Continuous Points of Functions on an Interval

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Abstract

In this paper, we address the problem of finding functions with predetermined continuous points and Lipschitz continuous points. More precisely, given $A \subseteq B \subseteq [0,1]$, we are interested in the existence of function $f : [0,1] \to \mathbb{R}$ which is Lipschitz continuous exactly on A and continuous exactly on B. We will give examples, existence theorems and non-existence theorems, which partially answer the question.

Contents

1 Introduction

For $f : [0,1] \to \mathbb{R}$, we denote

$$\mathcal{C}(f) = \{x \in [0,1] \,|\, f \text{ continuous at } x\} \tag{1.1}$$

and

$$\mathcal{L}(f) = \{x \in [0,1] \,|\, f \text{ Lipschitz continuous at } x\}. \tag{1.2}$$

This paper is motivated by an unsuccessful attempt to construct a function on $[0,1]$ with $\mathcal{C}(f) = [0,1] \cap \mathbb{Q}$. Given $S \subseteq [0,1]$, can we construct a function

$$f : [0,1] \to \mathbb{R} \tag{1.3}$$

such that $\mathcal{C}(f) = S$? Many beginners in analysis have (at least briefly) considered about this. So does the author. For S finite or $[0,1]\backslash S$ finite, the answer is immediate. For $S = [0,1]\backslash\mathbb{Q}$, the answer is standard. Now consider $S = [0,1] \cap \mathbb{Q}$. We will prove the follows.

1) There is no function $f : [0,1] \to \mathbb{R}$ with $\mathcal{C}(f) = [0,1] \cap \mathbb{Q}$.

2) There is a function $f : [0,1] \to \mathbb{R}$ such that $\mathcal{L}(f) = [0,1] \cap \mathbb{Q}$ and $\mathcal{C}(f)$ is a null set.

We will equally prove some related results. In particular, we prove that

3) Given $S \subseteq [0,1]$, if both S and $[0,1]\backslash S$ are dense, then there is no function on $[0,1]$ with $\mathcal{C}(f) = \mathcal{L}(f) = S$.

Here 1) is standard. But we still give a proof.

This paper is organized as follows. In §2, we give some examples of functions with predetermined continuous points. In §3, we introduce the oscillation function which will be used in latter sections. In §4, we address the problem of finding functions with predetermined continuous points. In §5, we introduce Liptchitz continuous points. In 6, we further address the problem of finding functions with predetermined Lipschitz continuous points. In the appendices §7,8, we introduce some tools from measure theory and topology.

2 Illustrative Examples

Most examples in this section are standard. We will skip some proofs.

For a bounded function $f : [0,1] \to \mathbb{R}$, we denote

$$\mathcal{C}(f) = \{x \in [0,1] \,|\, f \text{ continuous at } x\} \tag{2.1}$$

and

$$\mathcal{D}(f) = \{x \in [0,1] \,|\, f \text{ discontinuous at } x\}. \tag{2.2}$$

Let $f_1 : [0,1] \to \mathbb{R}$ be such that

$$f_1(x) = 0 \text{ for any } x. \tag{2.3}$$

We have $\mathcal{D}(f_1) = \emptyset$.

Let $f_2 : [0,1] \to \mathbb{R}$ be such that

$$f_2(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{else.} \end{cases} \tag{2.4}$$

We have $\mathcal{C}(f_2) = \emptyset$.

Let $a_1, \cdots, a_n \in [0,1]$ be distinct numbers.

Let $f_3 : [0,1] \to \mathbb{R}$ be such that

$$f_3(x) = \begin{cases} 1 & \text{if } x \in \{a_1, \cdots, a_n\}, \\ 0 & \text{else.} \end{cases} \tag{2.5}$$

We have $\mathcal{D}(f_3) = \{a_1, \cdots, a_n\}$.

Let $f_4 : [0,1] \to \mathbb{R}$ be such that

$$f_4(x) = f_2(x) \prod_{k=1}^{n} |x - a_k|. \tag{2.6}$$

We have $\mathcal{C}(f_4) = \{a_1, \cdots, a_n\}$.

In the examples above, either $\mathcal{C}(f)$ or $\mathcal{D}(f)$ is discrete. In the sequel, we consider functions $f : [0,1] \to \mathbb{R}$ such that both $\mathcal{C}(f)$ and $\mathcal{D}(f)$ are dense.

Let $A \subseteq [0,1]$ be countable and dense. We denote $A = \{a_k \mid k \in \mathbb{N}\}$.

Let $f_5 : [0,1] \to \mathbb{R}$ be such that

$$f_5(x) = \begin{cases} 2^{-k} & \text{if } x = a_k \text{ for certain } k \in \mathbb{N}, \\ 0 & \text{else.} \end{cases} \tag{2.7}$$

Proposition 2.1. We have

$$\mathcal{D}(f_5) = A. \tag{2.8}$$

Proof. Since the zero points of f_5 are dense and $f_5\big|_A > 0$, f_5 is discontinuous at any point in A.

Now we show that f_5 is continuous on $[0,1]\backslash A$. Fix $x_0 \in [0,1]\backslash A$. For any $\varepsilon > 0$, let $n_\varepsilon \in \mathbb{N}$ be such that $2^{-n_\varepsilon} \leq \varepsilon$. Let $x_0 \in U_\varepsilon$ be an open neighborhood such that $a_k \notin U_\varepsilon$ for $k = 0, \cdots, n_\varepsilon$. Then we have

$$0 \leq f_5\big|_{U_\varepsilon} < 2^{-n_\varepsilon} \leq \varepsilon. \tag{2.9}$$

Hence f_5 is continuous at x_0. □

Here $\mathcal{D}(f_5)$ is a null set (see Definition 7.1). We can equally construct a function f such that $\mathcal{C}(f)$ is a null set.

Let $f_6 : [0,1] \to \mathbb{R}$ be such that

$$f_6(x) = \inf_{k \in \mathbb{N}} 2^k |x - a_k|. \tag{2.10}$$

For a bounded function $f : [0,1] \to \mathbb{R}$, We denote

$$\mathcal{Z}(f) = \big\{ x \in [0,1] \,\big|\, f(x) = 0 \big\}. \tag{2.11}$$

Proposition 2.2. We have

$$A \subsetneq \mathcal{C}(f_6) = \mathcal{Z}(f_6). \tag{2.12}$$

And $\mathcal{C}(f_6)$ is an uncountable null set.

Proof. For convenience, we denote

$$\varphi_k(x) = 2^k|x - a_k|. \tag{2.13}$$

Then we have

$$f_6 = \inf_{k\in\mathbb{N}} \varphi_k. \tag{2.14}$$

First we show that

$$\mathcal{C}(f_6) = \mathcal{Z}(f_6). \tag{2.15}$$

Fix an arbitrary $x_0 \in [0,1]$. We need to show that

$$f_6 \text{ is continuous at } x_0 \;\Leftrightarrow\; f_6(x_0) = 0. \tag{2.16}$$

By our construction, we have

$$f_6 \geq 0 \text{ and } f_6\big|_A = 0. \tag{2.17}$$

Since A is dense and $f_6\big|_A = 0$, if $f_6(x_0) > 0$ then f_6 is not continuous at x_0. So we have proved the ($\Rightarrow$) side. Now we assume that $f_6(x_0) = 0$. By equation (2.14), for any $\varepsilon > 0$, there exists certain k such that $\varphi_k(x_0) < \varepsilon$. Since φ_k is continuous, there exists a neighborhood $x_0 \in U_\varepsilon$ such that

$$\varphi_k\big|_{U_\varepsilon} < \varepsilon. \tag{2.18}$$

Again, by equation (2.14), we have

$$0 \leq f_6 \leq \varphi_k. \tag{2.19}$$

Combining equation (2.18) and equation (2.19), we get

$$0 \leq f_6\big|_{U_\varepsilon} < \varepsilon. \tag{2.20}$$

Hence f_6 is continuous at x_0. So we have proved the ($\Leftarrow$) side.

Now we show that $\mathcal{Z}(f_6)$ is a null set. We denote

$$I_{k,\varepsilon} = (a_k - 2^{-k}\varepsilon, a_k + 2^{-k}\varepsilon) \cap [0,1]. \tag{2.21}$$

We remark that

$$I_{k,\varepsilon} = \big\{x \in [0,1] \,\big|\, \varphi_k(x) < \varepsilon\big\}. \tag{2.22}$$

By equation (2.14) and equation (2.22),

$$\mathcal{Z}(f_6) \subseteq \bigcup_{k\in\mathbb{N}} I_{k,\varepsilon} \text{ for any } \varepsilon > 0. \tag{2.23}$$

We also have

$$\sum_{k\in\mathbb{N}} \big|I_{k,\varepsilon}\big| = \sum_{k\in\mathbb{N}} 2^{1-k}\varepsilon = 4\varepsilon. \tag{2.24}$$

Hence $\mathcal{Z}(f_6)$ is a null set.

Now we show that $\mathcal{Z}(f_6)$ is uncountable. By equation (2.14) and equation (2.22), we have

$$\mathcal{Z}(f_6) = \bigcap_{n=1}^{\infty} \bigcup_{k\in\mathbb{N}} I_{k,1/n}, \tag{2.25}$$

which is uncountable by Proposition 8.6.

Now we have proved that $\mathcal{C}(f_6) = \mathcal{Z}(f_6)$. We have also proved that $\mathcal{Z}(f_6)$ is an uncountable null set. We obviously have $A \subseteq \mathcal{Z}(f_6)$. Since A is countable, we have $A \subsetneq \mathcal{C}(f_6)$. ∎

3 Continuous Points, Zero Points and Oscillation Function

For all the examples in §2, we have $f \geq 0$ and $\mathcal{C}(f) \subseteq \mathcal{Z}(f)$. This is not simply a coincidence. The key point lies in the oscillation function.

Definition 3.1. For a bounded function $f : [0,1] \to \mathbb{R}$ and $x \in [0,1]$, we denote

$$\omega(f)(x) = \inf_{x \in I} \Big(\sup_{w \in I} f(w) - \inf_{w \in I} f(w) \Big), \tag{3.1}$$

where I runs over open intervals containing x. We call $\omega(f)$ the oscillation function of f.

We will establish several properties of $\omega(f)$. Particularly, the continuous points of f are exactly the zero points of $\omega(f)$. This result will be repeatedly used in this paper.

Definition 3.2. We say that a function $f : [0,1] \to \mathbb{R}$ is lower semicontinuous if for any $x \in [0,1]$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(y) < f(x) + \varepsilon \text{ for any } y \text{ satisfying } |y - x| < \delta. \tag{3.2}$$

Proposition 3.3. If $f : [0,1] \to \mathbb{R}$ is lower semicontinuous, then for any $a \in \mathbb{R}$, the subset $\big\{x \in [0,1] \,\big|\, f(x) < a\big\}$ is open in $[0,1]$.

Proof. This is an immediate consequence of the definition. □

Proposition 3.4. The function $\omega(f) : [0,1] \to \mathbb{R}$ is lower semicontinuous.

Proof. Fix $x \in [0,1]$. By the construction of $\omega(f)$, for any $\varepsilon > 0$, there exists an open interval $I \ni x$ such that

$$\sup_{w \in I} f(w) - \inf_{w \in I} f(w) < \omega(f)(x) + \varepsilon. \tag{3.3}$$

Then, by the definition of $\omega(f)$, we have $\omega(f)(y) < \omega(f)(x) + \varepsilon$ for any $y \in I$. Hence $\omega(f)$ is lower semicontinuous. □

Proposition 3.5. For a bounded function $f : [0,1] \to \mathbb{R}$, we have

$$\mathcal{C}(f) = \mathcal{Z}(\omega(f)). \tag{3.4}$$

Moreover, if $\mathcal{C}(f)$ is dense, we have

$$\mathcal{C}(f) = \mathcal{Z}(\omega(f)) = \mathcal{C}(\omega(f)). \tag{3.5}$$

Proof. Equation (3.4) follows directly from the definition of $\omega(f)$. We only prove equation (3.5). Since $\omega(f)$ is non negative and lower semicontinuous, $\omega(f)$ is continuous on $\mathcal{Z}(\omega(f))$. On the other hand, since $\mathcal{Z}(\omega(f))$ is dense, $\omega(f)$ is discontinuous on $[0,1] \backslash \mathcal{Z}(\omega(f))$. Hence $\mathcal{Z}(\omega(f)) = \mathcal{C}(\omega(f))$. □

Theorem 3.6. Consider a dense subset $S \subseteq [0,1]$. If there exists $f : [0,1] \to \mathbb{R}$ satisfying $\mathcal{C}(f) = S$, then there exists $g : [0,1] \to \mathbb{R}$ satisfying

$$g \geq 0,\ \mathcal{C}(g) = S,\ \mathcal{Z}(g) = S. \tag{3.6}$$

Proof. Take $g = \omega(f)$. By Proposition 3.5, g satisfies the desired properties. □

Now we prove a result for latter use.

Proposition 3.7. Consider a bounded function $f : [0,1] \to \mathbb{R}$ such that $\mathcal{C}(f)$ is dense. Then $\mathcal{D}(f)$ is of first category (see Definition 8.3).

Proof. Set $U_n = \{x \in [0,1] \,|\, \omega(f) < 1/n\}$. By Proposition 3.3 and Proposition 3.4, U_n is open in $[0,1]$. On the other hand, by Proposition 3.5, we have

$$\mathcal{C}(f) = \mathcal{Z}(\omega(f)) = \bigcap_{n>0} U_n. \tag{3.7}$$

Thus $U_n \supseteq \mathcal{C}(f)$. Since $\mathcal{C}(f)$ is dense, U_n is open and dense. Set $V_n = [0,1] \backslash U_n$, which is nowhere dense (see Definition 8.1). Taking the complement of equation (3.7), we get

$$\mathcal{D}(f) = \bigcup_{n>0} V_n. \tag{3.8}$$

Hence $\mathcal{D}(f)$ is of first category. □

4 Functions with Predetermined Continuous Points

Let $A \subseteq [0,1]$ be countable and dense. We denote $A = \{a_k \,|\, k \in \mathbb{N}\}$. In this section, we show that there does not exist $f : [0,1] \to \mathbb{R}$ such that $\mathcal{C}(f) = A$. In particular, there does not exist $f : [0,1] \to \mathbb{R}$ such that $\mathcal{C}(f) = [0,1] \cap \mathbb{Q}$.

Theorem 4.1. Consider a dense subset $S \subseteq [0,1]$. The follows are equivalent.

- There exists $f : [0,1] \to \mathbb{R}$ satisfying $\mathcal{C}(f) = S$.
- There exists a countable family of open sets $(U_k)_{k \in \mathbb{N}}$ such that $S = \bigcap_{k \in \mathbb{N}} U_k$.

Proof. The ($\Rightarrow$) part follows from equation (3.7). Now we prove the ($\Leftarrow$) part. We may assume that $U_k \supseteq U_{k+1}$ for each k. Set

$$f(x) = \begin{cases} 0 & \text{if } x \in S, \\ 1/k & \text{if } x \in U_k \backslash U_{k+1} \text{ with } k \geq 1, \\ 1 & \text{else.} \end{cases} \tag{4.1}$$

Since f vanishes on the dense subset S, f is discontinuous at any non zero point. Hence $\mathcal{C}(f) \subseteq S$. On the other hand, for any $x \in S$ and any $\varepsilon > 0$, taking $n \in \mathbb{N}$ such that $1/n < \varepsilon$, we have $0 \leq f\big|_{U_n} \leq \varepsilon$ and $x \in U_n$. Hence $\mathcal{C}(f) \supseteq S$. □

Theorem 4.2. Consider $S \subseteq [0,1]$. If S is countable and dense, then there does not exist a bounded function $f : [0,1] \to \mathbb{R}$ such that $\mathcal{C}(f) = S$.

Proof. This follows from Theorem 4.1 and Proposition 8.6. □

5 Lipschitz Continuous Points

In this section, we turn to study the set of Lipschitz continuous points, which enjoys very different property.

Definition 5.1. Consider a bounded function $f : [0,1] \to \mathbb{R}$. We say that f is Lipschitz continuous at $x \in [0,1]$ if there exists $C_x > 0$ such that

$$\big|f(y) - f(x)\big| \leq C_x \big|y - x\big| \text{ for any } y. \tag{5.1}$$

We denote

$$\mathcal{L}(f) = \big\{x \in [0,1] \,\big|\, f \text{ Lipschitz continuous at } x\big\}. \tag{5.2}$$

Obviously, we have $\mathcal{L}(f) \subseteq \mathcal{C}(f)$.

For $S \subseteq \mathbb{R}$, we denote by $m^*(S)$ the outer measure (see Definition 7.2) of S.

Now we prove a result for latter use.

Lemma 5.2. Consider a bounded function $f : [0,1] \to \mathbb{R}$. If f is Lipschitz continuous at $b \in [0,1]$ and $f(b) = 0$, then there exists $\alpha > 0$ such that for any open interval $J \ni b$, we have

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} m^*\Big(\big\{x \in J \,\big|\, f(x) < \varepsilon\big\}\Big) \geq \alpha. \tag{5.3}$$

Proof. Since f is Lipschitz continuous at $b \in [0,1]$ and $f(b) = 0$, there exists $C > 0$ such that $f(x) \leq C|x - b|$ for any x. Thus we have

$$\big\{x \in J \,\big|\, f(x) < \varepsilon\big\} \supseteq [0,1] \cap J \cap \big(b - \varepsilon/C, b + \varepsilon/C\big). \tag{5.4}$$

Using equation (7.5), we can show that

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} m^*\Big([0,1] \cap J \cap \big(b - \varepsilon/C, b + \varepsilon/C\big)\Big) \geq 1/C. \tag{5.5}$$

By equation (5.4), equation (5.5) and equation (7.3), the inequality (5.3) holds with $\alpha = 1/C$. □

6 Functions with predetermined Lipschitz continuous points

Let $f_6 : [0,1] \to \mathbb{R}$ be as in §2.

Proposition 6.1. We have $\mathcal{L}(f_6) = A$. Here $A = \{a_k \,|\, k \in \mathbb{N}\}$ as is defined in §2.

Proof. Fix $a_k \in A$. We have $f_6(a_k) = 0$ and

$$0 \leq f_6(y) \leq 2^k \big|y - a_k\big| \text{ for any } y. \tag{6.1}$$

Hence f_6 is Lipschitz continuous at a_k.

Let φ_k be as in equation (2.13). Let $I_{k,\varepsilon}$ be as in equation (2.21).

Fix $b \in [0,1] \backslash A$. For $n \in \mathbb{N}$, let $J_n \ni b$ be an open interval such that $a_k \notin \overline{J_n}$ for $k \leq n$. Then there exists $\kappa_n > 0$ such that

$$\varphi_k\big|_{J_n} > \kappa_n \text{ for } k \leq n. \tag{6.2}$$

By equation (2.14), equation (2.22) and (6.2), for $0 < \varepsilon < \kappa_n$, we have

$$\{x \in J_n \mid f_6(x) < \varepsilon\} \subseteq \bigcup_{k>n} I_{k,\varepsilon}. \tag{6.3}$$

Thus, for $0 < \varepsilon < \kappa_n$, we have

$$m^*\Big(\{x \in J_n \mid f_6(x) < \varepsilon\}\Big) \le \sum_{k>n} |I_{k,\varepsilon}| = 2^{1-n}\varepsilon. \tag{6.4}$$

As a consequence, we have

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon} m^*\Big(\{x \in J_n \mid f_6(x) < \varepsilon\}\Big) \le 2^{1-n}. \tag{6.5}$$

Since n is arbitrary, by Lemma 5.2, f_6 is not Lipschitz continuous at b. □

In particular, taking $A = [0,1] \cap \mathbb{Q}$, we obtain a function which is Lipschitz continuous exactly on $[0,1] \cap \mathbb{Q}$.

It is natural to ask if we can find certain f such that $\mathcal{L}(f) = \mathcal{C}(f)$. The answer is always negative as long as both $\mathcal{C}(f)$ and $\mathcal{D}(f)$ are dense.

Theorem 6.2. Consider a bounded function $f : [0,1] \to \mathbb{R}$ such that both $\mathcal{C}(f)$ and $\mathcal{D}(f)$ are dense. Then $\mathcal{L}(f)$ is of first category. As a consequence, we have $\mathcal{L}(f) \subsetneq \mathcal{C}(f)$.

Proof. For any $b \in \mathcal{L}(f)$, set

$$C_b = \sup_{y\neq b} \left|\frac{f(y) - f(b)}{y - b}\right|, \tag{6.6}$$

which is finite. For $n \in \mathbb{N}$, set

$$V_n = \{b \in \mathcal{L}(f) \mid C_b \le n\}. \tag{6.7}$$

Then we have $\bigcup_{n\in\mathbb{N}} V_n = \mathcal{L}(f)$.

Now we show that V_n is closed. Let $(b_k)_{k\in\mathbb{N}}$ be a convering sequence in V_n. Let c be its limit. We need to show that $c \in V_n$. We have

$$f(b_k) - n|x - b_k| \le f(x) \le f(b_k) + n|x - b_k|, \tag{6.8}$$

which yields

$$\omega(f)(c) \le 2n|c - b_k|. \tag{6.9}$$

Taking $k \to \infty$, we get $\omega(f)(c) = 0$. Thus f is continuous at c. In particular, we have $f(b_k) \to f(c)$ as $k \to \infty$. Now, taking $k \to \infty$ in equation (6.8), we get

$$f(c) - n|x - c| \le f(x) \le f(c) + n|x - c|. \tag{6.10}$$

Hence $c \in V_n$.

Since $\mathcal{D}(f)$ is dense and $V_n \subseteq \mathcal{C}(f)$, $[0,1]\backslash V_n$ is dense. Moreover, V_n is closed. Then, by Proposition 8.2, V_n is nowhere dense. Hence $\mathcal{L}(f) = \bigcup_{n\in\mathbb{N}} V_n$ is of first category.

We have the trivial identity

$$[0,1] = \mathcal{L}(f) \cup \Big(\mathcal{C}(f)\backslash\mathcal{L}(f)\Big) \cup \mathcal{D}(f). \tag{6.11}$$

We have proved that $\mathcal{L}(f)$ is of first category. On the other hand, by Proposition 3.7, $\mathcal{D}(f)$ is of first category. Then, by Proposition 8.5, $\mathcal{C}(f)\backslash\mathcal{L}(f)$ is not empty. □

Theorem 6.3. Given $S \subseteq [0,1]$, if both S and $[0,1]\backslash S$ are dense, then there does not exist a bounded function $f : [0,1] \to \mathbb{R}$ such that $\mathcal{L}(f) = \mathcal{C}(f) = S$.

Proof. This is an immediate consequence of Theorem 6.2. □

Now we summarize what we get. Consider bounded functions $f : [0,1] \to \mathbb{R}$ such that both $\mathcal{C}(f)$ and $\mathcal{D}(f)$ are dense. The set $\mathcal{C}(f)$ must be the intersection of a countable family of open sets. As a consequence, we cannot find f with $\mathcal{C}(f) = [0,1] \cap \mathbb{Q}$. On the other hand, $\mathcal{L}(f)$ can be any countable set. In particular, there exists f with $\mathcal{L}(f) = [0,1] \cap \mathbb{Q}$. Moreover, the set $\mathcal{C}(f)$ is strictly larger than $\mathcal{L}(f)$. In fact, the sets $\mathcal{L}(f)$ and $\mathcal{D}(f)$ are of first category. Hence f is continuous but not Lipschitz continuous 'at almost all points' (in the sense of topology).

7 Appendix A. Measure Theoretic Tools

In this section, we introduce outer measure. For more details, see [Oxt80, §3].

Definition 7.1. A subset $S \subseteq \mathbb{R}$ is called a null set if for any $\varepsilon > 0$ there exists a countable family of open intervals $\left(I_k\right)_{k\in\mathbb{N}}$ such that $S \subseteq \bigcup_{k\in\mathbb{N}} I_k$ and

$$\sum_{k\in\mathbb{N}} \left|I_k\right| < \varepsilon. \tag{7.1}$$

In particular, a countable subset of $\mathbb{R}$ is a null set.

Definition 7.2. Let $S \subseteq \mathbb{R}$. The outer measure of S is defined by

$$m^*(S) = \inf\left\{ \sum_{k\in\mathbb{N}} \left|I_k\right| \,\middle|\, \left(I_k\right)_{k\in\mathbb{N}} \text{ open intervals such that } S \subseteq \bigcup_{k\in\mathbb{N}} I_k \right\}. \tag{7.2}$$

The follows are equivalent.

- S is a null set.
- $m^*(S) = 0$.

Proposition 7.3. For $A \subseteq B$, we have

$$m^*(A) \leq m^*(B). \tag{7.3}$$

For a countable family $\left(A_k\right)_{k\in\mathbb{N}}$, we have

$$m^*\Big(\bigcup_{k\in\mathbb{N}} A_k\Big) \leq \sum_{k\in\mathbb{N}} m^*(A_k). \tag{7.4}$$

For $a < b$, we have

$$m^*((a,b)) = m^*([a,b]) = |b-a|. \tag{7.5}$$

Proof. The inequalities (7.3) and (7.4) follow from of the definition of outer measure. Now we prove equation (7.5). Since any open cover of $[a,b]$ admits a finite sub cover, we have $m^*([a,b]) = |b-a|$. Taking $A = [a+\varepsilon, b-\varepsilon]$ and $B = (a,b)$ in inequality (7.3) with ε tending to 0, we get $m^*((a,b)) \geq |b-a|$. On the other hand, taking $A = (a,b)$ and $B = [a,b]$ in inequality (7.3), we get $m^*((a,b)) \leq |b-a|$. □

8 Appendix B. Topological Tools

In this section, we introduce set of first category. For more details, see [Oxt80, §4].

Definition 8.1. We say that a subset $S \subseteq \mathbb{R}$ is nowhere dense if for any open interval $I \subseteq \mathbb{R}$ there exists an open interval $J \subseteq I$ such that $J \cap S = \emptyset$.

Proposition 8.2. Let $S \subseteq \mathbb{R}$. The following statements are mutually equivalent.

1) S is nowhere dense.
2) $\overline{S}$ is nowhere dense.
3) $\mathbb{R}\backslash\overline{S}$ is dense.

Proof. 2)⇒1) Obvious.

1)⇒3) Let $I \subseteq \mathbb{R}$ be an arbitrary open interval. We need to show that $I \cap (\mathbb{R}\backslash\overline{S}) = I\backslash\overline{S} \neq \emptyset$. Since S is nowhere dense, there exists an open interval $J \subseteq I$ such that $J \cap S = \emptyset$. Since J is open, we have $J \cap \overline{S} = \emptyset$. Hence $J \subseteq I\backslash\overline{S}$.

3)⇒2) Let $I \subseteq \mathbb{R}$ be an arbitrary open interval. We need to find an open interval $J \subseteq I$ such that $J \cap \overline{S} = \emptyset$. Since $\mathbb{R}\backslash\overline{S}$ is dense, there exists $x \in I \cap (\mathbb{R}\backslash\overline{S}) = I\backslash\overline{S}$. Then $x \notin \overline{S}$. Since $\overline{S}$ is closed, there exists an open interval $J' \ni x$ such that $J' \cap \overline{S} = \emptyset$. We take $J = J' \cap I$. □

Definition 8.3. We say that a subset $S \subseteq \mathbb{R}$ is of first category if there exists a countable family of nowhere dense subsets $\left(A_k\right)_k$ such that $S = \bigcup_{k\in\mathbb{N}} A_k$. In particular, if S is countable, then S is of first category.

Proposition 8.4. If $\left(A_k\right)_{k\in\mathbb{N}}$ is a family of subsets of $\mathbb{R}$ of first category, then $\bigcup_{k\in\mathbb{N}} A_k$ is of first category.

Proof. This is an immediate consequence of the definition. □

Proposition 8.5. The interval $[0, 1]$ is not of first category. As a consequence, $[0, 1]$ cannot be covered by a countable family of subsets of first category.

Proof. Assume the contrary, i.e., $[0, 1] = \bigcup_{k\in\mathbb{N}} A_k$ where each A_k is nowhere dense. Set $B_k = \mathbb{R}\backslash\overline{A}_k$. Then we have $\bigcap_{k\in\mathbb{N}} B_k = \emptyset$. By our construction, each B_k is open. Moreover, by Proposition 8.2, each B_k is dense. Now we construct a decreasing sequence of non degenerated closed intervals

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \tag{8.1}$$

by induction. Since B_0 is open and dense, there exists a non degenerated closed interval $I_0 \subseteq B_0$. Assume that I_n is constructed. Since B_{n+1} is open and dense, there exists a non degenerated closed interval $I_{n+1} \subseteq B_{n+1} \cap I_n$. By our construction, we have $I_n \subseteq B_n$ for each $n \in \mathbb{N}$. Let $c \in \bigcap_{k\in\mathbb{N}} I_n$. Then we have $c \in \bigcap_{k\in\mathbb{N}} B_n$. But we have proved that $\bigcap_{k\in\mathbb{N}} B_n = \emptyset$. Contradiction.

Now we turn to prove the second part. Assume the contrary, i.e., [0,1] is the union of a countable family of subsets of first category. Then, by Proposition 8.4, $[0, 1]$ itself is of first category. Contradiction. □

Proposition 8.6. Let A be a countable dense subset of $[0,1]$. For any countable family of open sets $\left(U_k\right)_{k\in\mathbb{N}}$ in $[0,1]$, we have

$$A \neq \bigcap_{k\in\mathbb{N}} U_k. \tag{8.2}$$

Proof. If $A \nsubseteq U_k$ for certain k, then equation (8.2) obviously holds. In the rest of the proof, we assume that $A \subseteq U_k$ for each k. Then each U_k is open and dense in $[0,1]$. Set $V_k = [0,1]\backslash U_k$, which is nowhere dense. Since A is countable, A is of first category. Then, by Proposition 8.4, $A \cup \bigcup_{k\in\mathbb{N}} V_k$ is of first category. By Proposition 8.5, we have

$$A \cup \bigcup_{k\in\mathbb{N}} V_k \neq [0,1], \tag{8.3}$$

which is equivalent to equation (8.2). □

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The topic of this paper was independently selected by the author, who was inspired by his studies in mathematical analysis, particularly the concept of continuous points of functions on an interval. While considering various problems in mathematical analysis, the author became intrigued by the properties of functions concerning continuity. The idea of exploring continuous points of functions led to a deeper inquiry into related concepts such as limits, convergence, and differentiability. Motivated by the challenge of understanding functions that are continuous at certain points but not necessarily over their entire domain, the author decided to focus on this topic, seeing it as an opportunity to gain more profound insights into the general theory of functions.

All examples and counterexamples presented in the paper were independently conceived by the author. These examples were carefully crafted to illustrate specific properties of functions with varying degrees of continuity. The author sought to highlight various scenarios, including functions that are continuous at isolated points, functions that are continuous on dense sets but not everywhere, and functions that are nowhere continuous.

The author undertook the task of analyzing the examples and performing the necessary computations to demonstrate their properties. This process involved applying rigorous mathematical techniques to ensure that each example met the specified criteria for continuity or discontinuity. Throughout the writing of this paper, Mr. Lu was deeply involved in providing feedback on both the content and structure. The author took responsibility for drafting the initial versions, while Mr. Lu reviewed these drafts and offered detailed comments on how to improve the presentation of the material. His advice was crucial in enhancing the clarity and flow of the paper, ensuring that complex mathematical ideas were communicated effectively to the intended audience. Mr. Lu also assisted in refining mathematical arguments, providing feedback on the use of terminology, notation, and style, and suggesting ways to improve the exposition of key results.

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