$右 = k\sinh(k\eta)\cosh(kh^-)e^{ikx} + k\cosh(k\eta)\sinh(kh^-)e^{ikx} - ik\cosh(k\eta)\cosh(kh^-)\eta_x e^{ikx} - ik\sinh(k\eta)\sinh(kh^-)\eta_x e^{ikx}$$

$$右 = \cosh(k\eta)\left(k\sinh(kh^-) - ik\cosh(kh^-)\eta_x\right)e^{ikx} + \sinh(k\eta)\left(k\cosh(kh^-) - ik\sinh(kh^-)\eta_x\right)e^{ikx}$$

$$右 = \sum_{j\ \mathrm{even}} \frac{1}{j!}(k\eta)^j\left(k\sinh(kh^-) - ik\cosh(kh^-)\eta_x\right)e^{ikx} + \sum_{j\ \mathrm{odd}} \frac{1}{j!}(k\eta)^j\left(k\cosh(kh^-) - ik\sinh(kh^-)\eta_x\right)e^{ikx}$$

$$右 = \sum_{j\ \mathrm{even}} \frac{1}{j!}(k\eta)^j\left(k\sin(kh^-) - ik\eta_x\cosh(kh^-)\right)e^{ikx} + \sum_{j\ \mathrm{odd}} \frac{1}{j!}(k\eta)^j\left(k\cosh(kh^-) - ik\eta_x\sinh(kh^-)\right)e^{ikx}$$

$$\left(\sum_{l=0}^{\infty} G_l(\eta)\right)\left(\sum_{j\ \mathrm{even}} \frac{1}{j!}(k\eta)^j\cosh(kh^-)e^{ikx} + \sum_{j\ \mathrm{odd}} \frac{1}{j!}(k\eta)^j\sinh(kh^-)e^{ikx}\right) — 左$$

$$= \sum_{j\ \mathrm{even}} \frac{1}{j!}(k\eta)^j\left(k\sinh(kh^-) - ik\eta_x\cos(kh^-)\right)e^{ikx} + \sum_{j\ \mathrm{odd}} \frac{1}{j!}(k\eta)^j\left(k\cosh(kh^-) - ik\eta_x\sinh(kh^-)\right)e^{ikx} — 右$$

For $j>0$ 柯西乘积公式 $\left(\sum_{n=0}^{\infty} a_n\right)\cdot\left(\sum_{n=0}^{\infty} b_n\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_n b_{n-k}$

$$\left(\sum_{l=0}^{\infty} a_l\right)\cdot\left(\sum_{j=0}^{\infty} b_j\right)=\sum_{j=0}^{\infty}\sum_{l=0}^{j} a_l b_{j-l}$$

$$左=\sum_{j=0}^{\infty}\sum_{l=0}^{j} a_l^-\left[\underbrace{\frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^-)e^{ikx}}_{(j-l)\text{为even}}+\underbrace{\frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^-)e^{ikx}}_{(j-l)\text{为odd}}\right]$$

$$左=\sum_{j=0}^{\infty}\left[\underbrace{\sum_{l<j}}_{l<j} G_l^-\left[\frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^-)e^{ikx}+\frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^-)e^{ikx}\right]+\underbrace{G_j^-}_{l=j}\cosh(kh^-)e^{ikx}\right]$$

当 $j>0$，j 为 even 时

$$\underbrace{\sum_{l<j,\ l\text{ even}}}_{j-l\text{ even}} G_l^-\frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^-)e^{ikx}+\underbrace{\sum_{l<j,\ l\text{ odd}}}_{j-l\text{ odd}}\frac{1}{(j-l)!}G_l^-(k\eta)^{j-l}\sinh(kh^-)e^{ikx}+G_j^-\cosh(kh^-)e^{ikx}$$

当 $j>0$，j 为 odd 时

$$\underbrace{\sum_{l<j,\ l\text{ even}}}_{j-l\text{ odd}} G_l^-\frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^-)e^{ikx}+\underbrace{\sum_{l<j,\ l\text{ odd}}}_{j-l\text{ even}}\frac{1}{(j-l)!}G_l^-(k\eta)^{j-l}\cosh(kh^-)e^{ikx}+G_j^-\cosh(kh^-)e^{ikx}$$

$$右 = \sum_{j\,\text{even}} \frac{1}{j!}(k\eta)^j\left(k\sinh(kh^-) - ik\eta_x\cosh(kh^-)\right)e^{ikx} + \sum_{j\,\text{odd}} \frac{1}{j!}(k\eta)^j\left(k\cosh(kh^-) - ik\eta_x\sinh(kh^-)\right)e^{ikx}$$

当 $j>0$，j 是 even

$$\sum_{l<j,\,l\,\text{even}} G_l^- \frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^-)e^{ikx} + \sum_{l<j,\,l\,\text{odd}} \frac{1}{(j-l)!} G_l^-(k\eta)^{j-l}\sinh(kh^-)e^{ikx} + G_j^-\cosh(kh^-)e^{ikx}$$

$$= \frac{1}{j!}(k\eta)^j k\sinh(kh^-)e^{ikx} + \frac{1}{(j-1)!}(k\eta)^{j-1}(-ik)\eta_x\sinh(kh^-)e^{ikx}$$

$$G_j^-\cosh(kh^-)e^{ikx} = \frac{1}{j!}\eta^j k^{j+1}\sinh(kh^-)e^{ikx} - i\frac{1}{(j-1)!}\eta^{j-1}\eta_x k^j\sinh(kh^-)e^{ikx}$$

$$- \sum_{l<j,\,l\,\text{even}} G_l^- \frac{1}{(j-l)!} k^{j-l}\eta^{j-l}\cosh(kh^-)e^{ikx} - \sum_{l<j,\,l\,\text{odd}} \frac{1}{(j-l)!} G_l^-\eta^{j-l}k^{j-l}\sinh(kh^-)e^{ikx}$$

$$G_j^- e^{ikx} = \frac{1}{j!}\eta^j k^{j+1}\tanh(kh^-)e^{ikx} - i\frac{1}{(j-1)!}\eta^{j-1}\eta_x k^j\tanh(kh^-)e^{ikx}$$

$$- \sum_{l<j,\,l\,\text{even}} \frac{1}{(j-l)!} G_l^- k^{j-l}\eta^{j-l}e^{ikx} - \sum_{l<j,\,l\,\text{odd}} \frac{1}{(j-l)!} G_l^-\eta^{j-l}k^{j-l}\tanh(kh^-)e^{ikx}$$

$$G_j^- e^{ikx} = \frac{1}{j!}\eta^j k^{j+1}\tanh(kh^-)e^{ikx} - i\frac{1}{j!}(\eta^j)_x k^j \tanh(kh^-)e^{ikx}$$

$$\frac{1}{j}(\eta^j)_x = \frac{1}{j} j \eta^{j-1}\eta_x$$

$$-\sum_{l<j,\, l\,\text{even}} \frac{1}{(j-l)!} G_l^- \eta^{j-l} k^{j-l} e^{ikx} - \sum_{l<j,\, l\,\text{odd}} \frac{1}{(j-l)!} G_l^- \eta^{j-l} k^{j-l}\tanh(kh^-)e^{ikx}$$

$$\boxed{\begin{aligned} G_j^- &= \frac{1}{j!}\eta^j D^{j+1}\tanh(h^- D) - i\frac{1}{j!}(\eta^j)_x D^j \tanh(h^- D) \\ &\quad -\sum_{l<j,\, l\,\text{even}} \frac{1}{(j-l)!} G_l^- \eta^{j-l} D^{j-l} - \sum_{l<j,\, l\,\text{odd}} \frac{1}{(j-l)!} G_l^- \eta^{j-l} D^{j-l}\tanh(Dh^-) \end{aligned}}$$

j even

$G^- D{+}N{-}6$

当 $j>0$，j 是 odd

$$\sum_{l<j,\, l\ \text{even}} G_l^- \frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^-)e^{ikx} + \sum_{l<j,\, l\ \text{odd}} \frac{1}{(j-l)!} G_l^-(k\eta)^{j-l}\cosh(kh^-)e^{ikx} + G_j^-\cosh(kh^-)e^{ikx}$$

$$= \frac{1}{(j-1)!}(k\eta)^{j-1}(-ik)\eta_x\cosh(kh^-)e^{ikx} + \frac{1}{j!}(k\eta)^j k\cosh(kh^-)e^{ikx}$$

$$G_j^-\cosh(kh^-)e^{ikx} = -\frac{1}{(j-1)!}\eta^{j-1}k^{j-1}k\eta_x\cosh(kh^-)e^{ikx} + \frac{1}{j!}\eta^j k^{j+1}\cosh(kh^-)e^{ikx}$$

$$-\sum_{l<j,\, l\ \text{even}} G_l^- \frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^-)e^{ikx} - \sum_{l<j,\, l\ \text{odd}} \frac{1}{(j-l)!} G_l^-(k\eta)^{j-l}\cosh(kh^-)e^{ikx}$$

$$G_j^- e^{ikx} = -\frac{1}{(j-1)!}\eta^{j-1}k^{j+1}\eta_x \cancel{\cos}\, e^{ikx} + \frac{1}{j!}\eta^j k^{j+1} e^{ikx}$$

$$-\sum_{l<j,\, l\ \text{even}} G_l^- \frac{1}{(j-l)!}\eta^{j-l}k^{j-l}\tanh(kh^-)e^{ikx} - \sum_{l<j,\, l\ \text{odd}} \frac{1}{(j-l)!} G_l^-\eta^{j-l}k^{j-l}e^{ikx}$$

$$G_j^- e^{ikx} = -\frac{1}{j!}(\eta^j)_x k^j e^{ikx} + \frac{1}{j!}\eta^j k^{j+1} e^{ikx}$$

$$-\sum_{l<j,\, l\ \text{even}} G_l \frac{1}{(j-l)!}\eta^{j-l} k^{j-l}\tanh(kh^-) e^{ikx} - \sum_{l<j,\, l\ \text{odd}} \frac{1}{(j-l)!} G_l^- \eta^{j-l} k^{j-l} e^{ikx}$$

$$\boxed{\begin{aligned} G_j^- &= -\frac{1}{j!}(\eta^j)_x D^j + \frac{1}{j!}\eta^j D^{j+1} \\ &\quad - \sum_{l<j,\, l\ \text{even}} G_l \frac{1}{(j-l)!}\eta^{j-l} D^{j-l}\tanh(Dh^-) - \sum_{l<j,\, l\ \text{odd}} \frac{1}{(j-l)!} G_l^- \eta^{j-l} D^{j-l} \end{aligned}}$$

j odd

C- D.N-8

$$\text{☆}\begin{cases}\phi^+_{xx}+\phi^+_{yy}=0, & \eta<y<h^+\\ \phi^+_y\big|_{y=h^+}=0, & y=h^+\\ \phi^+=\xi^+, & y=\eta\end{cases}$$

上

$\phi^+_k=\cosh(k(y-h^+))e^{ikx}$ 是(☆)的一个解

D+N定义： $G^+\xi^+_k=G^+\phi^+_k(x,y=\eta)=\left(\eta_x\frac{\partial\phi^+_k}{\partial x}-\frac{\partial\phi^+_k}{\partial y}\right)\Big|_{y=\eta}$

$$G^+\phi^+_k(x,y=\eta)=\left(\eta_x\frac{\partial\phi^+_k}{\partial x}-\frac{\partial\phi^+_k}{\partial y}\right)\Big|_{y=\eta}$$

$G^+(\eta)=\sum_{n=0}^{\infty}G^+_n(\eta)$， n为η的次数

左边： $G^+\phi^+_k(x,y=\eta)=G^+\cosh(k\eta-kh^+)e^{ikx}=G^+[\cosh(k\eta)\cosh(kh^+)-\sinh(k\eta)\sinh(kh^+)]e^{ikx}$

泰勒展开 $\left(\sum_{l=0}^{\infty}G_l(\eta)\right)\left(\sum_{j\text{ even}}\frac{1}{j!}(k\eta)^j\cosh(kh^+)e^{ikx}-\sum_{j\text{ odd}}\frac{1}{j!}(k\eta)^j\sinh(kh^+)e^{ikx}\right)$

右边： $\left(\eta_x\frac{\partial\phi^+_k}{\partial x}-\frac{\partial\phi^+_k}{\partial y}\right)_{y=\eta}=\left[\eta_x\frac{\partial}{\partial x}\cosh(ky-kh^+)e^{ikx}-\frac{\partial}{\partial y}\cosh(ky-kh^+)e^{ikx}\right]_{y=\eta}$

右 $=\left[ik\eta_x\cosh(ky-kh^+)e^{ikx}-k\sinh(ky-kh^+)e^{ikx}\right]_{y=\eta}$

右 $=\left[ik\eta_x\cosh(k\eta-kh^+)e^{ikx}-k\sinh(k\eta-kh^+)e^{ikx}\right]$

右 $=\big[ik\eta_x\cosh(k\eta)\cosh(kh^+)e^{ikx}-ik\eta_x\sinh(k\eta)\sinh(kh^+)e^{ikx}$
$\quad-k\sinh(k\eta)\cosh(kh^+)e^{ikx}+k\cosh(k\eta)\sinh(kh^+)e^{ikx}\big]$

G^+ D+N-1

For $j=0$，让 $\eta=0$

$$G^+_{(0)}\cosh(kh^+)e^{ikx}=k\sinh(kh^+)e^{ikx} \Rightarrow G^+_{(0)}e^{ikx}=k\tanh(kh^+)e^{ikx}$$

定义 Fourier 变换对 $\begin{cases}\zeta(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\zeta}e^{ikx}dk\\ \hat{\zeta}(x)=\int_{-\infty}^{+\infty}\zeta(x)e^{-ikx}dx\end{cases}$

$$G^+_{(0)}\zeta(x)=\frac{1}{2\pi}G^+_{(0)}\int_{-\infty}^{+\infty}\hat{\zeta}e^{ikx}dk=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\zeta}G^+_{(0)}e^{ikx}dk=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\zeta}(k)\,k\tanh(kh^+)e^{ikx}dk$$

于是 $\widehat{G^+_{(0)}}=k\tanh(kh^+)$ 我们让 $\boxed{D=-i\partial_x}$ $\hat{D}=\widehat{-i\partial_x}=k$

$$G^+_{(0)}=D\tanh(Dh^+) \Rightarrow \boxed{G_0^+=D\tanh(h^+D)}$$

G^+ $D+N-$

$$右 = ik\eta_x \cosh(k\eta)\cosh(kh^+)e^{ikx} - ik\eta_x \sinh(k\eta)\sinh(kh^+)e^{ikx} - k\sinh(k\eta)\cosh(kh^+)e^{ikx} + k\cosh(k\eta)\sinh(kh^+)e^{ikx}$$

$$右 = \cosh(k\eta)\left(ik\eta_x \cosh(kh^+) + k\sinh(kh^+)\right)e^{ikx} - \sinh(k\eta)\left(ik\eta_x \sinh(kh^+) + k\cosh(kh^+)\right)e^{ikx}$$

$$右 = \sum_{j\ \mathrm{even}} \frac{(k\eta)^j}{j!}\left(ik\eta_x \cosh(kh^+) + k\sinh(kh^+)\right)e^{ikx} - \sum_{j\ \mathrm{odd}} \frac{(k\eta)^j}{j!}\left(ik\eta_x \sinh(kh^+) + k\cosh(kh^+)\right)e^{ikx}$$

左=右

$$\boxed{\begin{aligned}&\left(\sum_{l=0}^{\infty} G_l^+(\eta)\right)\left(\sum_{j\ \mathrm{even}} \frac{1}{j!}(k\eta)^j \cosh(kh^+)e^{ikx} - \sum_{j\ \mathrm{odd}} \frac{1}{j!}(k\eta)^j \sinh(kh^+)e^{ikx}\right) \to 左\\ &= \sum_{j\ \mathrm{even}} \frac{(k\eta)^j}{j!}\left(ik\eta_x \cosh(kh^+) + k\sinh(kh^+)\right)e^{ikx} - \sum_{j\ \mathrm{odd}} \frac{(k\eta)^j}{j!}\left(ik\eta_x \sinh(kh^+) + k\cosh(kh^+)\right)e^{ikx}\end{aligned}}$$

G^+ DtN-3

For $j>0$，栖西乘积公式 $\left(\sum_{n=0}^{\infty} a_n\right)\cdot\left(\sum_{n=0}^{\infty} b_n\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_n b_{n-k}$

$$\boxed{\left(\sum_{l=0}^{\infty} a_l\right)\cdot\left(\sum_{j=0}^{\infty} b_j\right)=\sum_{j=0}^{\infty}\sum_{l=0}^{j} a_l b_{j-l}}$$

$$左=\sum_{j=0}^{\infty}\sum_{l=0}^{j} G_l^+\left[\frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^+)e^{ikx}-\frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^+)e^{ikx}\right]$$

$j-l$ 为 even　　$j-l$ 为 odd

$$左=\sum_{j=0}^{\infty}\left[\sum_{l<j} G_l^+\left[\frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^+)e^{ikx}-\frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^+)e^{ikx}\right]+G_j^+\cosh(kh^+)e^{ikx}\right]$$

当 $j>0$，j 是 even

$$\sum_{l<j,\, l\,\text{even}} G_l^+\frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^+)e^{ikx}-\sum_{l<j,\, l\,\text{odd}} G_l^+\frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^+)e^{ikx}+G_j^+\cosh(kh^+)e^{ikx}$$

$j-l$ even　　$j-l$ odd

当 $j>0$，j 是 odd

$$\sum_{l<j,\, l\,\text{even}} -G_l^+\frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^+)e^{ikx}+\sum_{l<j,\, l\,\text{odd}} G_l^+\frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^+)e^{ikx}+G_j^+\cosh(kh^+)e^{ikx}$$

$j-l$ odd　　$j-l$ even

G_i^+ $D+N-\varphi$

$$右=\sum_{j\,even}\frac{1}{j!}(k\eta)^j\left(ik\eta_x\cosh(kh^+)+k\sinh(kh^+)\right)e^{ikx}-\sum_{j\,odd}\frac{1}{j!}(k\eta)^j\left(ik\eta_x\sinh(kh^+)+k\cosh(kh^+)\right)e^{ikx}$$

当 $j>0$，j是 even

$$\sum_{l<j,\,l\,even}G_l^+\frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^+)e^{ikx}-\sum_{l<j,\,l\,odd}G_l^+\frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^+)e^{ikx}+\underline{G_j^+\cosh(kh^+)e^{ikx}}$$

$$=\frac{1}{j!}(k\eta)^j k\sinh(kh^+)e^{ikx}-\frac{1}{(j-1)!}(k\eta)^{j-1}ik\eta_x\sinh(kh^+)e^{ikx}$$

$$G_j^+\cosh(kh^+)e^{ikx}=\frac{1}{j!}\eta^j k^{j+1}\sinh(kh^+)e^{ikx}-i\frac{1}{(j-1)!}\eta^{j-1}\eta_x k^j\sinh(kh^+)e^{ikx}$$

$$-\sum_{l<j,\,l\,even}G_l^+\frac{1}{(j-l)!}\eta^{j-l}k^{j-l}\cosh(kh^+)e^{ikx}+\sum_{l<j,\,l\,odd}G_l^+\frac{1}{(j-l)!}\eta^{j-l}k^{j-l}\sinh(kh^+)e^{ikx}$$

$$G_j^+e^{ikx}=\frac{1}{j!}\eta^j k^{j+1}\tanh(kh^+)e^{ikx}-i\frac{1}{j!}(\eta^j)_x k^j\tanh(kh^+)e^{ikx}$$

$$-\sum_{l<j,\,l\,even}G_l^+\frac{1}{(j-l)!}\eta^{j-l}k^{j-l}e^{ikx}+\sum_{l<j,\,l\,odd}G_l^+\frac{1}{(j-l)!}\eta^{j-l}k^{j-l}\tanh(kh^+)e^{ikx}$$

G^+ D.N-L

$$\boxed{\begin{aligned}G_j^+ &= \frac{1}{j!}\eta^j D^{j+1}\tanh(Dh^+) - i\frac{1}{j!}(\eta^j)_x D^j \tanh(Dh^+) \qquad \textcolor{red}{j \text{ even}} \\ &\quad - \sum_{l<j,\, l\, \text{even}} G_l^+ \frac{1}{(j-l)!}\eta^{j-l} D^{j-l} + \sum_{l<j,\, l\, \text{odd}} G_l^+ \frac{1}{(j-l)!}\eta^{j-l} D^{j-l}\tanh(Dh^+)\end{aligned}}$$

当$j>0$，j是odd

$$\sum_{l<j,\, l\,\text{even}} -G_l^+ \frac{1}{(j-l)!}(k\eta)^{j-l}\sinh(kh^+)e^{ikx} + \sum_{l<j,\, l\,\text{odd}} G_l^+ \frac{1}{(j-l)!}(k\eta)^{j-l}\cosh(kh^+)e^{ikx} + \underline{G_j^+\cosh(kh^+)e^{ikx}}$$

$$= \frac{1}{(j-1)!}\eta^{j-1}k^{j-1}ik\eta_x\cosh(kh^+)e^{ikx} - \frac{1}{j!}(k\eta)^j k\cosh(kh^+)e^{ikx}$$

$$G_j^+\cosh(kh^+)e^{ikx} = i\frac{1}{(j-1)!}\eta^{j-1}\cancel{\eta}_x k^j\cosh(kh^+)e^{ikx} - \frac{1}{j!}(k\eta)^j k\cosh(kh^+)e^{ikx}$$

$$+\sum_{l<j,\, l\,\text{even}} G_l^+\frac{1}{(j-l)!}\eta^{j-l}k^{j-l}\sinh(kh^+)e^{ikx} - \sum_{l<j,\, l\,\text{odd}} G_l^+\frac{1}{(j-l)!}\eta^{j-l}k^{j-l}\cosh(kh^+)e^{ikx}$$

$$G_j^+e^{ikx} = i\frac{1}{(j-1)!}(\eta^{j})_x k^j e^{ikx} - \frac{1}{j!}\eta^j k^{j+1}\cancel{\varepsilon}\, e^{ikx}$$

$$+\sum_{l<j,\, l\,\text{even}} G_l^+\frac{1}{(j-l)!}\eta^{j-l}k^{j-l}\tanh(kh^+)e^{ikx} - \sum_{l<j,\, l\,\text{odd}} G_l^+\frac{1}{(j-l)!}\eta^{j-l}k^{j-l}e^{ikx}$$

$$\boxed{\begin{aligned} G_j^+ &= i\frac{1}{j!}(\eta^j)_x D^j - \frac{1}{j!}\eta^j D^{j+1} \qquad j\ \text{odd} \\ &+\sum_{l<j,\, l\,\text{even}} G_l^+\frac{1}{(j-l)!}\eta^{j-l}D^{j-l}\tanh(Dh^+) - \sum_{l<j,\, l\,\text{odd}} G_l^+\frac{1}{(j-l)!}\eta^{j-l}D^{j-l}\end{aligned}}$$

G^+ D,N-7

浅水(下层)—浅水(上层)模型 KdV

G

定义小参数 $\mu = \frac{1}{\lambda}$, $h = O(1)$ $\boxed{D = -i\partial_x}$

$\partial_x = O(\mu)$, $\partial_t = O(\mu)$, $\eta = O(\mu^2)$, $\xi = O(\mu)$, $\rho = O(1)$, $B = O(1)$

$$G_0^- = D\tanh(D) = D\left(D - \frac{D^3}{3} + O(\mu^5)\right) = D^2 - \frac{D^4}{3} + O(\mu^6)$$

$$= (-i\partial_x)^2 - \frac{1}{3}(-i\partial_x)^4 + O(\mu^6) = -\partial_{xx} - \frac{1}{3}\partial_{xxxx} + O(\mu^6)$$

$$G_1^- = D\eta D - \underset{\mu}{D}\underset{\mu}{\tanh(D)}\underset{\mu^2}{\eta}\underset{\mu}{\tanh(D)}\underset{\mu}{D} \Rightarrow O(\mu^6)$$

$$= (-i\partial_x)\eta(-i\partial_x) + O(\mu^6) = -\partial_x\eta\partial_x + O(\mu^6)$$

$$G_2^- = -\frac{1}{2}\underset{\mu^2}{D^2}\underset{\mu^4}{\eta^2}\underset{\mu}{\tanh(D)}\underset{\mu}{D} - \frac{1}{2}D\tanh(D)\eta^2D^2 + D\tanh(D)\eta D\tanh(D)\eta D\tanh(D)$$

$$G^- = G_0^- + G_1^- = -\partial_{xx} - \frac{1}{3}\partial_{xxxx} - \partial_x\eta\partial_x + O(\mu^6)$$

$$G_0^+ = D\tanh(hD) = D\left(hD - \frac{h^3D^3}{3} + O(\mu^5)\right) = hD^2 - \frac{h^3D^4}{3} + O(\mu^6)$$

$$= h(-i\partial_x)^2 - \frac{h^3}{3}(-i\partial_x)^4 + O(\mu^6) = -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} + O(\mu^6)$$

$$G_1^+ = -D\eta D + \underset{\mu}{D}\underset{\mu}{\tanh(hD)}\underset{\mu^2}{\eta}\underset{\mu}{\tanh(hD)}\underset{\mu}{D} \Rightarrow O(\mu^6)$$

$$= -(-i\partial_x)\eta(-i\partial_x) + O(\mu^6) = \partial_x\eta\partial_x + O(\mu^6)$$

$$G_2^+ = -\frac{1}{2}\underset{\mu^2}{D^2}\underset{\mu^4}{\eta^2}\underset{\mu}{\tanh(hD)}\underset{\mu}{D} - \frac{1}{2}D\tanh(hD)\eta^2D^2 + D\tanh(hD)\eta D\tan(hD)\eta D\tanh(hD)$$

$$= O(\mu^8)$$

$$G^+ = G_0^+ + G_1^+ = -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} + \partial_x\eta\partial_x + O(\mu^6)$$

$$G^+ + \rho G^- = -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} + \partial_x\eta\partial_x - \rho\partial_{xx} - \frac{1}{3}\rho\partial_{xxxx} - \rho\partial_x\eta\partial_x + O(\mu^6)$$

$$= -(h+\rho)\partial_{xx} - \frac{1}{3}(h^3+\rho)\partial_{xxxx} + (1-\rho)\partial_x\eta\partial_x + O(\mu^6)$$

$$= -(h+\rho)\partial_{xx}\left[1 + \frac{h^3+\rho}{3(h+\rho)}\partial_{xx} - \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + O(\mu^4)\right]$$

$\boxed{\text{算子求逆公式 } (AB)^{-1} = B^{-1}A^{-1}}$

$$(G^+ + \rho G^-)^{-1} = -\frac{1}{h+\rho}\left[1 + \frac{h^3+\rho}{3(h+\rho)}\partial_{xx} - \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + O(\mu^4)\right]^{-1}(\partial_{xx})^{-1}$$

$$= -\frac{1}{h+\rho}\frac{1}{1+\left(\frac{h^3+\rho}{3(h+\rho)}\partial_{xx} - \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + O(\mu^4)\right)}\partial_{xx}^{-1}$$

$$\frac{1}{1+x} = 1 - x + \cdots$$

$$= -\frac{1}{h+\rho}\left(1 - \frac{h^3+\rho}{3(h+\rho)}\partial_{xx} + \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + O(\mu^4)\right)\partial_{xx}^{-1}$$

$$= -\frac{1}{h+\rho}\left(\partial_{xx}^{-1} - \frac{h^3+\rho}{3(h+\rho)} + \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x^{-1} + O(\mu^2)\right)$$

$$= -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3(h+\rho)^2} - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x^{-1} + O(\mu^2)$$

总结：

$$G^- = -\partial_{xx} - \frac{1}{3}\partial_{xxxx} - \partial_x\eta\partial_x + O(\mu^6)$$

$$G^+ = -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} + \partial_x\eta\partial_x + O(\mu^6)$$

$$G^+ + \rho G^- = -(h+\rho)\partial_{xx}\left[1 + \frac{h^3+\rho}{3(h+\rho)}\partial_{xx} - \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + O(\mu^6)\right]$$

$$(G^+ + \rho G^-)^{-1} = -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3(h+\rho)^2} - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x^{-1} + O(\mu^2)$$

kd 11-7

$$\xi^- = (G^+ + \rho G^-)^{-1} G^+ \xi$$

$$\xi^- = \left[-\frac{\overset{\mu^{-2}}{\partial_{xx}^{-1}}}{h+\rho} + \frac{\overset{1}{h^3+\rho}}{3(h+\rho)^2} - \frac{1-\rho}{(h+\rho)^2} \overset{1}{\partial_x^{-1} \eta \partial_x^{-1}} + O(\mu^2) \right] \left(-\overset{\mu^2}{h\partial_{xx}} - \frac{1}{3} \overset{\mu^4}{h^3 \partial_{xxxx}} + \overset{\mu^4}{\partial_x \eta \partial_x} + O(\mu^6)\right) \xi$$

$$\boxed{\xi^- = \frac{h}{h+\rho} \xi + O(\mu^2)}$$

$$\xi^+ = -(G^+ + \rho G^-)^{-1} G^- \xi$$

$$\xi^+ = -\left[-\frac{\overset{\mu^{-2}}{\partial_{xx}^{-1}}}{h+\rho} + \frac{\overset{1}{h^3+\rho}}{3(h+\rho)^2} - \frac{\overset{1}{1-\rho}}{(h+\rho)^2} \partial_x^{-1} \eta \partial_x^{-1} + O(\mu^2) \right] \left(-\overset{\mu^2}{\partial_{xx}} - \frac{1}{3} \overset{\mu^4}{\partial_{xxxx}} - \overset{\mu^4}{\partial_x \eta \partial_x} + O(\mu^6)\right) \xi$$

$$\boxed{\xi^+ = -\frac{1}{h+\rho} \xi}$$

$\eta_t = G^-(G^+ + \rho G^-)^{-1} G^+ \xi$ 运动学边界条件

$$(G^+ + \rho G^-)^{-1} G^+ \xi = \Big(-\overset{\mu^{-2}}{\frac{\partial_{xx}^{-1}}{h+\rho}} + \overset{1}{\frac{h^3+\rho}{3(h+\rho)^2}} - \overset{1}{\frac{1-\rho}{(h+\rho)^2} \partial_x^{-1}\eta \partial_x^{-1}} + O(\mu^2)\Big)\Big(-\overset{\mu^2}{h\partial_{xx}} - \overset{\mu^4}{\tfrac{1}{3}h^3\partial_{xxxx}} + \overset{\mu^4}{\partial_x\eta\partial_x} + O(\mu^6)\Big)\xi$$

$$(G^+ + \rho G^-)^{-1} G^+ \xi = \Big[\frac{h}{h+\rho} + \frac{1}{3}\frac{h^3}{h+\rho}\partial_{xx} - \frac{1}{h+\rho}\partial_x^{-1}\eta\partial_x - \frac{h(h^3+\rho)}{3(h+\rho)^2}\partial_{xx} + \frac{h(1-\rho)}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x + O(\mu^4)\Big]\xi$$

$$(G^+ + \rho G^-)^{-1} G^+ \xi = \Big[\frac{h}{h+\rho} + \frac{1}{3}\frac{1}{(h+\rho)^2}\big(h^3(h+\rho) - h(h^3+\rho)\big)\partial_{xx} - \frac{1}{h+\rho}\partial_x^{-1}\eta\partial_x + \frac{h(1-\rho)}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x + O(\mu^4)\Big]\xi$$

$$(G^+ + \rho G^-)^{-1} G^+ \xi = \Big[\frac{h}{h+\rho} + \frac{1}{3}\frac{1}{(h+\rho)^2}(\rho h^3 - \rho h)\partial_{xx} + \frac{h(1-\rho)-(h+\rho)}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x + O(\mu^4)\Big]\xi$$

$$(G^+ + \rho G^-)^{-1} G^+ \xi = \Big[\frac{h}{h+\rho} + \frac{1}{3}\frac{\rho h(h^2-1)}{(h+\rho)^2}\partial_{xx} + \frac{-\rho h-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x + O(\mu^4)\Big]\xi$$

$$\boxed{G^-}(G^+ + \rho G^-)^{-1} G^+ \xi = \Big(-\overset{\mu^2}{\partial_{xx}} - \overset{\mu^4}{\tfrac{1}{3}\partial_{xxxx}} - \overset{\mu^4}{\partial_x\eta\partial_x} + O(\mu^6)\Big)\Big[\overset{1}{\frac{h}{h+\rho}} + \overset{\mu^2}{\frac{1}{3}\frac{\rho h(h^2-1)}{(h+\rho)^2}\partial_{xx}} - \overset{\mu^2}{\frac{\rho(h+1)}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x} + O(\mu^4)\Big]\xi$$

$$G^-(G^+ + \rho G^-)^{-1} G^+ \xi = \Big\{-\frac{h}{h+\rho}\partial_{xx} - \frac{1}{3}\frac{\rho h(h^2-1)}{(h+\rho)^2}\partial_{xxxx} + \frac{\rho(h+1)}{(h+\rho)^2}\partial_x\eta\partial_x - \frac{1}{3}\frac{h}{h+\rho}\partial_{xxxx} - \frac{h}{h+\rho}\partial_x\eta\partial_x + O(\mu^6)\Big\}\xi$$

$$G^-(G^+ + \rho G^-)^{-1} G^+ \xi = \Big\{-\frac{h}{h+\rho}\partial_{xx} - \frac{1}{3}\frac{1}{(h+\rho)^2}\big[\rho h(h^2-1) + h(h+\rho)\big]\partial_{xxxx} + \frac{1}{(h+\rho)^2}\big[\rho(h+1) - h(h+\rho)\big]\partial_x\eta\partial_x + O(\mu^6)\Big\}\xi$$

$$\boxed{G^-(G^+ + \rho G^-)^{-1} G^+ \xi = \Big\{-\frac{h}{h+\rho}\partial_{xx} - \frac{1}{3}\frac{h^2(1+\rho h)}{(h+\rho)^2}\partial_{xxxx} + \frac{\rho-h^2}{(h+\rho)^2}\partial_x\eta\partial_x + O(\mu^6)\Big\}\xi}$$

$$\eta_t = G^-(G^+ + \rho G^-)^{-1} G^+ \xi$$

$$\Rightarrow \eta_t = -\frac{h}{h+\rho}\partial_{xx}\xi - \frac{1}{3}\frac{h^2(1+\rho h)}{(h+\rho)^2}\partial_{xxxx}\xi + \frac{\rho - h^2}{(h+\rho)^2}\partial_x \eta \partial_x \xi$$

$$\Rightarrow \boxed{\eta_t + \frac{h}{h+\rho}\partial_{xx}\xi + \frac{h^2(1+\rho h)}{3(h+\rho)^2}\partial_{xxxx}\xi + \frac{h^2-\rho}{(h+\rho)^2}\partial_x(\eta \partial_x \xi) = 0}$$

$$\underset{\mu^2}{\xi_t} + \frac{1}{2}\left[\underset{\mu^4}{(\xi_x^-)^2} - \underset{\mu^6}{\frac{(\eta_t + \eta_x \xi_x^-)^2}{1+\eta_x^2}}\right] - \frac{\rho}{2}\left[\underset{\mu^4}{(\xi_x^+)^2} - \underset{\mu^6}{\frac{(\eta_t + \eta_x \xi_x^+)^2}{1+\eta_x^2}}\right] + \underset{\mu^2}{(1-\rho)\eta} - \underset{\mu^4}{\frac{B\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}} = 0$$ 动力学边界条件

$$\xi_t + \frac{1}{2}(\xi_x^-)^2 - \frac{\rho}{2}(\xi_x^+)^2 + (1-\rho)\eta - B\eta_{xx} = 0$$

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$$\xi_t + \frac{1}{2}\left(\frac{h}{h+\rho}\xi_x\right)^2 - \frac{\rho}{2}\left(-\frac{1}{h+\rho}\xi_x\right)^2 + (1-\rho)\eta - B\eta_{xx} = 0$$

$$\xi_t + (1-\rho)\eta - B\eta_{xx} + \frac{1}{2}\frac{1}{(h+\rho)^2}(h^2-\rho)(\xi_x)^2 = 0$$

$$\boxed{\xi_t + (1-\rho)\eta - B\eta_{xx} + \frac{(h^2-\rho)}{2(h+\rho)^2}(\xi_x)^2 = 0}$$

忽略后两项小量 $\Rightarrow \eta = -\frac{1}{1-\rho}\xi_t$

$$\begin{cases} \eta_t + \dfrac{h}{h+\rho}\partial_{xx}\xi + \dfrac{h^2(1+\rho h)}{3(h+\rho)^2}\partial_{xxxx}\xi + \dfrac{h^2-\rho}{(h+\rho)^2}\partial_x(\eta\partial_x\xi) = 0 & ① \\ \underset{\mu^2}{\xi_t} + \underset{\mu^2}{(1-\rho)\eta} - \underset{\mu^4}{B\eta_{xx}} + \underset{\mu^4}{\dfrac{h^2-\rho}{2(h+\rho)^2}(\xi_x)^2} = 0 & ② \end{cases}$$

$$② \longrightarrow \eta = -\frac{1}{1-\rho}\xi_t + O(\mu^4)$$

$$\frac{\partial ②}{\partial t} \Rightarrow \xi_{tt} + (1-\rho)\eta_t - B\eta_{txx} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t[(\xi_x)^2] = 0$$

$$\underset{\mu^3}{\xi_{tt}} + (1-\rho)\left[-\underset{\mu^3}{\frac{h}{h+\rho}\partial_{xx}\xi} - \underset{\mu^5}{\frac{h^2(1+\rho h)}{3(h+\rho)^2}\partial_{xxxx}\xi} + \underset{\mu^5}{\frac{h^2-\rho}{(h+\rho)^2}\partial_x(\eta\partial_x\xi)}\right]$$

$$- \underset{\mu^2}{B\partial_{xx}}\left[-\underset{\mu^3}{\frac{h}{h+\rho}\partial_{xx}\xi} - \cancel{\underset{\mu^5}{\frac{h^2(1+\rho h)}{3(h+\rho)^2}\partial_{xxxx}\xi}} - \cancel{\underset{\mu^5}{\frac{h^2-\rho}{(h+\rho)^2}\partial_x(\eta\partial_x\xi)}}\right] + \underset{\mu^5}{\frac{h^2-\rho}{2(h+\rho)^2}\partial_t[(\xi_x)^2]} = 0$$

$$\xi_{tt} - \frac{h(1-\rho)}{h+\rho}\partial_{xx}\xi - \underline{\frac{h^2(1-\rho)(1+\rho h)}{3(h+\rho)^2}\partial_{xxxx}\xi} - \frac{(h^2-\rho)(1-\rho)}{(h+\rho)^2}\partial_x(\eta\partial_x\xi) + \underline{\frac{Bh}{h+\rho}\partial_{xxxx}\xi} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t[(\xi_x)^2] = 0$$

$$\xi_{tt} - \frac{h(1-\rho)}{h+\rho}\partial_{xx}\xi + \frac{h}{h+\rho}\left[B - \frac{h(1-\rho)(1+\rho h)}{3(h+\rho)}\right]\xi_{xxxx} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t[(\xi_x)^2] - \frac{(h^2-\rho)(1-\rho)}{(h+\rho)^2}\partial_x(\eta\partial_x\xi) = 0$$

其中 $\eta = -\frac{1}{1-\rho}\xi_t$

动力学边界条件忽略小量后 $\eta=-\frac{1}{1-\rho}\xi_t$

$$\xi_{tt}-\frac{h(1-\rho)}{h+\rho}\partial_{xx}\xi+\frac{h}{h+\rho}\left[B-\frac{h(1-\rho)(1+\rho h)}{3(h+\rho)}\right]\partial_{xxxx}\xi+\frac{h^2-\rho}{2(h+\rho)^2}\partial_t\left[(\xi_x)^2\right]-\frac{(h^2-\rho)}{(h+\rho)^2}(1-\rho)\partial_x\left(-\frac{1}{1-\rho}\xi_t\xi_x\right)=$$

$$\xi_{tt}-\frac{h(1-\rho)}{h+\rho}\partial_{xx}\xi+\frac{h}{h+\rho}\left[B-\frac{h(1-\rho)(1+\rho h)}{3(h+\rho)}\right]\partial_{xxxx}\xi+\frac{h^2-\rho}{2(h+\rho)^2}\partial_t\left[(\xi_x)^2\right]+\frac{h^2-\rho}{(h+\rho)^2}\partial_x(\xi_t\xi_x)=0$$

令 $c^2=\frac{h(1-\rho)}{h+\rho}$ $\quad\alpha=\frac{h}{h+\rho}\left[B-\frac{h(1-\rho)(1+\rho h)}{3(h+\rho)}\right]$

$$\xi_{tt}-c^2\xi_{xx}+\alpha\xi_{xxxx}+\frac{h^2-\rho}{2(h+\rho)^2}\partial_t\left[(\xi_x)^2\right]+\frac{h^2-\rho}{(h+\rho)^2}\partial_x(\xi_t\xi_x)=0$$

令 $X=x-ct$ $\quad \tau=\mu^3t$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial X}\qquad \frac{\partial^2}{\partial x^2}=\frac{\partial^2}{\partial X^2}\qquad \frac{\partial^3}{\partial x^3}=\frac{\partial^3}{\partial X^3}\qquad \frac{\partial^4}{\partial x^4}=\frac{\partial^4}{\partial X^4}$$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial X}\frac{\partial X}{\partial t}+\frac{\partial}{\partial\tau}\frac{\partial\tau}{\partial t}=-c\frac{\partial}{\partial X}+\mu^3\frac{\partial}{\partial\tau}\qquad \frac{\partial^2}{\partial t^2}=\left(-c\frac{\partial}{\partial X}+\mu^3\frac{\partial}{\partial\tau}\right)\left(-c\frac{\partial}{\partial X}+\mu^3\frac{\partial}{\partial\tau}\right)=c^2\frac{\partial^2}{\partial X^2}-2\mu^3c\frac{\partial^2}{\partial X\partial\tau}+\mu^6\frac{\partial^2}{\partial\tau^2}$$

$$\frac{\partial^2}{\partial t^2}\xi = C^2\frac{\partial^2\xi}{\partial x^2} - 2\mu^3 C\frac{\partial^2\xi}{\partial x\partial\tau} + \cancel{\mu^6\frac{\partial^2\xi}{\partial\tau^2}}_{\mu^7}$$

多重尺度法 $\tau = \overset{O(1)}{\mu_1\tau_1} + \overset{O(1)}{\mu_2\tau_2} + \mu_3\tau_3 + \cdots$

$\frac{\partial}{\partial\tau} = O(1)$

$$-C^2\xi_{xx} = -C^2\xi_{xx}$$

$$\alpha\,\xi_{xxxx} = \alpha\,\xi_{xxxx}$$

$$\frac{h^2-\rho}{2(h+\rho)^2}\,\overset{-C\frac{\partial}{\partial x}+\mu^3\frac{\partial}{\partial\tau}}{\partial_t}\left[(\xi_x)^2\right] = \frac{h^2-\rho}{2(h+\rho)^2}\left(-C\partial_x(\xi_x^2) + \cancel{\mu^3\partial_\tau(\xi_x)^2}_{\mu^7}\right)$$

$$\frac{h^2-\rho}{(h+\rho)^2}\,\partial_x(\xi_t\,\xi_x) = \frac{h^2-\rho}{(h+\rho)^2}\,\partial_x\left[\left(-C\partial_x\xi + \cancel{\mu^3\partial_\tau\xi}_{\mu^7}\right)\xi_x\right]$$

$$\cancel{C^2\frac{\partial^2\xi}{\partial x^2}} - 2\mu^3 C\frac{\partial^2\xi}{\partial x\partial\tau} - \cancel{C^2\xi_{xx}} + \alpha\,\xi_{xxxx} - \frac{C(h^2-\rho)}{2(h+\rho)^2}\partial_x(\xi_x^2) - \frac{C(h^2-\rho)}{(h+\rho)^2}\partial_x(\xi_x^2) = 0$$

$$-2\mu^3 C\frac{\partial}{\partial\tau}\xi_x + \alpha\,(\xi_x)_{xxx} - \frac{3C(h^2-\rho)}{2(h+\rho)^2}\partial_x(\xi_x^2) = 0$$

令 $H = \xi_x$

$$-2\mu^3 C\frac{\partial}{\partial\tau}H + \alpha H_{xxx} - \frac{3C(h^2-\rho)}{2(h+\rho)^2}\partial_x(H^2) = 0.$$

$\partial_x(H^2) = 2HH_x$

$$-2\mu^3 C\frac{\partial}{\partial\tau}H + \alpha H_{xxx} - \frac{3C(h^2-\rho)}{(h+\rho)^2}HH_x = 0.$$

$$\mu^3\frac{\partial H}{\partial\tau}-\frac{\alpha}{2c}H_{xxx}+\frac{3(h^2-\rho)}{2(h+\rho)^2}HH_x=0$$

而 $\eta=\frac{h}{c(h+\rho)H}\rightarrow H=\frac{c(h+\rho)}{h}\eta$

$$\Rightarrow \mu^3\frac{\partial}{\partial\tau}\left(\frac{c(h+\rho)}{h}\eta\right)-\frac{\alpha}{2c}\left(\frac{c(h+\rho)}{h}\eta\right)_{xxx}+\frac{3(h^2-\rho)}{2(h+\rho)^2}\frac{c(h+\rho)}{h}\eta\frac{c(h+\rho)}{\cancel{h}}\eta_x=0$$

$$\mu^3\eta_\tau-\frac{\alpha}{2c}\eta_{xxx}+\frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x=0.$$

运动学边界条件忽略小量

$$\eta_t+\frac{h}{h+\rho}\xi_{xx}=0$$

$$-c\eta_x+\mu^3\eta_\tau+\frac{h}{h+\rho}\xi_{xx}=0$$

$H=\xi_x$

$$-c\eta_x+\frac{h}{h+\rho}H_x=0$$

$$c\eta_x=\frac{h}{h+\rho}H_x$$

$$\eta_x=\frac{h}{c(h+\rho)}H_x$$

$$\eta=\frac{h}{c(h+\rho)}H$$

kdV-9

浅水(下)－浅水(上)模型 5th kdV $\boxed{D=-i\partial_x}$

$$\boxed{\partial_x = O(\mu),\ \partial_t = O(\mu),\ \eta = O(\mu^4),\ \xi = O(\mu^3),\ \alpha = O(\mu^2)}$$

H:

$$G_0^- = D\tanh(D) = D\left(D-\frac{D^3}{3}+\frac{2D^5}{15}+O(\mu^7)\right) = D^2-\frac{D^4}{3}+\frac{2D^6}{15}+O(\mu^8) = (-i\partial_x)^2-\frac{1}{3}(-i\partial_x)^4+\frac{2}{15}(-i\partial_x)^6+O(\mu^8)$$

$$G_0^- = -\partial_{xx}-\frac{1}{3}\partial_{xxxx}-\frac{2}{15}\partial_{xxxxxx}+O(\mu^8)$$

$$G_1^- = D\eta D-\underset{\mu}{D}\underset{\mu}{\tanh(D)}\underset{\mu^4}{\eta}\underset{\mu}{\tanh(D)}\underset{\mu\ \Rightarrow\mu^8}{D} = (-i\partial_x)\eta(-i\partial_x)+O(\mu^8) = -\partial_x\eta\partial_x+O(\mu^8)$$

$$\boxed{G^- = G_0^-+G_1^- = -\partial_{xx}-\frac{1}{3}\partial_{xxxx}-\partial_x\eta\partial_x-\frac{2}{15}\partial_{xxxxxx}+O(\mu^8)}$$

$$G_0^+ = D\tanh(hD) = D\left(hD-\frac{h^3D^3}{3}+\frac{2h^5D^5}{15}+O(\mu^7)\right) = hD^2-\frac{h^3}{3}D^4+\frac{2}{15}h^5D^6+O(\mu^8) = h(-i\partial_x)^2-\frac{1}{3}h^3(-i\partial_x)^4+\frac{2}{15}h^5(-i\partial_x)^6+O(\mu^8)$$

$$G_0^+ = -h\partial_{xx}-\frac{1}{3}h^3\partial_{xxxx}-\frac{2}{15}h^5\partial_{xxxxxx}+O(\mu^8)$$

$$G_1^+ = -D\eta D+\underset{\mu}{D}\underset{\mu}{\tanh(hD)}\underset{\mu^4}{\eta}\underset{\mu}{\tanh(hD)}\underset{\mu\ \Rightarrow\mu^8}{D} = -(-i\partial_x)\eta(-i\partial_x)+O(\mu^8) = \partial_x\eta\partial_x+O(\mu^8)$$

$$\boxed{G^+ = G_0^++G_1^+ = -h\partial_{xx}-\frac{1}{3}h^3\partial_{xxxx}+\partial_x\eta\partial_x-\frac{2}{15}h^5\partial_{xxxxxx}+O(\mu^8)}$$

$$G^+ + \rho G^- = -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} + \partial_x\eta\partial_x - \frac{2}{15}h^5\partial_{xxxxxx} + \rho\left(-\partial_{xx} - \frac{1}{3}\partial_{xxxx} - \partial_x\eta\partial_x - \frac{2}{15}\partial_{xxxxxx}\right) + O(\mu^8)$$

$$G^+ + \rho G^- = \underline{-h\partial_{xx}} - \frac{1}{3}h^3\partial_{xxxx} + \underline{\partial_x\eta\partial_x} - \frac{2}{15}h^5\partial_{xxxxxx} * \underline{-\rho\partial_{xx}} - \frac{1}{3}\rho\partial_{xxxx} - \underline{\rho\partial_x\eta\partial_x} - \frac{2}{15}\rho\partial_{xxxxxx} + O(\mu^8)$$

$$G^+ + \rho G^- = -(h+\rho)\partial_{xx} - \frac{1}{3}(h^3+\rho)\partial_{xxxx} + (1-\rho)\partial_x\eta\partial_x - \frac{2}{15}(h^5+\rho)\partial_{xxxxxx} + O(\mu^8)$$

$$\boxed{G^+ + \rho G^- = -(h+\rho)\partial_{xx}\left[1 + \frac{h^3+\rho}{3(h+\rho)}\partial_{xx} - \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + \frac{2}{15}\frac{h^5+\rho}{h+\rho}\partial_{xxxx} + O(\mu^6)\right]}$$

算子求逆公式 $(AB)^{-1} = B^{-1}A^{-1}$ $\quad B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$

$$(G^+ + \rho G^-)^{-1} = -\frac{1}{h+\rho}\left[1 + \frac{h^3+\rho}{3(h+\rho)}\partial_{xx} - \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + \frac{2}{15}\frac{h^5+\rho}{h+\rho}\partial_{xxxx} + O(\mu^6)\right]^{-1}\partial_{xx}^{-1}$$

$$(G^+ + \rho G^-)^{-1} = -\frac{1}{h+\rho}\,\frac{1}{1 + \frac{h^3+\rho}{3(h+\rho)}\partial_{xx} - \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + \frac{2}{15}\frac{h^5+\rho}{h+\rho}\partial_{xxxx} + O(\mu^6)}\,\partial_{xx}^{-1}$$

$$\boxed{\frac{1}{1+x} = 1 - x + \cdots}$$

$$(G^+ + \rho G^-)^{-1} = -\frac{1}{h+\rho}\left(1 - \frac{h^3+\rho}{3(h+\rho)}\partial_{xx} + \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x - \frac{2}{15}\frac{h^5+\rho}{h+\rho}\partial_{xxxx} + O(\mu^6)\right)\partial_{xx}^{-1}$$

$$= -\frac{1}{h+\rho}\left(\partial_{xx}^{-1} - \frac{h^3+\rho}{3(h+\rho)} + \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x^{-1} - \frac{2}{15}\frac{h^5+\rho}{h+\rho}\partial_{xx} + O(\mu^4)\right)$$

$$\boxed{(G^+ + \rho G^-)^{-1} = -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3(h+\rho)^2} - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x^{-1} + \frac{2}{15}\frac{h^5+\rho}{(h+\rho)^2}\partial_{xx} + O(\mu^4)}$$

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总结：

$$G^- = -\partial_{xx} - \frac{1}{3}\partial_{xxxx} - \frac{2}{15}\partial_{xxxxxx} - \partial_x \eta \partial_x + O(\mu^8)$$

$$G^+ = -h\partial_{xx} - \frac{1}{3}h^3\partial_{xxxx} - \frac{2}{15}h^5\partial_{xxxxxx} + \partial_x\eta\partial_x + O(\mu^8)$$

$$G^+ + \rho G^- = -(h+\rho)\partial_{xx}\left[1 + \frac{h^3+\rho}{3(h+\rho)}\partial_{xx} - \frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x + \frac{2}{15}\frac{h^5+\rho}{h+\rho}\partial_{xxxx} + O(\mu^6)\right]$$

$$(G^+ + \rho G^-)^{-1} = -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3(h+\rho)^2} + \frac{2}{15}\frac{h^5+\rho}{(h+\rho)^2}\partial_{xx} - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x^{-1} + O(\mu^4)$$

$$(G^+ + \rho G^-)^{-1} = -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3(h+\rho)^2} - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x^{-1} + \frac{2}{15}\frac{h^5+\rho}{(h+\rho)^2}\partial_{xx} + O(\mu^4)$$

$\partial_x = O(\mu)$ $\quad \eta = O(\mu^4)$ $\quad \xi = O(\mu^3)$

$\partial_x = O(\mu)$, $\eta = O(\mu^4)$, $\xi = O(\mu^3)$

$$\eta_t = G^-(G^+ + \rho G^-)^{-1} G^+ \xi$$

$$(G^+ + \rho G^-)^{-1} G^+ = \Big[\underbrace{-\overset{\mu^{-2}}{\frac{\partial_{xx}^{-1}}{h+\rho}}}_{①} + \underbrace{\overset{1}{\frac{h^3+\rho}{3(h+\rho)^2}}}_{②} - \underbrace{\overset{\mu^2}{\frac{1-\rho}{(h+\rho)^2} \partial_x^{-1} \eta \partial_x^{-1}}}_{③} + \underbrace{\overset{\mu^2}{\frac{2}{15} \frac{h^5+\rho}{(h+\rho)^2} \partial_{xx}}}_{④} + O(\mu^4) \Big]$$

$$\Big[-\overset{\mu^2}{h\partial_{xx}} - \overset{\mu^4}{\frac{1}{3} h^3 \partial_{xxxx}} - \overset{\mu^6}{\frac{2}{15} h^5 \partial_{xxxxxx}} + \overset{\mu^6}{\partial_x \eta \partial_x} + O(\mu^8) \Big]$$

$$(G^+ + \rho G^-)^{-1} G^+ = \frac{h}{h+\rho} + \frac{1}{3} \frac{h^3}{h+\rho} \partial_{xx} + \frac{2}{15} \frac{h^5}{h+\rho} \partial_{xxxx} - \frac{1}{h+\rho} \partial_x^{-1} \eta \partial_x \quad ①$$

$$- \frac{h(h^3+\rho)}{3(h+\rho)^2} \partial_{xx} - \frac{h^3(h^3+\rho)}{9(h+\rho)^2} \partial_{xxxx} \quad ②$$

$$+ \frac{(1-\rho)h}{(h+\rho)^2} \partial_x^{-1} \eta \partial_x \quad ③$$

$$- \frac{2}{15} \frac{(h^5+\rho)h}{(h+\rho)^2} \partial_{xxxx} + O(\mu^6) \quad ④$$

$$(G^+ + \rho G^-)^{-1} G^+ = \frac{h}{h+\rho} + \left(\frac{1}{3}\frac{h^3}{h+\rho} - \frac{h(h^3+\rho)}{3(h+\rho)^2}\right)\partial_{xx} + \left[-\frac{1}{h+\rho} + \frac{(1-\rho)h}{(h+\rho)^2}\right]\partial_x^{-1}\eta\partial_x$$
$$+ \left[\frac{2}{15}\frac{h^5}{h+\rho} - \frac{h^3(h^3+\rho)}{9(h+\rho)^2} - \frac{2}{15}\frac{(h^5+\rho)h}{(h+\rho)^2}\right]\partial_{xxxx} + O(\mu^6)$$

$$(G^+ + \rho G^-)^{-1} G^+ = \frac{h}{h+\rho} + \frac{\overbrace{h^3(h+\rho) - h(h^3+\rho)}^{\cancel{h^4} + h^3\rho - \cancel{h^4} - h\rho}}{3(h+\rho)^2}\partial_{xx} + \frac{\overbrace{-(h+\rho) + (1-\rho)h}^{-\cancel{h} - \rho + \cancel{h} - \rho h}}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x$$
$$+ \left[\frac{2}{15}\frac{\overbrace{h^5(h+\rho) - (h^5+\rho)h}^{\cancel{h^6} + h^5\rho - \cancel{h^6} - \rho h}}{(h+\rho)^2} - \frac{h^3(h^3+\rho)}{9(h+\rho)^2}\right]\partial_{xxxx} + O(\mu^6)$$

$$\boxed{(G^+ + \rho G^-)^{-1} G^+ = \frac{h}{h+\rho} + \frac{h^3\rho - h\rho}{3(h+\rho)^2}\partial_{xx} + \frac{-\rho - \rho h}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x + \left[\frac{2}{15}\frac{h^5\rho - \rho h}{(h+\rho)^2} - \frac{h^3(h^3+\rho)}{9(h+\rho)^2}\right]\partial_{xxxx}}$$

$$(G^+ + \rho G^-)^{-1} G^+ = \frac{h}{h+\rho} + \frac{\rho h(h^2-1)}{3(h+\rho)^2}\partial_{xx} - \frac{\rho(1+h)}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x + \left[\frac{2}{15}\frac{\rho h(h^4-1)}{(h+\rho)^2} - \frac{h^3(h^3+\rho)}{9(h+\rho)^2}\right]\partial_{xxxx} + O(\mu^6)$$

$\partial_x = O(\mu)$, $\eta = O(\mu^4)$, $\xi = O(\mu^3)$

$$G^-(G^+ + \rho G^-)^{-1} G^+ = \Big(\underbrace{-\overset{\mu^2}{\partial_{xx}}}_{①} \underbrace{- \frac{1}{3}\overset{\mu^4}{\partial_{xxxx}}}_{②} \underbrace{- \frac{2}{15}\overset{\mu^6}{\partial_{xxxxxx}}}_{③} \underbrace{- \overset{\mu^6}{\partial_x\eta\,\partial_x}}_{④} + O(\mu^8)\Big)$$

$$\left[\overset{1}{\frac{h}{h+\rho}} + \frac{\rho h(h^2-1)}{3(h+\rho)^2}\overset{\mu^2}{\partial_{xx}} - \frac{\rho(1+h)}{(h+\rho)^2}\overset{\mu^4}{\partial_x^{-1}\eta\,\partial_x} + \left(\frac{2}{15}\frac{\rho h(h^4-1)}{(h+\rho)^2} - \frac{h^3(h^3+\rho)}{9(h+\rho)^2}\right)\overset{\mu^4}{\partial_{xxxx}} + O(\mu^6)\right]$$

$$G^-(G^+ + \rho G^-)^{-1} G^+ = -\frac{h}{h+\rho}\partial_{xx} - \frac{\rho h(h^2-1)}{3(h+\rho)^2}\partial_{xxxx} + \frac{\rho(1+h)}{(h+\rho)^2}\partial_x\eta\,\partial_x - \left[\frac{2}{15}\frac{\rho h(h^4-1)}{(h+\rho)^2} - \frac{h^3(h^3+\rho)}{9(h+\rho)^2}\right]\partial_{xxxxxx} \quad ①$$

$$-\frac{1}{3}\frac{h}{h+\rho}\partial_{xxxx} - \frac{\rho h(h^2-1)}{9(h+\rho)^2}\partial_{xxxxxx} \quad ②$$

$$-\frac{2}{15}\frac{h}{h+\rho}\partial_{xxxxxx} \quad ③$$

$$-\frac{h}{h+\rho}\partial_x\eta\,\partial_x + O(\mu^8) \quad ④$$

$$G^-(G^+ + \rho G^-)^{-1}G^+ = -\frac{h}{h+\rho}\partial_{xx} + \left[-\frac{\rho h(h^2-1)}{3(h+\rho)^2} - \frac{1}{3}\frac{h}{h+\rho}\right]\partial_{xxxx} + \left[\frac{\rho(1+h)}{(h+\rho)^2} - \frac{h}{(h+\rho)}\right]\partial_x \eta \partial_x$$

$$+\left\{-\left[\frac{2}{15}\frac{\rho h(h^4-1)}{(h+\rho)^2} - \frac{h^3(h^3+\rho)}{9(h+\rho)^2}\right] - \frac{\rho h(h^2-1)}{9(h+\rho)^2} - \frac{2}{15}\frac{h}{h+\rho}\right\}\partial_{xxxxxx} + O(\mu^8)$$

$$G^-(G^+ + \rho G^-)^{-1}G^+ = -\frac{h}{h+\rho}\partial_{xx} - \frac{\overbrace{\rho h(h^2-1)+h(h+\rho)}^{\to\ \rho h^3 - \cancel{\rho h} + h^2 + \cancel{\rho h}}}{3(h+\rho)^2}\partial_{xxxx} + \frac{\overbrace{\rho(1+h) - h(h+\rho)}^{\rho + \cancel{\rho h} - h^2 - \cancel{\rho h}}}{(h+\rho)^2}\partial_x \eta \partial_x$$

$$+\left\{-\frac{2}{15}\frac{\rho h(h^4-1)}{(h+\rho)^2} + \frac{h^3(h^3+\rho)}{9(h+\rho)^2} - \frac{\rho h(h^2-1)}{9(h+\rho)^2} - \frac{2}{15}\frac{h}{h+\rho}\right\}\partial_{xxxxxx} + O(\mu^8)$$

$$G^-(G^+ + \rho G^-)^{-1}G^+ = -\frac{h}{h+\rho}\partial_{xx} - \frac{h^2(1+\rho h)}{3(h+\rho)^2}\partial_{xxxx} + \frac{\rho - h^2}{(h+\rho)^2}\partial_x \eta \partial_x$$

$$+\left\{-\frac{2}{15}\frac{\underbrace{\rho h^5 - \rho h + h(h+\rho)}_{\rho h^5 - \cancel{\rho h} + h^2 + \cancel{\rho h}}}{(h+\rho)^2} + \frac{h^6 + \cancel{h^3\rho} - \cancel{\rho h^3} - \rho h}{9(h+\rho)^2}\right\}\partial_{xxxxxx} + O(\mu^8)$$

$$G^{-}(G^{+}+\rho G^{-})^{-1}G^{+} = -\frac{h}{h+\rho}\partial_{xx} - \frac{h^2(1+\rho h)}{3(h+\rho)^2}\partial_{xxxx} + \frac{\rho-h^2}{(h+\rho)^2}\partial_x \eta \partial_x + \left[-\frac{2}{15}\frac{\rho h^5+h^2}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2}\right]\partial_{xxxxxx} + O(\mu^8)$$

$$\eta_t = G^{-}(G^{+}+\rho G^{-})^{-1}G^{+}\xi$$

$$\eta_t = \left\{-\frac{h}{h+\rho}\partial_{xx} - \frac{h^2(1+\rho h)}{3(h+\rho)^2}\partial_{xxxx} + \frac{\rho-h^2}{(h+\rho)^2}\partial_x\eta\partial_x + \left[-\frac{2}{15}\frac{\rho h^5+h^2}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2}\right]\partial_{xxxxxx} + O(\mu^8)\right\}\xi$$

$$\boxed{\begin{aligned}\eta_t = &-\frac{h}{h+\rho}\partial_{xx}\xi - \frac{1}{3}\frac{h^2(1+\rho h)}{(h+\rho)^2}\partial_{xxxx}\xi + \frac{\rho-h^2}{(h+\rho)^2}\partial_x\eta\partial_x\xi \\ &+ \left[-\frac{2}{15}\frac{\rho h^5+h^2}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2}\right]\partial_{xxxxxx}\xi\end{aligned}}$$

$$\xi^- = (G^+ + \rho G^-)^{-1} G^+ \xi \qquad \eta = O(\mu^4) \quad \partial_x = O(\mu) \quad \xi = O(\mu^3)$$

$$\xi^- = \Big[-\overset{\mu^{-2}}{\frac{\partial_{xx}^{-1}}{h+\rho}} + \overset{1}{\frac{h^3+\rho}{3(h+\rho)^2}} + \frac{2}{15}\frac{h^5+\rho}{(h+\rho)^2}\overset{\mu^2}{\partial_{xx}} - \frac{1-\rho}{(h+\rho)^2}\overset{\mu^2}{\partial_x^{-1}\eta\,\partial_x^{-1}} + O(\mu^4) \Big]$$

$$\Big(\overset{\mu^2}{-h\partial_{xx}} - \frac{h^3}{3}\overset{\mu^4}{\partial_{xxxx}} - \frac{2}{15}h^5\overset{\mu^6}{\partial_{xxxxxx}} + \overset{\mu^6}{\partial_x\eta\,\partial_x} + O(\mu^8)\Big] \xi$$

$$\boxed{\xi^- = \frac{h}{h+\rho}\,\xi + O(\mu^5)}$$

$$\xi^+ = -(G^+ + \rho G^-)^{-1} G^- \xi$$

$$= -\Big[-\overset{\mu^{-2}}{\frac{\partial_{xx}^{-1}}{h+\rho}} + \overset{1}{\frac{h^3+\rho}{3(h+\rho)^2}} + \frac{2}{15}\frac{h^5+\rho}{(h+\rho)^2}\overset{\mu^2}{\partial_{xx}} - \frac{1-\rho}{(h+\rho)^2}\overset{\mu^2}{\partial_x^{-1}\eta\,\partial_x^{-1}} + O(\mu^4) \Big]$$

$$\Big(\overset{\mu^2}{-\partial_{xx}} - \frac{1}{3}\overset{\mu^4}{\partial_{xxxx}} - \frac{2}{15}h^5\overset{\mu^6}{\partial_{xxxxxx}} - \overset{\mu^6}{\partial_x\eta\,\partial_x} + O(\mu^8)\Big) \xi$$

$$\boxed{\xi^+ = -\frac{1}{h+\rho}\,\xi + O(\mu^5)}$$

$$\xi_t + \frac{1}{2}\left[(\xi_x^-)^2 - \frac{(\eta_t + \eta_x \xi_x^-)^2}{(1+\eta_x^2)} \right] - \frac{\rho}{2}\left[(\xi_x^+)^2 - \frac{(\eta_t + \eta_x \xi_x^+)^2}{(1+\eta_x^2)} \right] + (1-\rho)\eta - \frac{B\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$\partial_x = O(\mu)$, $\eta = O(\mu^4)$, $\xi = O(\mu^2)$

$$\xi_t + \frac{1}{2}(\xi_x^-)^2 - \frac{1}{2}\rho(\xi_x^+)^2 + (1-\rho)\eta - B\eta_{xx} = 0$$

$$\xi_x^- = \frac{h}{h+\rho}\xi_x + O(\mu^6)$$

$$\xi_x^+ = -\frac{1}{h+\rho}\xi_x + O(\mu^6)$$

$$\xi_t + \frac{1}{2}\frac{h^2}{(h+\rho)^2}(\xi_x)^2 - \frac{1}{2}\rho\frac{1}{(h+\rho)^2}(\xi_x)^2 + (1-\rho)\eta - B\eta_{xx} = 0$$

$$\boxed{\underbrace{\xi_t}_{\mu^4} + \underbrace{(1-\rho)\eta}_{\mu^4} - \underbrace{B\eta_{xx}}_{\mu^6} - \underbrace{\frac{h^2-\rho}{2(h+\rho)^2}(\xi_x)^2}_{\mu^6} = 0} \Rightarrow \eta = -\frac{\xi_t}{1-\rho}$$

总结:

$$\boxed{\begin{aligned}
&\eta_t = -\frac{h}{h+\rho}\partial_{xx}\xi - \frac{1}{3}\frac{h^2(1+\rho h)}{(h+\rho)^2}\overset{\mu^4}{\partial_{xxxx}}\overset{\mu^3}{\xi} + \frac{\rho-h^2}{(h+\rho)^2}\overset{\mu}{\partial_x}\overset{\mu^4}{\eta}\overset{\mu}{\partial_x}\overset{\mu^3}{\xi} \\
&\qquad + \left[-\frac{2}{15}\frac{\rho h^5+h^2}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2}\right]\overset{\mu^9}{\partial_{xxxxxx}}\xi \qquad (1) \\
&\xi_t + (1-\rho)\eta - B\eta_{xx} + \frac{h^2-\rho}{2(h+\rho)^2}(\xi_x)^2 = 0 \qquad (2) \\
&\eta = -\frac{\xi_t}{1-\rho} + O(\mu^6) \qquad (3)
\end{aligned}}$$

$\partial_x = O(\mu)$

$\eta = O(\mu^4)$

$\xi = O(\mu^3)$

(1) $\eta_t = -\frac{h}{h+\rho}\partial_{xx}\xi - \frac{1}{3}\frac{h^2(1+\rho h)}{(h+\rho)^2}\partial_{xxxx}\xi + O(\mu^9)$

5th KdV – II

$$\frac{\partial (2)}{\partial t} \Rightarrow \quad \xi_{tt} + (1-\rho)\eta_t - B\eta_{txx} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t(\xi_x)^2 = 0$$

$$\xi_{tt} + (1-\rho)\Big\{ -\frac{h}{h+\rho}\partial_{xx}\xi - \frac{1}{3}\frac{h^2(1+\rho h)}{(h+\rho)^2}\partial_{xxxx}\xi + \frac{\rho-h^2}{(h+\rho)^2}\partial_x(\eta\,\xi_x)$$

$$+\left[-\frac{2}{15}\frac{\rho h^5+h^2}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2}\right]\partial_{xxxxxx}\xi \Big\} \quad \mu^6 \quad \mu^3 \Rightarrow \mu^9$$

$$-B\Big\{ -\frac{h}{h+\rho}\partial_{xx}\xi - \frac{1}{3}\frac{h^2(1+\rho h)}{(h+\rho)^2}\partial_{xxxx}\xi\Big\}_{xx} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t(\xi_x)^2 = 0 \quad \mu^4 \quad \mu^3 \Rightarrow \mu^9 \quad \mu^2$$

$$\xi_{tt} - \frac{h(1-\rho)}{h+\rho}\xi_{xx} - \frac{1}{3}\frac{h^2(1+\rho h)(1-\rho)}{(h+\rho)^2}\partial_{xxxx}\xi + \frac{Bh}{h+\rho}\partial_{xxxx}\xi + (1-\rho)\frac{\rho-h^2}{(h+\rho)^2}\partial_x(\eta\,\xi_x)$$

$$+\frac{h^2-\rho}{2(h+\rho)^2}\partial_t(\xi_x^2) + \left[-\frac{2}{15}\frac{(\rho h^5+h^2)(1-\rho)}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2}\right]\partial_{xxxxxx}\xi$$

$$+\frac{B}{3}\frac{h^2(1+\rho h)}{(h+\rho)^2}\partial_{xxxxxx}\xi = 0$$

$$\xi_{tt}-\frac{h(1-\rho)}{h+\rho}\xi_{xx}+\frac{h}{h+\rho}\left[B-\frac{h(1+\rho h)(1-\rho)}{3(h+\rho)}\right]\partial_{xxxx}\xi+\frac{h^2-\rho}{2(h+\rho)^2}\partial_t(\xi_x^2)+(1-\rho)\frac{\rho-h^2}{(h+\rho)^2}\partial_x(\eta\,\xi_x)$$

$$+\left[-\frac{2}{15}\frac{(\rho h^5+h^2)(1-\rho)}{(h+\rho)^2}+\frac{h(h^5+\rho)}{9(h+\rho)^2}+\frac{B}{3}\frac{h^2(1-\rho h)}{(h+\rho)^2}\right]\partial_{xxxxxx}\xi=0$$

$$\eta=-\frac{1}{1-\rho}\xi_t+O(\mu^6)$$

$$\xi_{tt}-\frac{h(1-\rho)}{h+\rho}\partial_{xx}\xi+\frac{h}{h+\rho}\left[B-\frac{h(1+\rho h)(1-\rho)}{3(h+\rho)}\right]\partial_{xxxx}\xi+\frac{h^2-\rho}{2(h+\rho)^2}\partial_t(\xi_x^2)+(1-\rho)\frac{\rho-h^2}{(h+\rho)^2}\partial_x\left(-\frac{1}{1-\rho}\xi_t\xi_x\right)$$

$$+\left[-\frac{2}{15}\frac{(\rho h^5+h^2)(1-\rho)}{(h+\rho)^2}+\frac{h(h^5+\rho)}{9(h+\rho)^2}+\frac{B}{3}\frac{h^2(1-\rho h)}{(h+\rho)^2}\right]\partial_{xxxxxx}\xi=0$$

$$\xi_{tt}-\frac{h(1-\rho)}{h+\rho}\xi_{xx}+\frac{h}{h+\rho}\left[B-\frac{h(1+\rho h)(1-\rho)}{3(h+\rho)}\right]\xi_{xxxx}+\frac{h^2-\rho}{2(h+\rho)^2}\partial_t(\xi_x^2)+\frac{h^2-\rho}{(h+\rho)^2}\partial_x(\xi_t\xi_x)$$

$$+\left[-\frac{2}{15}\frac{(\rho h^5+h^2)(1-\rho)}{(h+\rho)^2}+\frac{h(h^5+\rho)}{9(h+\rho)^2}+\frac{B}{3}\frac{h^2(1-\rho h)}{(h+\rho)^2}\right]\xi_{xxxxxx}=0$$

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其中 $c^2 = \dfrac{h(1-\rho)}{h+\rho}$ $\qquad \alpha = \dfrac{h}{h+\rho}\left[B - \dfrac{h(1-\rho)(1+\rho h)}{3(h+\rho)}\right]$

5th kdV -14

$$\xi_{tt} - c^2\xi_{xx} + \alpha\,\xi_{xxxx} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t(\xi_x^2) + \frac{h^2-\rho}{(h+\rho)^2}\partial_x(\xi_t\xi_x)$$
$$+\left[-\frac{2}{15}\frac{(\rho h^5+h^2)(1-\rho)}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2} + \frac{B}{3}\frac{h^2(1-\rho h)}{(h+\rho)^2}\right]\xi_{xxxxxx} = 0.$$

令 $\cancel{X} = x - ct$, $\overset{O(1)}{\tau} = \mu^3 t$ $\quad t = \dfrac{\tau}{\mu^3} \gg 1$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X}, \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial X^2}$$

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial X} + \mu^3\frac{\partial}{\partial \tau}, \quad \frac{\partial^2}{\partial t^2} = \left(-c\frac{\partial}{\partial X} + \mu^3\frac{\partial}{\partial \tau}\right)\left(-c\frac{\partial}{\partial X} + \mu^3\frac{\partial}{\partial \tau}\right) = c^2\frac{\partial^2}{\partial X^2} - 2\mu^3 c\frac{\partial^2}{\partial X\partial\tau} + \cancel{\mu^6\frac{\partial^2}{\partial \tau^2}}$$

$$\xi_{tt} = c^2\frac{\partial^2}{\partial X^2}\xi - 2\mu^3 c\frac{\partial^2}{\partial X\partial\tau}\xi = c^2\xi_{XX} - 2\mu^3 c\,\xi_{X\tau}$$

$$-c^2\xi_{xx} = -c^2\xi_{XX}$$

$$\xi_x = \xi_X$$

$$\xi_t = -c\,\xi_X + \mu^3\xi_\tau$$

$$\xi_{tt} = c^2 \xi_{xx} - 2\mu^3 c\, \xi_{x\tau}$$

$$-c^2 \xi_{xx} = -c^2 \xi_{xx} \qquad \alpha \xi_{xxxx} = \alpha \xi_{xxxx}$$

$$\frac{h^2-\rho}{2(h+\rho)^2} \partial_t (\xi_x)^2 = \frac{h^2-\rho}{2(h+\rho)^2}\left[-c\,\partial_x (\xi_x^2) + \cancel{\mu^3 \partial_\tau (\xi_x)^2}^{\,\mu^{11}} \right]$$

$$\frac{h^2-\rho}{(h+\rho)^2} \partial_x (\xi_t \xi_x) = \frac{h^2-\rho}{(h+\rho)^2} \overset{\mu}{\partial_x} \left[(-c\,\xi_x + \overset{\mu^3}{\mu^3} \overset{\mu^3}{\xi_\tau})\, \overset{\mu^4}{\xi_x} \right] = \frac{h^2-\rho}{2(h+\rho)^2} \partial_x (-c\,\xi_x^2)$$

$$\left[-\frac{2}{15} \frac{(\rho h^5 + h^2)(1-\rho)}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2} + \frac{B}{3} \frac{h^2(1-\rho h)}{(h+\rho)^2} \right] \xi_{xxxxxx}$$

$$= \left[-\frac{2}{15} \frac{(\rho h^5 + h^2)(1-\rho)}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2} + \frac{B}{3} \frac{h^2(1-\rho h)}{(h+\rho)^2} \right] \xi_{xxxxxx}$$

$$\cancel{c^2\xi_{xx}} - 2\mu^3 c\,\xi_{x\tau} - \cancel{c^2\xi_{xx}} + \frac{h^2-\rho}{2(h+\rho)^2}(-c)\,\partial_x(\xi_x^2) + \frac{h^2-\rho}{(h+\rho)^2}\,\partial_x(-c\,\xi_x^2) + \alpha\,\xi_{xxxx}$$
$$+\left[-\frac{2}{15}\frac{(\rho h^5+h^2)(1-\rho)}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2} + \frac{B}{3}\frac{h^2(1-\rho h)}{(h+\rho)^2}\right]\xi_{xxxxxx} = 0.$$

$$-2\mu^3 c\,\xi_{x\tau} - \frac{h^2-\rho}{2(h+\rho)^2}c\,\partial_x(\xi_x^2) - \frac{h^2-\rho}{(h+\rho)^2}c\,\partial_x(\xi_x^2) + \alpha\,\xi_{xxxx}$$
$$+\left[-\frac{2}{15}\frac{(\rho h^5+h^2)(1-\rho)}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2} + \frac{B}{3}\frac{h^2(1-\rho h)}{(h+\rho)^2}\right]\xi_{xxxxxx} = 0.$$

令 $H = \xi_x$

$$-2\mu^3 c\,H_\tau + \alpha H_{xxx} - \frac{3c(h^2-\rho)}{2(h+\rho)^2}\,\partial_x H^2$$
$$+\Bigg[\underbrace{-\frac{2}{15}\frac{(\rho h^5+h^2)(1-\rho)}{(h+\rho)^2} + \frac{h(h^5+\rho)}{9(h+\rho)^2} + \frac{Bh^2(1-\rho h)}{(h+\rho)^2}}_{=M}\Bigg]H_{xxxxx} = 0$$

$$-2\mu^3 c H_\tau + \alpha H_{xxx} - \frac{3c(h^2-\rho)}{2(h+\rho)^2}\partial_x H^2 + M H_{xxxxx} = 0$$

$$\xi_t + (1-\rho)\eta - B\eta_{xx} = 0 \quad , \quad -c\,\xi_x + (1-\rho)\eta - B\eta_{xx} = 0$$

$$\Rightarrow \ -cH = -(1-\rho)\eta + B\eta_{xx} \Longrightarrow H = \frac{1-\rho}{c}\eta - \frac{B}{c}\eta_{xx}$$

$$-2\mu^3 c\left(\frac{1-\rho}{c}\eta_\tau\right) + \frac{1-\rho}{c}\alpha\,\eta_{xxx} - \frac{B\alpha}{c}\eta_{xxxxx} - \frac{3c(h^2-\rho)}{2(h+\rho)^2}\partial_x\left(\left(\frac{1-\rho}{c}\right)^2\eta^2\right) + M\frac{1-\rho}{c}\eta_{xxxxx} = 0$$

$$-2\mu^3 c\,\eta_\tau + \alpha\eta_{xxx} - \frac{B\alpha}{1-\rho}\eta_{xxxxx} - \frac{3c(h^2-\rho)}{2(h+\rho)^2}\frac{1-\rho}{c}\partial_x\eta^2 + M\eta_{xxxxx} = 0$$

$$-2\mu^3 c\,\eta_\tau + \alpha\eta_{xxx} - \frac{3c(h^2-\rho)}{2(h+\rho)^2}\frac{1-\rho}{c}2\eta\eta_x + \underbrace{\left(M - \frac{B\alpha}{1-\rho}\right)}_{=-\beta}\eta_{xxxxx} = 0$$

$\cancel{\mu^3}\eta_t$

$$\mu^3\eta_t - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{2(h+\rho)^2}\frac{1-\rho}{c^2}\eta\eta_x + \frac{\beta}{2c}\eta_{xxxxx} = 0$$

$$\boxed{c^2 = \frac{h(1-\rho)}{h+\rho} \qquad \frac{1-\rho}{c^2} = \frac{h+\rho}{h}}$$

$$\mu^3\eta_t - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x + \frac{\beta}{2c}\eta_{xxxxx} = 0$$

5th kdV-1i

浅水(下)－浅水(上)模型 mkdV

$$\boxed{D=-i\partial_x}$$

Ⅰ $\boxed{\partial_x = O(\mu) \quad \partial_t = O(\mu) \quad \eta = O(\mu) \quad \xi = O(1)}$

$$G_0^- = D\tanh(D) = D\left(D-\frac{D^3}{3}+O(\mu^5)\right) = D^2-\frac{D^4}{3}+O(\mu^6)$$
$$= (-i\partial_x)^2-\frac{1}{3}(-i\partial_x)^4+O(\mu^6) = -\partial_{xx}-\frac{1}{3}\partial_{xxxx}+O(\mu^6)$$

$$G_1^- = D\eta D - \underset{\mu}{D}\tanh\underset{\mu}{(D)}\underset{\mu}{\eta}\tanh\underset{\mu}{(D)}\underset{\mu}{D} \to O(\mu^5)$$
$$= -i\partial_x\eta(-i\partial_x)+O(\mu^5) = -\partial_x\eta\partial_x+O(\mu^5)$$

$$G_2^- = -\frac{1}{2}D^2\eta^2\tanh(D)D-\frac{1}{2}D\tanh(D)\eta^2D^2+D\tanh(D)\eta D\tanh(D)\eta D\tanh(D)$$

$$G^- = G_0^-+G_1^- = -\partial_{xx}-\frac{1}{3}\partial_{xxxx}-\partial_x\eta\partial_x+O(\mu^5)$$

$$G_0^+ = D\tanh(hD) = D\left(hD-\frac{h^3D^3}{3}+O(\mu^5)\right) = hD^2-\frac{h^3D^4}{3}+O(\mu^6)$$
$$= h(-i\partial_x)^2-\frac{h^3}{3}(-i\partial_x)^4+O(\mu^6) = -h\partial_{xx}-\frac{1}{3}h^3\partial_{xxxx}+O(\mu^6)$$

$$G_1^+ = -D\eta D+\underset{\mu}{D}\tanh\underset{\mu}{(hD)}\underset{\mu}{\eta}\tanh\underset{\mu}{(hD)}\underset{\mu}{D} \to O(\mu^5)$$
$$= -(-i\partial_x)\eta(-i\partial_x)+O(\mu^5) = \partial_x\eta\partial_x+O(\mu^5)$$

$$G_2^+ = -\frac{1}{2}\underset{\mu^2}{D^2}\underset{\mu^2}{\eta^2}\tanh\underset{\mu}{(hD)}\underset{\mu}{D} \to \mu^6 -\frac{1}{2}D\tanh(hD)\eta^2D^2+D\tanh(hD)\eta D\tanh(hD)\eta D\tanh(hD)$$
$$= O(\mu^6)$$

$$G^+ = G_0^++G_1^+ = -h\partial_{xx}-\frac{1}{3}h^3\partial_{xxxx}+\partial_x\eta\partial_x+O(\mu^5)$$

$$G^{+}+\rho G^{-}=-h\partial_{xx}-\frac{1}{3}h^{3}\partial_{xxxx}+\partial_x\eta\partial_x+\rho(-\partial_{xx}-\frac{1}{3}\partial_{xxxx}-\partial_x\eta\partial_x)+O(\mu^{5})$$

$$G^{+}+\rho G^{-}=\underline{-h\partial_{xx}}-\underline{\frac{1}{3}h^{3}\partial_{xxxx}}\,\underline{+\partial_x\eta\partial_x}\;\underline{-\rho\partial_{xx}}-\underline{\frac{1}{3}\rho\partial_{xxxx}}\,\underline{-\rho\partial_x\eta\partial_x}+O(\mu^{5})$$

$$G^{+}+\rho G^{-}=-(\rho+h)\partial_{xx}-\frac{1}{3}(\rho+h^{3})\partial_{xxxx}+(1-\rho)\partial_x\eta\partial_x+O(\mu^{5})$$

$$G^{+}+\rho G^{-}=-(h+\rho)\partial_{xx}\left[1+\frac{h^{3}+\rho}{3(h+\rho)}\partial_{xx}-\frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x+O(\mu^{3})\right]$$

$\partial_x=O(\mu)$, $\partial_t=O(\mu)$, $\eta=O(\mu)$, $\xi=O(1)$

算子求逆法则 $(AB)^{-1}=B^{-1}A^{-1}$

$$(G^{+}+\rho G^{-})^{-1}=-\frac{1}{h+\rho}\left[1+\frac{h^{3}+\rho}{3(h+\rho)}\overset{\mu^2}{\partial_{xx}}-\frac{(1-\rho)}{h+\rho}\partial_x^{-1}\overset{\mu}{\eta}\partial_x+O(\mu^{3})\right]^{-1}\partial_{xx}^{-1}$$

一个泰勒展开

$$(1+x)^{-1}=1-x+x^{2}+\cdots\cdots$$

$$\left[1+\frac{h^{3}+\rho}{3(h+\rho)}\partial_{xx}-\frac{(1-\rho)}{h+\rho}\partial_x^{-1}\eta\partial_x+O(\mu^{3})\right]^{-1}$$

$$=1-\frac{h^{3}+\rho}{3(h+\rho)}\partial_{xx}+\frac{(1-\rho)}{h+\rho}\partial_x^{-1}\eta\partial_x+\left(\frac{h^{3}+\rho}{3(h+\rho)}\overset{\mu^2}{\partial_{xx}}-\frac{(1-\rho)}{h+\rho}\partial_x^{-1}\overset{\mu}{\eta}\partial_x\right)^{2}+O(\mu^{3})$$

$\hookrightarrow$ 稍 μ^2

$$=1-\frac{h^{3}+\rho}{3(h+\rho)}\partial_{xx}+\frac{(1-\rho)}{h+\rho}\partial_x^{-1}\eta\partial_x+\frac{(1-\rho)^{2}}{(h+\rho)^{2}}\partial_x^{-1}\eta\underline{\partial_x\partial_x^{-1}}\eta\partial_x+O(\mu^{3})$$

$$=1-\frac{h^{3}+\rho}{3(h+\rho)}\partial_{xx}+\frac{(1-\rho)}{h+\rho}\partial_x^{-1}\eta\partial_x+\frac{(1-\rho)^{2}}{(h+\rho)^{2}}\partial_x^{-1}\eta^{2}\partial_x+O(\mu^{3})$$

$$\Rightarrow(G^{+}+\rho G^{-})^{-1}=-\frac{1}{h+\rho}\left[1-\frac{h^{3}+\rho}{3(h+\rho)}\partial_{xx}+\frac{1-\rho}{h+\rho}\partial_x^{-1}\eta\partial_x+\frac{(1-\rho)^{2}}{(h+\rho)^{2}}\partial_x^{-1}\eta\partial_x+O(\mu^{3})\right]\partial_{xx}^{-1}$$

$$(G^+ + \rho G^-)^{-1} = -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3(h+\rho)^2} - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x^{-1} - \frac{(1-\rho)^2}{(h+\rho)^3}\partial_x^{-1}\eta\,\partial_x^{-1} + O(\mu)$$

总结：

$$\boxed{\begin{aligned} G^- &= -\partial_{xx} - \partial_x\eta\,\partial_x - \frac{1}{3}\partial_{xxxx} + O(\mu^5) \\ G^+ &= -h\,\partial_{xx} + \partial_x\eta\,\partial_x - \frac{1}{3}h^3\partial_{xxxx} + O(\mu^5) \\ (G^+ + \rho G^-)^{-1} &= -\frac{\partial_{xx}^{-1}}{h+\rho} + \frac{h^3+\rho}{3(h+\rho)^2} - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x^{-1} - \frac{(1-\rho)^2}{(h+\rho)^3}\partial_x^{-1}\eta^2\partial_x^{-1} + O(\mu) \end{aligned}}$$

$\partial_x = O(\mu)$

$\partial_t = O(\mu)$

$\eta = O(\mu)$

$\xi = O(1)$

$$\xi^- = (G^+ + \rho G^-)^{-1} G^+ \xi$$

$$\xi^- = \left[-\overset{\mu^{-2}}{\frac{\partial_{xx}^{-1}}{h+\rho}} + \overset{1}{\frac{h^3+\rho}{3(h+\rho)^2}} - \overset{\mu^{-1}}{\frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x^{-1}} - \overset{1}{\frac{(1-\rho)^2}{(h+\rho)^3}\partial_x^{-1}\eta^2\partial_x^{-1}} + O(\mu)\right]\left(-\overset{\mu^2}{h\,\partial_{xx}} + \overset{\mu^3}{\partial_x\eta\,\partial_x} - \frac{1}{3}\overset{\mu^4}{h^3\partial_{xxxx}} + O(\mu^5)\right)\xi$$

$$\xi^- = \left[\frac{h}{h+\rho} - \frac{\partial_{xx}^{-1}}{h+\rho}\partial_x\eta\,\partial_x + \frac{(1-\rho)h}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x^{-1}\partial_{xx}\right]\xi + O(\mu^2)$$

$$\xi^- = \left[\frac{h}{h+\rho} - \frac{1}{h+\rho}\partial_x^{-1}\eta\,\partial_x + \frac{(1-\rho)h}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x\right]\xi + O(\mu^2)$$

$$\xi^- = \frac{h}{h+\rho}\xi + \frac{\cancel{-h}-\rho\cancel{+h}-\rho h}{(h+\rho)^2}\partial_x^{-1}\eta\,\xi_x + O(\mu^2)$$

$$\xi^- = \frac{h}{h+\rho}\xi - \frac{\rho(1+h)}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x + O(\mu^2)$$

$$\xi_x^- = \partial_x\xi^- = \partial_x\left(\frac{h}{h+\rho}\xi - \frac{\rho(1+h)}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x\xi + O(\mu^2)\right)$$

$$\boxed{\xi_x^- = \frac{h}{h+\rho}\xi_x - \frac{\rho(1+h)}{(h+\rho)^2}\eta\,\xi_x}$$

mkdV-3

$$\xi^+ = -(a^+ + \rho a^-)^{-1} a^- \xi \qquad \partial_x = O(\mu),\ \partial_t = O(\mu),\ \eta = O(\mu),\ \xi = O(1)$$

$$\xi^+ = -\left[\overset{\mu^{-2}}{-\frac{\partial_{xx}^{-1}}{h+\rho}} + \overset{1}{\frac{h^3+\rho}{3(h+\rho)^2}} - \overset{\mu^{-1}}{\frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x^{-1}} - \overset{1}{\frac{(1-\rho)^2}{(h+\rho)^2}\partial_x^{-1}\eta^2\partial_x^{-1}} + O(\mu)\right]\left(\overset{\mu^2}{-\partial_{xx}} - \overset{\mu^3}{\partial_x\eta\partial_x} - \overset{\mu^4}{\frac{1}{3}h^3\partial_{xxxx}} + O(\mu^5)\right)\xi$$

$$\xi^+ = -\left[+\frac{1}{h+\rho} + \frac{1}{h+\rho}\partial_x^{-1}\eta\partial_x + \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x + O(\mu^2)\right]\xi$$

$$\xi^+ = -\left[\frac{1}{h+\rho} + \frac{h+\rho+1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x + O(\mu^2)\right]\xi$$

$$\xi^+ = -\frac{1}{h+\rho}\xi - \frac{1+h}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x\xi + O(\mu^2)$$

$$\xi_x^+ = \partial_x\xi^+ = -\frac{1}{h+\rho}\xi_x - \frac{1+h}{(h+\rho)^2}\eta\partial_x\xi + O(\mu^3)$$

$$\boxed{\xi_x^+ = -\frac{1}{h+\rho}\xi_x - \frac{1+h}{(h+\rho)^2}\eta\xi_x + O(\mu^3)}$$

mkdv-4

$$\eta_t = G^-(G^+ + \rho G^-)^{-1} G^+ \xi \qquad \boxed{\partial_x = O(\mu),\ \partial t = O(\mu),\ \eta = O(\mu),\ \xi = O(1)}$$

$$(G^+ + \rho G^-)^{-1} G^+ = \Big[\underbrace{-\frac{\partial_{xx}^{-1}}{h+\rho}}_{①}\overset{\mu^{-2}}{} + \underbrace{\frac{h^3+\rho}{3(h+\rho)^2}}_{②}\overset{1}{} - \underbrace{\frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x^{-1}}_{③}\overset{\mu^{-1}}{} - \underbrace{\frac{(1-\rho)^2}{(h+\rho)^3}\partial_x^{-1}\eta^2\partial_x^{-1}}_{④}\overset{1}{} + O(\mu)\Big]\big(\overset{\mu^2}{-h\partial_{xx}} + \overset{\mu^3}{\partial_x\eta\,\partial_x} - \overset{\mu^4}{\tfrac{1}{3}h^3\partial_{xxxx}} + O(\mu$$

$$\begin{aligned}(G^+ + \rho G^-)^{-1} G^+ = {} & \frac{h}{h+\rho} - \frac{1}{h+\rho}\partial_x^{-1}\eta\,\partial_x + \frac{h^3}{3(h+\rho)}\partial_{xx} \quad ① \\ & - \frac{h(h^3+\rho)}{3(h+\rho)^2}\partial_{xx} \quad ② \\ & + \frac{h(1-\rho)}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta\,\partial_x^{-1}\partial_x\eta\,\partial_x \quad ③ \\ & + \frac{h(1-\rho)^2}{(h+\rho)^3}\partial_x^{-1}\eta^2\partial_x^{-1}\partial_{xx} \quad ④ \quad + O(\mu^3)\end{aligned}$$

$$\begin{aligned}(G^+ + \rho G^-)^{-1} G^+ = {} & \frac{h}{h+\rho} + \Big[\frac{h^3}{3(h+\rho)} - \frac{h(h^3+\rho)}{3(h+\rho)^2}\Big]\partial_{xx} + \Big[\frac{h(1-\rho)}{(h+\rho)^2} - \frac{1}{h+\rho}\Big]\partial_x^{-1}\eta\,\partial_x \\ & - \frac{1-\rho}{(h+\rho)^2}\partial_x^{-1}\eta^2\partial_x + \frac{h(1-\rho)^2}{(h+\rho)^3}\partial_x^{-1}\eta^2\partial_x + O(\mu^3)\end{aligned}$$

$$(a^+ + \rho a^-)^{-1} a^+ = \frac{h}{h+\rho} + \frac{1}{3}\frac{\rho h(h^2-1)}{(h+\rho)^2}\partial_{xx} - \frac{\rho(h+1)}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x + \frac{1-\rho}{(h+\rho)^3}\left(-h-\rho+h(1-\rho)\right)\partial_x^{-1}\eta^2\partial_x$$

$$(a^+ + \rho a^-)^{-1} a^+ = \frac{h}{h+\rho} + \frac{1}{3}\frac{\rho h(h^2-1)}{(h+\rho)^2}\partial_{xx} - \frac{\rho(h+1)}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x - \frac{(1-\rho)\rho}{(h+\rho)^3}(1+h)\partial_x^{-1}\eta^2\partial_x$$

$$a^-(a^+ + \rho a^-)^{-1} a^+ = \Big(\underbrace{-\overset{\mu^2}{\partial_{xx}}}_{①} - \underbrace{\overset{\mu^3}{\partial_x\eta\partial_x}}_{②} - \frac{1}{3}\overset{\mu^4}{\partial_{xxxx}} + O(\mu^5)\Big)\Big[\overset{1}{\frac{h}{h+\rho}} + \frac{1}{3}\overset{\mu^2}{\frac{\rho h(h^2-1)}{(h+\rho)^2}\partial_{xx}} - \overset{\mu}{\frac{\rho(h+1)}{(h+\rho)^2}\partial_x^{-1}\eta\partial_x} - \overset{\mu^2}{\frac{(1-\rho)\rho}{(h+\rho)^3}(1+h)\partial_x^{-1}\eta^2\partial_x} + O(\ldots$$

$$= -\frac{h}{h+\rho}\partial_{xx} - \frac{1}{3}\frac{\rho h(h^2-1)}{(\rho+h)^2}\partial_{xxxx} + \frac{\rho(h+1)}{(h+\rho)^2}\partial_x\eta\partial_x + \frac{(1-\rho)\rho}{(h+\rho)^3}(1+h)\partial_x\eta^2\partial_x \quad ①$$

$$-\frac{h}{h+\rho}\partial_x\eta\partial_x + \frac{\rho(h+1)}{(h+\rho)^3}\partial_x\eta\partial_x\underbrace{\partial_x^{-1}\eta\partial_x}_{\to\ \partial_x\eta^2\partial_x} \quad ②$$

$$-\frac{1}{3}\frac{h}{h+\rho}\partial_{xxxx} + O(\mu^5) \quad ③$$

$$= -\frac{h}{h+\rho}\partial_{xx} - \frac{1}{3}\frac{1}{(h+\rho)^2}\left[\rho h(h^2-1) + h(h+\rho)\right]\partial_{xxxx} + \frac{1}{(h+\rho)^2}\left[\rho h + \rho - h^2 - \rho h\right]\partial_x\eta\partial_x$$
$$+ \frac{\rho(1+h)}{(h+\rho)^3}(1\cancel{-\rho}+h\cancel{+\rho})\partial_x\eta^2\partial_x$$

mkdV-4

$$a^-(a^+ + \rho a^-)^{-1} a^+ = -\frac{h}{h+\rho}\partial_{xx} - \frac{1}{3}\frac{h^2(1+\rho h)}{(h+\rho)^2}\partial_{xxxx} + \frac{\rho - h^2}{(h+\rho)^2}\partial_x \eta \partial_x + \frac{\rho(1+h)^2}{(h+\rho)^3}\partial_x \eta^2 \partial_x$$

$$\eta_t - a^-(a^+ + \rho a^-)^{-1} a^+ \xi = 0$$

$$\boxed{\eta_t + \frac{h}{h+\rho}\xi_{xx} + \frac{h^2(1+\rho h)}{3(h+\rho)^2}\xi_{xxxx} + \frac{h^2-\rho}{(h+\rho)^2}\partial_x\eta\partial_x\xi - \frac{\rho(1+h)^2}{(h+\rho)^3}\partial_x\eta^2\partial_x\xi = 0}$$

$\partial_x = O(\mu)$

$\partial_t = O(\mu)$

$\eta = O(\mu)$

$\xi = O(1)$

$$\underset{\mu}{\xi_t} + \frac{1}{2}\Big[\underset{\mu^2}{(\xi_x^-)^2} - \underset{\mu^4}{\cancel{\frac{(\eta_t + \eta_x \xi_x^-)^2}{1+\eta_x^2}}}\Big] - \frac{\rho}{2}\Big[\underset{\mu}{(\xi_x^+)^2} - \underset{\mu^4}{\cancel{\frac{(\eta_t + \eta_x \xi_x^+)^2}{1+\eta_x^2}}}\Big] + (1-\rho)\eta - B\underset{\mu^6}{\frac{\overset{\mu^3}{\eta_{xx}}}{(1+\eta_x^2)^{\frac{3}{2}}}} = 0$$

$$\xi_x^- = \frac{h}{h+\rho}\xi_x - \frac{\rho(1+h)}{(h+\rho)^2}\eta\xi_x \qquad \xi_x^+ = -\frac{1}{h+\rho}\xi_x - \frac{1+h}{(h+\rho)^2}\eta\xi_x + \cancel{O(\mu^3)}$$

$$(\xi_x^-)^2 = \left(\frac{h}{h+\rho}\xi_x - \frac{\rho(1+h)}{(h+\rho)^2}\eta\xi_x\right)^2 = \frac{h^2}{(h+\rho)^2}\xi_x^2 - \frac{2\rho h(1+h)}{(h+\rho)^3}\eta\xi_x^2 + O(\mu^4)$$

$$(\xi_x^+)^2 = \left(-\frac{1}{h+\rho}\xi_x - \frac{1+h}{(h+\rho)^2}\eta\xi_x\right)^2 = \frac{1}{(h+\rho)^2}\xi_x^2 + \frac{2(1+h)}{(h+\rho)^3}\eta\xi_x^2 + O(\mu^4)$$

$$\xi_t + \frac{1}{2}(\xi_x^-)^2 - \frac{\rho}{2}(\xi_x^+)^2 + (1-\rho)\eta - B\eta_{xx} = 0$$

$$\rho h + \rho h^2 + \rho + \rho h = \rho(h^2 + 2h + 1) = \rho(h+1)^2$$

$$\xi_t + \frac{h^2}{2(h+\rho)^2}\xi_x^2 - \frac{\rho h(1+h)}{(h+\rho)^3}\eta\xi_x^2 - \frac{\rho}{2}\frac{1}{(h+\rho)^2}\xi_x^2 - \frac{\rho(1+h)}{(h+\rho)^3}\eta\xi_x^2 + (1-\rho)\eta - B\eta_{xx} = 0$$

$$\boxed{\xi_t + (1-\rho)\eta - B\eta_{xx} + \frac{h^2-\rho}{2(h+\rho)^2}\xi_x^2 - \frac{\rho(1+h)^2}{(h+\rho)^3}\eta\xi_x^2 = 0}$$

$$\boxed{\begin{aligned}&\eta_t+\frac{h}{h+\rho}\xi_{xx}+\frac{h^2(1+\rho h)}{3(h+\rho)^2}\xi_{xxxx}+\frac{h^2-\rho}{(h+\rho)^2}(\eta\xi_x)_x-\frac{\rho(1+h)^2}{(h+\rho)^3}(\eta^2\xi_x)_x=0 \quad ①\\ &\xi_t+(1-\rho)\eta-B\eta_{xx}+\frac{h^2-\rho}{2(h+\rho)^2}\xi_x^2-\frac{\rho(1+h)^2}{(h+\rho)^3}\eta\xi_x^2=0 \quad ②\end{aligned}}$$

$\partial_x = O(\mu)$
$\partial_t = O(\mu)$
$\eta = O(\mu)$
$\xi = O(1)$

Note: $\eta_t+\frac{h}{h+\rho}\xi_{xx}=0 \qquad \xi_t+(1-\rho)\eta=0$

$$\frac{\partial ②}{\partial t}: \quad \xi_{tt}+(1-\rho)\eta_t-B\eta_{txx}+\frac{h^2-\rho}{2(h+\rho)^2}\partial_t\xi_x^2-\frac{\rho(1+h)^2}{(h+\rho)^3}\partial_t(\eta\xi_x^2)=0$$

$$\xi_{tt}+(1-\rho)\left[-\frac{h}{h+\rho}\xi_{xx}-\frac{h^2(1+\rho h)}{3(h+\rho)^2}\xi_{xxxx}-\frac{h^2-\rho}{(h+\rho)^2}(\eta\xi_x)_x+\frac{\rho(1+h)^2}{(h+\rho)^3}\partial_t(\eta\xi_x^2)\right]$$

$$-B\underset{\mu^2}{\partial_{xx}}\left[-\frac{h}{h+\rho}\underset{\mu^2}{\xi_{xx}}-\cancel{\frac{h^2(1+\rho h)}{3(h+\rho)^2}\underset{\mu^4}{\xi_{xxxx}}}-\cancel{\frac{h^2-\rho}{(h+\rho)^2}\underset{\mu^3}{(\eta\xi_x)_x}}+\cancel{\frac{\rho(1+h)^2}{(h+\rho)^3}\underset{\mu^4}{\partial_t(\eta\xi_x^2)}}\right]$$

$$+\frac{h^2-\rho}{2(h+\rho)^2}\partial_t\xi_x^2-\frac{\rho(1+h)^2}{(h+\rho)^3}\partial_t(\eta\xi_x^2)=0$$

$$\xi_{tt} + (1-\rho)\left(-\frac{h}{h+\rho}\right)\xi_{xx} - \frac{h^2(1-\rho)(1+\rho h)}{3(h+\rho)^2}\xi_{xxxx} - \frac{(1-\rho)(h^2-\rho)}{(h+\rho)^2}(\eta\xi_x)_x + \frac{\rho(1-\rho)(1+h)^2}{(h+\rho)^3}\partial_t(\eta\xi_x^2)$$
$$+ \frac{Bh}{h+\rho}\xi_{xxxx} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t\xi_x^2 - \frac{\rho(1+h)^2}{(h+\rho)^3}\partial_t(\eta\xi_x^2) = 0$$

$$\xi_{tt} - \frac{h(1-\rho)}{h+\rho}\xi_{xx} + \frac{h}{h+\rho}\left[B - \frac{h(1-\rho)(1+\rho h)}{3(h+\rho)}\right]\xi_{xxxx} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t\xi_x^2 - \frac{(h^2-\rho)(1-\rho)}{(h+\rho)^2}(\eta\xi_x)_x$$
$$+ \frac{\rho(1+h)^2}{(h+\rho)^3}(-1+1-\rho)\partial_t(\eta\xi_x^2) = 0$$

$$\xi_{tt} - \frac{h(1-\rho)}{h+\rho}\xi_{xx} + \frac{h}{h+\rho}\left[B - \frac{h(1-\rho)(1+\rho h)}{3(h+\rho)}\right]\xi_{xxxx} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t\xi_x^2 - \frac{(h^2-\rho)(1-\rho)}{(h+\rho)^2}(\eta\xi_x)_x$$
$$- \frac{\rho^2(1+h)^2}{(h+\rho)^3}\partial_t(\eta\xi_x^2) = 0$$

令 $C^2 = \frac{h(1-\rho)}{h+\rho}$ $\qquad \alpha = \frac{h}{h+\rho}\left[B - \frac{h(1-\rho)(1+\rho h)}{3(h+\rho)}\right]$

$$\xi_{tt} - C^2\xi_{xx} + \alpha\xi_{xxxx} + \frac{h^2-\rho}{2(h+\rho)^2}\partial_t(\xi_x^2) - \frac{(h^2-\rho)(1-\rho)}{(h+\rho)^2}(\eta\xi_x)_x - \frac{\rho^2(1+h)^2}{(h+\rho)^3}\partial_t(\eta\xi_x^2) = 0$$

$$\xi_t=-(1-\rho)\eta \Rightarrow \eta=-\frac{\xi_t}{1-\rho}$$

$$\underbrace{\xi_{tt}}_{①}-\underbrace{c^2\xi_{xx}}_{②}+\underbrace{\alpha\,\xi_{xxxx}}_{③}+\underbrace{\frac{h^2-\rho}{2(h+\rho)^2}\partial_t\left(\xi_x^2\right)}_{④}+\underbrace{\frac{h^2-\rho}{(h+\rho)^2}\left(\xi_t\xi_x\right)_x}_{⑤}+\underbrace{\frac{\rho^2(1+h^2)}{(h+\rho)^3(1-\rho)}\partial_t\left(\xi_t\xi_x^2\right)}_{⑥}=0$$

令 $X=x-ct$ ， $\tau=\mu^3 t$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial X},\quad \frac{\partial^2}{\partial x^2}=\frac{\partial^2}{\partial X^2},\quad \frac{\partial^3}{\partial x^3}=\frac{\partial^3}{\partial X^3}\cdots\qquad \frac{\partial}{\partial t}=-c\frac{\partial}{\partial X}+\mu^3\frac{\partial}{\partial \tau},\quad \frac{\partial^2}{\partial t^2}=c^2\frac{\partial^2}{\partial X^2}-2\mu^3 c\frac{\partial^2}{\partial X\partial\tau}\cancel{+\mu^6\frac{\partial^2}{\partial\tau^2}}$$

①: $\xi_{tt}=c^2\xi_{XX}-2\mu^3 c\,\xi_{X\tau}$

②: $-c^2\xi_{xx}=-c^2\xi_{XX}$

$$\partial_t\left(\left(\partial_t\xi\right)\xi_x^2\right)=\left(-c\partial_X\cancel{+\mu^3\partial_\tau}\right)\left[\left(-c\xi_X\cancel{+\mu^3\xi_\tau}\right)\xi_X^2\right]$$

③: $\alpha\,\xi_{xxxx}=\alpha\,\xi_{XXXX}$

$$=c^2\partial_X\left(\xi_X^3\right)$$

④: $\frac{h^2-\rho}{2(h+\rho)^2}\partial_t\left(\xi_x^2\right)=\frac{h^2-\rho}{2(h+\rho)^2}\left(-c\frac{\partial}{\partial X}+\mu^3\frac{\partial}{\partial\tau}\right)\left(\xi_X^2\right)=-\frac{(h^2-\rho)c}{2(h+\rho)^2}\frac{\partial}{\partial X}\left(\xi_X^2\right)$

⑤: $\frac{h^2-\rho}{(h+\rho)^2}\left(\xi_t\xi_x\right)_x=\frac{h^2-\rho}{(h+\rho)^2}\partial_X\left(\left(-c\xi_X+\mu^3\xi_\tau\right)\xi_X\right)=-\frac{(h^2-\rho)c}{(h+\rho)^2}\frac{\partial}{\partial X}\left(\xi_X^2\right)$

⑥: $\frac{\rho^2(1+h^2)}{(h+\rho)^3(1-\rho)}\partial_t\left(\xi_t\xi_x^2\right)=\frac{\rho^2(1+h^2)c^2}{(h+\rho)^3(1-\rho)}\partial_X\left(\xi_X^3\right)$

$$\cancel{c^2\xi_{xx}} - 2\mu^3 c\,\xi_{x\tau} - \cancel{c^2\xi_{xx}} + \alpha\,\xi_{xxxx} - \underline{\frac{(h^2-\rho)c}{2(h+\rho)^2}\frac{\partial}{\partial x}(\xi_x^2)} - \underline{\frac{(h^2-\rho)c}{(h+\rho)^2}\frac{\partial}{\partial x}(\xi_x^2)} + \frac{\rho^2(1+h)^2c^2}{(h+\rho)^3(1-\rho)}\partial_x(\xi_x^3) = 0$$

令 $H = \xi_x$

$$-2\mu^3 c\frac{\partial H}{\partial \tau} + \alpha\,\cancel{\xi}^{H}_{xxx} - \frac{3c(h^2-\rho)}{2(h+\rho)^2}\partial_x(H^2) + \frac{\rho^2(1+h)^2c^2}{(h+\rho)^3(1-\rho)}\partial_x(H^3) = 0$$

$$\eta = -\frac{\xi_t}{1-\rho} = \frac{c\,\xi_x}{1-\rho} \qquad \xi_x = \frac{1-\rho}{c}\eta \Rightarrow H = \frac{1-\rho}{c}\eta$$

$$\boxed{c^2 = \frac{h(1-\rho)}{h+\rho}}$$

$$-2\mu^3 c\frac{1-\rho}{c}\frac{\partial \eta}{\partial \tau} + \frac{\alpha(1-\rho)}{c}\eta_{xxx} - \frac{3c(h^2-\rho)}{2(h+\rho)^2}\left(\frac{1-\rho}{c}\right)^2\partial_x(\eta^2) + \frac{\rho^2(1+h)^2c^2}{(h+\rho)^3(1-\rho)}\left(\frac{1-\rho}{c}\right)^3\partial_x(\eta^3) = 0$$

$$-2\mu^3\cancel{(1-\rho)}\frac{\partial \eta}{\partial \tau} + \frac{\alpha\cancel{(1-\rho)}}{c}\eta_{xxx} - \frac{3(h^2-\rho)(1-\rho)^{\cancel{2}}c}{2(h+\rho)^2c^2}\partial_x(\eta^2) + \frac{\rho^2(1+h)^2(1-\rho)^{\cancel{2}}}{c(h+\rho)^3}\partial_x(\eta^3) = 0$$

$$-2\mu^3\frac{\partial \eta}{\partial \tau} + \frac{\alpha}{c}\eta_{xxx} - \frac{3(h^2-\rho)(1-\rho)c}{2(h+\rho)^2c^2}\partial_x\eta^2 + \frac{\rho^2c(1+h)^2}{h(h+\rho)^2}\partial_x(\eta^3) = 0$$

$$-2\mu^3\frac{\partial \eta}{\partial \tau} + \frac{\alpha}{c}\eta_{xxx} - \frac{3c(h^2-\rho)}{2(h+\rho)h}\partial_x\eta^2 + \frac{\rho^2c(1+h)^2}{h(h+\rho)^2}\partial_x(\eta^3) = 0$$

$$\mu^3\frac{\partial \eta}{\partial \tau} - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{4(h+\rho)h}2\eta\eta_x - \frac{\rho^2c(h+1)^2}{2(h+\rho)^2h}\cdot 3\eta^2\eta_x = 0$$

$$\mu^3\frac{\partial \eta}{\partial \tau} - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{4(h+\rho)h}\eta\eta_x - \frac{3\rho^2c(1+h)^2}{2(h+\rho)^2h}\eta^2\eta_x = 0$$

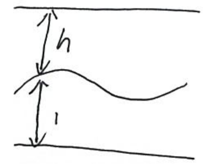
mkdV-11

浅水(下) — 深水(上)

假设上层水深 $h>>1$, $h\sim O(\lambda)$

定义小参数: $\mu=\frac{1}{\lambda}<<1$, $h=O(\frac{1}{\mu})$

且 $B=O(\frac{1}{\mu})$ $\quad kh=O(\mu)\cdot O(\frac{1}{\mu})=O(1)$



$\boxed{D=-i\partial x}$

$$\boxed{\partial x=O(\mu),\ \partial t=O(\mu),\ \eta=O(\mu),\ \xi=O(1),\ \rho=O(1),\ B=O(\tfrac{1}{\mu})}$$

$$G_0^-=D\tanh(D)=D(D-\frac{1}{3}D^3+O(\mu^5))=D^2-\frac{D^4}{3}+O(\mu^6)$$

$$=-\partial_{xx}-\frac{1}{3}\partial_{xxxx}+O(\mu^6)$$

$$G_1^-=D\eta D-\underset{\mu}{D}\tanh\underset{\mu}{(D)}\underset{\mu}{\eta}\tanh\underset{\mu}{(D)}\underset{\mu}{D}=D\eta D+O(\mu^5)$$

$\Rightarrow O(\mu^5)$

$$=(-i\partial_x)\eta(-i\partial_x)+O(\mu^5)=-\partial_x\eta\partial_x+O(\mu^5)$$

$$G_2^-=-\frac{1}{2}\underset{\mu^2}{D^2}\underset{\mu^2}{\eta^2}\tanh(D)D-\frac{1}{2}\overset{\mu}{D}\tanh\underset{\mu}{(D)}\eta^2D^2+D\tanh(D)\eta D\tanh(D)\eta D\tanh(D)$$

$\mu\ \mu^2\ \mu^2\Rightarrow O(\mu^6)$ $\quad \mu\ \mu\Rightarrow O(\mu^6)$

$$\boxed{G^-=G_0^-+G_1^-=-\partial_{xx}-\frac{1}{3}\partial_{xxxx}-\partial_x\eta\partial_x+O(\mu^5)}$$

$$G_0^+=\underset{\mu}{D}\tanh(hD)\qquad \boxed{h=O(\tfrac{1}{\mu})\quad D=O(\mu)\Rightarrow hD=O(1)}$$

$$G_1^+=-\underset{\mu\mu\mu}{D\eta D}+D\tanh(hD)\underset{\mu}{\eta}\tanh(hD)\underset{\mu}{D}\rightarrow\mu^3$$

$$=-(-i\partial_x)\eta(-i\partial_x)+G_0^+\eta G_0^+=\partial_x\eta\partial_x+G_0^+\eta G_0^+$$

$$G_2^+=-\frac{1}{2}\underset{\mu^2}{D^2}\underset{\mu^2}{\eta^2}\tanh(hD)\underset{\mu}{D}-\frac{1}{2}D\tanh(hD)\eta^2D^2+D\tanh(hD)\eta D\tanh(hD)\eta D\tanh(hD)$$

$$=O(\mu^5)$$

$$\boxed{G^+=G_0^++G_1^+=G_0^++\partial_x\eta\partial_x+G_0^+\eta G_0^++O(\mu^5)}$$

$$G^+ + \rho G^- = \underbrace{G_0^+}_{\mu} + \underbrace{\cancel{\partial_x\eta\partial_x}}_{\mu^3} + \underbrace{\cancel{G_0^+\eta G_0^+}}_{\mu^3} - \underbrace{\rho\partial_{xx}}_{\mu^2} - \underbrace{\cancel{\tfrac{1}{3}\rho\partial_{xxxx}}}_{\mu^4} - \underbrace{\cancel{\rho\partial_x\eta\partial_x}}_{\mu^3} + O(\mu^5)$$

$$G^+ + \rho G^- = G_0^+ - \rho\partial_{xx} + O(\mu^3)$$

$$(G^+ + \rho G^-)^{-1} = [G_0^+ - \rho\partial_{xx} + O(\mu^3)]^{-1}$$

$$= \cancel{[G_0^+(1+\rho(G_0^+)^{-1}\partial_{xx} + O(\mu^2))]^{-1}}$$

$$= [G_0^+(1-\rho(G_0^+)^{-1}\partial_{xx} + O(\mu^2))]^{-1}$$

算子求逆

$(AB)^{-1} = B^{-1}A^{-1}$

$$= [1-\rho(G_0^+)^{-1}\partial_{xx} + O(\mu^2)]^{-1}(G_0^+)^{-1}$$

$$= [1+\rho(G_0^+)^{-1}\partial_{xx} + O(\mu^2)](G_0^+)^{-1}$$

$(1+x)^{-1} = 1-x+\cdots$

$$(G^+ + \rho G^-)^{-1} = (G_0^+)^{-1} + \rho(G_0^+)^{-1}\partial_{xx}(G_0^+)^{-1} + O(\mu)$$

$\partial_x = O(\mu),\ \partial_t = O(\mu),\ \eta = O(\mu),\ \xi = O(1)$

总结:

$$\boxed{\begin{aligned} G^- &= -\partial_{xx} - \partial_x\eta\partial_x - \tfrac{1}{3}\partial_{xxxx} - O(\mu^5) \\ G_0^+ &= D\tanh(hD) \\ G^+ &= G_0^+ + G_0^+\eta G_0^+ + \partial_x\eta\partial_x + O(\mu^5) \\ (G^+ + \rho G^-)^{-1} &= (G_0^+)^{-1} + \rho(G_0^+)^{-1}\partial_{xx}(G_0^+)^{-1} + O(\mu) \end{aligned}}$$

$$\xi^- = (G^+ + \rho G^-)^{-1}G^+\xi = [\overset{\mu^{-1}}{(G_0^+)^{-1}} + \overset{1}{\rho(G_0^+)^{-1}\partial_{xx}(G_0^+)^{-1}} + O(\mu)](\overset{\mu}{G_0^+} + \overset{\mu^3}{G_0^+\eta G_0^+} + \overset{\mu^3}{\partial_x\eta\partial_x})\overset{1}{\xi}$$

$$= (G_0^+)^{-1}G_0^+\xi + O(\mu) = \xi + O(\mu)$$

$$\xi^+ = -(G^+ + \rho G^-)^{-1}G^-\xi = [\overset{\mu^{-1}}{(G_0^+)^{-1}} + \overset{1}{\rho(G_0^+)^{-1}\partial_{xx}(G_0^+)^{-1}} + O(\mu)](-\overset{\mu^2}{\partial_{xx}} - \overset{\mu^3}{\partial_x\eta\partial_x} - \overset{\mu^4}{\tfrac{1}{3}\partial_{xxxx}} \cdots)\xi$$

$$= 0 + O(\mu)$$

$$\boxed{\begin{aligned} \xi^- &= \xi + O(\mu) \\ \xi^+ &= 0 + O(\mu) \end{aligned}}$$

$$\eta_t - G^-(G^+ + \rho G^-)^{-1}G^+\xi = 0$$

$\partial_x = O(\mu)$ $\partial_t = O(\mu)$

$\eta = O(\mu)$ $\xi = O(1)$

$\xi = \xi^- - \rho\xi^+$

$$(G^+ + \rho G^-)^{-1}G^+ = \Big[\underbrace{\overset{\mu^{-1}}{(G_0^+)^{-1}}}_{①} + \underbrace{\rho (G_0^+)^{-1}\overset{1}{\partial_{xx}}(G_0^+)^{-1}}_{②} + O(\mu)\Big]\big(\overset{\mu}{G_0^+} + \overset{\mu^3}{G_0^+\eta G_0^+} + \overset{\mu^3}{\partial_x\eta\partial_x}\big)$$

$$= 1 + \rho(G_0^+)^{-1}\partial_{xx} + O(\mu^2)$$

$$G^-(G_0^+ + \rho G^-)^{-1}G^+ = \big(\underbrace{-\overset{\mu^2}{\partial_{xx}}}_{①} - \underbrace{\overset{\mu^3}{\partial_x\eta\partial_x}}_{②} - \overset{\mu^4}{\tfrac{1}{3}\partial_{xxxx}} - O(\mu^5)\big)\big(1 + \rho\overset{\mu}{(G_0^+)^{-1}\partial_{xx}} + O(\mu^2)\big)$$

$$= -\partial_{xx} - \rho(G_0^+)^{-1}\partial_{xxxx} - \partial_x\eta\partial_x + O(\mu^5)$$

Note: $G_0^+ = D\tanh(Dh)$, $D = -i\partial_x$

So: $(G_0^+)^{-1} = \coth(Dh)D^{-1}$ $\boxed{kh = O(1)}$

$$\widehat{(G_0^+)^{-1}} = \coth(kh)k^{-1}$$

$$\Rightarrow \widehat{(G_0^+)^{-1}\partial_{xxxx}} = \coth(kh)k^{-1}(ik)^4 = \coth(kh)k^3$$

$$= -k\coth(kh)(-k^2)$$

定义 $\hat{K} = k\coth(kh)$

$$\Rightarrow \widehat{(G_0^+)^{-1}\partial_{xxxx}} = -\hat{K}(-k^2)$$

$$\Rightarrow (G_0^+)^{-1}\partial_{xxxx} = -K\partial_{xx}$$

$$G^-(G^+ + \rho G^-)^{-1}G^+ = -\partial_{xx} + \rho K\partial_{xx} - \partial_x\eta\partial_x + O(\mu^5)$$

$$\eta_t - G^-(G^+ + \rho G^-)^{-1}G^+\xi = 0$$

$$\boxed{\eta_t + \partial_{xx}\xi - \rho K[\xi_{xx}] + \partial_x\eta\partial_x\xi = 0}$$

$$\underbrace{\xi_t}_{\mu} + \frac{1}{2}\left[\underbrace{(\xi_x^-)^2}_{\mu^2} - \frac{(\eta_t + \eta_x \xi_x^-)^2}{\cancel{1+\eta_x^2}} \right]^{=1M} - \frac{\rho}{2}\left[\overbrace{(\xi_x^+)^2}^{\mu^2} - \frac{(\eta_t+\eta_x \xi_x^+)^2}{1+\eta_x^2}^{\Rightarrow \mu^4} \right] + \underbrace{(1-\rho)\overbrace{\eta}^{\mu}}_{O(\frac{1}{\mu})} - \underbrace{B\frac{\eta_{xx}^{\ \mu^3}}{(1+\eta_x^2)^{\frac{3}{2}}}}_{\mu^4} = 0$$

$\partial_x = O(\mu)$

$\partial_t = O(\mu)$

$\eta = O(\mu)$

$\xi = O(1)$

$B = O(\frac{1}{\mu})$

Benimain − 4

$$\xi_t + \frac{1}{2}(\xi_x^-)^2 - \frac{1}{2}\rho(\xi_x^+)^2 + \underbrace{(1-\rho)\eta}_{O(\frac{1}{\mu})} - \underbrace{B\eta_{xx}}_{O(\mu^3)} = 0 \quad \Rightarrow O(\mu^2)$$

$$\xi^- = \xi + O(\mu) \qquad \xi^+ = 0$$

$$\xi^{\#} = \quad \xi_t + \frac{1}{2}\xi_x^2 - B\eta_{xx} + (1-\rho)\eta = 0$$

$$\boxed{\begin{aligned} &\eta_t + \partial_{xx}\xi + \partial_x\eta\,\partial_x\xi - \rho K[\xi_{xx}] = 0 \quad ① \\ &\xi_t + \frac{1}{2}\xi_x^2 + (1-\rho)\eta - B\eta_{xx} = 0 \quad ② \end{aligned}} \longrightarrow \eta = -\frac{\xi_t}{1-\rho}$$

$$\frac{\partial ②}{\partial t} = \xi_{tt} + (1-\rho)\eta_t + \frac{1}{2}\partial_t(\xi_x^2) - B\eta_{txx} = 0$$

$$\xi_{tt} + (1-\rho)\eta_t - B\eta_{txx} + \frac{1}{2}\partial_t(\xi_x^2) = 0$$

$\partial_x = O(\mu),\ \partial_t = O(\mu)$

$\eta = O(\mu),\ \xi = O(1)$

$B = O(\frac{1}{\mu}),\ kh = O(1)$

$\hat{K} = k\coth(kh)$

$$\xi_{tt} + (1-\rho)\left(-\xi_{xx} - \partial_x\eta\,\xi_x + \rho K[\xi_{xx}]\right)$$
$$-B\partial_{xx}\left(-\underbrace{\xi_{xx}}_{\mu^2} - \cancel{\underbrace{\partial_x\eta\,\xi_x}_{\mu^5}} + \rho K[\cancel{\underbrace{\xi_{xx}}_{\mu^3}}]\right) + \frac{1}{2}\partial_t(\xi_x^2) = 0$$

$$\xi_{tt} - (1-\rho)\xi_{xx} - (1-\rho)\partial_x(\eta\xi_x) + (1-\rho)\rho K[\xi_{xx}] + B\xi_{xxxx} + \frac{1}{2}\partial_t(\xi_x^2) = 0$$

$\eta = -\frac{\xi_t}{1-\rho}$

$$\xi_{tt} - (1-\rho)\xi_{xx} + \cancel{(1-\rho)}\partial_x\left(\frac{\xi_t}{\cancel{1-\rho}}\xi_x\right) + (1-\rho)\rho K[\xi_{xx}] + B\xi_{xxxx} + \frac{1}{2}\partial_t(\xi_x^2) = 0$$

$$\xi_{tt} - (1-\rho)\xi_{xx} + \partial_x(\xi_t\xi_x) + \frac{1}{2}\partial_t(\xi_x^2) + B\xi_{xxxx} + (1-\rho)\rho K[\xi_{xx}] = 0.$$

令 $1-\rho = c^2$

$$\underset{①}{\xi_{tt}} - \underset{②}{c^2\xi_{xx}} + \underset{③}{\partial_x(\xi_t\xi_x)} + \underset{④}{\frac{1}{2}\partial_t(\xi_x^2)} + \underset{⑤}{B\xi_{xxxx}} + \underset{⑥}{\rho c^2 K[\xi_{xx}]} = 0$$

令 $X = x - ct$, $T = \mu t$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X}, \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial X^2} \quad \cdots$$

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial X} + \mu\frac{\partial}{\partial T}$$

$$\frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial X^2} + \cancel{\mu^2\frac{\partial^2}{\partial T^2}} - 2c\mu\frac{\partial^2}{\partial X\partial T}$$

Benjmain-5

① $\xi_{tt} = c^2\xi_{xx} - 2c\mu\xi_{x\tau} + \mu^2\xi_{\tau\tau}$

② $-c^2\xi_{xx} = -c^2\xi_{xx}$

③ $\partial_x(\xi_t\xi_x) = \partial_x\left[(-c\xi_x + \mu\xi_\tau)\xi_x\right] = -c\,\partial_x(\xi_x^2)$

④ $\frac{1}{2}\partial_t(\xi_x^2) = \frac{1}{2}\left(-c\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial\tau}\right)(\xi_x^2) = -\frac{1}{2}c\,\partial_x(\xi_x^2)$

⑤ $B\xi_{xxxx} = B\xi_{xxxx}$

⑥ $\rho c^2 K[\xi_{xx}] = \rho c^2 K[\xi_{xx}]$

$\Rightarrow \cancel{c^2\xi_{xx}} - 2c\mu\xi_{x\tau} - \cancel{c^2\xi_{xx}} - c\,\partial_x(\xi_x^2) - \frac{1}{2}c\,\partial_x(\xi_x^2) + B\xi_{xxxx} + \rho c^2 K[\xi_{xx}] = 0$

$\mu\xi_{x\tau} + \frac{3}{4}\partial_x(\xi_x^2) - \frac{B}{2c}\xi_{xxxx} - \frac{\rho c}{2}K[\xi_{xx}] = 0$

令 $H = \xi_x$

$\Rightarrow \cancel{\mu\xi}\ \mu H_\tau + \frac{3}{4}\partial_x H^2 - \frac{B}{2c}H_{xxx} - \frac{\rho c}{2}K[H_x] = 0$

$$\boxed{\begin{aligned} &\eta = -\frac{\xi_t}{1-\rho} = \frac{c\xi_x}{1-\rho} = \frac{c\xi_x}{c^2} = \frac{\xi_x}{c} = \frac{H}{c} \\ &\Rightarrow H = c\eta \end{aligned}}$$

$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial x}$

$\xi_t = -c\,\xi_x$

$\Rightarrow \mu(c\eta)_\tau + \frac{3}{4}\partial_x(c^2\eta^2) - \frac{B}{2c}(c\eta)_{xxx} - \frac{\rho c}{2}K[c\eta_x] = 0$

$$\Rightarrow \mu(c\eta_t) + \frac{3}{2}c^2\eta_x\eta - \frac{B}{2c}c\,\eta_{xxx} - \frac{\rho c}{2}c\,K[\eta_x] = 0$$

$$\mu\eta_t + \frac{3c}{2}\eta\eta_x - \frac{B}{2c}\eta_{xxx} - \frac{\rho c}{2}K(\eta_x) = 0$$

浅水(下)一深水(上) 假设上层无限深，$h=\infty$

$\boxed{\mu=\frac{1}{\lambda}<<1}$

$\boxed{\partial_x=O(\mu),\ \partial_t=O(\mu),\ \eta=O(1),\ \xi=O(\tfrac{1}{\mu}),\ B=O(\tfrac{1}{\mu})}$

$\boxed{D=-i\partial_x}$

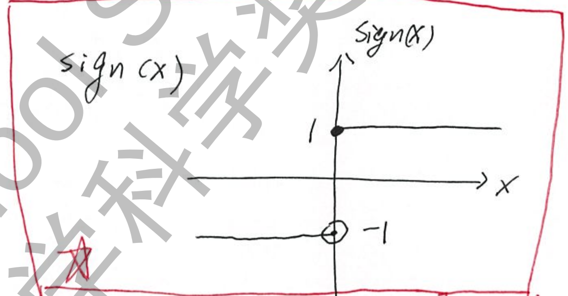
$$G_0^-=D\tanh(D)=D(D-O(\mu^3))=D^2+O(\mu^4)=-\partial_{xx}+O(\mu^4)$$

$$G_1^-=\underset{\mu}{D}\underset{1}{\eta}\underset{\mu}{D}-\underset{\mu}{D}\underset{\mu}{\tanh(D)}\underset{1}{\eta}\underset{\mu}{\tanh(D)}\underset{\mu}{D}\Rightarrow\mu^4=D\eta D+O(\mu^4)=-\partial_x\eta\partial_x+O(\mu^4)$$

$$G_2^-=-\frac{1}{2}\underset{\mu^2}{D^2}\eta^2\underset{\mu}{\tanh(D)}\underset{\mu}{D}-\frac{1}{2}\underset{\mu}{D}\underset{\mu}{\tanh(D)}\eta^2\underset{\mu^2}{D^2}+\underset{\mu}{D}\underset{\mu}{\tanh(D)}\eta\underset{\mu}{D}\underset{\mu}{\tanh(D)}\eta\underset{\mu}{D}\underset{\mu}{\tanh(D)}$$

$$=O(\mu^4)\qquad\Rightarrow\mu^4\qquad\Rightarrow\mu^4\qquad\Rightarrow\mu^6$$

$\boxed{G^-=G_0^-+G_1^-=-\partial_{xx}-\partial_x\eta\partial_x+O(\mu^4)}$



注意，我们假设 $h=\infty$

$$G_0^+=D\tanh(hD)\Big|_{h=\infty}=D\,\mathrm{sign}(D)=|D|$$

$$|D|=(-\partial_{xx})^{\frac{1}{2}}$$

$$\widehat{|D|}=|k|$$

$$G_1^+=-D\eta D+D\tanh(hD)\eta\tanh(hD)D\Big|_{h=\infty}$$

$$=-\underset{\mu}{D}\eta\underset{\mu}{D}+\underset{\mu}{|D|}\eta\underset{\mu}{|D|}=-(-i\partial_x)\eta(-i\partial_x)+|D|\eta|D|$$

$$=\partial_x\eta\partial_x+|D|\eta|D|$$

$$|D|=(-\partial_{xx})^{\frac{1}{2}}$$

$$\widehat{|D|}=\widehat{(-\partial_{xx})^{\frac{1}{2}}}=(+k^2)^{\frac{1}{2}}=|k|$$

$$G_2^+=-\frac{1}{2}\underset{\mu^2}{D^2}\eta^2\tanh(hD)\underset{\mu}{D}-\frac{1}{2}\underset{\mu}{D}\tanh(hD)\eta^2\underset{\mu^2}{D^2}+\underset{\mu}{D}\tanh(hD)\eta\underset{\mu}{D}\tanh(hD)\eta\underset{\mu}{D}\tanh(hD)$$

$$=O(\mu^3)$$

$\boxed{G^+=G_0^++G_1^+=|D|+\partial_x\eta\partial_x+|D|\eta|D|+O(\mu^3)}$

$\boxed{G^++\rho G^-=|D|+\partial_x\eta\partial_x+|D|\eta|D|-\rho\partial_{xx}-\rho\partial_x\eta\partial_x+O(\mu^3)}$

$$(G^+ + \rho G^-) = |D|\left(1 + |D|^{-1}\partial_x\eta\partial_x + \eta|D| - \rho|D|^{-1}\partial_{xx} - \rho|D|^{-1}\partial_x\eta\partial_x + O(\mu^2)\right)$$

算子求逆 $(AB)^{-1} = B^{-1}A^{-1}$ $\quad (1+X)^{-1} = 1 - X + \cdots$

$$(G^+ + \rho G^-)^{-1} = \left[1 + |D|^{-1}\partial_x\eta\partial_x + \eta|D| - \rho|D|^{-1}\partial_{xx} - \rho|D|^{-1}\partial_x\eta\partial_x + O(\mu^2)\right]^{-1}|D|^{-1}$$

$$= \left[1 - |D|^{-1}\partial_x\eta\partial_x - \eta|D| + \rho|D|^{-1}\partial_{xx} + \rho|D|^{-1}\partial_x\eta\partial_x + O(\mu^2)\right]|D|^{-1}$$

$$= |D|^{-1} - \underline{|D|^{-1}\partial_x\eta\partial_x|D|^{-1}} - \eta + \rho|D|^{-1}\partial_{xx}|D|^{-1} + \underline{\rho|D|^{-1}\partial_x\eta\partial_x|D|^{-1}} + O(\mu)$$

$$= |D|^{-1} - \eta - (1-\rho)|D|^{-1}\partial_x\eta\partial_x|D|^{-1} + \rho|D|^{-1}\partial_{xx}|D|^{-1} + O(\mu)$$

Note $\widehat{|D|^{-1}\partial_{xx}|D|^{-1}} = \frac{1}{|k|}(-k^2)\frac{1}{|k|} = -\frac{k^2}{k^2} = -1$ $\quad D = -i\partial_x$, $\hat{D} = k$, $\hat{\partial_x} = ik$

$\Rightarrow |D|^{-1}\partial_{xx}|D|^{-1} = -1$

$$\boxed{\begin{aligned}|D|^{-1}\partial_{xx}|D|^{-1}f &= \frac{1}{2\pi}\int \widehat{|D|^{-1}\partial_{xx}|D|^{-1}f}\, e^{ikx}\,dk \\ &= \frac{1}{2\pi}\int -1\,\hat{f}\, e^{ikx}\,dk = -\frac{1}{2\pi}\int \hat{f}\, e^{ikx}\,dk = -f\end{aligned}}$$

$$\Rightarrow (G^+ + \rho G^-)^{-1} = |D|^{-1} - \eta - (1-\rho)|D|^{-1}\partial_x\eta\partial_x|D|^{-1} - \rho + O(\mu)$$

$$= |D|^{-1} - \eta - \rho - (1-\rho)|D|^{-1}\partial_x\eta\partial_x|D|^{-1} + O(\mu)$$

总结:

$$\boxed{\begin{aligned}G^+ &= |D| + \partial_x\eta\partial_x + |D|\eta|D| + O(\mu^3) \\ G^- &= -\partial_{xx} - \partial_x\eta\partial_x + O(\mu^4) \\ (G^+ + \rho G^-)^{-1} &= |D|^{-1} - \eta - \rho - (1-\rho)|D|^{-1}\partial_x\eta\partial_x|D|^{-1} + O(\mu)\end{aligned}}$$

$\xi^- = (G^+ + \rho G^-)^{-1} G^+ \xi$ $\qquad$ $\partial_x = O(\mu),\ \partial_t = O(\mu),\ \eta = O(1),\ \xi = O(\frac{1}{\mu}),\ B = O(\frac{1}{\mu}),\ D = O(\mu)$

$$\xi^- = \Big[\overset{\mu^{-1}}{|D|^{-1}} - \overset{1}{\eta} - \overset{1}{\rho} - (1-\rho)\overset{1}{|D|^{-1}\partial_x\eta\partial_x|D|^{-1}} + O(\mu)\Big]\Big(\overset{\mu}{|D|} + \overset{\mu^2}{\partial_x\eta\partial_x} + \overset{\mu^2}{|D|\eta|D|} + O(\mu^3)\Big)\xi$$

$$= \Big\{1 + |D|^{-1}\partial_x\eta\partial_x + \cancel{\eta|D|} - \cancel{\eta|D|} - \rho|D| - (1-\rho)|D|^{-1}\partial_x\eta\partial_x|D|^{-1}|D| + O(\mu^2)\Big\}\xi$$

$$= \Big\{1 - \rho|D| + \cancel{|D|^{-1}\partial_x\eta\partial_x} - \cancel{|D|^{-1}\partial_x\eta\partial_x} + \rho|D|^{-1}\partial_x\eta\partial_x + O(\mu^2)\Big\}\xi$$

$$\boxed{\xi^- = \xi - \rho|D|\xi + \rho|D|^{-1}\partial_x\eta\partial_x\xi + O(\mu)}$$

$$\xi^+ = -(G^+ + \rho G^-)^{-1} G^- \xi$$

$$\xi^+ = -\Big[\overset{\mu^{-1}}{|D|^{-1}} - \overset{1}{\eta} - \overset{1}{\rho} - (1-\rho)\overset{1}{|D|^{-1}\partial_x\eta\partial_x|D|^{-1}} + O(\mu)\Big]\Big(-\overset{\mu^2}{\partial_{xx}} - \overset{\mu^2}{\partial_x\eta\partial_x}\Big)\xi$$

$$= -\Big[-|D|^{-1}\partial_{xx} - |D|^{-1}\partial_x\eta\partial_x + O(\mu^2)\Big]\xi$$

$$= -\Big[|D| - |D|^{-1}\partial_x\eta\partial_x + O(\mu^2)\Big]\xi \longleftarrow \boxed{\begin{aligned}&|D|^2 = -\partial_{xx}\\ &|D| = -|D|^{-1}\partial_{xx}\end{aligned}}$$

$$\boxed{\xi^+ = -|D|\xi + |D|^{-1}\partial_x\eta\partial_x\xi + O(\mu)}$$

$$\eta_t - G^-(G^+ + \rho G^-)^{-1} G^+ \xi = 0 \qquad \partial_x = O(\mu),\ |D| = O(\mu),\ \eta = O(1)$$

$$(G^+ + \rho G^-)^{-1} G^+ = \Big(\overset{\mu^{-1}}{|D|^{-1}} - \overset{1}{\eta} - \overset{1}{\rho} - (1-\rho)\overset{1}{|D|^{-1}\partial_x \eta \partial_x |D|^{-1}} + O(\mu)\Big)\Big(\overset{\mu}{|D|} + \overset{\mu^2}{\partial_x \eta \partial_x} + \overset{\mu^2}{|D|\eta|D|} + O(\mu^3)\Big)$$

$$= 1 + |D|^{-1}\partial_x \eta \partial_x + \cancel{\eta|D|} - \cancel{\eta|D|} - \rho|D| - (1-\rho)|D|^{-1}\partial_x \eta \partial_x + O(\mu^2)$$

$$= 1 - \rho|D| + \rho|D|^{-1}\partial_x \eta \partial_x + O(\mu^2)$$

$$G^-(G^+ + \rho G^-)^{-1} G^+ = \Big(\underbrace{\overset{\mu^2}{-\partial_{xx}}}_{①} \underbrace{\overset{\mu^2}{- \partial_x \eta \partial_x}}_{②} + O(\mu^4)\Big)\Big(1 - \rho\overset{\mu}{|D|} + \rho\overset{\mu}{|D|^{-1}\partial_x \eta \partial_x} + O(\mu^3)\Big)$$

$$= -\partial_{xx} + \rho\partial_{xx}|D| - \rho\partial_{xx}|D|^{-1}\partial_x \eta \partial_x$$

$$- \partial_x \eta \partial_x + \rho\partial_x \eta \partial_x |D| - \rho\partial_x \eta \underline{\partial_x |D|^{-1} \partial_x} \eta \partial_x$$

$$|D|^{-1}\partial_{xx} = -|D|$$

可交换顺序

$\partial_x f = \frac{1}{2\pi}\int ik \hat{f} e^{ikx} dk$	$\lvert D\rvert^{-1}\partial_{xx} = -\lvert D\rvert$
$\lvert D\rvert^{-1}\partial_x f = \frac{1}{2\pi}\int \frac{1}{\lvert k\rvert} ik \hat{f} e^{ikx} dk$	$\widehat{\lvert D\rvert} = \lvert k\rvert$
$\lvert D\rvert^{-1} f = \frac{1}{2\pi}\int \frac{1}{\lvert k\rvert} \hat{f} e^{ikx} dk$	$\widehat{\lvert D\rvert^2} = k^2$
$\partial_x \lvert D\rvert^{-1} f = \frac{1}{2\pi}\int ik \frac{1}{\lvert k\rvert} \hat{f} e^{ikx} dk$	$\widehat{\partial_{xx}} = -k^2$
	$-\partial_{xx} = \lvert D\rvert^2$

$$= -\partial_{xx} + \rho|D|\partial_{xx} + \rho|D|\partial_x \eta \partial_x$$

$$- \partial_x \eta \partial_x + \rho\partial_x \eta \partial_x |D| + \rho\partial_x \eta |D| \eta \partial_x$$

$$G^-(G^+ + \rho G^-)^{-1} G^+ = -\partial_{xx} - \partial_x \eta \partial_x + \rho |D| \partial_{xx} + \rho \partial_x \left(|D| \eta \partial_x + \eta |D| \partial_x + \eta |D| \eta \partial_x \right)$$

$\partial_x = O(\mu),\ |D| = O(\mu),\ \eta = O(1),\ \xi = O(\frac{1}{\mu})$

$B = O(\frac{1}{\mu})$

5-5

$$G^-(G^+ + \rho G^-)^{-1} G^+ \xi = -\partial_{xx} \xi - \partial_x \eta \partial_x \xi + \rho |D| \partial_{xx} \xi + \rho \partial_x \left(|D| \eta \partial_x \xi + \eta |D| \partial_x \xi + \eta |D| \eta \partial_x \xi \right)$$

$$\eta_t - G^-(G^+ + \rho G^-)^{-1} G^+ \xi = 0$$

$$\eta_t - \xi_{xx} + (\eta \xi_x)_x - \rho |D| \xi_{xx} - \rho \partial_x \left(|D| \eta \partial_x \xi + \eta |D| \partial_x \xi + \eta |D| \eta \partial_x \xi \right) = 0$$

令 $H = \xi_x$

$$\boxed{\eta_t - H_x + (\eta H)_x - \rho |D| H_x - \rho \partial_x \left(|D| \eta H + \eta |D| H + \eta |D| \eta H \right) = 0}$$

$$\underset{1}{\xi_t} + \frac{1}{2}\left[(\overset{1}{\xi_x^-})^2 - \frac{\cancel{(\eta_t + \eta_x \xi_x^-)^2}^{\mu^2}}{(1+\eta_x^2)} \right] - \frac{\rho}{2}\left[(\overset{1}{\xi_x^+})^2 - \frac{\cancel{(\eta_t + \eta_x \xi_x^+)^2}^{\mu^2}}{1+\eta_x^2} \right] + (1-\rho)\overset{\mu}{\eta} - \overset{\frac{1}{\mu}}{B} \frac{\overset{\mu^2}{\eta_{xx}}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$\boxed{\xi_t + \frac{1}{2}(\xi_x^-)^2 - \frac{1}{2}\rho (\xi_x^+)^2 + (1-\rho)\eta - B\eta_{xx} = 0}$$

$$\xi_x^- = \xi_x - \rho |D| \xi_x + \rho |D|^{-1} \partial_{xx} \eta \partial_x \xi + O(\mu^2)$$

$$\boxed{\xi_x^- = \xi_x - \rho |D| \xi_x - \rho |D| \eta \partial_x \xi + O(\mu^2)}$$

$$\xi_x^+ = -|D| \xi_x + |D|^{-1} \partial_{xx} \eta \partial_x \xi + O(\mu^2)$$

$$\boxed{\xi_x^+ = -|D| \xi_x - |D| \eta \partial_x \xi + O(\mu^2)}$$

$$\boxed{\begin{aligned} -|D|^2 &= \partial_{xx} \\ |D|^{-1} \partial_{xx} &= -|D| \end{aligned}}$$

$$\frac{1}{2}(\xi_x^-)^2 = \frac{1}{2}\left(\overset{1}{\xi_x} - \rho \overset{\mu}{|D|}\xi_x - \rho \overset{\mu}{|D|}\eta \partial_x \xi\right)^2 \qquad \partial_x = O(\mu),\ |D| = O(\mu),\ \eta = O(1),\ \xi = O\left(\frac{1}{\mu}\right),\ B = O\left(\frac{1}{\mu}\right)$$

$$= \frac{1}{2}\left(\overset{1}{\xi_x}{}^2 - 2(\xi_x)\cdot \rho \overset{\mu}{|D|}\xi_x - 2(\xi_x)\cdot \rho \overset{\mu}{|D|}\eta \partial_x \xi + O(\mu^2)\right)$$

$$= \frac{1}{2}\xi_x^2 - \xi_x \rho |D| \xi_x - \xi_x \rho |D| \eta \xi_x + O(\mu^2)$$

$$\frac{\rho}{2}(\xi_x^+)^2 = \frac{\rho}{2}\left(-\overset{\mu}{|D|}\xi_x - \overset{\mu}{|D|}\eta \partial_x \xi + O(\mu^2)\right)^2 = 0 + O(\mu^2)$$

$$\xi_t + \frac{1}{2}\xi_x^2 - \xi_x \rho |D| \xi_x - \xi_x \rho |D| \eta \xi_x + (1-\rho)\eta - B\eta_{xx} = 0$$

$$\Rightarrow \xi_{xt} + (1-\rho)\eta_x - B\eta_{xxx} + \frac{1}{2}(\xi_x^2)_x - \partial_x(\xi_x \rho |D| \xi_x + \xi_x \rho |D| \eta \xi_x) = 0$$

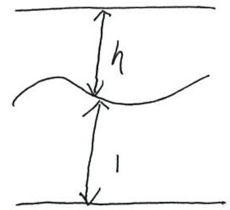
令 $H = \xi_x$

$$\Rightarrow \boxed{H_t + (1-\rho)\eta_x - B\eta_{xxx} + \cancel{\tfrac{1}{2}} HH_x - \partial_x(H\rho |D| H) - \partial_x(H\rho |D| \eta H) = 0}$$

$$\eta_t + H_x + (\eta H)_x - \rho |D| H - \rho \partial_x(|D|\eta H + \eta |D| H + \eta |D| \eta H) = 0$$

L

深水(下)—浅水(上)



$$\boxed{h<<1，定义\ \mu^2=\frac{h}{\lambda}\quad \lambda=O(1)}$$

$$\boxed{\begin{aligned}&\partial x=O(1),\ \partial t=O(\mu),\ \eta=O(\mu^2)\\ &\xi=O(\mu),\ h=O(\mu^2),\ D=O(1)\end{aligned}}$$

$$G_0^- = D\tanh(D)$$

$$G_1^- = D\eta D - D\tanh(hD)\eta\tanh(hD)D = (-i\partial_x)\eta(-i\partial_x) - G_0^-\eta G_0^-$$

$$= -\partial_x\eta\partial_x - G_0^-\eta G_0^-$$

$$G_2^- = -\frac{1}{2}\underset{\mu^4}{D^2\eta^2}\tanh(D)D - \frac{1}{2}D\tanh(D)\underset{\mu^4}{\eta^2}D^2 + D\tanh(D)\underset{\mu^2}{\eta}D\tanh(D)\underset{\mu^2}{\eta}D\tanh(D)$$

$$\boxed{G^- = G_0^- + G_1^- = G_0^- - \partial_x\eta\partial_x - G_0^-\eta G_0^- + O(\mu^4)}$$

$$G_0^+ = D\tanh(hD) = D\left(hD - \frac{h^3D^3}{3} + O(\mu^{10})\right) = \overset{\mu^2}{hD^2} - \overset{\mu^6}{\frac{h^3}{3}D^4} + O(\mu^5)$$

$$= h(-i\partial_x)^2 + O(\mu^6) = -h\partial_{xx} + O(\mu^6)$$

$$G_1^+ = -D\eta D + D\tanh(hD)\eta\tanh(hD)D = -(i\partial_x)\eta(-i\partial_x) + O(\mu^4)$$

$$= \partial_x\eta\partial_x + O(\mu^4)$$

$\tanh = x - \frac{x^3}{3} + \cdots \longrightarrow$ $hD = O(\mu^2\cdot 1) = O(\mu^2)$

$\tanh(hD) = O(\mu^2)$

$$G_2^+ = -\frac{1}{2}\underset{\mu^4}{D^2\eta^2}\underset{\mu^2}{\tanh(hD)}D - \frac{1}{2}D\underset{\mu^2}{\tanh(hD)}\underset{\mu^4}{\eta^2}D^2 + D\underset{\mu^2}{\tanh(hD)}\underset{\mu^2}{\eta}D\underset{\mu^2}{\tanh(hD)}\underset{\mu^2}{\eta}D\underset{\mu^2}{\tanh(h}$$

$$\boxed{G^+ = G_0^+ + G_1^+ = -h\partial_{xx} + \partial_x\eta\partial_x + O(\mu^4)}$$

$$\boxed{G^+ + \rho G^- = -h\partial_{xx} + \partial_x\eta\partial_x + \rho(G_0^- - \partial_x\eta\partial_x - G_0^-\eta G_0^-) + O(\mu^4)}$$

$$G^+ + \rho G^- = \rho G_0^- - \rho \partial_x \eta \partial_x - \rho G_0^- \eta G_0^- - h\partial_{xx} + \partial_x \eta \partial_x + O(\mu^4)$$

$\partial_x = O(1),\ \partial_t = O(\mu),\ \eta = O(\mu^2)$
$\xi = O(\mu),\ h = O(\mu^2),\ D = O(1)$

$$= \rho G_0^- \Big[1 - \underbrace{(G_0^-)^{-1}\partial_x \eta \partial_x}_{\mu^2} - \underbrace{\eta G_0^-}_{\mu^2} - \underbrace{h \rho^{-1}(G_0^-)^{-1}}_{\mu^2} + \underbrace{\rho^{-1}(G_0^-)^{-1}\partial_x \eta \partial_x}_{\mu^2} + O(\mu^4) \Big]$$

算子求逆 $(AB)^{-1} = B^{-1}A^{-1}$

$$(G^+ + \rho G^-)^{-1} = \rho^{-1}\Big[1 - \underbrace{\cancel{(G_0^-)^{-1}\partial_x \eta \partial_x}}_{\mu^2} - \underbrace{\cancel{\eta G_0^-}}_{\mu^2} - \underbrace{\cancel{h\rho^{-1}(G_0^-)^{-1}}}_{\mu^2} + \underbrace{\cancel{\rho^{-1}(G_0^-)^{-1}\partial_x \eta \partial_x}}_{\mu^2} + O(\mu^4) \Big]^{-1} (G_0^-)^{-1}$$

$$\boxed{(G^+ + \rho G^-)^{-1} = \rho^{-1}(G_0^-)^{-1} + O(\mu^2) = \frac{1}{\rho}(G_0^-)^{-1} + O(\mu^2)}$$

$$\xi^- = (G^+ + \rho G^-)^{-1} G^+ \xi = \Big[\frac{1}{\rho} \underbrace{(G_0^-)^{-1}}_{1} \big(\underbrace{-h\partial_{xx}}_{\mu^2} + \underbrace{\partial_x \eta \partial_x}_{\mu^2} + O(\mu^4) \big) \Big] \underbrace{\xi}_{\mu} = 0 + O(\mu^3)$$

$$\xi^+ = -(G^+ + \rho G^-)^{-1} G^- \xi = -\Big[\frac{1}{\rho} \underbrace{(G_0^-)^{-1}}_{1} + O(\mu^2) \Big] \big(\underbrace{G_0^-}_{1} - \underbrace{\cancel{\partial_x \eta \partial_x}}_{\mu^2} - \underbrace{\cancel{G_0^- \eta G_0^-}}_{\mu^2} + O(\mu^4) \big) \xi$$

$$\boxed{\begin{aligned} \xi^+ &= -\frac{1}{\rho}\xi + O(\mu^2) \\ \xi^- &= 0 \end{aligned}}$$

B-2

$$\eta_t = G^-(G^+ + \rho G^-)^{-1} G^+ \xi$$

$\partial x = O(1)$, $\partial t = O(\mu)$, $\eta = O(\mu^2)$, $\xi = O(\mu)$

$h = O(\mu^2)$ $\quad G_0^- = O(1)$

$$(G^+ + \rho G^-)^{-1} G^+ = \Big[\frac{1}{\rho}\underbrace{(G_0^-)^{-1}}_{1} + O(\mu^2)\Big]\big(-\underbrace{h}_{\mu^2}\partial_{xx} + \underbrace{\partial_x \eta}_{\mu^2} \partial_x + O(\mu^4)\big)$$

$$= -\frac{1}{\rho}(G_0^-)^{-1} h\partial_{xx} + \frac{1}{\rho}(G_0^-)^{-1}\partial_x\eta\partial_x + O(\mu^4)$$

$$G^-(G^+ + \rho G^-)^{-1}G^+ = (G_0^- - \partial_x\eta\partial_x - G_0^-\eta G_0^- + O(\mu^4))\big(-\frac{1}{\rho}(G_0^-)^{-1}h\partial_{xx} + \frac{1}{\rho}(G_0^-)^{-1}\partial_x\eta\partial_x + O(\mu^4)\big)$$

$$= -\frac{1}{\rho}h\partial_{xx} + \frac{1}{\rho}\partial_x\eta\partial_x + O(\mu^4)$$

$$\eta_t - G^-(G^+ + \rho G^-)^{-1}G^+\xi = 0$$

$$\Rightarrow \boxed{\eta_t + \frac{1}{\rho}h\xi_{xx} - \frac{1}{\rho}\partial_x\eta\,\xi_x = 0}$$

$$\underset{\mu^2}{\xi_t} + \frac{1}{2}\left[\underset{\mu^2}{(\xi_x^-)^2} - \cancel{\frac{(\eta_t + \eta_x \xi_x^-)^2}{1+\eta_x^2}}_{\mu^6}\right] - \frac{\rho}{2}\left[(\xi_x^+)^2 - \cancel{\frac{(\eta_t + \eta_x \xi_x^+)^2}{1+\eta_x^2}}_{\mu^6}\right] + \underset{\mu^2}{(1-\rho)\eta} - B\frac{\overset{\mu^2}{\eta_{xx}}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0$$

$$\xi_t + (1-\rho)\eta - B\eta_{xx} + \frac{1}{2}(\xi_x^-)^2 - \frac{\rho}{2}(\xi_x^+)^2 = 0$$

$$\boxed{\begin{cases} \xi^+ = -\frac{1}{\rho}\xi + O(\mu^2) \\ \xi^- = 0 + O(\mu^3) \end{cases}}$$

$\partial_x = O(1)$

$\partial_t = O(\mu)$

$\eta = O(\mu^2)$

$\xi = O(\mu)$

$h = O(\mu^2)$

$$\boxed{\xi_t + (1-\rho)\eta - B\eta_{xx} - \frac{\rho}{2}\frac{1}{\rho^2}(\xi_x^2) = 0}$$

$$\boxed{\eta_t + \frac{1}{\rho}h\,\xi_{xx} - \frac{1}{\rho}\partial_x(\eta\,\xi_x) = 0}$$

B-1

$$\begin{cases} \eta_t + \frac{h}{\rho}\xi_{xx} - \frac{1}{\rho}\partial_x(\eta\xi_x) = 0 \\ \xi_t + (1-\rho)\eta - B\eta_{xx} - \frac{1}{2\rho}\xi_x^2 = 0 \end{cases}$$

令 $\eta = h(1-\Lambda)$ $\xi_x = -\rho\sqrt{h}\,U$ $t = \frac{\tau}{\sqrt{h}}$ $\tau = \sqrt{h}\,t$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} = \cancel{\sqrt{h}\frac{\partial}{\partial t}}\ \sqrt{h}\frac{\partial}{\partial \tau}$$

$$\eta_t = \sqrt{h}\frac{\partial}{\partial \tau}(h(1-\Lambda)) = -h^{\frac{3}{2}}\frac{\partial \Lambda}{\partial \tau}$$

$$\frac{h}{\rho}\xi_{xx} = \frac{h}{\rho}(-\rho\sqrt{h}\,U)_x = -h^{\frac{3}{2}}U_x$$

$$-\frac{1}{\rho}\partial_x(\eta\xi_x) = -\frac{1}{\rho}\partial_x[h(1-\Lambda)(-\rho\sqrt{h})U] = h^{\frac{3}{2}}\partial_x[(1-\Lambda)U] = \cancel{h^{\frac{3}{2}}\partial}\ h^{\frac{3}{2}}\partial_x(U - \Lambda U) = U_x - \partial_x(\Lambda U)$$

$$\Rightarrow -h^{\frac{3}{2}}\frac{\partial \Lambda}{\partial \tau} - \cancel{h^{\frac{3}{2}}U_x} + \cancel{h^{\frac{3}{2}}U_x} - h^{\frac{3}{2}}\partial_x(\Lambda U) = 0 \Rightarrow \boxed{\Lambda_\tau - \partial_x(\Lambda U) = 0}$$

$$\xi_t + (1-\rho)\eta - B\eta_{xx} - \frac{1}{2\rho}\xi_x^2 = 0$$

$$\xi_{xt} + (1-\rho)\eta_x - B\eta_{xxx} - \frac{1}{\rho}\xi_x\xi_{xx} = 0$$

$$\xi_{xt} = \frac{\partial \xi_x}{\partial t} = \frac{\partial \xi_x}{\partial \tau}\sqrt{h} = \sqrt{h}\frac{\partial}{\partial \tau}(-\rho\sqrt{h}\,U) = -\rho h U_\tau$$

$$(1-\rho)\eta_x = (1-\rho)(h(1-\Lambda))_x = -(1-\rho)h\Lambda_x$$

$$-B\eta_{xxx} = -B(h(1-\Lambda))_{xxx} = Bh\Lambda_{xxx}$$

$$-\frac{1}{\rho}\xi_x\xi_{xx} = -\frac{1}{\rho}(-\rho\sqrt{h}\,U)(-\rho\sqrt{h}\,U)_x = -\rho h U U_x$$

$$-\rho h U_\tau - (1-\rho)h\Lambda_x + Bh\Lambda_{xxx} - \rho h U U_x = 0$$

$$\boxed{\rho(U_\tau + UU_x) - B\Lambda_{xxx} + (1-\rho)\Lambda_x = 0}$$

$$\begin{cases} \Lambda_\tau + \partial_x(\Lambda U) = 0 \\ \rho(U_\tau + UU_x) - B\Lambda_{xxx} + (1-\rho)\Lambda_x = 0 \end{cases}$$

· Numerical Computation

$$\eta_t + \underbrace{\mathcal{L}_1[\eta]}_{\text{线性部分}} + \underbrace{\mathcal{N}[\eta]}_{\text{非线性部分}} = 0$$

令 $X = x - ct \Rightarrow \eta_t = -c\eta_x$

$\Rightarrow -c\eta_x + \mathcal{L}_1[\eta] + \mathcal{N}[\eta] = 0$ 令 $\mathcal{L}[\eta] = -c\eta_x + \mathcal{L}_1[\eta]$

$\Rightarrow \mathcal{L}[\eta] + \mathcal{N}[\eta] = 0 \Rightarrow$

作 Fourier 变换

$$\hat{\mathcal{L}}\hat{\eta} + \widehat{\mathcal{N}[\eta]} = 0 \Rightarrow \hat{\eta} = -\frac{\widehat{\mathcal{N}[\eta]}}{\hat{\mathcal{L}}} = P[\hat{\eta}]$$

迭代格式 $$\hat{\eta}_{n+1} = \left(\frac{\int |\hat{\eta}_n|^2 dk}{\int \hat{\eta}_n^* P[\hat{\eta}_n] dk}\right)^m P[\hat{\eta}_n]$$

收敛性判定 $|\hat{\eta}_{n+1} - \hat{\eta}_n| < \varepsilon = 1\times 10^{-9}$

Num-1

$$\text{KdV}: \eta_\tau - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x = 0 \qquad X = X - V\tau$$

$$-V\eta_x - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x = 0 \qquad \mathcal{L} = -V\partial_x - \frac{\alpha}{2c}\partial_{xxx}$$

$$\Rightarrow \hat{\mathcal{L}} = -iVk + \frac{\alpha}{2c}ik^3$$

$$\Rightarrow \left(-iVk + \frac{\alpha}{2c}ik^3\right)\hat{\eta} + \mathcal{F}\left[\frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x\right] = 0$$

$$\hat{\eta} = -\frac{1}{-iVk + \frac{\alpha}{2c}ik^3}\mathcal{F}\left[\frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x\right] =: P_{kdv}[\hat{\eta}]$$

$$\hat{\eta}_{n+1} = \left(\frac{\int |\hat{\eta}_n|^2 dk}{\int \hat{\eta}_n^* P_{kdv}[\hat{\eta}_n]dk}\right)^m P_{kdv}[\hat{\eta}_n]$$

Num 2

5th KdV

$$\eta_\tau - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x + \frac{\beta}{2c}\eta_{xxxxx} = 0$$

令 $X = x - c\tau$

$$-v\eta_x - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x + \frac{\beta}{2c}\eta_{xxxxx} = 0$$

$$-vik\hat{\eta} + i\frac{\alpha}{2c}k^3\hat{\eta} + i\frac{\beta}{2c}k^5\hat{\eta} + \mathcal{F}\left[\frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x\right] = 0$$

$$\hat{\eta} = -\frac{1}{-ivk + i\frac{\alpha}{2c}k^3 + i\frac{\beta}{2c}k^5}\mathcal{F}\left[\frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x\right] =: P_{5th\,kdv}[\hat{\eta}]$$

$$\hat{\eta}_{n+1} = \left(\frac{\int |\hat{\eta}_n|^2 dk}{\int \hat{\eta}_n^* P_{5th\,kdv}[\hat{\eta}_n]\,dk}\right)^m P_{5th\,kdv}[\hat{\eta}_n]$$

Num-3

m kdV

$$\eta_\tau - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x - \frac{3c\rho(1+h)^2}{h(h+\rho)^2}\eta^2\eta_x = 0$$

令 $X = x - V\tau$

$$-V\eta_x - \frac{\alpha}{2c}\eta_{xxx} + \frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x - \frac{3c\rho(1+h)^2}{h(h+\rho)^2}\eta^2\eta_x = 0$$

$$-Vik\hat{\eta} + i\frac{\alpha}{2c}k^3\hat{\eta} + \mathcal{F}\left[\frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x - \frac{3c\rho(1+h)^2}{h(h+\rho)}\eta^2\eta_x\right] = 0$$

$$\hat{\eta} = -\frac{1}{-iVk + i\frac{\alpha}{2c}k^3}\mathcal{F}\left[\frac{3c(h^2-\rho)}{2h(h+\rho)}\eta\eta_x - \frac{3c\rho(1+h)^2}{h(h+\rho)^2}\eta^2\eta_x\right] =: P_{5th\,kdV}[\hat{\eta}]$$

$$\hat{\eta}_{n+1} = \left(\frac{\int|\hat{\eta}_n|^2 dk}{\int\hat{\eta}_n^* P_{5th\,kdV}[\hat{\eta}_n]dk}\right)^m P_{5th\,kdV}[\hat{\eta}_n]$$

Num-4

Benjamin

$$\eta_t + \frac{3c}{2}\eta\eta_x - \frac{B}{2c}\eta_{xxx} - \frac{\rho c}{2}K[\eta_x] = 0$$

令 $x = x - Vt$

$$\Rightarrow -V\eta_x + \frac{3c}{2}\eta\eta_x - \frac{B}{2c}\eta_{xxx} - \frac{\rho c}{2}K[\eta_x] = 0$$

$$-Vik\hat{\eta} + i\frac{B}{2c}k^3\hat{\eta} - i\frac{\rho c}{2}k^2\coth(kh)\hat{\eta} + \mathcal{F}\left[\frac{3c}{2}\eta\eta_x\right] = 0$$

$$\hat{\eta} = -\frac{1}{-ivk + i\frac{B}{2c}k^3 - i\frac{\rho c}{2}k^2\coth(kh)}\mathcal{F}\left(\frac{3c}{2}\eta\eta_x\right) =: P_{Benjamin}[\hat{\eta}]$$

$$\hat{\eta}_{n+1} = \left(\frac{\int |\hat{\eta}_n|^2 dk}{\int \hat{\eta}_n^* P_{Benjamin}[\hat{\eta}_n]\,dk}\right)^m P_{Benjamin}[\hat{\eta}_n]$$

Num-5

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