参赛学生姓名: 姚贺

中学: 杭州狄邦文理学校

省份: 浙江省

国家/地区: 中国

指导老师姓名: 冯景行

指导老师单位: 杭州狄邦文理学校

论文题目: Coloring Problems on the Triangular Lattice

Coloring Problems on the Triangular Lattice

He Yao

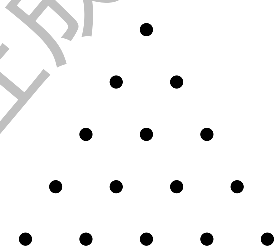
Abstract

Denote the triangular lattice with n rows by T_n. Let $f(n)$ be the minimum number of colors needed to color the n-row triangular lattice such that no three points constituting the vertices of an equilateral triangle each receive the same color. In a recent work of Brouwer et al. [1], they show that the limits $\lim_{n\to\infty} f(n)$ diverges to ∞ and $\lim_{n\to\infty} \frac{f(n)}{n} \leq \frac{1}{3}$. In this paper, we show that $f(n) \geq \frac{1}{4}\log\log n$.

Keywords: Triangular Lattice, Minimal coloring

1 Introduction

Denote the triangular lattice with n rows by T_n.



Recently, Brouwer et al. [1] studied the following problem:

Question 1. *What is the minimum number of colors needed to color the n-row triangular lattice such that no three points constituting the vertices of an equilateral triangle each receive the same color?*

Let $f(n)$ be the minimum number of colors needed to color the points of n-row triangular lattice T_n such that no three points constituting the vertices of an equilateral triangle each receive the same color.

It is easy to find that $f(2) = f(3) = 2$ and the first non-trivial case is $n = 4$ for f-function.

Theorem 1 ([1]). *$f(4) = 3$.*

The following coloring of T_n will give $f(n) \leq \lfloor \frac{n}{2} \rfloor + 1$([1]). One color class consists of points in the middle column of T_n; Other color classes consist of points lying on pairs of lines which make a $\frac{pi}{3}$ angle with the horizontal and intersect in a point in the middle column. See Figure 1 for such a coloring of T_5.

Brouwer et al. [1] construct a $(\frac{n}{3} + O(1))$-coloring of T_n without monochromatic equilateral triangle, which shows that $f(n) \leq \frac{n}{3} + O(1)$, an non-trivial upper bound $f(n)$.

Theorem 2 ([1]). *$\lim_{n\to\infty} \frac{f(n)}{n} \leq \frac{1}{3}$.*

In the triangular lattice of T_n, we call an equilateral triangle horizontal, if its bottom side is horizontal.

Definition 1. *Let $h(n)$ be the minimum number of colors needed to color the points of n-row triangular lattice T_n such that no three points constituting the vertices of a horizontal equilateral triangle each receive the same color.*

In an REU [2] conducted by Neil Lyall and Akos Magyar in 2005, $h(n)$ is proposed as a potential topic for student research.

By the definition of $f(n)$ and $g(n)$, we have $h(n) \leq f(n)$.

It is obvious that $h(2) = h(3) = h(4) = 2$ and the first non-trivial case is $n = 5$ for g-function. We will show that $h(5) = 3$.

The famous van der Waerden's Theorem states that for any positive integers r and k, there exists an integer N, such that any coloring of the integers

$1, 2, \ldots, N$ with r colors is guaranteed to contain a monochromatic arithmetic progression of length k. The minimum such N is typically denoted as the van der Waerden number $W(r, k)$. Brouwer et al. [1] show that the limit $\lim_{n\to\infty} f(n)$ diverges to ∞ by van der Waerden's Theorem. The same proof also holds for h-function.

Theorem 3 ([1])**.** *The limits* $\lim_{n\to\infty} f(n)$ *and* $\lim_{n\to\infty} h(n)$ *diverges to* ∞*.*

Proof. We will prove that for any positive integer r, there exists an integer $N = N_r$ such that $h(N) \geq r$ by induction on r. The base case $r = 1$ is obviously true. Assume the statement holds for $r-1$, that is, there exists an integer N_{r-1} such that $h(N_{r-1}) \geq r-1$. Let $N_r = W(r-1, N_{r-1}+1)$. To prove that $h(N_r) \geq r$, we just need to prove that for any $(r-1)$-coloring of T_{N_r}, there must exits a monochromatic equilateral triangle. Now consider the N_r-st layer L_{N_r}, by van der Waerden's Theorem, there are $N_{r-1}+1$ points of L_{N_r} whose coordinates form a monochromatic arithmetic progression of length $N_{r-1}+1$. Denote such points set by P and the color used in P by c. Also, there is a unique isomorphic copy Q of $T_{N_{r-1}+1}$ containing P. $Q \setminus P$ forms a copy of $T_{N_{r-1}}$ and the points of $Q \setminus P$ could not be colored by c, otherwise such a point with two point from P will form a monochromatic equilateral triangle. Thus, we may assume that the points of $Q \setminus P$ are colored by $r-2$ colors. By induction hypothesis, there is a monochromatic equilateral triangle in $Q \setminus P$.

□

Graham and Solymosi [3] proved that for any coloring of the N × N grid using fewer than $\log\log N$ colors, one can always find a monochromatic equilateral right triangle, a triangle with vertex coordinates (x, y), $(x+d, y)$, and $(x, y+d)$. Their method can be used to show that $h(n) \geq \frac{1}{4}\log\log n$. It follows that $f(n) \geq \frac{1}{4}\log\log n$.

2 Our Results

Theorem 4. $h(5) = 3$.

Proof. It is obvious that T_5 can be colored with three colors such that there is no monochromatic horizontal equilateral triangles (See Figure 1).

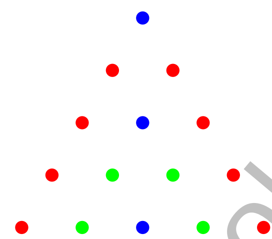


Figure 1: A three-coloring of T_5 without monochromatic horizontal equilateral triangles

Suppose on the contrary that $h(5) < 3$, and T_5 can be colored with two colors red and blue such that there is no monochromatic horizontal equilateral triangles.

Claim 1. *The coordinates of the monochromatic points in the* 5*-th layer,* L_5*, can not form an arithmetic progression of length at least* 3*.*

Proof. Suppose there are three points v_1, v_2, v_3 colored with red and their coordinates form an arithmetic progression. There is a unique homomorphism copy of T_3 containing v_1, v_2, v_3. Let the other three points be u_1, u_2 and u_3.

None of u_1, u_2 and u_3 can be colored with red, otherwise such a red point with two points from $\{v_1, v_2, v_3\}$ will form a red horizontal equilateral triangle. Thus, all of u_1, u_2 and u_3 must be colored with blue and form a blue horizontal equilateral triangle.

□

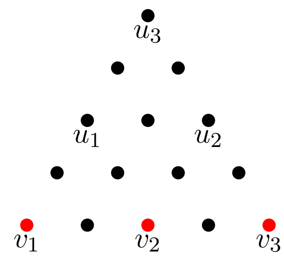


Figure 2: An example for Claim 1

Now without loss of generality, we assume that the most popular color used in L_5 is red. By symmetry and Claim 1, we only need discuss the following four cases depending on the coloring of L_5.

Case 1. There are four points in L_5 are colored with red and only the third point is colored with blue (See Figure 3).

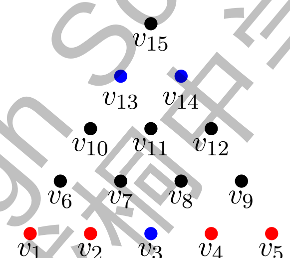


Figure 3: Case 1

Since v_1, v_4, v_{13} form a horizontal equilateral triangle, and v_1, v_4 are colored with red, it follows that v_{13} must be colored with blue. Since v_2, v_5, v_{14} form a horizontal equilateral triangle, and v_2, v_5 are colored with red, it follows that v_{14} must be colored with blue. Now in the horizontal equilateral triangle $v_1v_5v_{15}$, the point v_{15} must be colored with blue. But in the horizontal equilateral triangle $v_{13}v_{14}v_{15}$, the point v_{15} must be colored with red, which is a contradiction.

Case 2. There are three points in L_5 colored with red and the first and third points are colored with blue (See Figure 4).

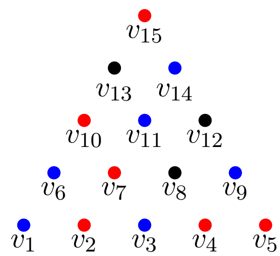


Figure 4: Case 2

Since v_1, v_3, v_{10} form a horizontal equilateral triangle, and v_1, v_3 are colored with blue, it follows that v_{10} must be colored with red. Since v_2, v_4, v_{11} form a horizontal equilateral triangle, and v_2, v_4 are colored with red, it follows that v_{11} must be colored with blue. Since v_4, v_5, v_9 form a horizontal equilateral triangle, and v_4, v_5 are colored with red, it follows that v_9 must be colored with blue. Since v_2, v_5, v_{14} form a horizontal equilateral triangle, and v_2, v_5 are colored with red, it follows that v_{14} must be colored with blue. Since v_7, v_9, v_{14} form a horizontal equilateral triangle, and v_9, v_{14} are colored with blue, it follows that v_7 must be colored with red. Since v_6, v_7, v_{10} form a horizontal equilateral triangle, and v_7, v_{10} are colored with red, it follows that v_6 must be colored with blue. Since v_6, v_9, v_{15} form a horizontal equilateral triangle, and v_6, v_9 are colored with blue, it follows that v_{15} must be colored with red. Now in the horizontal equilateral triangle $v_{10}v_{12}v_{15}$, the point v_{12} must be colored with blue. But in the horizontal equilateral triangle $v_{11}v_{12}v_{14}$, the point v_{12} must be colored with red, which is a contradiction.

Case 3. There are three points in L_5 are colored with red and the first and fourth points are colored with blue (See Figure 5).

Since v_1, v_4, v_{13} form a horizontal equilateral triangle, and v_1, v_4 are colored with blue, it follows that v_{13} must be colored with red. Since v_2, v_3, v_7 form a horizontal equilateral triangle, and v_2, v_3 are colored with red, it follows that v_7 must be colored with blue. Since v_2, v_5, v_{14} form a horizontal equilateral triangle, and v_2, v_5 are colored with red, it follows that v_{14} must be colored with blue. Since v_3, v_5, v_{12} form a horizontal equilateral triangle, and v_3, v_5

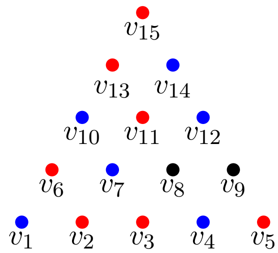


Figure 5: Case 3

are colored with red, it follows that v_{12} must be colored with blue. Since v_{11}, v_{12}, v_{14} form a horizontal equilateral triangle, and v_{12}, v_{14} are colored with blue, it follows that v_{11} must be colored with red. Since v_{10}, v_{11}, v_{13} form a horizontal equilateral triangle, and v_{11}, v_{13} are colored with red, it follows that v_{10} must be colored with blue. Since v_6, v_7, v_{10} form a horizontal equilateral triangle, and v_7, v_{10} are colored with blue, it follows that v_6 must be colored with red. Since v_{10}, v_{12}, v_{15} form a horizontal equilateral triangle, and v_{10}, v_{12} are colored with blue, it follows that v_{15} must be colored with red.

Now in the horizontal equilateral triangle $v_7v_9v_{14}$, the point v_9 must be colored with blue. But in the horizontal equilateral triangle $v_6v_9v_{15}$, the point v_9 must be colored with red, which is a contradiction.

Case 4. There are three points in L_5 are colored with red and the second and third points are colored with blue (See Figure 6).

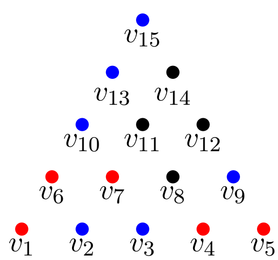


Figure 6: Case 4

Since v_2, v_3, v_7 form a horizontal equilateral triangle, and v_2, v_3 are colored with blue, it follows that v_7 must be colored with red. Since v_1, v_4, v_{13} form a horizontal equilateral triangle, and v_1, v_4 are colored with red, it follows that v_{13} must be colored with blue. Since v_1, v_5, v_{15} form a horizontal equilateral triangle, and v_1, v_5 are colored with red, it follows that v_{15} must be colored with blue. Since v_4, v_5, v_9 form a horizontal equilateral triangle, and v_4, v_5 are colored with red, it follows that v_9 must be colored with blue. Since v_6, v_9, v_{15} form a horizontal equilateral triangle, and v_9, v_{15} are colored with blue, it follows that v_6 must be colored with red. Since v_6, v_7, v_{10} form a horizontal equilateral triangle, and v_6, v_7 are colored with red, it follows that v_{10} must be colored with blue.

Now in the horizontal equilateral triangles $v_{10}v_{11}v_{13}$, $v_{10}v_{12}v_{15}$, and $v_{13}v_{14}v_{15}$, all the points v_{11}, v_{12} and v_{14} must be colored with red, and we find a red horizontal equilateral triangles $v_{11}v_{12}v_{14}$, which is a contradiction.

□

Theorem 5. *$h(n) \geq \Omega(\log\log n)$.*

Proof. For $1 \leq i \leq n$, denote the points set of the i-th layer by L_i. Suppose that the points of T_n are colored by r colors and there is no monochromatic horizontal equilateral triangle. Examine the coloring of the points in L_n.

Select the most popular color, denoted by c_1.

The set of points in L_n with color c_1 is denoted by S_1. $|S_1| \geq \frac{n}{r}$.

Let R_1 be the set of points which can form a horizontal equilateral triangle with two points from S_1. Since there is no monochromatic horizontal equilateral triangle, the points of R_1 can not be colored by c_1. Also, $|R_1| = \binom{|S_1|}{2}$.

Now, there is a layer which contains many points of R_1. Let L_{j_1} be a layer, where $1 \leq j_1 \leq n-1$, such that

$$|L_{j_1} \cap R_1| \geq \frac{|R_1|}{n}.$$

Select the most popular color used in $L_{j_1} \cap R_1$, say c_2. The set of points in $L_{j_1} \cap R_1$ with color c_2 is denoted by S_2. Since the points set R_1 avoids the color c_1, we have

$$|S_2| \geq \frac{|L_{j_1} \cap R_1|}{r-1} \geq \frac{|R_1|}{n(r-1)}.$$

Let R_2 be the set of points which can form a horizontal equilateral triangle with two points from S_2. Note that $R_2 \subset R_1$. Since there is no monochromatic horizontal equilateral triangle, the points of R_2 can not be colored by c_1 and c_2. Also, $|R_2| = \binom{|S_2|}{2}$.

For $i \geq 1$, we define the color c_{i+1}, the point sets S_{i+1}, and R_{i+1} recursively, based on c_i, S_i and R_i.

Suppose the points set R_i avoids the colors $c_1, c_2, \ldots, c_i$.

There is a layer which contains many points of R_i. Let L_{j_i} be a layer, where $1 \leq j_i \leq j_{i-1} - 1$, such that

$$|L_{j_i} \cap R_i| \geq \frac{|R_i|}{n}.$$

Select the most popular color used in $L_{j_i} \cap R_i$, say c_{i+1}. The set of points in $L_{j_i} \cap R_i$ with color c_{i+1} is denoted by S_{i+1}. Since the points set R_i avoids the color $c_1, c_2, \ldots, c_i$, we have

$$|S_{i+1}| \geq \frac{|L_{j_i} \cap R_i|}{r-i} \geq \frac{|R_i|}{n(r-i)}.$$

Let R_{i+1} be the set of points which can form a horizontal equilateral triangle with two points from S_{i+1}. Note that $R_{i+1} \subset R_i$. Since there is no monochromatic horizontal equilateral triangle, the points of R_{i+1} cannot be colored by $c_1, c_2, \ldots, c_{i+1}$. Also, $|R_{i+1}| = \binom{|S_{i+1}|}{2}$.

Claim 2. *We have $|S_r| < 2$.*

Proof. If $|S_r| \geq 2$, then $|R_r| = \binom{|S_r|}{2} \geq 1$, but there is no colors for the points of R_i, which is a contradiction. □

Claim 3. $n \leq (2r)^{2^r}$.

Proof. Note that $|S_1| \geq 2$ and $|S_r| < 2$. Let $k = \min\{i : |S_i| < 2\}$. We have for all integer $1 \leq i \leq k-1$, $|S_i| \geq 2$. It follows that

$$\binom{|S_i|}{2} = \frac{|S_i|(|S_i|-1)}{2} \geq \frac{|S_i|^2}{4}.$$

Thus, for $1 \leq i \leq k-1$, we have

$$|S_{i+1}| \geq \frac{|R_i|}{n(r-i)} \geq \frac{|S_i|^2}{4rn}.$$

It follows that

$$\begin{aligned} |S_k| &\geq \frac{|S_{k-1}|^2}{4rn} \geq \frac{1}{4rn}\frac{1}{(4rn)^2}|S_{k-2}|^{2^2} \\ &\geq \frac{1}{4rn}\frac{1}{(4rn)^2}\frac{1}{(4rn)^{2^2}}|S_{k-3}|^{2^3} \\ &\quad \cdots\cdots \\ &\geq \frac{1}{4rn}\frac{1}{(4rn)^2}\frac{1}{(4rn)^{2^2}}\cdots\frac{1}{(4rn)^{2^{k-2}}}|S_1|^{2^{k-1}} \\ &= \frac{1}{(4rn)^{1+2+2^2+\cdots+2^{k-2}}}|S_1|^{2^{k-1}} \\ &\geq \frac{1}{(4rn)^{2^{k-1}-1}}\left(\frac{n}{r}\right)^{2^{k-1}} \\ &= \frac{n}{2^{2^k-2}r^{2^{k-1}}}. \end{aligned}$$

Thus, we have $n \leq 2^{2^k-2}r^{2^{k-1}}|S_k| \leq 2^{2^k-1}r^{2^{k-1}} \leq (2r)^{2^r}$.

□

By Claim 3, we have $r \geq \frac{1}{4}\log\log n$.

□

3 Some Counting Results

Proposition 1. *The number of equilateral triangles in T_n is $\frac{n^4+2n^3-n^2-2n}{24}$.*

Proof. The number of equilateral triangles which contains the k-th point of the n-th layer is $k(n-k+1)-1$.

The number of equilateral triangles which contains two point of the n-th layer is $\binom{n}{2}$.

Thus, the number of equilateral triangles which contains at least one point of the n-th layer is

$$\sum_{k=1}^{n}[k(n-k+1)-1]-\binom{n}{2}=\frac{n^3-n}{6}.$$

Thus, the number of equilateral triangles in T_n is

$$\sum_{k=1}^{n}\frac{k^3-k}{6}=\frac{n^4+2n^3-n^2-2n}{24}.$$

□

Proposition 2. *The number of horizontal equilateral triangles in T_n is $\frac{n^3-n}{6}$.*

Proof. For $2 \le k \le n$, the number of horizontal equilateral triangles whose bottom points are in the k-th layer is $\binom{k}{2}$.

Thus, the number of horizontal equilateral triangles in T_n is $\sum_{k=2}^{n}\binom{k}{2}=\frac{n^3-n}{6}$.

□

4 Further Questions

The equilateral triangle can be viewed as a isomorphic copy of T_2, then similar questions can be proposed for T_k where $k \ge 2$.

Problem 1. *How many isomorphic copies of T_k are there in T_n?*

Problem 2. *What is the minimum number of colors needed to color the points of n-row triangular lattice T_n such that there is no monochromatic T_k?*

References

[1] G. A. Brouwer, J. Joe, A. A. Noble and N. Matt, Problems on the Triangular Lattice, arXiv: 2405.12321.

[2] N. Lyall, A. Magyar, Ramsey Theory REU Summer, 2005, http://alpha.math.uga.edu/~lyall/REU/.

[3] R. Graham and J. Solymosi, Monochromatic equilateral right triangles on the integer grid, Topics in Discrete Mathematics, Springer (2006), 129-132.

[4] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, Nieuw Arch. Wisk. 15(1927), 212-216.

Acknowledgements

I have come to understand that the field of extremal combinatorics has developed rapidly in recent years. Following the suggestion of my supervisor, I came across the paper Problems on the Triangular Lattice by Brouwer et al. on arXiv. This paper investigates the minimum number of colors, denoted as $f(n)$, required to color the vertices of Tn such that no monochromatic equilateral triangle exists. While the authors provided an upper bound for $f(n)$, they did not present any clear lower bound with an explicit expression, which prompted me to explore this problem.

Initially, while drawing figures, I observed that the maximum number of vertices in Tn that could avoid forming an equilateral triangle would not exceed $2n$. If this conjecture were true, dividing the number of vertices of T_n, $\frac{n(n-1)}{2}$, by $2n$ would yield a lower bound for $f(n)$ of $\frac{n}{4}$. This would be a promising result, as the upper bound provided by Brouwer et al. is $\frac{n}{3}$, which would make the upper and lower bounds quite close. However, after spending a long time trying to prove my conjecture, I realized it was a very challenging task. I then began to carefully review the references in this paper, hoping to find a method to solve this problem.

In the fourth reference, I found that Graham and Solymosi used a clever technique to establish a logarithmic lower bound, $\log\log n$, for the minimum number of colors required to color lattice without monochromatic equilateral triangles. It suddenly occurred to me that this method could also be applied to the coloring problem of T_n, which led to the main conclusion of this paper. Under the guidance of my supervisor, I also learned the basic techniques of using LaTeX to edit the paper, and I eventually completed this paper.