参赛学生姓名: 李润博

中学: 中国人民大学附属中学

省份: 北京

国家/地区: 中国

指导老师姓名: 郭振宇

指导老师单位: 西安交通大学

指导老师姓名: 崔旺

指导老师单位: 中国人民大学附属中学

论文题目: On Chen's theorem, Goldbach's conjecture and applications of sieve methods

ON CHEN'S THEOREM, GOLDBACH'S CONJECTURE AND APPLICATIONS OF SIEVE METHODS

RUNBO LI

Abstract. Let N denote a sufficiently large even integer, we define $D_{1,2}(N)$ as the same as those in previous articles about Chen's theorem. In this paper, we show that $D_{1,2}(N) \geqslant 1.733 \frac{C(N)N}{(\log N)^2}$, improving previous record about 93%. We also get some new results on twin prime problem and additive representations of integers.

1. Introduction

Let N denote a sufficiently large even integer, p denote a prime number, and P_2 denote an integer with at most two prime factors counted with multiplicity. We define

$$D_{1,2}(N) := |\{p : p \leqslant N, N - p = P_2\}|. \tag{1}$$

In 1973 Chen [8] established his remarkable Chen's theorem:

$$D_{1,2}(N) \geqslant 0.67 \frac{C(N)N}{(\log N)^2}, \tag{2}$$

where

$$C(N) := \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right). \tag{3}$$

Chen's constant 0.67 was improved successively to

$$0.689,\ 0.7544,\ 0.81,\ 0.8285,\ 0.836,\ 0.867,\ 0.899$$

by Halberstam and Richert [14] [13], Chen [10] [9], Cai and Lu [7], Wu [29], Cai [3] and Wu [30] respectively. Chen [11] announced a better constant 0.9, but this work has not been published. In this paper, we obtain the following sharper result.

Theorem 1.1.

$$D_{1,2}(N) \geqslant 1.733 \frac{C(N)N}{(\log N)^2}.$$

One important significance of our Theorem 1.1 is to make us truly achieve and exceed the constant 0.9 claimed by Chen [11]. Our constant 1.733 gives a 92.7% refinement of Wu's prior record 0.899. This is the greatest refinement on the problem since Chen [8] from 1973.

Furthermore, for two relatively prime square-free positive integers a, b, let M denote a sufficiently large integer that is relatively prime to both a and b, $a, b < M^{\varepsilon}$ and let M be even if a and b are both odd. Let $R_{a,b}(M)$ denote the number of primes p such that ap and $M - ap$ are both square-free, $b \mid (M - ap)$, and $\frac{M-ap}{b} = P_2$. In 1976, Ross [[26], Chapter 3] established that

$$R_{a,b}(M) \geqslant 0.608 \frac{C(abM)M}{ab(\log M)^2}, \tag{4}$$

where

$$C(abM) := \prod_{\substack{p \mid abM \\ p>2}} \frac{p-1}{p-2} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right), \tag{5}$$

Key words and phrases. Chen's theorem, Weighted sieve, Distribution level.

and the constant 0.608 was improved successively to 0.68 and 0.8671 by Li [16] and Li [19] respectively. By using the same sieve process and methods in [19], we have the following sharper.

Theorem 1.2.

$$R_{a,b}(M) \geqslant 1.733\frac{C(abM)M}{ab(\log M)^2}.$$

Let x denote a sufficiently large integer and define

$$\pi_{1,2}(x) := |\{p : p \leqslant x, p+2 = P_2\}|. \tag{6}$$

In 1973 Chen [8] showed simultaneously that

$$\pi_{1,2}(x) \geqslant 0.335\frac{C_2 x}{(\log x)^2}, \tag{7}$$

where

$$C_2 := 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right), \tag{8}$$

and the constant 0.608 was improved successively to

$$0.3445,\ 0.3772,\ 0.405,\ 0.71,\ 1.015,\ 1.05,\ 1.0974,\ 1.104,\ 1.123,\ 1.13,\ 1.145$$

by Halberstam [13], Chen [10] [9], Fouvry and Grupp [12], Liu [24], Wu [27], Cai [2], Wu [29], Cai [3], Cai [4] and Wu [30] respectively. In this paper, we get the following sharper.

Theorem 1.3.

$$\pi_{1,2}(x) \geqslant 1.238\frac{C_2 x}{(\log x)^2}.$$

Chen's theorem with small primes was first studied by Cai [1]. For $0 < \theta \leqslant 1$, we define

$$D_{1,r}^{\theta}(N) := \left|\left\{p : p \leqslant N^{\theta}, N-p = P_r\right\}\right|. \tag{9}$$

Then it is proved in [1] that for $0.95 \leqslant \theta \leqslant 1$, we have

$$D_{1,2}^{\theta}(N) \gg \frac{C(N)N^{\theta}}{(\log N)^2}. \tag{10}$$

Cai's range $0.95 \leqslant \theta \leqslant 1$ was extended successively to $0.945 \leqslant \theta \leqslant 1$ in [6] and to $0.941 \leqslant \theta \leqslant 1$ in [5]. In this paper, we obtain the following sharper.

Theorem 1.4. *For $0.9409 \leqslant \theta \leqslant 1$, we have*

$$D_{1,2}^{\theta}(N) \gg \frac{C(N)N^{\theta}}{(\log N)^2}.$$

Again, one can get parallel results on arithmetic progressions or on integers of form $ap + bP_2$ with small p. One can see [19] for a detailed explanation.

The results proved in the present paper are mainly come from the author's papers [19] and [21]. One can also see the author's other papers [17], [18] and [20] for more applications of sieve method.

2. Levels of distribution of primes

We first provide some distribution lemmas which will be used in bounding various sieve error terms later. The first two lemmas come from Pan and Pan's book [25] which are generalized versions of Bombieri–Vinogradov theorem.

Lemma 2.1. *([25], p. 192, Corollary 8.2]). Let*

$$\pi(x; k, d, l) = \sum_{\substack{kp \leq x \\ kp \equiv l (\bmod d)}} 1$$

and let $g(k)$ be a real function, $g(k) \ll 1$. Then, for any given constant $A > 0$, there exists a constant $B = B(A) > 0$ such that

$$\sum_{d \leqslant x^{1/2}(\log x)^{-B}} \max_{y \leqslant x} \max_{(l,d)=1} \left| \sum_{\substack{k \leqslant E(x) \\ (k,d)=1}} g(k) H(y; k, d, l) \right| \ll \frac{x}{\log^A x},$$

where

$$H(y; k, d, l) = \pi(y; k, d, l) - \frac{1}{\varphi(d)} \pi(y; k, 1, 1) = \sum_{\substack{kp \leqslant y \\ kp \equiv l (\bmod d)}} 1 - \frac{1}{\varphi(d)} \sum_{kp \leqslant y} 1,$$

$$\frac{1}{2} \leqslant E(x) \ll x^{1-\alpha}, \quad 0 < \alpha \leqslant 1, \quad B(A) = \frac{3}{2} A + 17.$$

Lemma 2.2. *([25], p. 195–196, Corollary 8.3 and 8.4]). Let $r_1(y)$ be a positive function depending on x and satisfying $r_1(y) \ll x^{\alpha}$ for $y \leqslant x$. Then under the conditions in Lemma 2.1, we have*

$$\sum_{d \leqslant x^{1/2}(\log x)^{-B}} \max_{y \leqslant x} \max_{(l,d)=1} \left| \sum_{\substack{k \leqslant E(x) \\ (k,d)=1}} g(k) H\left(k r_1(y); k, d, l\right) \right| \ll \frac{x}{\log^A x}.$$

Let $r_2(k)$ be a positive function depending on x and y such that $k r_2(k) \ll x$ for $k \leqslant E(x)$, $y \leqslant x$. Then under the conditions in Lemma 2.1, we have

$$\sum_{d \leqslant x^{1/2}(\log x)^{-B}} \max_{y \leqslant x} \max_{(l,d)=1} \left| \sum_{\substack{k \leqslant E(x) \\ (k,d)=1}} g(k) H\left(k r_2(k); k, d, l\right) \right| \ll \frac{x}{\log^A x}.$$

The next lemma was first proven by Wu [28], and it is a ”short interval” versions of Lemmas 2.1–2.2.

Lemma 2.3. *([28], Theorem 2]). Let $g(k)$ be a real function such that*

$$\sum_{k \leqslant x} \frac{g^2(k)}{k} \ll \log^C x$$

for some $C > 0$. Then, for any given constant $A > 0$, there exists a constant $B = B(A, C) > 0$ such that

$$\sum_{d \leqslant x^{t-1/2}(\log x)^{-B}} \max_{x/2 \leqslant y \leqslant x} \max_{(l,d)=1} \max_{h \leqslant x^t} \left| \sum_{\substack{k \leqslant x^{\beta} \\ (k,d)=1}} g(k) \bar{H}(y, h, k, d, l) \right| \ll \frac{x^t}{\log^A x},$$

where

$$\begin{aligned} \bar{H}(y, h, k, d, l) &= (\pi(y+h; k, d, l) - \pi(y; k, d, l)) \\ &\quad - \frac{1}{\varphi(d)} (\pi(y+h; k, 1, 1) - (\pi(y; k, 1, 1)) \\ &= \sum_{\substack{y < kp \leqslant y+h \\ kp \equiv l (\bmod d)}} 1 - \frac{1}{\varphi(d)} \sum_{y < kp \leqslant y+h} 1, \end{aligned}$$

$$\frac{3}{5} < t \leqslant 1, \quad 0 \leqslant \beta < \frac{5t-3}{2}, \quad B(A, C) = 3A + C + 34.$$

In [5], Cai said that we faced the difficulty which cannot be surmounted that our Lemma 2.3 are not sufficient to deal with some of the sieve error terms involved. Actually the function $g(k)$ cannot be well-defined to control the sieve error terms involved in evaluation of some terms, so we need a new distribution theorem to overcome that. The next lemma is a new distribution theorem for products of large primes over short intervals and it was first proven by Cai [5].

Lemma 2.4. *([5], Lemma 8]). Let*

$$\mathcal{F}_1=\Big\{mp_1p_2p_3p_4:(p_1p_2p_3p_4,N)=1,N^{\frac{1}{14}}\leqslant p_1<p_2<p_3<p_4<N^{\frac{1}{8.8}},$$
$$\frac{N-N^\theta}{p_1p_2p_3p_4}\leqslant m\leqslant\frac{N}{p_1p_2p_3p_4},\left(m,p_1^{-1}NP\left(p_2\right)\right)=1\Big\},$$
$$\mathcal{F}_2=\Big\{mp_1p_2p_3p_4:(p_1p_2p_3p_4,N)=1,N^{\frac{1}{14}}\leqslant p_1<p_2<p_3<N^{\frac{1}{8.8}}\leqslant p_4<N^{\frac{4.5863}{14}}p_3^{-1},$$
$$\frac{N-N^\theta}{p_1p_2p_3p_4}\leqslant m\leqslant\frac{N}{p_1p_2p_3p_4},\left(m,p_1^{-1}NP\left(p_2\right)\right)=1\Big\},$$

then for $j=1,2$ and any given constant $A>0$, there exists a constant $B=B(A)>0$ such that

$$\sum_{d\leqslant x^{0.9409-1/2}(\log x)^{-B}}\max_{(l,d)=1}\left|\sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j\\ mp_1p_2p_3p_4\equiv l(\bmod d)}}1-\frac{1}{\varphi(d)}\sum_{\substack{mp_1p_2p_3p_4\in\mathcal{F}_j\\ (mp_1p_2p_3p_4,d)=1}}1\right|\ll\frac{x^{0.9409}}{\log^A x}.$$

3. Lichtman's new distribution levels

In 2023, Lichtman [23] got some new distribution levels on some special conditions. He also used this to get great improvement on the upper bound of Goldbach representations. The distribution level of his results is much better than Lemma 2.1 and Lemma 2.2. In this paper we focus on the lower bound instead, but his results are still very useful.

We present the new results of Lichtman [23] in this section. Put $A,B>0$, $\theta=\frac{7}{32}$ from Kim–Sarnak [15], and we define the functions $\boldsymbol{\vartheta}_\alpha(t_1)$ and $\boldsymbol{\vartheta}_\alpha(t_1,t_2,t_3)$ with $\alpha=0$ or 1 as the same as those in [23]. We consider the analogous set of well–factorable vectors $\mathbf{D}_r^{\text{well}}$:

$$\mathbf{D}_r^{\text{well}}(D)=\left\{(D_1,\ldots,D_r):D_1\cdots D_{m-1}D_m^2<D\text{ for all }m\leqslant r\right\}.$$

Lemma 3.1. *Let $(D_1,\ldots,D_r)\in\mathbf{D}_r^{well}(D)$ and write $D=N^{\boldsymbol{\vartheta}},D_i=N^{t_i}$ for $i\leqslant r$. If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1)-\varepsilon$, then*

$$\sum_{\substack{b=p_1\cdots p_r\\ D_i<p_i\leqslant D_i^{1+\varepsilon^9}}}\sum_{\substack{q=bc\leqslant D\\ c\mid P(p_r)\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N)-\frac{\pi(N)}{\varphi(q)}\right)\ll\frac{N}{(\log N)^A}.\tag{i}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4}$ and $r\geqslant3$, then (i) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1)-\varepsilon$ and $r=2$, then

$$\sum_{\substack{b=p_1p_2\\ D_1<p_1\leqslant D_1^{1+\varepsilon^9}\\ D_2<p_2\leqslant D_2^{1+\varepsilon^9}}}\sum_{\substack{q=bc\leqslant D\\ c\mid P(N^u)\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N)-\frac{\pi(N)}{\varphi(q)}\right)\ll\frac{N}{(\log N)^A}.\tag{ii}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4}$, then (ii) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1,t_2,u)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1)-\varepsilon$ and $r=1$, then

$$\sum_{\substack{b=p_1\\ D_1<p_1\leqslant D_1^{1+\varepsilon^9}}}\sum_{\substack{q=bc\leqslant D\\ c\mid P(N^u)\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N)-\frac{\pi(N)}{\varphi(q)}\right)\ll\frac{N}{(\log N)^A}.\tag{iii}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4}$, then (iii) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1,u,u)-\varepsilon$.

If $r=0$ and $u=\frac{1}{500}$, this simplifies as

$$\sum_{\substack{q\leqslant N^{\frac{19101}{32000}}\\ q\mid P\left(N^{1/500}\right)\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\pi(N;q,N)-\frac{\pi(N)}{\varphi(q)}\right)\ll\frac{N}{(\log N)^A}.\tag{iv}$$

Lemma 3.2. *Let $(D_1,\dots,D_r)\in\mathbf{D}_r^{well}(D)$ and write $D=x^{\boldsymbol{\vartheta}}, D_i=x^{t_i}$ for $i\leqslant r$. If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1)-\varepsilon$, then*

$$\sum_{\substack{b=p_1\cdots p_r\\ D_i<p_i\leqslant D_i^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(p_r)\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2)-\frac{\pi(x)}{\varphi(q)}\right)\ll\frac{x}{(\log x)^A}. \tag{v}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4-3\theta}$ and $r\geqslant 3$, then (v) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1)-\varepsilon$ and $r=2$, then

$$\sum_{\substack{b=p_1p_2\\ D_1<p_1\leqslant D_1^{1+\varepsilon^9}\\ D_2<p_2\leqslant D_2^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(x^u)\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2)-\frac{\pi(x)}{\varphi(q)}\right)\ll\frac{x}{(\log x)^A}. \tag{vi}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4-3\theta}$, then (vi) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1,t_2,u)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1)-\varepsilon$ and $r=1$, then

$$\sum_{\substack{b=p_1\\ D_1<p_1\leqslant D_1^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(x^u)\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2)-\frac{\pi(x)}{\varphi(q)}\right)\ll\frac{x}{(\log x)^A}. \tag{vii}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4-3\theta}$, then (vii) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1,u,u)-\varepsilon$.

If $r=0$ and $u=\frac{1}{500}$, this simplifies as

$$\sum_{\substack{q\leqslant x^{\frac{16483}{26750}}\\ q\mid P\left(x^{1/500}\right)\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\pi(x;q,-2)-\frac{\pi(x)}{\varphi(q)}\right)\ll\frac{x}{(\log x)^A}. \tag{viii}$$

Lemma 3.3. *Let $(D_1,\dots,D_r)\in\mathbf{D}_r^{well}(D)$ and write $D=N^{\boldsymbol{\vartheta}}, D_i=N^{t_i}$ for $i\leqslant r$. Let $\varepsilon>0$ and real numbers $\varepsilon_1,\dots,\varepsilon_k\geqslant\varepsilon$ such that $\sum_{i\leqslant k}\varepsilon_i=1$, and let $\Delta=1+(\log N)^{-B}$. If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1)-\varepsilon$, then*

$$\sum_{\substack{b=p_1'\cdots p_r'\\ D_i<p_i'\leqslant D_i^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(p_r')\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{N}{(\log N)^A}. \tag{ix}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4}$ and $r\geqslant 3$, then (ix) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1)-\varepsilon$ and $r=2$, then

$$\sum_{\substack{b=p_1'p_2'\\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}\\ D_2<p_2'\leqslant D_2^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(N^u)\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{N}{(\log N)^A}. \tag{x}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4}$, then (x) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1,t_2,u)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1)-\varepsilon$ and $r=1$, then

$$\sum_{\substack{b=p_1'\\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(N^u)\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{N}{(\log N)^A}. \tag{xi}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4}$, then (xi) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_1(t_1,u,u)-\varepsilon$.

If $r=0$ and $u=\frac{1}{500}$, this simplifies as

$$\sum_{\substack{q\leqslant N^{\frac{19101}{32000}}\\ q\mid P\left(N^{1/500}\right)\\ (q,N)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv N(\bmod q)\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,N)=1\\ N^{\varepsilon_i}/\Delta<p_i\leqslant N^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{N}{(\log N)^A}. \tag{xii}$$

Lemma 3.4. *Let $(D_1,\ldots,D_r)\in\mathbf{D}_r^{well}(D)$ and write $D=x^{\boldsymbol{\vartheta}}, D_i=x^{t_i}$ for $i\leqslant r$. Let $\varepsilon>0$ and real numbers $\varepsilon_1,\ldots,\varepsilon_k\geqslant\varepsilon$ such that $\sum_{i\leqslant k}\varepsilon_i=1$, and let $\Delta=1+(\log x)^{-B}$. If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1)-\varepsilon$, then*

$$\sum_{\substack{b=p_1'\cdots p_r'\\ D_i<p_i'\leqslant D_i^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P\left(p_r'\right)\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q)\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,2)=1\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{x}{(\log x)^A}. \tag{xiii}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4-3\theta}$ and $r\geqslant 3$, then (xiii) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1)-\varepsilon$ and $r=2$, then

$$\sum_{\substack{b=p_1'p_2'\\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}\\ D_2<p_2'\leqslant D_2^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(x^u)\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q)\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,2)=1\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{x}{(\log x)^A}. \tag{xiv}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4-3\theta}$, then (xiv) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1,t_2,u)-\varepsilon$.

If $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1)-\varepsilon$ and $r=1$, then

$$\sum_{\substack{b=p_1'\\ D_1<p_1'\leqslant D_1^{1+\varepsilon^9}}}\ \sum_{\substack{q=bc\leqslant D\\ c\mid P(x^u)\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q)\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,2)=1\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{x}{(\log x)^A}. \tag{xv}$$

Moreover if $t_1\leqslant\frac{1-\theta}{4-3\theta}$, then (xv) holds if $\boldsymbol{\vartheta}\leqslant\boldsymbol{\vartheta}_0(t_1,u,u)-\varepsilon$.

If $r=0$ and $u=\frac{1}{500}$, this simplifies as

$$\sum_{\substack{q\leqslant x^{\frac{16483}{26750}}\\ q\mid P\left(x^{1/500}\right)\\ (q,2)=1}}\widetilde{\lambda}^{\pm}(q)\left(\sum_{\substack{p_1\cdots p_k\equiv 2(\bmod q)\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1-\frac{1}{\varphi(q)}\sum_{\substack{(p_1\cdots p_k,2)=1\\ x^{\varepsilon_i}/\Delta<p_i\leqslant x^{\varepsilon_i}\ \forall i\leqslant k}}1\right)\ll\frac{x}{(\log x)^A}. \tag{xvi}$$

4. Weighted Sieve Method

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ denote finite sets of positive integers, $\mathcal{P}$ denote an infinite set of primes and $z\geqslant 2$. Put

$$\mathcal{A}=\{N-p:p\leqslant N\},\quad \mathcal{B}=\{p+2:p\leqslant x\},\quad \mathcal{C}=\left\{N-p:p\leqslant N^{0.9409}\right\},$$

$$\mathcal{P}=\{p:(p,2)=1\},\quad \mathcal{P}(q)=\{p:p\in\mathcal{P},(p,q)=1\},$$

$$P(z)=\prod_{\substack{p\in\mathcal{P}\\ p<z}}p,\quad \mathcal{A}_d=\{a:a\in\mathcal{A},a\equiv 0(\bmod d)\},\quad S(\mathcal{A};\mathcal{P},z)=\sum_{\substack{a\in\mathcal{A}\\ (a,P(z))=1}}1.$$

Lemma 4.1. *([30], Lemma 2.2). We have*

$$
\begin{aligned}
4D_{1,2}(N) \geqslant\ & 3S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{13.27}}\right)+S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{8.24}}\right)\\
&-2\sum_{\substack{N^{\frac{1}{13.27}}\leqslant p<N^{\frac{25}{128}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right)\\
&-2\sum_{\substack{N^{\frac{25}{128}}\leqslant p<N^{\frac{1}{4}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right)\\
&-2\sum_{\substack{N^{\frac{1}{4}}\leqslant p<N^{\frac{57}{224}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right)\\
&-\sum_{\substack{N^{\frac{57}{224}}\leqslant p<N^{\frac{1}{3}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right)\\
&-\sum_{\substack{N^{\frac{57}{224}}\leqslant p<N^{\frac{1}{2}-\frac{3}{13.27}}\\(p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right)\\
&+\sum_{\substack{N^{\frac{1}{13.27}}\leqslant p_2<p_1<N^{\frac{1}{8.24}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{13.27}}\right)\\
&+\sum_{\substack{N^{\frac{1}{13.27}}\leqslant p_2<N^{\frac{1}{8.24}}\leqslant p_1<N^{\frac{25}{128}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{13.27}}\right)\\
&+\sum_{\substack{N^{\frac{1}{13.27}}\leqslant p_2<N^{\frac{1}{8.24}}<N^{\frac{25}{128}}\leqslant p_1<N^{\frac{57}{224}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{13.27}}\right)\\
&+\sum_{\substack{N^{\frac{1}{13.27}}\leqslant p_2<N^{\frac{1}{8.24}}<N^{\frac{57}{224}}\leqslant p_1<N^{\frac{1}{2}-\frac{3}{13.27}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{13.27}}\right)\\
&-2\sum_{\substack{N^{\frac{1}{2}-\frac{3}{13.27}}\leqslant p_1<p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),p_2\right)\\
&-\sum_{\substack{N^{\frac{1}{13.27}}\leqslant p_1<N^{\frac{1}{3}}\leqslant p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),p_2\right)\\
&-\sum_{\substack{N^{\frac{1}{8.24}}\leqslant p_1<N^{\frac{1}{2}-\frac{3}{13.27}}\leqslant p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),\left(\frac{N}{p_1p_2}\right)^{\frac{1}{2}}\right)\\
&-\sum_{\substack{N^{\frac{1}{13.27}}\leqslant p_1<p_2<p_3<p_4<N^{\frac{1}{8.24}}\\(p_1p_2p_3p_4,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(N),p_2\right)\\
&-\sum_{\substack{N^{\frac{1}{13.27}}\leqslant p_1<p_2<p_3<N^{\frac{1}{8.24}}\leqslant p_4<N^{\frac{1}{2}-\frac{2}{13.27}}p_3^{-1}\\(p_1p_2p_3p_4,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3p_4};\mathcal{P}(N),p_2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+O\left(N^{\frac{12.27}{13.27}}\right)\\
=&\ 3S_1+S_2-2S_3-2S_4-2S_5-S_6-S_7+S_8+S_9\\
&+S_{10}+S_{11}-2S_{12}-S_{13}-S_{14}-S_{15}-S_{16}+O\left(N^{\frac{12.27}{13.27}}\right).
\end{aligned}
$$

Lemma 4.2. *([4], Lemma 3.2]). We have*

$$
\begin{aligned}
4\pi_{1,2}(x) \geqslant&\ 3S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{12}}\right)+S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{7.2}}\right)\\
&+\sum_{x^{\frac{1}{12}}\leqslant p_2<p_1<x^{\frac{1}{7.2}}} S\left(\mathcal{B}_{p_1p_2};\mathcal{P},x^{\frac{1}{12}}\right)\\
&+\sum_{x^{\frac{1}{12}}\leqslant p_2<x^{\frac{1}{7.2}}\leqslant p_1<x^{\frac{25}{107}}} S\left(\mathcal{B}_{p_1p_2};\mathcal{P},x^{\frac{1}{12}}\right)\\
&+\sum_{x^{\frac{1}{12}}\leqslant p_2<x^{\frac{1}{7.2}}<x^{\frac{25}{107}}\leqslant p_1<\min(x^{\frac{2}{7}},x^{\frac{17}{42}}p_2^{-1})} S\left(\mathcal{B}_{p_1p_2};\mathcal{P},x^{\frac{1}{12}}\right)\\
&-2\sum_{x^{\frac{1}{12}}\leqslant p<x^{\frac{25}{107}}} S\left(\mathcal{B}_{p};\mathcal{P},x^{\frac{1}{12}}\right)-2\sum_{x^{\frac{25}{107}}\leqslant p<x^{\frac{2}{7}-\varepsilon}} S\left(\mathcal{B}_{p};\mathcal{P},x^{\frac{1}{12}}\right)\\
&-\sum_{x^{\frac{2}{7}-\varepsilon}\leqslant p<x^{\frac{2}{7}}} S\left(\mathcal{B}_{p};\mathcal{P},x^{\frac{1}{12}}\right)-\sum_{x^{\frac{2}{7}-\varepsilon}\leqslant p<x^{\frac{29}{100}}} S\left(\mathcal{B}_{p};\mathcal{P},x^{\frac{1}{12}}\right)\\
&-\sum_{x^{\frac{29}{100}}\leqslant p<x^{\frac{1}{3}-\varepsilon}} S\left(\mathcal{B}_{p};\mathcal{P},x^{\frac{1}{12}}\right)-\sum_{x^{\frac{1}{3}-\varepsilon}\leqslant p<x^{\frac{1}{3}}} S\left(\mathcal{B}_{p};\mathcal{P},x^{\frac{1}{12}}\right)\\
&-\sum_{x^{\frac{1}{12}}\leqslant p_1<x^{\frac{1}{3}}\leqslant p_2<\left(\frac{x}{p_1}\right)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{x^{\frac{1}{7.2}}\leqslant p_1<x^{\frac{2}{7}}\leqslant p_2<\left(\frac{x}{p_1}\right)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1p_2};\mathcal{P}(p_1),\left(\frac{x}{p_1p_2}\right)^{\frac{1}{2}}\right)\\
&-2\sum_{x^{\frac{2}{7}}\leqslant p_1<p_2<\left(\frac{x}{p_1}\right)^{\frac{1}{2}}} S\left(\mathcal{B}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{x^{\frac{1}{12}}\leqslant p_1<p_2<p_3<p_4<x^{\frac{1}{7.2}}} S\left(\mathcal{B}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{x^{\frac{1}{12}}\leqslant p_1<p_2<p_3<x^{\frac{5}{42}}<x^{\frac{1}{7.2}}<p_4<x^{\frac{2}{7}}} S\left(\mathcal{B}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{x^{\frac{1}{12}}\leqslant p_1<p_2<x^{\frac{5}{42}}\leqslant p_3<x^{\frac{1}{7.2}}\leqslant p_4<x^{\frac{17}{42}}p_3^{-1}} S\left(\mathcal{B}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{x^{\frac{1}{12}}\leqslant p_1<x^{\frac{5}{42}}\leqslant p_2<p_3<x^{\frac{1}{7.2}}\leqslant p_4<x^{\frac{17}{42}}p_3^{-1}} S\left(\mathcal{B}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{x^{\frac{5}{42}}\leqslant p_1<p_2<p_3<x^{\frac{1}{7.2}}\leqslant p_4<x^{\frac{17}{42}}p_3^{-1}} S\left(\mathcal{B}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&+O\left(x^{\frac{11}{12}}\right)\\
=&\ 3S_1'+S_2'+S_3'+S_4'+S_5'-2S_6'-2S_7'-S_8'-S_9'-S_{10}'-S_{11}'\\
&-S_{12}'-S_{13}'-2S_{14}'-S_{15}'-S_{16}'-S_{17}'-S_{18}'-S_{19}'+O\left(x^{\frac{11}{12}}\right).
\end{aligned}
$$

Lemma 4.3. *([5], Lemma 9]). We have*

$$
\begin{aligned}
4D_{1,2}^{\theta}(N) \geqslant\ & 3S\left(\mathcal{C};\mathcal{P},N^{\frac{1}{14}}\right)+S\left(\mathcal{C};\mathcal{P},N^{\frac{1}{8.8}}\right)\\
&+\sum_{\substack{N^{\frac{1}{14}}\leqslant p_1<p_2<N^{\frac{1}{8.8}}\\(p_1p_2,N)=1}} S\left(\mathcal{C}_{p_1p_2};\mathcal{P},N^{\frac{1}{14}}\right)\\
&+\sum_{\substack{N^{\frac{1}{14}}\leqslant p_1<N^{\frac{1}{8.8}}\leqslant p_2<N^{\frac{4.5863}{14}}p_1^{-1}\\(p_1p_2,N)=1}} S\left(\mathcal{C}_{p_1p_2};\mathcal{P},N^{\frac{1}{14}}\right)\\
&-\sum_{\substack{N^{\frac{1}{14}}\leqslant p<N^{\frac{4.08631}{14}}\\(p,N)=1}} S\left(\mathcal{C}_{p};\mathcal{P},N^{\frac{1}{14}}\right)\\
&-\sum_{\substack{N^{\frac{1}{14}}\leqslant p<N^{\frac{3.5863}{14}}\\(p,N)=1}} S\left(\mathcal{C}_{p};\mathcal{P},N^{\frac{1}{14}}\right)\\
&-\sum_{\substack{N^{\frac{1}{14}}\leqslant p_1<N^{\frac{1}{3.1}}\leqslant p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{C}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\substack{N^{\frac{1}{8.8}}\leqslant p_1<N^{\frac{1}{3.7}}\leqslant p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{C}_{p_1p_2};\mathcal{P}(p_1),\left(\frac{N}{p_1p_2}\right)^{\frac{1}{2}}\right)\\
&-\sum_{\substack{N^{\frac{4.08631}{14}}\leqslant p<N^{\frac{1}{3.1}}\\(p,N)=1}} S\left(\mathcal{C}_{p};\mathcal{P},N^{\frac{1}{14}}\right)\\
&-\sum_{\substack{N^{\frac{3.5863}{14}}\leqslant p<N^{\frac{1}{3.7}}\\(p,N)=1}} S\left(\mathcal{C}_{p};\mathcal{P},N^{\frac{1}{8.8}}\right)\\
&-\sum_{\substack{N^{\frac{1}{14}}\leqslant p_1<p_2<p_3<p_4<N^{\frac{1}{8.8}}\\(p_1p_2p_3p_4,N)=1}} S\left(\mathcal{C}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-\sum_{\substack{N^{\frac{1}{14}}\leqslant p_1<p_2<p_3<N^{\frac{1}{8.8}}\leqslant p_4<N^{\frac{4.5863}{14}}p_3^{-1}\\(p_1p_2p_3p_4,N)=1}} S\left(\mathcal{C}_{p_1p_2p_3p_4};\mathcal{P}(p_1),p_2\right)\\
&-2\sum_{\substack{N^{\frac{1}{3.1}}\leqslant p_1<p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{C}_{p_1p_2};\mathcal{P}(p_1),p_2\right)\\
&-2\sum_{\substack{N^{\frac{1}{3.7}}\leqslant p_1<p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{C}_{p_1p_2};\mathcal{P}(p_1),p_2\right)+O\left(N^{\frac{13}{14}}\right)\\
=\ & 3S_1''+S_2''+S_3''+S_4''-S_5''-S_6''-S_7''\\
&-S_8''-S_9''-S_{10}''-S_{11}''-S_{12}''-2S_{13}''-2S_{14}''+O\left(N^{\frac{13}{14}}\right).
\end{aligned}
$$

5. Proof of Theorem 1.1

In this section, sets $\mathcal{A}$ and $\mathcal{P}$ are defined respectively. Let γ denote the Euler's constant, $F(s)$ and $f(s)$ are determined by the following differential-difference equation

$$\begin{cases} F(s)=\frac{2e^{\gamma}}{s}, \quad f(s)=0, & 0<s\leqslant 2,\\ (sF(s))'=f(s-1), \quad (sf(s))'=F(s-1), & s\geqslant 2, \end{cases}$$

and $\omega(u)$ denote the Buchstab function determined by the following differential-difference equation

$$\begin{cases} \omega(u)=\frac{1}{u}, & 1\leqslant u\leqslant 2,\\ (u\omega(u))'=\omega(u-1), & u\geqslant 2. \end{cases}$$

We first consider S_1 and S_2. By Buchstab's identity, we have

$$\begin{aligned} S_1 = S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{13.27}}\right) &= S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{500}}\right) - \sum_{\substack{N^{\frac{1}{500}}\leqslant p<N^{\frac{1}{13.27}}\\ (p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{500}}\right)\\ &+ \sum_{\substack{N^{\frac{1}{500}}\leqslant p_2<p_1<N^{\frac{1}{13.27}}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{500}}\right)\\ &- \sum_{\substack{N^{\frac{1}{500}}\leqslant p_3<p_2<p_1<N^{\frac{1}{13.27}}\\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_3\right) \end{aligned} \tag{11}$$

and

$$\begin{aligned} S_2 = S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{8.24}}\right) &= S\left(\mathcal{A};\mathcal{P}(N),N^{\frac{1}{500}}\right) - \sum_{\substack{N^{\frac{1}{500}}\leqslant p<N^{\frac{1}{8.24}}\\ (p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{500}}\right)\\ &+ \sum_{\substack{N^{\frac{1}{500}}\leqslant p_2<p_1<N^{\frac{1}{8.24}}\\ (p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(N),N^{\frac{1}{500}}\right)\\ &- \sum_{\substack{N^{\frac{1}{500}}\leqslant p_3<p_2<p_1<N^{\frac{1}{8.24}}\\ (p_1p_2p_3,N)=1}} S\left(\mathcal{A}_{p_1p_2p_3};\mathcal{P}(N),p_3\right). \end{aligned} \tag{12}$$

By Lemma 3.1, Iwaniec's linear sieve method and arguments in [22] and [23] we have

$$\begin{aligned} S_1 \geqslant (1+o(1))\frac{2}{e^{\gamma}}\Bigg(& 500f\left(500\boldsymbol{\vartheta}_{\frac{1}{500}}\right) - 500\int_{\frac{1}{500}}^{\frac{1}{13.27}} \frac{F(500(\boldsymbol{\vartheta}_1(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt\\ &+500\int_{\frac{1}{500}}^{\frac{1}{13.27}}\int_{\frac{1}{500}}^{t_1} \frac{f(500(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1\\ &-\int_{\frac{1}{500}}^{\frac{1}{13.27}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2} \end{aligned} \tag{13}$$

and

$$\begin{aligned} S_2 \geqslant (1+o(1))\frac{2}{e^{\gamma}}\Bigg(& 500f\left(500\boldsymbol{\vartheta}_{\frac{1}{500}}\right) - 500\int_{\frac{1}{500}}^{\frac{1}{8.24}} \frac{F(500(\boldsymbol{\vartheta}_1(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt\\ &+500\int_{\frac{1}{500}}^{\frac{1}{8.24}}\int_{\frac{1}{500}}^{t_1} \frac{f(500(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1 \end{aligned}$$

$$
-\int_{\frac{1}{500}}^{\frac{1}{8.24}} \int_{\frac{1}{500}}^{t_1} \int_{\frac{1}{500}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1 \Bigg) \frac{C(N)N}{(\log N)^2}, \tag{14}
$$

where $\boldsymbol{\vartheta}_{\frac{1}{500}} = \frac{19101}{32000}$. By numerical calculations we get that

$$
S_1 \geqslant 14.901125 \frac{C(N)N}{(\log N)^2} \tag{15}
$$

and

$$
S_2 \geqslant 9.228483 \frac{C(N)N}{(\log N)^2}. \tag{16}
$$

For S_3, we can either use Buchstab's identity and Lichtman's method to estimate S_3 with a better distribution level as in [23] or use Chen's double sieve technique as in [30]. The first option leads to

$$
\begin{aligned}
\sum_{\substack{p \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}(N), N^{\frac{1}{13.27}}\right) = & \sum_{\substack{p \\ (p,N)=1}} S\left(\mathcal{A}_p; \mathcal{P}(N), N^{\frac{1}{k}}\right) \\
& - \sum_{\substack{p_1 \\ N^{\frac{1}{k}} \leqslant p_2 < N^{\frac{1}{13.27}} \\ (p_1 p_2, N)=1}} S\left(\mathcal{A}_{p_1 p_2}; \mathcal{P}(N), N^{\frac{1}{k}}\right) \\
& + \sum_{\substack{p_1 \\ N^{\frac{1}{k}} \leqslant p_3 < p_2 < N^{\frac{1}{13.27}} \\ (p_1 p_2 p_3, N)=1}} S\left(\mathcal{A}_{p_1 p_2 p_3}; \mathcal{P}(N), p_3\right)
\end{aligned} \tag{17}
$$

for some $k \geqslant 13.27$, while the second option creates a small saving on S_3 itself. We can also use Chen's double sieve on the first two sums on the right–hand side of (17) after applying Buchstab's identity. We don't know which of these options gives a smaller value, hence we take a minimum. By Lemma 3.1, Iwaniec's linear sieve method and arguments in [22] and [23] we have

$$
\begin{aligned}
S_3 \leqslant & (1+o(1)) \frac{2}{e^\gamma} \Bigg(\int_{\frac{1}{13.27}}^{\frac{25}{128}} \min \Bigg(13.27 \frac{F(13.27(\boldsymbol{\vartheta}_1(t_1, \frac{1}{13.27}, \frac{1}{13.27}) - t_1))}{t_1} \\
& - \frac{26.54 e^\gamma H(13.27(\frac{1}{2} - t_1))}{(13.27(\frac{1}{2} - t_1))t_1}, \min_{13.27 \leqslant k \leqslant 500} \Bigg(k \frac{F(k(\boldsymbol{\vartheta}_1(t_1, \frac{1}{k}, \frac{1}{k}) - t_1))}{t_1} \\
& - \frac{2k e^\gamma H(k(\frac{1}{2} - t_1))}{(k(\frac{1}{2} - t_1))t_1} - k \int_{\frac{1}{k}}^{\frac{1}{13.27}} \frac{f(k(\boldsymbol{\vartheta}_1(t_1, t_2, \frac{1}{k}) - t_1 - t_2))}{t_1 t_2} dt_2 \\
& - 2k e^\gamma \int_{\frac{1}{k}}^{\frac{1}{13.27}} \frac{h(k(\frac{1}{2} - t_1 - t_2))}{(k(\frac{1}{2} - t_1 - t_2))t_1 t_2} dt_2 \\
& + \int_{\frac{1}{k}}^{\frac{1}{13.27}} \int_{\frac{1}{k}}^{t_2} \frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 \Bigg)\Bigg) dt_1 \Bigg) \frac{C(N)N}{(\log N)^2} \\
\leqslant & 14.192163 \frac{C(N)N}{(\log N)^2},
\end{aligned} \tag{18}
$$

where we choose $k = 14.4$ and $H(s) = H_{1/2}(s)$ and $h(s) = h_{1/2}(s)$ are defined as the same in [30]. We have used the following lower bounds of $H(s)$ and $h(s)$ for $2.0 \leqslant s \leqslant 4.9$. These values can be found at Tables 1 and 2 of [30]. We remark that we have $H_{\boldsymbol{\vartheta}}(s) \geqslant H_{1/2}(s)$ and $h_{\boldsymbol{\vartheta}}(s) \geqslant h_{1/2}(s)$ for $\boldsymbol{\vartheta} > \frac{1}{2}$.

$$H(s) \geqslant \begin{cases} 0.0223939, & 2.0 < s \leqslant 2.2, \\ 0.0217196, & 2.2 < s \leqslant 2.3, \\ 0.0202876, & 2.3 < s \leqslant 2.4, \\ 0.0181433, & 2.4 < s \leqslant 2.5, \\ 0.0158644, & 2.5 < s \leqslant 2.6, \\ 0.0129923, & 2.6 < s \leqslant 2.7, \\ 0.0100686, & 2.7 < s \leqslant 2.8, \\ 0.0078162, & 2.8 < s \leqslant 2.9, \\ 0.0072943, & 2.9 < s \leqslant 3.0, \\ 0.0061642, & 3.0 < s \leqslant 3.1, \\ 0.0052233, & 3.1 < s \leqslant 3.2, \\ 0.0044073, & 3.2 < s \leqslant 3.3, \\ 0.0036995, & 3.3 < s \leqslant 3.4, \\ 0.0030860, & 3.4 < s \leqslant 3.5, \end{cases} \begin{cases} 0.0025551, & 3.5 < s \leqslant 3.6, \\ 0.0020972, & 3.6 < s \leqslant 3.7, \\ 0.0017038, & 3.7 < s \leqslant 3.8, \\ 0.0013680, & 3.8 < s \leqslant 3.9, \\ 0.0010835, & 3.9 < s \leqslant 4.0, \\ 0.0008451, & 4.0 < s \leqslant 4.1, \\ 0.0006482, & 4.1 < s \leqslant 4.2, \\ 0.0004882, & 4.2 < s \leqslant 4.3, \\ 0.0003602, & 4.3 < s \leqslant 4.4, \\ 0.0002592, & 4.4 < s \leqslant 4.5, \\ 0.0001803, & 4.5 < s \leqslant 4.6, \\ 0.0001187, & 4.6 < s \leqslant 4.7, \\ 0.0000702, & 4.7 < s \leqslant 4.8, \\ 0.0000313, & 4.8 < s \leqslant 4.9, \end{cases} \tag{19}$$

$$h(s) \geqslant \begin{cases} 0.0232385, & s = 2.0, \\ 0.0211041, & 2.0 < s \leqslant 2.1, \\ 0.0191556, & 2.1 < s \leqslant 2.2, \\ 0.0173631, & 2.2 < s \leqslant 2.3, \\ 0.0157035, & 2.3 < s \leqslant 2.4, \\ 0.0141585, & 2.4 < s \leqslant 2.5, \\ 0.0127132, & 2.5 < s \leqslant 2.6, \\ 0.0113556, & 2.6 < s \leqslant 2.7, \\ 0.0100756, & 2.7 < s \leqslant 2.8, \\ 0.0088648, & 2.8 < s \leqslant 2.9, \\ 0.0077612, & 2.9 < s \leqslant 3.0, \\ 0.0066236, & 3.0 < s \leqslant 3.1, \\ 0.0055818, & 3.1 < s \leqslant 3.2, \\ 0.0046164, & 3.2 < s \leqslant 3.3, \\ 0.0037529, & 3.3 < s \leqslant 3.4, \end{cases} \begin{cases} 0.0030123, & 3.4 < s \leqslant 3.5, \\ 0.0023901, & 3.5 < s \leqslant 3.6, \\ 0.0018997, & 3.6 < s \leqslant 3.7, \\ 0.0015336, & 3.7 < s \leqslant 3.8, \\ 0.0012593, & 3.8 < s \leqslant 3.9, \\ 0.0010120, & 3.9 < s \leqslant 4.0, \\ 0.0008099, & 4.0 < s \leqslant 4.1, \\ 0.0006440, & 4.1 < s \leqslant 4.2, \\ 0.0005084, & 4.2 < s \leqslant 4.3, \\ 0.0003980, & 4.3 < s \leqslant 4.4, \\ 0.0003085, & 4.4 < s \leqslant 4.5, \\ 0.0002365, & 4.5 < s \leqslant 4.6, \\ 0.0001791, & 4.6 < s \leqslant 4.7, \\ 0.0001396, & 4.7 < s \leqslant 4.8, \\ 0.0000981, & 4.8 < s \leqslant 4.9. \end{cases} \tag{20}$$

Similarly, for S_4 and S_5 we have

$$\begin{aligned} S_4 \leqslant (1+o(1))\frac{2}{e^\gamma}\Bigg(\int_{\frac{25}{128}}^{\frac{1}{4}} \min\Bigg(& 13.27\frac{F(13.27(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} - \frac{26.54e^\gamma H(13.27(\frac{1}{2}-t_1))}{(13.27(\frac{1}{2}-t_1))t_1}, \\ & \min_{13.27 \leqslant k \leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1} - \frac{2ke^\gamma H(k(\frac{1}{2}-t_1))}{(k(\frac{1}{2}-t_1))t_1} \\ & - \int_{\frac{1}{k}}^{\frac{1}{13.27}} \frac{f\left(\frac{(\boldsymbol{\vartheta}_1(t_1)-t_1-t_2)}{t_2}\right)}{t_1 t_2^2} dt_2\Bigg)\Bigg) dt_1\Bigg) \frac{C(N)N}{(\log N)^2} \\ \leqslant{}& 3.721794\frac{C(N)N}{(\log N)^2}, \end{aligned} \tag{21}$$

$$S_5 \leqslant (1+o(1))\frac{2}{e^\gamma}\Bigg(\int_{\frac{1}{4}}^{\frac{57}{224}} \min\Bigg(13.27\frac{F(13.27(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1},$$

$$\min_{13.27\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_1(t_1)-t_1))}{t_1}-\int_{\frac{1}{k}}^{\frac{1}{13.27}}\frac{f\left(\frac{(\boldsymbol{\vartheta}_1(t_1)-t_1-t_2)}{t_2}\right)}{t_1t_2^2}dt_2\Bigg)\Bigg)\,dt_1\Bigg)\frac{C(N)N}{(\log N)^2}$$

$$\leqslant 0.282907\frac{C(N)N}{(\log N)^2}. \tag{22}$$

We shall use Chen's double sieve to gain a small saving on S_6. By the discussion in [30], we know that [[30], Proposition 4.4] can be used to handle the following sum:

$$\sum_{\substack{N^{\frac{1}{2}-\frac{2.9}{13.27}}\leqslant p<N^{\frac{1}{3}}\\ (p,N)=1}} S\left(\mathcal{A}_p;\mathcal{P}(N),N^{\frac{1}{13.27}}\right). \tag{23}$$

By the same process as in [30] we get that

$$\begin{aligned}S_6&\leqslant(1+o(1))\frac{2}{e^\gamma}\left(13.27\int_{\frac{57}{224}}^{\frac{1}{3}}\frac{F(13.27(\frac{1}{2}-t))}{t}dt\right)\frac{C(N)N}{(\log N)^2}-G_1\\ &\leqslant(5.265577-0.031029)\frac{C(N)N}{(\log N)^2}\\ &\leqslant 5.234548\frac{C(N)N}{(\log N)^2},\end{aligned} \tag{24}$$

where

$$\begin{aligned}G_1=8\Bigg(&\log\left(\frac{\frac{9.2}{13.27}}{1-\frac{4.6}{13.27}}\right)\Psi_2(2.3)+\sum_{4\leqslant i\leqslant 5}\log\left(\frac{(2+0.1i)\left(1-\frac{3.8+0.2i}{13.27}\right)}{(1.9+0.1i)\left(1-\frac{4+0.2i}{13.27}\right)}\right)\Psi_2(2+0.1i)\\ &+\sum_{6\leqslant i\leqslant 9}\log\left(\frac{(2+0.1i)\left(1-\frac{3.8+0.2i}{13.27}\right)}{(1.9+0.1i)\left(1-\frac{4+0.2i}{13.27}\right)}\right)\Psi_1(2+0.1i)\Bigg)\frac{C(N)N}{(\log N)^2},\end{aligned} \tag{25}$$

where $\Psi_1(s)$ and $\Psi_2(s)$ are defined as the same in [[29], Lemmas 5.1–5.2] and we have used the following lower bounds of them. These values can be found at Table 1 of [29].

$$\Psi_2(s)\geqslant\begin{cases}0.015247971, & s=2.3,\\ 0.013898757, & s=2.4,\\ 0.011776059, & s=2.5,\end{cases}\quad \Psi_1(s)\geqslant\begin{cases}0.009405211, & s=2.6,\\ 0.006558950, & s=2.7,\\ 0.003536751, & s=2.8,\\ 0.001056651, & s=2.9.\end{cases} \tag{26}$$

Similarly, for S_7 we have

$$\begin{aligned}S_7&\leqslant(1+o(1))\frac{2}{e^\gamma}\left(13.27\int_{\frac{57}{224}}^{\frac{1}{2}-\frac{3}{13.27}}\frac{F(13.27(\frac{1}{2}-t))}{t}dt\right)\frac{C(N)N}{(\log N)^2}\\ &\leqslant 1.256371\frac{C(N)N}{(\log N)^2}.\end{aligned} \tag{27}$$

For S_8 we can take a maximum of the lower bounds obtained by those two methods we used on the estimation of S_3.

$$\begin{aligned}S_8\geqslant(1+o(1))\frac{2}{e^\gamma}\Bigg(\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\int_{\frac{1}{13.27}}^{t_1}\max\Bigg(&13.27\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{13.27})-t_1-t_2))}{t_1t_2}\\ &+\frac{26.54e^\gamma h(13.27(\frac{1}{2}-t_1-t_2))}{(13.27(\frac{1}{2}-t_1-t_2))t_1t_2},\max_{13.27\leqslant k\leqslant 500}\Bigg(k\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}\\ &+\frac{2ke^\gamma h(k(\frac{1}{2}-t_1-t_2))}{(k(\frac{1}{2}-t_1-t_2))t_1t_2}-\int_{\frac{1}{k}}^{\frac{1}{13.27}}\frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3\Bigg)\Bigg)\,dt_2dt_1\Bigg)\frac{C(N)N}{(\log N)^2}\end{aligned}$$

$$\geqslant 1.691493\frac{C(N)N}{(\log N)^2}. \tag{28}$$

Similarly, for S_9–S_{11} we have

$$S_9 \geqslant (1+o(1))\frac{2}{e^\gamma}\left(\int_{\frac{1}{8.24}}^{\frac{25}{128}}\int_{\frac{1}{13.27}}^{\frac{1}{8.24}} \max\left(13.27\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{13.27})-t_1-t_2))}{t_1t_2}\right.\right.$$

$$+\frac{26.54e^\gamma h(13.27(\frac{1}{2}-t_1-t_2))}{(13.27(\frac{1}{2}-t_1-t_2))t_1t_2}, \max_{13.27\leqslant k\leqslant 500}\left(k\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}\right.$$

$$\left.\left.\left.+\frac{2ke^\gamma h(k(\frac{1}{2}-t_1-t_2))}{(k(\frac{1}{2}-t_1-t_2))t_1t_2}-\int_{\frac{1}{k}}^{\frac{1}{13.27}}\frac{F\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3\right)\right)dt_2dt_1\right)\frac{C(N)N}{(\log N)^2}$$

$$\geqslant 3.367923\frac{C(N)N}{(\log N)^2}, \tag{29}$$

$$S_{10}+S_{11} \geqslant (1+o(1))\frac{2}{e^\gamma}\left(13.27\int_{\frac{25}{128}}^{\frac{57}{224}}\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\frac{f(13.27(\boldsymbol{\vartheta}_1(t_1)-t_1-t_2))}{t_1t_2}dt_2dt_1\right.$$

$$\left.+13.27\int_{\frac{57}{224}}^{\frac{1}{2}-\frac{3}{13.27}}\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\frac{f(13.27(\frac{1}{2}-t_1-t_2))}{t_1t_2}dt_2dt_1\right)\frac{C(N)N}{(\log N)^2}+G_2$$

$$\geqslant (1.462958+0.041633)\frac{C(N)N}{(\log N)^2}$$

$$\geqslant 1.504591\frac{C(N)N}{(\log N)^2}, \tag{30}$$

where

$$G_2 = 4\left(13.27\int_{\frac{25}{128}}^{\frac{1}{2}-\frac{2}{8.24}}\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\frac{h(13.27(\frac{1}{2}-t_1-t_2))}{(13.27(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2dt_1\right.$$

$$\left.+13.27\int_{\frac{1}{2}-\frac{2}{8.24}}^{\frac{1}{2}-\frac{3}{13.27}}\int_{\frac{1}{13.27}}^{\frac{1.5}{13.27}}\frac{h(13.27(\frac{1}{2}-t_1-t_2))}{(13.27(\frac{1}{2}-t_1-t_2))t_1t_2}dt_2dt_1\right)\frac{C(N)N}{(\log N)^2}. \tag{31}$$

For the remaining terms, we can use Chen's switching principle together with Lichtman's distribution level to estimate them. Namely, for S_{12} we have

$$S_{12} = \sum_{\substack{N^{\frac{1}{2}-\frac{3}{13.27}}\leqslant p_1<p_2<(\frac{N}{p_1})^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathcal{A}_{p_1p_2};\mathcal{P}(Np_1),p_2\right) = S\left(\mathcal{A}';\mathcal{P}(N),N^{\frac{1}{2}}\right), \tag{32}$$

where the set $\mathcal{A}'$ is defined as

$$\mathcal{A}' = \left\{N-p_1p_2m : N^{\frac{1}{2}-\frac{3}{13.27}} \leqslant p_1 < p_2 < (N/p_1)^{\frac{1}{2}},\ p' \mid m \Rightarrow p' > p_2 \text{ or } p' = p_1\right\}.$$

We note that each m above must be a prime number since $\frac{1}{2}-\frac{3}{13.27} > \frac{1}{4}$. By Buchstab's identity, we have

$$S_{12} = S\left(\mathcal{A}';\mathcal{P}(N),N^{\frac{1}{2}}\right) \leqslant S\left(\mathcal{A}';\mathcal{P}(N),N^{\frac{25}{128}}\right)$$

$$= S\left(\mathcal{A}';\mathcal{P}(N),N^{\frac{1}{500}}\right) - \sum_{\substack{N^{\frac{1}{500}}\leqslant p'<N^{\frac{25}{128}}\\(p',N)=1}} S\left(\mathcal{A}'_{p'};\mathcal{P}(N),N^{\frac{1}{500}}\right)$$

$$+ \sum_{\substack{N^{\frac{1}{500}}\leqslant p_2'<p_1'<N^{\frac{25}{128}}\\(p_1'p_2',N)=1}} S\left(\mathcal{A}'_{p_1'p_2'};\mathcal{P}(N),N^{\frac{1}{500}}\right)$$

$$
-\sum_{\substack{N^{\frac{1}{500}}\leqslant p_3'<p_2'<p_1'<N^{\frac{25}{128}}\\(p_1'p_2'p_3',N)=1}} S\left(\mathcal{A}'_{p_1'p_2'p_3'};\mathcal{P}(N),p_3'\right). \tag{33}
$$

Then by Lemma 3.3, Iwaniec's linear sieve method and arguments in [22] and [23] we have

$$
\begin{aligned}
S_{12} \leqslant{}& (1+o(1))\frac{2C(N)\left|\mathcal{A}'\right|}{e^{\gamma}\log N}\left(500F\left(500\boldsymbol{\vartheta}_{\frac{1}{500}}\right)-500\int_{\frac{1}{500}}^{\frac{25}{128}}\frac{f(500(\boldsymbol{\vartheta}_1(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt\right.\\
&+500\int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1}\frac{F(500(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1\\
&\left.-\int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{f\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\right)\\
\leqslant{}& (1+o(1))\frac{2G_3}{e^{\gamma}}\left(\int_2^{\frac{1927}{727}}\frac{\log(t-1)}{t}dt\right)\frac{C(N)N}{(\log N)^2}\\
\leqslant{}& 0.498525\frac{C(N)N}{(\log N)^2},
\end{aligned} \tag{34}
$$

where

$$
\begin{aligned}
G_3 ={}& 500F\left(500\boldsymbol{\vartheta}_{\frac{1}{500}}\right)-500\int_{\frac{1}{500}}^{\frac{25}{128}}\frac{f(500(\boldsymbol{\vartheta}_1(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt\\
&+500\int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1}\frac{F(500(\boldsymbol{\vartheta}_1(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1\\
&-\int_{\frac{1}{500}}^{\frac{25}{128}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{f\left(\frac{(\boldsymbol{\vartheta}_1(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1.
\end{aligned} \tag{35}
$$

Similarly, for S_{13}–S_{16} we have

$$
\begin{aligned}
S_{13} \leqslant{}& (1+o(1))\frac{2G_3}{e^{\gamma}}\left(\int_2^{12.27}\frac{\log\left(2-\frac{3}{t+1}\right)}{t}dt\right)\frac{C(N)N}{(\log N)^2}\\
\leqslant{}& 4.514343\frac{C(N)N}{(\log N)^2},
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
S_{14} \leqslant{}& (1+o(1))\frac{2G_3}{e^{\gamma}}\left(\int_{\frac{1927}{727}}^{7.24}\frac{\log\left(\frac{1927}{727}-\frac{\frac{2654}{727}}{t+1}\right)}{t}dt\right)\frac{C(N)N}{(\log N)^2}\\
\leqslant{}& 4.576860\frac{C(N)N}{(\log N)^2},
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
S_{15} \leqslant{}& (1+o(1))\frac{2G_3}{e^{\gamma}}\left(\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\int_{t_1}^{\frac{1}{8.24}}\int_{t_2}^{\frac{1}{8.24}}\int_{t_3}^{\frac{1}{8.24}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C(N)N}{(\log N)^2}\\
\leqslant{}& 0.090595\frac{C(N)N}{(\log N)^2},
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
S_{16} \leqslant{}& (1+o(1))\frac{2G_3}{e^{\gamma}}\left(\int_{\frac{1}{13.27}}^{\frac{1}{8.24}}\int_{t_1}^{\frac{1}{8.24}}\int_{t_2}^{\frac{1}{8.24}}\int_{\frac{1}{8.24}}^{\frac{1}{2}-\frac{2}{13.27}-t_3}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C(N)N}{(\log N)^2}\\
\leqslant{}& 0.499530\frac{C(N)N}{(\log N)^2}.
\end{aligned} \tag{39}
$$

Finally, by Lemma 4.1 and (11)–(39) we get

$$3S_1+S_2+S_8+S_9+S_{10}+S_{11}\geqslant 60.495865\frac{C(N)N}{(\log N)^2},$$

$$2S_3+2S_4+2S_5+S_6+S_7+2S_{12}+S_{13}+S_{14}+S_{15}+S_{16}\leqslant 53.563025\frac{C(N)N}{(\log N)^2},$$

$$\begin{aligned}4D_{1,2}(N)\geqslant&(3S_1+S_2+S_8+S_9+S_{10}+S_{11})\\&-(2S_3+2S_4+2S_5+S_6+S_7+2S_{12}+S_{13}+S_{14}+S_{15}+S_{16})\\\geqslant&\ 6.932\frac{C(N)N}{(\log N)^2},\end{aligned}$$

$$D_{1,2}(N)\geqslant 1.733\frac{C(N)N}{(\log N)^2}.$$

Theorem 1.1 is proved. The detail of the proof of Theorem 1.2 is similar to those of Theorem 1.1 and Theorem 1.1 in [19] so we omit it in this paper.

6. Proof of Theorem 1.3

In this section, sets $\mathcal{B}$ and $\mathcal{P}$ are defined respectively. For S_1' and S_2', by Buchstab's identity, we have

$$\begin{aligned}S_1'=S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{12}}\right)=S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{500}}\right)&-\sum_{x^{\frac{1}{500}}\leqslant p<x^{\frac{1}{12}}}S\left(\mathcal{B}_p;\mathcal{P},x^{\frac{1}{500}}\right)\\&+\sum_{x^{\frac{1}{500}}\leqslant p_2<p_1<x^{\frac{1}{12}}}S\left(\mathcal{B}_{p_1p_2};\mathcal{P},x^{\frac{1}{500}}\right)\\&-\sum_{x^{\frac{1}{500}}\leqslant p_3<p_2<p_1<x^{\frac{1}{12}}}S\left(\mathcal{B}_{p_1p_2p_3};\mathcal{P},p_3\right)\end{aligned}\tag{40}$$

and

$$\begin{aligned}S_2'=S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{7.2}}\right)=S\left(\mathcal{B};\mathcal{P},x^{\frac{1}{500}}\right)&-\sum_{x^{\frac{1}{500}}\leqslant p<x^{\frac{1}{7.2}}}S\left(\mathcal{B}_p;\mathcal{P},x^{\frac{1}{500}}\right)\\&+\sum_{x^{\frac{1}{500}}\leqslant p_2<p_1<x^{\frac{1}{7.2}}}S\left(\mathcal{B}_{p_1p_2};\mathcal{P},x^{\frac{1}{500}}\right)\\&-\sum_{x^{\frac{1}{500}}\leqslant p_3<p_2<p_1<x^{\frac{1}{7.2}}}S\left(\mathcal{B}_{p_1p_2p_3};\mathcal{P},p_3\right).\end{aligned}\tag{41}$$

By Lemma 3.2, Iwaniec's linear sieve method and arguments in [22] and [23] we have

$$\begin{aligned}S_1'\geqslant&(1+o(1))\frac{1}{e^\gamma}\left(500f\left(500\boldsymbol{\vartheta}'_{\frac{1}{500}}\right)-500\int_{\frac{1}{500}}^{\frac{1}{12}}\frac{F(500(\boldsymbol{\vartheta}_0(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt\right.\\&+500\int_{\frac{1}{500}}^{\frac{1}{12}}\int_{\frac{1}{500}}^{t_1}\frac{f(500(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1\\&\left.-\int_{\frac{1}{500}}^{\frac{1}{12}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}\\\geqslant&\ 6.737438\frac{C_2x}{(\log x)^2}\end{aligned}\tag{42}$$

and

$$S_2'\geqslant(1+o(1))\frac{1}{e^\gamma}\left(500f\left(500\boldsymbol{\vartheta}'_{\frac{1}{500}}\right)-500\int_{\frac{1}{500}}^{\frac{1}{7.2}}\frac{F(500(\boldsymbol{\vartheta}_0(t,\frac{1}{500},\frac{1}{500})-t))}{t}dt\right.$$

$$
\begin{aligned}
&+500\int_{\frac{1}{500}}^{\frac{1}{7.2}}\int_{\frac{1}{500}}^{t_1}\frac{f(500(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{500})-t_1-t_2))}{t_1t_2}dt_2dt_1\\
&-\int_{\frac{1}{500}}^{\frac{1}{7.2}}\int_{\frac{1}{500}}^{t_1}\int_{\frac{1}{500}}^{t_2}\frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2}\\
\geqslant\ & 4.008831\frac{C_2x}{(\log x)^2},
\end{aligned}
\tag{43}
$$

where $\boldsymbol{\vartheta}'_{\frac{1}{500}}=\frac{16483}{26750}$. For S_3'–S_7', by Lemma 3.2, Iwaniec's linear sieve method and above discussion, we have

$$
\begin{aligned}
S_3'\geqslant\ &(1+o(1))\frac{1}{e^\gamma}\Bigg(\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{\frac{1}{12}}^{t_1}\max\Bigg(12\frac{f(12(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{12})-t_1-t_2))}{t_1t_2},\\
&\max_{12\leqslant k\leqslant 500}\Bigg(k\frac{f(k(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}\\
&-\int_{\frac{1}{k}}^{\frac{1}{12}}\frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3\Bigg)\Bigg)dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2}\\
\geqslant\ & 0.874702\frac{C_2x}{(\log x)^2},
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
S_4'\geqslant\ &(1+o(1))\frac{1}{e^\gamma}\Bigg(\int_{\frac{1}{7.2}}^{\frac{25}{107}}\int_{\frac{1}{12}}^{\frac{1}{7.2}}\max\Bigg(12\frac{f(12(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{12})-t_1-t_2))}{t_1t_2},\\
&\max_{12\leqslant k\leqslant 500}\Bigg(k\frac{f(k(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}\\
&-\int_{\frac{1}{k}}^{\frac{1}{12}}\frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3\Bigg)\Bigg)dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2}\\
\geqslant\ & 1.704764\frac{C_2x}{(\log x)^2},
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
S_5'\geqslant\ &(1+o(1))\frac{1}{e^\gamma}\Bigg(12\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{\frac{25}{107}}^{\min(\frac{2}{7},\frac{17}{42}-t_1)}\frac{f(12(\boldsymbol{\vartheta}_0(t_2)-t_1-t_2))}{t_1t_2}dt_2dt_1\Bigg)\frac{C_2x}{(\log x)^2}\\
\geqslant\ & 0.448166\frac{C_2x}{(\log x)^2},
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
S_6'\leqslant\ &(1+o(1))\frac{1}{e^\gamma}\Bigg(\int_{\frac{1}{12}}^{\frac{25}{107}}\min\Bigg(12\frac{F(12(\boldsymbol{\vartheta}_0(t_1,\frac{1}{12},\frac{1}{12})-t_1))}{t_1},\\
&\min_{12\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_0(t_1,\frac{1}{k},\frac{1}{k})-t_1))}{t_1}-k\int_{\frac{1}{k}}^{\frac{1}{12}}\frac{f(k(\boldsymbol{\vartheta}_0(t_1,t_2,\frac{1}{k})-t_1-t_2))}{t_1t_2}dt_2\\
&+\int_{\frac{1}{k}}^{\frac{1}{12}}\int_{\frac{1}{k}}^{t_2}\frac{F\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1t_2t_3^2}dt_3dt_2\Bigg)\Bigg)dt_1\Bigg)\frac{C_2x}{(\log x)^2}\\
\leqslant\ & 6.953322\frac{C_2x}{(\log x)^2},
\end{aligned}
\tag{47}
$$

$$
S_7'\leqslant(1+o(1))\frac{1}{e^\gamma}\Bigg(\int_{\frac{25}{107}}^{\frac{2}{7}}\min\Bigg(12\frac{F(12(\boldsymbol{\vartheta}_0(t_1)-t_1))}{t_1},
$$

$$\min_{12\leqslant k\leqslant 500}\Bigg(k\frac{F(k(\boldsymbol{\vartheta}_0(t_1)-t_1))}{t_1}-\int_{\frac{1}{k}}^{\frac{1}{12}}\frac{f\left(\frac{(\boldsymbol{\vartheta}_0(t_1)-t_1-t_2)}{t_2}\right)}{t_1t_2^2}dt_2\Bigg)\Bigg)\,dt_1\Bigg)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 1.390939\frac{C_2x}{(\log x)^2}. \tag{48}$$

For S'_{12}–S'_{19}, by Chen's switching principle, Lemma 3.4 and above arguments on estimating S_{12}–S_{16} we have

$$S'_{12}\leqslant(1+o(1))\frac{G_4}{e^\gamma}\left(\int_2^{11}\frac{\log\left(2-\frac{3}{t+1}\right)}{t}dt\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 1.981662\frac{C_2x}{(\log x)^2}, \tag{49}$$

$$S'_{13}\leqslant(1+o(1))\frac{G_4}{e^\gamma}\left(\int_{2.5}^{6.2}\frac{\log\left(2.5-\frac{3.5}{t+1}\right)}{t}dt\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 1.717054\frac{C_2x}{(\log x)^2}, \tag{50}$$

$$S'_{14}\leqslant(1+o(1))\frac{G_4}{e^\gamma}\left(\int_2^{2.5}\frac{\log(t-1)}{t}dt\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.153821\frac{C_2x}{(\log x)^2}, \tag{51}$$

$$S'_{15}\leqslant(1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{1}{12}}^{\frac{1}{7.2}}\int_{t_1}^{\frac{1}{7.2}}\int_{t_2}^{\frac{1}{7.2}}\int_{t_3}^{\frac{1}{7.2}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.051713\frac{C_2x}{(\log x)^2}, \tag{52}$$

$$S'_{16}\leqslant(1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{1}{12}}^{\frac{5}{42}}\int_{t_1}^{\frac{5}{42}}\int_{t_2}^{\frac{5}{42}}\int_{\frac{1}{7.2}}^{\frac{2}{7}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.101840\frac{C_2x}{(\log x)^2}, \tag{53}$$

$$S'_{17}\leqslant(1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{1}{12}}^{\frac{5}{42}}\int_{t_1}^{\frac{5}{42}}\int_{\frac{5}{42}}^{\frac{1}{7.2}}\int_{\frac{1}{7.2}}^{\frac{17}{42}-t_3}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.118478\frac{C_2x}{(\log x)^2}, \tag{54}$$

$$S'_{18}\leqslant(1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{1}{12}}^{\frac{5}{42}}\int_{\frac{5}{42}}^{\frac{1}{7.2}}\int_{t_2}^{\frac{1}{7.2}}\int_{\frac{1}{7.2}}^{\frac{17}{42}-t_3}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.042337\frac{C_2x}{(\log x)^2}, \tag{55}$$

$$S'_{19}\leqslant(1+o(1))\frac{G_4}{e^\gamma}\left(\int_{\frac{5}{42}}^{\frac{1}{7.2}}\int_{t_1}^{\frac{1}{7.2}}\int_{t_2}^{\frac{1}{7.2}}\int_{\frac{1}{7.2}}^{\frac{17}{42}-t_3}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\frac{C_2x}{(\log x)^2}$$
$$\leqslant 0.005901\frac{C_2x}{(\log x)^2}. \tag{56}$$

where

$$G_4 = 500F\left(500\boldsymbol{\vartheta}'_{\frac{1}{500}}\right) - 500\int_{\frac{1}{500}}^{\frac{21}{107}} \frac{f(500(\boldsymbol{\vartheta}_0(t, \frac{1}{500}, \frac{1}{500}) - t))}{t} dt$$
$$+ 500\int_{\frac{1}{500}}^{\frac{21}{107}} \int_{\frac{1}{500}}^{t_1} \frac{F(500(\boldsymbol{\vartheta}_0(t_1, t_2, \frac{1}{500}) - t_1 - t_2))}{t_1 t_2} dt_2 dt_1$$
$$- \int_{\frac{1}{500}}^{\frac{21}{107}} \int_{\frac{1}{500}}^{t_1} \int_{\frac{1}{500}}^{t_2} \frac{f\left(\frac{(\boldsymbol{\vartheta}_0(t_1,t_2,t_3)-t_1-t_2-t_3)}{t_3}\right)}{t_1 t_2 t_3^2} dt_3 dt_2 dt_1. \tag{57}$$

For the remaining terms, by the arguments in [4] and [30], we have

$$S'_8 \ll \frac{\varepsilon C_2 x}{(\log x)^2}, \tag{58}$$

$$S'_9 \leqslant (1+o(1))\frac{12}{e^\gamma}\left(\int_{(\frac{11}{20}-\frac{29}{100})12}^{(\frac{4}{7}-\frac{2}{7})12} \frac{F(t)}{2\times 12 - t} dt\right) \leqslant 0.111039\frac{C_2 x}{(\log x)^2}, \tag{59}$$

$$S'_{10} \leqslant (1+o(1))\frac{12}{e^\gamma}\left(\int_{(\frac{11}{20}-\frac{1}{3})12}^{(\frac{11}{20}-\frac{29}{100})12} \frac{F(t)}{\frac{11}{20}\times 12 - t} dt\right) \leqslant 1.169696\frac{C_2 x}{(\log x)^2}, \tag{60}$$

$$S'_{11} \ll \frac{\varepsilon C_2 x}{(\log x)^2}. \tag{61}$$

Finally, by Lemma 4.2 and (40)–(61) we get

$$3S'_1 + S'_2 + S'_3 + S'_4 + S'_5 \geqslant 27.248777\frac{C_2 x}{(\log x)^2},$$
$$2S'_6 + 2S'_7 + S'_8 + S'_9 + S'_{10} + S'_{11} + S'_{12} + S'_{13}$$
$$+ 2S'_{14} + S'_{15} + S'_{16} + S'_{17} + S'_{18} + S'_{19} \leqslant 22.295884\frac{C_2 x}{(\log x)^2},$$
$$4\pi_{1,2}(x) \geqslant (3S'_1 + S'_2 + S'_3 + S'_4 + S'_5)$$
$$- (2S'_6 + 2S'_7 + S'_8 + S'_9 + S'_{10} + S'_{11} + S'_{12} + S'_{13}$$
$$+ 2S'_{14} + S'_{15} + S'_{16} + S'_{17} + S'_{18} + S'_{19})$$
$$\geqslant 4.952\frac{C_2 x}{(\log x)^2},$$
$$\pi_{1,2}(x) \geqslant 1.238\frac{C_2 x}{(\log x)^2}.$$

Theorem 1.3 is proved.

7. Proof of Theorem 1.4

In this section, sets $\mathcal{C}$ and $\mathcal{P}$ are defined respectively. The details of calculation in this section can be found in [[19], Section 7]. Let $\theta = 0.9409$ in this section. For S''_1–S''_6, by Lemma 2.3 and Iwaniec's linear sieve we have

$$S''_1 \geqslant (1+o(1))\frac{8C(N)N^\theta}{\theta^2(\log N)^2}\left(\log(7\theta - 1) + \int_2^{7\theta-2} \frac{\log(s-1)}{s}\log\frac{7\theta-1}{s+1} ds\right)$$
$$\geqslant 16.70802\frac{C(N)N^\theta}{(\log N)^2}, \tag{62}$$

$$S''_2 \geqslant (1+o(1))\frac{8C(N)N^\theta}{\theta^2(\log N)^2}\left(\log(4.4\theta - 1) + \int_2^{4.4\theta-2} \frac{\log(s-1)}{s}\log\frac{4.4\theta-1}{s+1} ds\right)$$
$$\geqslant 10.340342\frac{C(N)N^\theta}{(\log N)^2}, \tag{63}$$

$$
\begin{aligned}
S_3'' + S_4'' &\geqslant (1+o(1))\frac{8C(N)N^\theta}{\theta(\log N)^2}\left(\int_{\frac{1}{14}}^{\frac{1}{8.8}}\int_{t_1}^{\frac{4.5863}{14}-t_1}\frac{\log\left((7\theta-1)-14\left(t_1+t_2\right)\right)}{t_1t_2\left(\theta-2\left(t_1+t_2\right)\right)}dt_2dt_1\right)\\
&\geqslant 5.914688\frac{C(N)N^\theta}{(\log N)^2}, \qquad (64)\\
S_5'' &\leqslant (1+o(1))\frac{8C(N)N^\theta}{\theta^2(\log N)^2}\left(\log\frac{4.08631(14\theta-2)}{14\theta-8.17262}\right.\\
&\quad+\int_2^{7\theta-2}\frac{\log(s-1)}{s}\log\frac{(7\theta-1)(7\theta-1-s)}{s+1}ds\\
&\quad\left.+\int_2^{7\theta-4}\frac{\log(s-1)}{s}ds\int_{s+2}^{7\theta-2}\frac{1}{t}\log\frac{t-1}{s+1}\log\frac{(7\theta-1)(7\theta-1-t)}{t+1}dt\right)\\
&\leqslant 24.63508\frac{C(N)N^\theta}{(\log N)^2}, \qquad (65)\\
S_6'' &\leqslant (1+o(1))\frac{8C(N)N^\theta}{\theta^2(\log N)^2}\left(\log\frac{3.5863(14\theta-2)}{14\theta-7.1726}\right.\\
&\quad+\int_2^{7\theta-2}\frac{\log(s-1)}{s}\log\frac{(7\theta-1)(7\theta-1-s)}{s+1}ds\\
&\quad\left.+\int_2^{7\theta-4}\frac{\log(s-1)}{s}ds\int_{s+2}^{7\theta-2}\frac{1}{t}\log\frac{t-1}{s+1}\log\frac{(7\theta-1)(7\theta-1-t)}{t+1}dt\right)\\
&\leqslant 21.808021\frac{C(N)N^\theta}{(\log N)^2}. \qquad (66)
\end{aligned}
$$

For S_7'', S_8'', S_{13}'' and S_{14}'', by Chen's switching principle, Lemma 2.3 and similar arguments as those in estimation of S_{12}, we have

$$
\begin{aligned}
S_7''+S_8'' &\leqslant (1+o(1))\frac{8C(N)N^\theta}{(2\theta-1)(\log N)^2}\left(\int_{2.1}^{13}\frac{\log\left(2.1-\frac{3.1}{s+1}\right)}{s}ds+\int_{2.7}^{7.8}\frac{\log\left(2.7-\frac{3.7}{s+1}\right)}{s}ds\right)\\
&\leqslant 13.953531\frac{C(N)N^\theta}{(\log N)^2}, \qquad (67)\\
S_{13}''+S_{14}'' &\leqslant (1+o(1))\frac{8C(N)N^\theta}{ab(2\theta-1)(\log N)^2}\left(\int_2^{2.1}\frac{\log(s-1)}{s}ds+\int_2^{2.7}\frac{\log(s-1)}{s}ds\right)\\
&\leqslant 0.771273\frac{C(N)N^\theta}{(\log N)^2}. \qquad (68)
\end{aligned}
$$

For S_{11}'' and S_{12}'', by Chen's switching principle and Lemma 2.4 we have

$$
\begin{aligned}
S_{11}'' &\leqslant (1+o(1))\frac{8C(N)N^\theta}{(2\theta-1)(\log N)^2}\left(\int_{\frac{1}{14}}^{\frac{1}{8.8}}\int_{t_1}^{\frac{1}{8.8}}\int_{t_2}^{\frac{1}{8.8}}\int_{t_3}^{\frac{1}{8.8}}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\\
&\leqslant 0.115178\frac{C(N)N^\theta}{(\log N)^2}, \qquad (69)\\
S_{12}'' &\leqslant (1+o(1))\frac{8C(N)N^\theta}{(2\theta-1)(\log N)^2}\left(\int_{\frac{1}{14}}^{\frac{1}{8.8}}\int_{t_1}^{\frac{1}{8.8}}\int_{t_2}^{\frac{1}{8.8}}\int_{\frac{1}{8.8}}^{\frac{4.5863}{14}-t_3}\frac{\omega\left(\frac{1-t_1-t_2-t_3-t_4}{t_2}\right)}{t_1t_2^2t_3t_4}dt_4dt_3dt_2dt_1\right)\\
&\leqslant 0.653492\frac{C(N)N^\theta}{(\log N)^2}. \qquad (70)
\end{aligned}
$$

For the remaining S_9'' and S_{10}'', by Lemma 2.4 and the methods used in [6] and [19] we have

$$S_9'' + S_{10}'' \leqslant (1+o(1))\frac{8G_5 C(N)N^\theta}{\theta(\log N)^2}\left(\int_{\frac{4.08631}{14}}^{\frac{1}{3.1}} \frac{dt}{t(\theta-2t)} + \int_{\frac{3.5863}{14}}^{\frac{1}{3.7}} \frac{dt}{t(\theta-2t)}\right)$$

$$\leqslant 3.67\frac{C(N)N^\theta}{(\log N)^2}, \tag{71}$$

where

$$G_5 = 1 + \int_2^{2.675} \frac{\log(t-1)}{t}dt - \frac{1}{2}\int_{1.5}^{2.675} \frac{\log\left(2.675 - \frac{3.675}{t+1}\right)}{t}dt + \frac{\theta\left(6.175\log\frac{3.675}{2.5} - 2.35\right)}{1.763(2\theta-1)}. \tag{72}$$

Finally, by Lemma 4.3 and (62)–(72) we get

$$3S_1'' + S_2'' + S_3'' + S_4'' \geqslant 66.379\frac{C(N)N^\theta}{(\log N)^2},$$

$$S_5'' + S_6'' + S_7'' + S_8'' + S_9'' + S_{10}''$$

$$+ S_{11}'' + S_{12}'' + 2S_{13}'' + 2S_{14}'' \leqslant 66.378\frac{C(N)N^\theta}{(\log N)^2},$$

$$4D_{1,2}^\theta(N) \geqslant (3S_1'' + S_2'' + S_3'' + S_4'')$$

$$- (S_5'' + S_6'' + S_7'' + S_8'' + S_9'' + S_{10}''$$

$$+ S_{11}'' + S_{12}'' + 2S_{13}'' + 2S_{14}'')$$

$$\geqslant 0.001\frac{C(N)N^\theta}{(\log N)^2},$$

$$D_{1,2}^\theta(N) \geqslant 0.00025\frac{C(N)N^\theta}{(\log N)^2}.$$

Theorem 1.4 is proved.

References

[1] Y. Cai. Chen's theorem with small primes. *Acta Math. Sin. (Engl. Ser.)*, 18(3):597–604, 2002.

[2] Y. Cai. A remark on Chen's theorem. *Acta Arith.*, 102(4):339–352, 2002.

[3] Y. Cai. On Chen's theorem. II. *J. Number Theory*, 128(5):1336–1357, 2008.

[4] Y. Cai. A remark on Chen's theorem (II). *Chinese Ann. Math. Ser. B*, 29(6):687–698, 2008.

[5] Y. Cai. A remark on Chen's theorem with small primes. *Taiwanese J. Math.*, 19(4):1183–1202, 2015.

[6] Y. Cai and Y. Li. Chen's theorem with small primes. *Chinese Ann. Math. Ser. B*, 32(3):387–396, 2011.

[7] Y. Cai and M. Lu. On Chen's theorem. In *Analytic number theory (Beijing/Kyoto, 1999)*, volume 6 of *Dev. Math.*, pages 99–119. Kluwer Acad. Publ., Dordrecht, 2002.

[8] J. R. Chen. On the representation of a larger even integer as the sum of a prime and the product of at most two primes. *Sci. Sinica*, 16:157–176, 1973.

[9] J. R. Chen. Further improvement on the constant in the proposition '1+2': On the representation of a large even integer as the sum of a prime and the product of at most two primes (II). *Sci. Sinica*, pages 477–494(in Chinese), 1978.

[10] J. R. Chen. On the representation of a large even integer as the sum of a prime and the product of at most two primes. II. *Sci. Sinica*, 21(4):421–430, 1978.

[11] J. R. Chen. On some problems in prime number theory. In *Séminaire de théorie des nombres, Paris 1979-80*, pages 167–170. Birkhäuser, Boston, 1981.

[12] E. Fouvry and F. Grupp. On the switching principle in sieve theory. *J. Reine Angew. Math.*, 1986(370):101–126, 1986.

[13] H. Halberstam. A proof of Chen's theorem. In *Journées Arithmétiques de Bordeaux (Conf., Univ. Bordeaux, 1974),*, Astérisque, No. 24-25,, pages 281–293. ,, 1975.

[14] H. Halberstam and H.-E. Richert. *Sieve methods*, volume No. 4. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1974.

[15] H. H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, with appendix 1 by D. Ramakrishnan and appendix 2 by H. H. Kim and P. Sarnak. *J. Amer. Math. Soc.*, 16:139–183, 2003.

[16] H. Li. Additive representations of natural numbers. *Ramanujan J.*, 60(4):999–1024, 2023.

[17] R. Li. A remark on the distribution of $\sqrt{p}$ modulo one involving primes of special type. *arXiv e-prints*, page arXiv:2401.01351, November 2023.

[18] R. Li. On the upper and lower bound orders of almost prime triples. *arXiv e-prints*, page arXiv:2401.01348, November 2023.

[19] R. Li. Remarks on additive representations of natural numbers. *arXiv e-prints*, page arXiv:2309.03218, September 2023.
[20] R. Li. A remark on large even integers of the form $p + P_3$. *arXiv e-prints*, page arXiv:2403.09691, February 2024.
[21] R. Li. On Chen's theorem, Goldbach's conjecture and almost prime twins. *arXiv e-prints*, page arXiv:2405.05727, May 2024.
[22] J. D. Lichtman. A modification of the linear sieve, and the count of twin primes. *arXiv e-prints*, page arXiv:2109.02851, September 2021.
[23] J. D. Lichtman. Primes in arithmetic progressions to large moduli, and Goldbach beyond the square-root barrier. *arXiv e-prints*, page arXiv:2309.08522, August 2023.
[24] H.-Q. Liu. On the prime twins problem. *Sci. Sinica*, 33(3):281–298, 1990.
[25] C. D. Pan and C. B. Pan. *Goldbach conjecture*. Science Press Beijing, Beijing, 1992. Translated and revised from the Chinese manuscript by the authors.
[26] P. M. Ross. On linear combinations of primes and numbers having at most two prime factors. *Ph.D. Thesis*, University of London, 1976.
[27] J. Wu. Sur la suite des nombres premiers jumeaux. *Acta Arith.*, 55(4):365–394, 1990.
[28] J. Wu. Théorèmes généralisés de Bombieri-Vinogradov dans les petits intervalles. *Quart. J. Math. Oxford Ser. (2)*, 44(173):109–128, 1993.
[29] J. Wu. Chen's double sieve, Goldbach's conjecture and the twin prime problem. *Acta Arith.*, 114(3):215–273, 2004.
[30] J. Wu. Chen's double sieve, Goldbach's conjecture and the twin prime problem. II. *Acta Arith.*, 131(4):367–387, 2008.

致谢:

1. 本文完全由本人研究撰写，论文结果整合了本人在 arXiv 上的两篇预印本的部分结果，在本文第一节已有叙述。文章中的全部数值计算过程均由本人完成，使用 Mathematica 14 软件辅助计算。

2. 本文的研究背景是筛法研究哥德巴赫猜想与陈景润的“1+2”定理，文章的主要目的是改进目前最好的下界系数与改进特殊素数情况下的陈景润定理。郭振宇老师为本文提供了一些参考文献，并对文章的写作作了一些建议。崔旺老师是本人的校内数学老师，提供无偿指导。本文作者对郭振宇老师提供的论文写作修改建议以及使用 Matematica 软件进行计算的建议表示衷心的感谢。

3. 本文的选题由作者本人确定，证明中的关键工具来自 2023 年 9 月 Lichtman 的文章。本文的全部计算推导过程均由作者本人独立完成。郭振宇老师在西安交通大学与本人进行了有关文章内容和新方法的讨论，并提供了一些该问题在此前的研究历史文献。崔旺老师在校内与作者进行了一些讨论，并提供了一些建议。本文全文由作者本人撰写，其中部分内

容来自作者本人此前的 arXiv 预印本。在本文的写作过程中，作者在整合此前自己的预印本中的结果时由于结果过于纷繁复杂，难以处理。因此作者决定简化证明过程，缩减了定理 1.4 证明的篇幅（证明中的详细推导可参考本人的预印本 [19] 第七节），最终完成了本文的写作。