第一页为封面页

参赛学生姓名：沈子原

中学：上海平和双语学校

省份：上海市

国家/地区：中国

指导老师姓名：丁煜宸

指导老师单位：扬州大学

论文题目：On a sign change problem on sums of divisors

论文修改增补情况说明

一、增加了论文封面页

二、论文结尾处例子拓展部分内容如下：

Example 1. *Let $s = \frac{1}{2}$ in Theorem 1. Let also*

$$a_1 n + b_1 = 4n + 1, \quad a_2 n + b_2 = 6n + 5, \quad a_3 n + b_3 = 12n + 7$$

and

$$c_1 n + d_1 = 4n + 3, \quad c_2 n + d_2 = 6n + 1, \quad c_3 n + d_3 = 12n + 11.$$

Then a_i, b_i, c_i and d_i satisfy the conditions in Theorem 1. Thus, by Theorem 1 there are infinitely many positive integers n satisfying that

$$\min\left\{ \sum_{d|4n+3} \sqrt{d}, \sum_{d|6n+1} \sqrt{d}, \sum_{d|12n+11} \sqrt{d} \right\} > \max\left\{ \sum_{d|4n+1} \sqrt{d}, \sum_{d|6n+5} \sqrt{d}, \sum_{d|12n+7} \sqrt{d} \right\}, \quad (1)$$

which, as one could see, is not trivial at all.

Furthermore, if we exchange the positions of $a_i n + b_i$ and $c_i n + d_i$, i.e.,

$$a_1 n + b_1 = 4n + 3, \quad a_2 n + b_2 = 6n + 1, \quad a_3 n + b_3 = 12n + 11$$

and

$$c_1 n + d_1 = 4n + 1, \quad c_2 n + d_2 = 6n + 5, \quad c_3 n + d_3 = 12n + 7.$$

Then these new a_i, b_i, c_i and d_i also satisfy the conditions in Theorem 1. Thus, by Theorem 1 there are infinitely many positive integers n so that

$$\max\left\{ \sum_{d|4n+3} \sqrt{d}, \sum_{d|6n+1} \sqrt{d}, \sum_{d|12n+11} \sqrt{d} \right\} < \min\left\{ \sum_{d|4n+1} \sqrt{d}, \sum_{d|6n+5} \sqrt{d}, \sum_{d|12n+7} \sqrt{d} \right\}. \quad (2)$$

Inequalities (1) and (2) can be viewed as a new example of the Chebyshev bias phenomenon on sums of divisors.

三、Acknowledgement 部分略有改动如下：

Acknowledgments

I have written and completed the paper and the proof. I would like to thank my supervisor Yuchen Ding for his help in advising me to study the paper of Professor Pongsriiam, which has provided the problem to solve in this paper; and my supervisor has guided me in studying the paper itself. He has also provided help and support for free on this paper, including providing potential problems to solve, guidance on how to write the abstract and introduction, and also reviewing the paper for mistakes.

On a sign change problem on sums of divisors

Ziyuan Shen

Abstract. Let $0 < s \leq 1$ and a_i, b_i, c_i, d_i be nonnegative integers with $a_i, c_i > 0$, $a_i d_i - b_i c_i \neq 0$ $(1 \leq i \leq \ell)$. Recently, Pongsriiam asked whether there are infinitely many positive integers n so that $\sigma_s(a_i n + b_i) < \sigma_s(c_i n + d_i)$ for all $i \in \{1, 2, \cdots, \ell\}$, where $\sigma_s(n) = \sum_{d|n} d^s$. In this note, we answer this problem affirmatively with a natural constraint on the admissible situation.

1. Introduction

In 1853, Chebyshev [2] observed that there appear to be more primes of the form $p \equiv 3 \pmod 4$ than those of the form $p \equiv 1 \pmod 4$, a phenomenon now known as the Chebyshev bias. Riemann, in his famous memoir [1], noted that $\pi(x)$, the number of primes less than or equal to x, is often less than $\mathrm{Li}(x)$, where $\mathrm{Li}(x)$ is the logarithmic integral defined by

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t}\, dt.$$

Following the significant results of Hadamard [6] and de la Vallée Poussin [14], proving the prime number theorem, it is now known that

$$\pi(x) \sim \mathrm{Li}(x) \quad \text{as } x \to \infty.$$

Riemann's observation represents a second example of the Chebyshev bias phenomenon. However, Littlewood [9] later disproved both Chebyshev's and Riemann's claims, showing that the inequality between $\pi(x)$ and $\mathrm{Li}(x)$ reverses infinitely often. Along with Hardy, Littlewood [10] also proved that

$$\pi(x; 4, 3) - \pi(x; 4, 1)$$

and

$$\mathrm{Li}(x) - \pi(x)$$

oscillate infinitely, meaning they change signs infinitely many times.

Another well-known instance of the Chebyshev bias occurs with Euler's totient function, $\varphi(n)$. In 1973, Jarden [8, page 65] noted that

$$\varphi(30n + 1) > \varphi(30n)$$

for all $n \leq 100{,}000$. By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many n for which this inequality holds. However, Newman [11] later demonstrated that there are also infinitely many n such that

$$\varphi(30n + 1) < \varphi(30n).$$

Therefore, the difference

$$\varphi(30n + 1) - \varphi(30n)$$

has infinitely many sign changes, as established by Dirichlet and Newman.

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Inspired by work from Erdős [5, 13], Wang, Chen [15], and Pongsriiam [12] studied arithmetic functions related to sums of divisors, specifically

$$\sigma_s(n) = \sum_{d|n} d^s,$$

where s is a real number. Pongsriiam [12] extended the study of the Chebyshev bias to these divisor functions, proving several results about sign changes. For instance, Pongsriiam [12, Theorem 2.4] showed that the difference

$$\sigma(30n) - \sigma(30n+1)$$

(σ without s means s=1) experiences infinitely many sign changes. He also demonstrated [12, Theorem 2.10] that for integers $a > c > 0$ and $b \geq d \geq 0$, there exists a threshold $s_0 > 1$ such that

$$\sigma_s(an+b) > \sigma_s(cn+d)$$

for all $s \geq s_0$ and $n \geq 1$. Additionally, Pongsriiam [12, Theorem 2.11] found that

$$\sigma_s(2m+5) < \sigma_s(6m+17) \quad \text{and} \quad \sigma_s(5m+4) < \sigma_s(6m+7)$$

hold for all $m \in \mathbb{N}$ when $s > 3$.

Pongsriiam concluded with several open questions for further exploration, one of which is central to this paper.

Problem 1 ([12], Problem 3.3). *Suppose $0 \leq s \leq 1$ and a_i, b_i, c_i, d_i are non-negative integers, a_i,c_i>0 and $a_i d_i - b_i c_i \neq 0$, for each $i = 1, 2, \cdots, \ell$. Are there infinitely many $n \in \mathbb{N}$ such that*

$$\sigma_s(a_i n + b_i) < \sigma_s(c_i n + d_i)$$

for all $i \in \{1, 2, \cdots, \ell\}$?

Motivated by the very recent solutions to some problems of Pongsriiam given by Ding, Pan and Sun [4]. We deal with Problem 1 mentioned above in a narrow but natural situation.

Theorem 1. *Suppose $0 \leq s \leq 1$ and a_i, b_i, c_i, d_i are non-negative integers, a_i,c_i>0 and $a_i d_i - b_i c_i \neq 0$, for each $i = 1, 2, \cdots, \ell$. Suppose that for any prime p there is at least one integer n_p such that p divides none of the $a_i n + b_i$, then there are $\gg x(\log x)^{-\ell}$ integers $n \leq x$ (hence infinitely many often n) such that*

$$\sigma_s(c_j n + d_j) > \max_{1 \leq i \leq \ell} \sigma_s(a_i n + b_i) \quad \big(\forall\, j \in \{1, 2, \cdots, \ell\}\big),$$

where the constant implied by $\gg$ depends only on the choices of a_i, b_i, c_i and d_i.

As a corollary, we immediately obtain the following partial solution to Problem 1.

Corollary 1. *The answer to Theorem 1 is positive provided that for any prime p there is at least one integer n_p such that p divides none of the $a_i n + b_i$ $(1 \leq i \leq \ell)$.*

2. Proof of Theorem 1

We first state some lemmas below before presenting the proof of Theorem 1.

The first two lemmas are standard results in number theory[3].

Lemma 1. *The sum of the reciprocals of prime numbers diverges. That is, for any given $M > 0$ there is some constant C_M so that*

$$\sum_{p \le C_M} \frac{1}{p} > M.$$

Lemma 2 (Chinese Remainder Theorem)**.** *Let $n_1, n_2, \dots, n_k$ be pairwise coprime integers, and let $N = n_1 n_2 \cdots n_k$. For any integers $a_1, a_2, \dots, a_k$, the system of congruences*

$$\begin{cases} x \equiv a_1 \pmod{n_1}, \\ x \equiv a_2 \pmod{n_2}, \\ \quad \vdots \\ x \equiv a_k \pmod{n_k} \end{cases}$$

has a unique solution modulo N. In other words, there exists an integer x such that $x \equiv a_i \pmod{n_i}$ for each $i = 1, 2, \dots, k$, and any two such solutions are congruent modulo N.

Our proof is based on the following deep result of Heath–Brown [7, Theorem 1].

Lemma 3 (Heath–Brown)**.** *Let $L_i(n) = a_i n + b_i$ $(1 \le i \le k)$ be linear functions with integer coefficients and $a_i > 0$. Suppose that for any prime p there is at least one integer n_p such that p divides none of the $L_i(n_p)$, then there are $\gg x(\log x)^{-k}$ integers $n \le x$ such that*

$$\max_{1 \le i \le k} \Omega(L_i(n)) \le G_k,$$

where the constant G_k is given by

$$G_k = \left\lfloor \log_2 \left\lfloor \frac{3k^2 + 4k + 4}{2} \right\rfloor \right\rfloor.$$

We now turn to the proof of Theorem 1.

Proof of Theorem 1. For any $1 \le i, j \le \ell$, let

$$L_i(n) = a_i n + b_i \quad \text{and} \quad H_j(n) = c_j n + d_j.$$

By the condition of Theorem 1, we know that for any prime p, there exists at least one integer n_p such that p divides none of the $L_i(n_p)$. Under these notations, it suffices to show that there are $\gg x(\log x)^{-\ell}$ integers $n \le x$ such that

$$\sigma_s(H_j(n)) > \max_{1 \le i \le \ell} \sigma_s(L_i(n)) \quad (\forall 1 \le j \le \ell).$$

Let p_f be the f-th prime. For any f, there exists some integer n_f satisfying

$$p_f \nmid \prod_{1 \le i \le \ell} L_i(n_f)$$

by our conditions. Set

$$A = \max_{1 \le i \le \ell} a_i,$$
$$C = \max_{1 \le j \le \ell} c_j$$

and

$$B = \max_{1 \le i, j \le \ell} |a_i d_j - b_i c_j|.$$

Now, let p_{f_1} be the smallest prime greater than $A+B+C+\ell$. Thus, thanks to Lemma 1, we could find out a series of numbers f_i (depending only on a_i, b_i, c_i, d_i and ℓ) satisfying

$$\sum_{f=f_v+1}^{f_{v+1}} \frac{1}{p_f^s} > 2^{G_\ell+3}A$$

for any $1 \leq v \leq \ell$.

We next consider the following congruence system:

$$\begin{cases} n \equiv n_f \pmod{p_f}, & (\forall 1 \leq f \leq f_1) \\ c_1 n + d_1 \equiv 0 \pmod{p_f}, & (\forall f_1 + 1 \leq f \leq f_2) \\ c_2 n + d_2 \equiv 0 \pmod{p_f}, & (\forall f_2 + 1 \leq f \leq f_3) \\ \quad \vdots \\ c_\ell n + d_\ell \equiv 0 \pmod{p_f}, & (\forall f_\ell + 1 \leq f \leq f_{\ell+1}). \end{cases}$$

By the Chinese Remainder Theorem, i.e., Lemma 2 there exactly one congruence $n \equiv n_0 \pmod{P}$ satisfying the system above, where

$$P = p_1 \cdots p_{f_1} \prod_{1 \leq v \leq \ell} \Big(\prod_{f_v+1 \leq f \leq f_{v+1}} p_f \Big).$$

Therefore, if $n \leq x$ with $n = mP + n_0$ then the number of m is clearly greater than

$$\frac{x - n_0}{P} - 1 \gg x.$$

For any $1 \leq i \leq \ell$, define $S_i(m)$ as

$$S_i(m) = L_i(mP + n_0) = a_i P m + a_i n_0 + b_i.$$

We are going to prove that for any prime p there exists an integer m_p such that p divides none of the $S_i(m_p)$ $(1 \leq i \leq \ell)$. First, set $m_p = 0$ for $p \leq p_{f_1}$. Since none of the $L_i(n_f)$ can be divisible by p_f, we can conclude by using the congruences system earlier that

$$S_i(0) \equiv a_i n_0 + b_i \equiv a_i n_f + b_i = L_i(n_f) \not\equiv 0 \pmod{p_f}$$

for any $p \leq p_{f_1}$. For primes $p_{f_1+1} \leq p \leq p_{f_{\ell+1}}$, we again choose $m_p = 0$. From the system of congruences, there exists some j such that $1 \leq j \leq \ell$ and $c_j n_0 + d_j \equiv 0 \pmod{p}$. Hence, we have

$$n_0 \equiv -c_j^{-1} d_j \pmod{p}$$

since p is larger than $C = \max_{1 \leq j \leq \ell} c_j$ which clearly means that $p \nmid c_j$. Recall that $B = \max_{1 \leq i,j \leq \ell} |a_i d_j - b_i c_j|$, it implies that

$$p \nmid a_i d_j - b_i c_j$$

for all $1 \leq i, j \leq \ell$ due to the fact $a_i d_j - b_i c_j \neq 0$. Thus, we can deduce that

$$S_i(0) \equiv a_i n_0 + b_i \equiv a_i(-c_j^{-1} d_j) + b_i \not\equiv 0 \pmod{p}.$$

Finally, for any $p > p_{f_{l+1}}$, the prime p and $a_i P$ are relatively prime, so there is only one solution $m \pmod{p}$ to $S_i(m) \equiv 0 \pmod{p}$ for each $S_i(m)$. Hence,

$$S_1(m) S_2(m) \cdots S_\ell(m) \equiv 0 \pmod{p}$$

has at most ℓ solutions. Since $p_{f_1} > A+B+C+\ell$, there must exist at least one integer m_p such that p divides none of the $S_i(n_p)$.

Employing Lemma 3, there are

$$\gg \frac{x-n_0}{P}\left(\log \frac{x-n_0}{P}\right)^{-\ell} \gg x(\log x)^{-\ell}$$

integers $m \le \frac{x-n_0}{P}$ satisfying that

$$\max_{1\le i\le \ell} \Omega(L_i(n)) = \max_{1\le i\le l} \Omega(S_i(m)) \le G_\ell.$$

And for such m (where $n = mP + n_0$), we will have

$$\max_{1\le i\le \ell} \sigma_s(L_i(n)) \le \max_{1\le i\le \ell} \sum_{d|L_i(n)} d^s \le 2^{G_\ell}\Big(An + \max_{1\le i\le \ell} b_i\Big)^s \le 2^{G_\ell}(2An)^s \le 2^{G_\ell+1}An^s,$$

providing that n is sufficiently large. Additionally, for these integers n, we know

$$H_j(n) = c_j(mP + n_0) + d_j \equiv c_j n_0 + d_j \equiv 0 \pmod p$$

for any $p_{f_j+1} \le p \le p_{f_{j+1}}$, following from the system of congruences above. Therefore, for any $1 \le j \le \ell$ we have

$$\sigma_s(H_j(n)) \ge \sum_{f=f_j+1}^{f_{j+1}} \left(\frac{c_j n + d_j}{p_f}\right)^s \ge \sum_{f=f_j+1}^{f_{j+1}} \left(\frac{n}{2p_f}\right)^s \ge \frac{n^s}{2} 2^{G_\ell+3}A = 2^{G_\ell+2}An^s.$$

Thus, combining this with the earlier bound, we conclude

$$\sigma_s(H_j(n)) \ge 2^{G_\ell+2}An^s > 2^{G_\ell+1}An^s \ge \max_{1\le i\le \ell} \sigma_s(L_i(n))$$

for any $1 \le j \le \ell$, which completes this part of the proof of the theorem.

To end up this short article, we provide an example which illustrates the strength of our Theorem 1.

Example 1. *Let $s = \frac{1}{2}$ in Theorem 1. Let also*

$$a_1n + b_1 = 4n + 1, \quad a_2n + b_2 = 6n + 5, \quad a_3n + b_3 = 12n + 7$$

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Then a_i, b_i, c_i and d_i satisfy the conditions in Theorem 1. Thus, by Theorem 1 there are infinitely many positive integers n satisfying that

$$\min\left\{\sum_{d|4n+3}\sqrt{d}, \sum_{d|6n+1}\sqrt{d}, \sum_{d|12n+11}\sqrt{d}\right\} > \max\left\{\sum_{d|4n+1}\sqrt{d}, \sum_{d|6n+5}\sqrt{d}, \sum_{d|12n+7}\sqrt{d}\right\}, \quad (1)$$

which, as one could see, is not trivial at all.

Furthermore, if we exchange the positions of $a_in + b_i$ and $c_in + d_i$, i.e.,

$$a_1n + b_1 = 4n + 3, \quad a_2n + b_2 = 6n + 1, \quad a_3n + b_3 = 12n + 11$$

and

$$c_1n + d_1 = 4n + 1, \quad c_2n + d_2 = 6n + 5, \quad c_3n + d_3 = 12n + 7.$$

Then these new a_i, b_i, c_i and d_i also satisfy the conditions in Theorem 1. Thus, by Theorem 1 there are infinitely many positive integers n so that

$$\max\left\{\sum_{d|4n+3}\sqrt{d}, \sum_{d|6n+1}\sqrt{d}, \sum_{d|12n+11}\sqrt{d}\right\} < \min\left\{\sum_{d|4n+1}\sqrt{d}, \sum_{d|6n+5}\sqrt{d}, \sum_{d|12n+7}\sqrt{d}\right\}. \quad (2)$$

Inequalities (1) and (2) can be viewed as a new example of the Chebyshev bias phenomenon on sums of divisors.

Acknowledgments

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(Shanghai, Huangyang Road 18, B902)
Email address: 18916802220@163.com