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THE SOLUTION TO SCHUBERT'S PROBLEM AND CONVEX POLYHEDRA

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Abstract. Consider given four lines in the space $\mathbf{R}^3$, how many lines are there that intersect all of them? This is a famous problem posed by Hermann Schubert in 1879. While Schubert's original solution to this problem was insightful and gave the correct answer, the rigorous foundation of his arguments had not emerged until the inventions of various 20th century mathematical concepts. In this paper, we give a detailed and comprehensive solution to Schubert's problem. Our method is based on Morse theory in order to study the topology of the Grassmannian $Gr_2(\mathbf{C}^4)$. We derive the answer by computing the cup products in $H^*(Gr_2(\mathbf{C}^4))$. Moreover, we remark that the problem can be related to the convexity theorem in symplectic geometry. We present an interesting graphic interpretation of the answer using a regular octahedron at the end of the paper.

Keywords. Schubert's calculus, Grassmannian, Morse theory, cohomology, intersection number, convex polyhedra.

Contents

1. Introduction

In [Sch79], Schubert considered various problems, that nowadays are viewed as marking the beginning of the flourishing of *enumerative geometry*. One of them is

. *Schubert's problem. Let $l_1, l_2, l_3, l_4 \subset \mathbf{R}^3$ be four given lines, how many lines $l \subset \mathbf{R}^3$ are there such that $l \cap l_i \neq \varnothing$, $\forall i = 1, 2, 3, 4$?*

Using some arguments that heavily rely on geometric intuition, Schubert (loc. cit.) gave a "proof" of the following.

Theorem 1.1. In *generic* cases, the answer is 2.

There could be other answers, when l_1, l_2, l_3, l_4 happen to be in some special position:

(1) When three of them are skew-parallel lines, then no line intersects them simultaneously, the answer is 0.
(2) When three of them are concurrent, then infinitely many lines intersect them simultaneously, the answer is ∞.
(3) The answer could also be 1: let l_1, l_2 be two parallel lines in a plane e, take two points $A, B \in e$ such that the line AB intersects l_1, l_2, and let l_3, l_4 be two lines that intersect e at A, B respectively, then only AB intersects all the four lines.

Remark 1.2. Actually one can show that the intersection number can only be 0, 1, 2 or ∞. But in this paper, we shall be constrained in generic cases.

If the four lines are in general position (to be defined precisely later), we say it is a generic case, and Schubert showed that there are exactly 2 lines intersecting all of them. Ever since, people had been trying to make sense of Schubert's heuristic proof. In 1900, Hilbert proposed his famous 23 problems ([Hil24]) as tasks for the 20th century mathematicians. One of them is

. *Hilbert's 15th problem. Reformulate Schubert's work in a rigorous manner.*

The culmination of the endeavors on this problem nowadays becomes part of a general theory called *Schubert's calculus*. The goal of this paper is roughly to complete the task in Hilbert's 15th problem. A significance difference from the existing expositions in the literature on Schubert's calculus more commonly based on algebraic geometry is that, our approach is topological in essence. Moreover, we will highlight the comparison between the modern treatment and Schubert's original proof, to really put an emphasis on the process of making sense of his arguments. To this end, we need to first understand what Schubert originally did.

The first step is to put the problem into a more general framework, that is, the study of positional relations of points, lines and planes. We use the following notations to denote some logical propositions or judgements about a line $l \subset \mathbf{R}^3$.

- $*$: l coincides with a given line.

- i: l intersects a given line.
- p: l passes through a given point.
- e: l is contained in a given plane.
- r: l belongs to a given pencil, i.e. l is in a given plane and passes through a given point on that plane.
- $\cdots$: a vacuously true proposition.

. *The inclusion relations. We denote, purely formally:*

$$* \hookrightarrow r \hookrightarrow p, e \hookrightarrow i \hookrightarrow \cdots. \tag{1.1}$$

To explain the notation $\hookrightarrow$, Schubert meant it as certain implication relations among the propositions, when the given data are in some special position. Consider p and i for example, if the given point happens to be on the given line, then p implies i, so we write $p \hookrightarrow i$. For e and i, if the given plane happens to contain the given line, then e "almost" implies i. We are ignoring the case when l is in that plane but parallel to the line, still writing $e \hookrightarrow i$. Similarly, if the plane or the point in r coincides with that in e or p, then we have the implication so $r \hookrightarrow p, e$. In any case, we can not deduce p and e from each other.

Remark 1.3. The above is not a rigorous "proof", nor a mathematically precise statement. In fact, to provide a rigorous context is exactly the aim of the later sections.

Alternatively we can interpret the inclusion relations as an ascending chain of "degree of freedom". We say that a given line has no degree of freedom, since it is given. A line in a pencil has 1 degree of freedom, since to determine it, we just need a direction which is a 1-dimensional datum or parameter. To determine a line in a plane, we need 2 parameters: the slope and the intercept (we are again ignoring some negligible cases). For p, we need a direction in the 3-dimensional space, which has 2 degrees of freedom. To determine a line satisfying i, we need an intersection point on the given line and a direction, so i has $1 + 2 = 3$ degrees of freedom. Lastly, a line $l \subset \mathbf{R}^3$ without any condition has $2 + 2 = 4$ degrees of freedom: we first choose a direction, then choose a point in the perpendicular plane.

. *The product relations. We denote, purely formally:*

$$i^2 = p + e. \tag{1.2}$$

$$i \cdot p = i \cdot e = r. \tag{1.3}$$

$$p^2 = e^2 = i \cdot r = *. \tag{1.4}$$

$$p \cdot e = 0. \tag{1.5}$$

These look rather mysterious, but in principle $+$ and $\cdot$ represent the logical operators $\vee$ (or) and $\wedge$ (and) respectively. For example, if l satisfies both i and p, then l is in a pencil (provided that the point in p is not on the line in i), so we write $i \cdot p = r$. Similarly, $i \cdot e = r$ means $i \wedge e \Rightarrow r$, provided that the line in i is not parallel to or contained in the plane in e.

The equations $p^2 = *$ and $e^2 = *$ essentially mean "two points determine a line" and "two planes intersect at a line". Similarly, the conditions i and r determine the line uniquely (modulo some degenerate positional cases). If the given point is not on the given plane, then p and e can not hold simultaneously so we denote $p \cdot e = 0$.

To explain $i^2 = p + e$, it involves quite some equivocation. Generically speaking, two given lines will be in skew position. Schubert argued, by invoking his "*principle of continuity*", that one could always reconsider the problem after first parallelly "shifting a line". We can thus shift the position so that the two lines intersect at a point, hence span a plane. It then follows that a line intersecting both of them either contains that point, or is contained in that plane.

. *Principle of continuity. The problem is invariant up to deformation. That is, while continuously varying the position of the given lines, as long as they are always in generic position, then the answer number is conserved (has no sudden change).*

Once the product relations are set up, the answer to Schubert's problem follows magically. We are studying the condition i^4, and this is eventually reduced to some purely algebraic manipulations:

$$i^4 = i^2 \cdot (p + e) = i \cdot (r + r) = 2(i \cdot r) = 2* \tag{1.6}$$

and the number 2 indicates that there are exactly two lines satisfying the condition!

Remark 1.4. By employing the principle of continuity for a specific deformation, the answer can also be derived geometrically. Suppose that there exists at least one line l satisfying the condition, then we can "slide the four lines along l", so that we make l_1 and l_2 intersect at a point A, l_3 and l_4 intersect at B. In such position, a line intersecting all the four lines has to be either AB (i.e. the line l), or the intersection of the planes spanned by $\{l_1, l_2\}$ and $\{l_3, l_4\}$. We thus recover the result that there are 2 lines.

Of course, a large part of this proof is hard to justify rigorously. We are thus left with the tasks below, that we are going to address in the later sections.

Question 1.5. What is "generic position", exactly?

Question 1.6. What is the "principle of continuity", exactly?

Question 1.7. What do the inclusion relations really mean?

Question 1.8. What do the product relations really mean?

The rest of the paper is organized as follows. In section 2 we translate Schubert's problem into one on the topological space $Gr_2(\mathbf{C}^4)$. Section 3 is preparatory, we list all the relevant results that we will later use. The key information is that we need to determine the structure of the *cohomology ring* of $Gr_2(\mathbf{C}^4)$. For this purpose, we perform a Morse theoretic computation in our section 4, which is the main technical part of the paper. The final proof of theorem 1.1 will appear in section 5, along with a comparison with the terms singled out in the four questions above. To the best of our knowledge, this is the first time such a comparison has been made transparent in the literature, though [Ron06] gives a very detailed historical account. Finally, section 6 can be viewed as an extension of our project. We reinterpret our results, in terms of a graphic presentation on a convex polyhedron. Throughout the paper, while some of the concepts might look fancy, when we are actually doing proofs, it involves no more knowledge than calculus, linear algebra and basic topology.

2. Reformulation of the problem

We will show that Schubert's problem is really a problem on the *Grassmannian* $Gr_2(\mathbf{C}^4)$ (or $Gr_2(\mathbf{R}^4)$). We first introduce the Grassmannians in general.

2.1. **The Grassmann manifolds.**

Definition 2.1. Let $\mathbf{F}$ be $\mathbf{R}$ or $\mathbf{C}$. For $0 < k < n$, denote by $Gr_k(\mathbf{F}^n)$ the topological space consisting of all the k-dimensional linear subspaces of $\mathbf{F}^n$.

So in particular, a point (an element) in $Gr_2(\mathbf{F}^4)$ represents a plane in $\mathbf{F}^4$ containing 0. Generally, one can show that (see [MS74], lemma 5.1) $Gr_k(\mathbf{F}^n)$ is a compact smooth manifold of dimension $k(n-k)$. We shall only describe a local coordinate system here, for the later computations. Recall that $\mathbf{R}^n$ and $\mathbf{C}^n$ are inner product spaces. Any linear subspace $V \subset \mathbf{F}^n$ admits an *orthogonal complement*, denoted as $V^\perp$.

Definition 2.2. The projection map $\mathbf{F}^n = V \oplus V^\perp \to V$ is denoted as π_V.

It follows that $V^\perp = ker(\pi_V)$. For $V \in Gr_k(\mathbf{F}^n)$, i.e. $\dim_{\mathbf{F}}(V) = k$, the result below gives a local coordinate chart at V in $Gr_k(\mathbf{F}^n)$, that will be of fundamental use.

Proposition 2.3. On the open neighborhood $U_V := \{W \in Gr_k(\mathbf{F}^n) | W \cap V^\perp = \{0\}\}$ of V, there is a homeomorphism defined as

$$\varphi_V : U_V \to Hom_{\mathbf{F}}(V, V^\perp) \cong \mathbf{F}^{k(n-k)},\ W \mapsto (\pi_{V^\perp}|_W) \circ (\pi_V|_W^{-1}). \tag{2.1}$$

Proof: We easily see that $V \in U_V$ since $V \cap V^\perp = \{0\}\}$. To see that U_V is open, consider the continuous map

$$D : Gr_k(\mathbf{F}^n) \to \mathbf{F},\ W \mapsto det(\pi_V|_W). \tag{2.2}$$

Then $U_V = D^{-1}(\mathbf{F} - \{0\})$ is open. To verify that φ_V is well defined, since $W \in U_V$ implies that $\pi_V|_W$ is a bijection, we have a map $\pi_V|_W^{-1} : V \to W$. Thus $\varphi_V(W)$ is defined as a composition

$$\varphi_V(W) : V \xrightarrow{\pi_V|_W^{-1}} W \xrightarrow{\pi_{V^\perp}|_W} V^\perp. \tag{2.3}$$

To show that φ_V is a homeomorphism, we construct its inverse directly. Given $\omega \in Hom_{\mathbf{F}}(V, V^{\perp})$, we check that

$$\varphi_V^{-1}(\omega) := \{v + \omega(v) | v \in V\} \in U_V \subset Gr_k(\mathbf{F}^n). \tag{2.4}$$

Here $v + \omega(v)$ means an element in $\mathbf{F}^n = V \oplus V^{\perp}$. As v ranges over V, the elements form a k-dimensional subspace because $id + \omega \in Hom_{\mathbf{F}}(V, V \oplus V^{\perp})$ is of full rank, so $\varphi_V^{-1}(\omega) = (id + \omega)(V) \in Gr_k(\mathbf{F}^n)$. Since $\varphi_V^{-1}(\omega) \cap V^{\perp} = \{0\}\}$, $\varphi_V^{-1}(\omega) \in U_V$. Clearly $id + \omega$ is continuous in ω hence $\varphi_V^{-1} : Hom_{\mathbf{F}}(V, V^{\perp}) \to U_V$ is continuous. It is then straightforward to check that $\varphi_V^{-1} \circ \varphi_V = id$ and $\varphi_V \circ \varphi_V^{-1} = id$, so φ_V^{-1} and φ_V are inverses of each other. □

For $\omega \in Hom_{\mathbf{F}}(V, V^{\perp})$, let $\omega^* \in Hom_{\mathbf{F}}(V^{\perp}, V)$ be its *adjoint operator*, characterized by the property that

$$(\omega(v), v') = (v, \omega^*(v')),\ \forall v \in V, v' \in V^{\perp} \tag{2.5}$$

where $(-, -)$ denotes the inner product in V or $V^{\perp}$ inherited from $\mathbf{F}^n$. More concretely, in terms of matrices, ω^* is the conjugate transpose of ω when $\mathbf{F} = \mathbf{C}$, and transpose when $\mathbf{F} = \mathbf{R}$. The next result is prepared for section 4.

Lemma 2.4. We have a commutative diagram, where all arrows are homeomorphisms:

$$\begin{array}{ccc} U_V & \xrightarrow{W \mapsto W^{\perp}} & U_{V^{\perp}} \\ {\scriptstyle \varphi_V}\downarrow & & \downarrow{\scriptstyle \varphi_{V^{\perp}}} \\ Hom_{\mathbf{F}}(V, V^{\perp}) & \xrightarrow{\omega \mapsto -\omega^*} & Hom_{\mathbf{F}}(V^{\perp}, V) \end{array}. \tag{2.6}$$

Proof: The upper arrow is well defined, because $(W \cap V^{\perp})^{\perp} = W^{\perp} \cap V$, so $W \cap V^{\perp} = 0 \iff W^{\perp} \cap V = 0$. Recall in the previous proof that

$$W = \{v + \omega(v) | v \in V\},\ \omega = \varphi_V(W). \tag{2.7}$$

It then suffices to show that

$$\varphi_{V^{\perp}}^{-1}(-\omega^*) = \{v' - \omega^*(v') | v' \in V^{\perp}\} \tag{2.8}$$

is the orthogonal complement of W. Since W and $\varphi_{V^{\perp}}^{-1}(-\omega^*)$ have complementary dimensions, it suffices to show that they are perpendicular to each other. That is,

$$(v + \omega(v), v' - \omega^*(v')) = 0,\ \forall v \in V, v' \in V^{\perp} \tag{2.9}$$

which is then reduced to equation 2.5. □

2.2. **Translating the problem.** When $k = 1$, $Gr_1(\mathbf{F}^n)$ is the projective space $\mathbf{FP}^{n-1}$, on which we have the *homogeneous coordinate* $[x_1, \cdots, x_n]$. Consider the obvious embedding

$$\mathcal{P} : \mathbf{R}^3 \hookrightarrow \mathbf{RP}^3,\ (x, y, z) \mapsto [1, x, y, z]. \tag{2.10}$$

Geometrically, what $\mathcal{P}$ does in $\mathbf{R}^4$ is to send a point A in the hyperplane $x_1 = 1$ (identified with $\mathbf{R}^3$) to the line OA, which is an element in $\mathbf{RP}^3$. It is then clear that $\mathcal{P}$ will take a line $l \subset \mathbf{R}$

to a 2-dimensional subspace in $\mathbf{R}^4$. That is, we have an embedding

$$\mathcal{P} : \{\text{lines in } \mathbf{R}^3\} \hookrightarrow Gr_2(\mathbf{R}^4) \tag{2.11}$$

and similarly an embedding

$$\mathcal{P} : \{\text{planes in } \mathbf{R}^3\} \hookrightarrow Gr_3(\mathbf{R}^4). \tag{2.12}$$

Example 2.5. As a concrete example, consider the plane $x + y + z = 1$ in $\mathbf{R}^3$, then $\mathcal{P}$ sends it to the 3-dimensional subspace spanned by $(1, 1, 0, 0)$, $(1, 0, 1, 0)$ and $(1, 0, 0, 1)$.

Example 2.6. For a line $l = AB$ in $\mathbf{R}^3$, $\mathcal{P}(l)$ is the 2-dimensional subspace spanned by $(1, A)$ and $(1, B)$.

Consider two lines $l, l' \subset \mathbf{R}^3$. As two 2-dimensional subspaces in $\mathbf{R}^4$, generically $\mathcal{P}(l)$ and $\mathcal{P}(l')$ intersect at $\{O\}$. This corresponds to the fact that two lines in the space generically do not intersect. We could have a 1-dimensional intersection $\mathcal{P}(l) \cap \mathcal{P}(l') = \mathcal{L}$. This happens if l and l' intersect at a point A, so $\mathcal{L} = \mathcal{P}(A)$. Therefore, via $\mathcal{P}$ we can view a line in $\mathbf{R}^3$ as a point in $Gr_2(\mathbf{R}^4)$, and translate any intersection statement in $\mathbf{R}^3$ to a statement in $Gr_2(\mathbf{R}^4)$ (or in the linear subspaces of $\mathbf{R}^4$).

Remark 2.7. One might worry that after the projectivization $\mathcal{P}$, two parallel lines would be wrongly counted as having an intersection. However, later we will see that we have a precise way to rule out such degenerate cases of our discussion, so that the translation into Grassmannian really preserves the intersection information.

Let's translate the logical propositions we have seen in section 1 about $l \subset \mathbf{R}^3$, to those below about $x \in Gr_2(\mathbf{R}^4)$.

- $*$: x is a given point.
- i: for a given $y \in Gr_2(\mathbf{R}^4)$, $\dim(x \cap y) > 0$.
- p: for a given 1-dimensional subspace $L \subset \mathbf{R}^4$, $L \subset x$.
- e: for a given 3-dimensional subspace $H \subset \mathbf{R}^4$, $x \subset H$.
- r: for a given 3-dimensional subspace $H \subset \mathbf{R}^4$ and a given 1-dimensional subspace $L \subset H \subset \mathbf{R}^4$, $L \subset x \subset H$.
- $\cdots$: a vacuously true proposition.

The last point is that we furthermore need to pass from the real numbers $\mathbf{R}$ to the complex numbers $\mathbf{C}$ (for a purely technical reason, see the remark below) under the canonical embedding

$$c : \mathbf{R}^4 \hookrightarrow \mathbf{R}^4 \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{C}^4. \tag{2.13}$$

Under c, a linear subspace $V \subset \mathbf{R}^4$ is sent to a linear subspace $c(V) := V \otimes_{\mathbf{R}} \mathbf{C} \subset \mathbf{C}^4$. Intuitively, what c does is to formally adjoin complex scalar multiplication of vectors. The complexification

c behaves well with other notions, in the sense that

$$\dim_{\mathbf{R}}(V) = \dim_{\mathbf{C}}(c(V)). \tag{2.14}$$

$$V_1 \subset V_2 \iff c(V_1) \subset c(V_2). \tag{2.15}$$

$$c(V_1 \cap V_2) = c(V_1) \cap c(V_2). \tag{2.16}$$

Thus what we have discussed earlier in this subsection holds verbatim et literatim, replacing $\mathbf{R}$ with $\mathbf{C}$.

Remark 2.8. From the topological perspective working over $\mathbf{C}$ as opposed to $\mathbf{R}$ has many advantages. For example, complex manifolds are canonically oriented. Moreover, complex dimensions are always converted to even real dimensions. As we will see later, this evenness property significantly simplifies the structure of the cohomology ring.

Finally, after the projectivization $\mathcal{P}$ followed by the complexification $c : Gr_2(\mathbf{R}^4) \hookrightarrow Gr_2(\mathbf{C}^4)$, the problem ends up being translated into the following.

. *Reformulation of Schubert's problem. For the given l_k $(k = 1, 2, 3, 4)$, denote $x_k := (c \circ \mathcal{P})(l_k) \in Gr_2(\mathbf{C}^4)$, and let $I_k \subset Gr_2(\mathbf{C}^4)$ be the subspace of the elements satisfying the previous proposition i for the given x_k, what is the cardinality of $I_1 \cap I_2 \cap I_3 \cap I_4$ generically?*

3. Preliminaries from topology and Morse theory

Throughout this section, M will be a closed, connected, oriented and smooth manifold of dimension n. The central result we are going to borrow from topology is the following. Roughly, it tells us that "the *cup product* encodes the information of intersections".

Theorem 3.1. ([Hut11].) Let $A, B \subset M$ be two submanifolds (assumed to be closed, oriented and smooth). Suppose that A intersects B *transversally* (denoted as $A \pitchfork B$), then

$$[A \cap B] = [A] \cup [B] \tag{3.1}$$

where $\cup$ denotes the cup product, $[X] \in H^k(M)$ is the cohomology class associated to X where $k = \operatorname{codim}(X)$. If $A \cap B = \varnothing$ then $[A] \cup [B] = 0$.

We will explain the terms showing up in this statement in the subsection below.

3.1. **Cohomology and cup product.** Recall that the singular homology ([Hat02], chapter 2) associates to each degree $0 \leq k \leq n$ and to M an abelian group $H_k(M)$. For $k = n$ in particular, the orientation of M induces an isomorphism $H_n(M) = \mathbf{Z}$, under which "1" corresponds to the *fundamental class* of M. For $X = A$ and B, the inclusion map induces a graded homomorphism $H_*(X) \to H_*(M)$, under which the fundamental class of X is sent to a homology class $x \in H_d(M)$ where $d = \dim(X)$. The Poincare dual ([Hat02], section 3.3) of x is a cohomology class at the degree $n - d := \operatorname{codim}(X)$, denoted as $[X]$.

For the cup product, in this paper we will not need its explicit definition or construction in [Hat02], section 3.2. The relevant fact for us is that, the cup product $\cup$ is an operation on the cohomology, that satisfies the following conditions:

- If $x \in H^k(M)$, $y \in H^l(M)$, then $x \cup y \in H^{k+l}(M)$.
- For x, y, z three cohomology classes, $x \cup (y \cup z) = (x \cup y) \cup z$.
- For $x \in H^k(M)$ and $y, z \in H^l(M)$, $x \cup (y + z) = x \cup y + x \cup z$.
- For $x \in H^k(M)$ and $y \in H^l(M)$, $x \cup y = (-1)^{kl} y \cup x$.

The message is that $\cup$ is really a product-like operation, subject to the associativity, distributivity and (graded) commutativity. One says that $\cup$ makes $H^*(M)$ into a graded commutative ring. These properties will be vital, to validate the algebraic manipulations in section 1.

Lastly, the transversality is an important property that also shows up in the next subsection. Informally, a transverse intersection is one with overlap "as little as possible".

Definition 3.2. We denote $A \pitchfork B$, if $A \cap B \neq \varnothing$ and $\forall p \in A \cap B$, the tangent spaces satisfy

$$T_pM = T_pA + T_pB := \{v_1 + v_2 | v_1 \in T_pA, v_2 \in T_pB\}. \tag{3.2}$$

Equivalently, if $\text{codim}(T_pA \cap T_pB) = \text{codim}(T_pA) + \text{codim}(T_pB)$.

The second characterization is easily generalized to the transversality of multiple objects:

$$\text{codim}(\cap_{i=1}^k T_pA_i) = \sum_{i=1}^k \text{codim}(T_pA_i). \tag{3.3}$$

The consequence of transversality is that $A \cap B$ is again a closed smooth submanifold with orientation induced by that of A, B and M by the convention in [Hut11], so that $[A \cap B]$ is defined, and theorem 3.1 holds. In particular when $\dim(A)$ and $\dim(B)$ are complementary, then $A \cap B$ is 0-dimensional. So A intersects B at some isolated points. Each intersection point has an orientation $+1$ or -1, and the total sum of these numbers is exactly $[A \cap B] \in H^n(M) = \mathbf{Z}$, called the *intersection number* of A and B.

The next result says that the transversality is a very mild condition, that always holds up to small perturbation. We shall only give a rough statement, see [Lee13], theorem 6.35 for a precise and more general version.

. *Thom's transversality theorem. When $\dim(A) + \dim(B) \geq n$ and $A \cap B \neq \varnothing$, one can always perturb them (arbitrarily) slightly, so that $A \pitchfork B$.*

3.2. **Morse theory and unstable manifolds.** Broadly speaking, Morse theory is a technique to extract topological information from studying certain smooth functions on a space. Specifically, it can be employed to derive the homology and cohomology.

Let f be a smooth function on M. At a point $x \in M$, the tangent map

$$f_* : T_xM \to T_{f(x)}\mathbf{R} = \mathbf{R} \tag{3.4}$$

is called the *gradient* of f at x, denoted as $\nabla_x f$. Usually one expresses $\nabla_x f$ in a local coordinate, as the Jacobian matrix

$$\nabla_x f = (\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x)). \tag{3.5}$$

When M is embedded in an ambient Euclidean space, or more generally possesses a Riemannian structure, we can speak of TM as a metric bundle, so any linear function on T_xM is obtained from taking the inner product with a fixed vector. In particular, we will also view $\nabla_x f$ as a tangent vector at x, in the sense that

$$\frac{\partial f}{\partial v} = f_*(v) = (v, \nabla_x f)_{T_xM}, \ \forall v \in T_xM. \tag{3.6}$$

As x varies, ∇f defines a smooth vector field on M, which then induces a flow on M called the *gradient flow*. Precisely, a gradient flow is a smooth curve $\gamma(t)$ on M satisfying

$$\gamma'(t) = \nabla_{\gamma(t)} f, \ \forall t \in \mathbf{R}. \tag{3.7}$$

By the existence and uniqueness theorem for ODE's, any point $x \in M$ is contained in a unique gradient flow, denoted as γ_x. For a gradient flow γ, by equation 3.6

$$(f \circ \gamma)'(t) = f_*(\gamma'(t)) = (\nabla_{\gamma(t)} f, \nabla_{\gamma(t)} f) = |\nabla_{\gamma(t)} f|^2 \geq 0, \ \forall t \in \mathbf{R}. \tag{3.8}$$

Hence f is non decreasing along gradient flow. A special case is when the gradient vanishes at x, then the flow generated by x is a constant curve.

Definition 3.3. If $\nabla_x f = 0$, x is said to be a *critical point* of f.

Definition 3.4. f is called a *Morse function*, if all its critical points are *non degenerate*.

We refer the definition of non degeneracy to [Mil63], section 1.2. Informally speaking, a critical point x being non degenerate means that at x locally, f has k directions of decreasing, l directions of increasing, and $k + l = n$. We say that k is the *index* of x. These notions are closely related to *stable / unstable manifolds*, as we shall now explain.

The fact $k + l = n$ implies that non degenerate critical points are isolated. Since M is assumed to be compact, a Morse function f has finitely many critical points. Let $\mathcal{C}(f)$ denote the set of critical points. For a non constant gradient flow γ, since f has bounded value and f monotonically increases along γ, $\gamma(t)$ must converge to critical points as $t \to \pm\infty$.

Definition 3.5. For $x \in \mathcal{C}(f)$, the unstable manifold of x is defined as

$$U(x) := \{y \in M | \lim_{t \to +\infty} \gamma_y(t) = x\} \tag{3.9}$$

and the stable manifold of x is defined as

$$S(x) := \{y \in M | \lim_{t \to -\infty} \gamma_y(t) = x\}. \tag{3.10}$$

It follows that M can be decomposed as a disjoint union

$$M = \coprod_{x \in \mathcal{C}(f)} U(x) = \coprod_{x \in \mathcal{C}(f)} S(x). \tag{3.11}$$

The next result is [ADE14], proposition 2.1.5. Actually we can use it to characterize the non degeneracy and the index:

Theorem 3.6. $U(x)$, $S(x)$ are submanifolds of M of dimension $k = \mathrm{ind}(x)$, and $l = n - k$ respectively. Moreover, we have the homeomorphisms

$$U(x) \cong e^k, \ S(x) \cong e^l \tag{3.12}$$

where $e^i \subset \mathbf{R}^i$ denotes the open unit disk, called an *open cell.*

This almost makes 3.11 into a CW decomposition. But we need a mild technical condition.

. *The Smale condition. We say f satisfies the Smale condition, if for any $U(x) \cap S(y) \neq \varnothing$, $U(x) \pitchfork S(y)$.*

Remark 3.7. By the definition, $x \in U(x)$ and $x \in S(x)$. By the monotonicity, if $U(x) \cap S(y) \neq \varnothing$, then $f(x) > f(y)$ or $x = y$. In the latter case, $U(x) \cap S(x) = \{x\}$ and the non degeneracy implies that $U(x) \pitchfork S(x)$. The Smale condition in particularly implies that if $U(x) \cap S(y) \neq \varnothing$, then $\mathrm{ind}(x) \geq \mathrm{ind}(y)$. Otherwise $\dim(U(x)) + \dim(S(y)) < n$, we can not have $U(x) \pitchfork S(y)$.

It turns out that the Smale condition ensures that every unstable manifold is attached along the boundary to the lower dimensional ones (see [ADE14], section 4.9):

Theorem 3.8. Suppose that f is a Morse function satisfying the Smale condition, then 3.11 is a CW decomposition. In particular, via the unstable manifold decomposition, as a CW complex, the cells of M of dimension i are in bijection with the critical points of f of index i, for all i.

This is also recognized as the fundamental theorem of Morse theory. In fact, section 4.9 of [ADE14] shows that not only we know the cellular decomposition, we also know what are the boundary maps in the cellular complex from analyzing the gradient flow of f. More precisely, there is a theory of *Morse homology / cohomology*, that one can compute the homology / cohomology groups of M using the Morse function f. However, we shall not present this theory here. As we will soon see, the cellular complex of complex Grassmannians has all boundary maps zero, so the example we consider in this paper is much simpler than the general theory.

4. The Morse gradient flow on $Gr_2(\mathbf{C}^4)$

The goal of this section is to apply the general theory sketched in the last subsection to the example $Gr_2(\mathbf{C}^4)$, or more generally any complex Grassmannian. We fix some notations: we use $*$ to represent a free variable, and we typically represent an element $V \in Gr_2(\mathbf{C}^4)$ as a full-rank 2×4 matrix $(v_1, v_2)^T$, meaning V is spanned by the vectors $v_1, v_2 \in \mathbf{C}^4$.

Theorem 4.1. There is a Morse-Smale function on $Gr_2(\mathbf{C}^4)$ with 6 critical points, such that the unstable manifolds (called the *Schubert cells* of $Gr_2(\mathbf{C}^4)$) are

- $e^0 := \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\right\}$.
- $e^2 := \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{pmatrix}\right\}$.
- $e^4_1 := \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{pmatrix}\right\}$.
- $e^4_2 := \left\{\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{pmatrix}\right\}$.
- $e^6 := \left\{\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}\right\}$.
- $e^8 := \left\{\begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}\right\}$.

Remark 4.2. One might recognize that these are exactly the *row canonical forms* in linear algebra. Every full-rank 2×4 matrix can be put into a such form uniquely, by a series of elementary row operations which do not change the underlying space $V \in Gr_2(\mathbf{C}^4)$. So it already follows that the above is a disjoint decomposition of $Gr_2(\mathbf{C}^4)$. With some efforts one can show by hand that it is indeed a CW decomposition. Nevertheless, we insist a Morse theoretic approach here, for twofold reasons. First, it is a general method. We can naturally derive this decomposition from analyzing the gradient flow, and invoke theorem 3.8 to show directly that it is a cellular one. Second, we will need some of the notions and results here for the later sections.

We break the proof into parts. The key part is to compute and analyze the gradient flow.

4.1. **Step one: construct the Morse function.** We do it for the general case. Fix an arbitrary sequence of real numbers $c_1 < c_2 < \cdots < c_n$ and denote the diagonal matrix $C := \mathrm{diag}(c_1, \cdots, c_n)$. Recall that for $V \in Gr_k(\mathbf{C}^n)$, π_V denotes the orthogonal projection. We can view π_V as a linear map $\mathbf{C}^n \to \mathbf{C}^n$ with image V.

Definition 4.3. Let f be a function on $Gr_k(\mathbf{C}^n)$ defined by

$$f(V) := Trace(C\pi_V). \tag{4.1}$$

A priori the value $f(V) \in \mathbf{C}$, but actually $f(V) \in \mathbf{R}$. This is because as an orthogonal projection, π_V is Hermitian (i.e. $\pi_V = \pi_V^*$) and so is C. Thus

$$f(V)^* = Tr((C\pi_V)^*) = Tr(\pi_V^* C^*) = Tr(\pi_V C) = Tr(C\pi_V) = f(V) \;\Rightarrow\; f(V) \in \mathbf{R}. \tag{4.2}$$

Since $V \mapsto \pi_V$ is smooth, f is a smooth function. It is often convenient to write $f(V)$ as

$$f(V) = \sum_{i=1}^{n} (C\pi_V e_i, e_i) \tag{4.3}$$

where $(-,-)$ is the standard Hermitian inner product on $\mathbf{C}^n$, $\{e_i\}$ is any orthonormal basis.

Example 4.4. For $k = 1$ and any $V \in \mathbf{CP}^{n-1}$ given by a homogeneous coordinate $[z_1, \cdots, z_n]$, take an orthonormal basis with $e_1 \in V$, $e_2, \cdots, e_n \in V^\perp$, then

$$f(V) = (C\pi_V e_1, e_1) = (Ce_1, e_1) = \frac{\sum_{i=1}^n c_i |z_i|^2}{\sum_{i=1}^n |z_i|^2}. \tag{4.4}$$

This is exactly the example Milnor considered in [Mil63], section 1.4.

From now on fix $\{e_i\}$ to be the standard orthonormal basis. For each $1 \le i_1 < \cdots < i_k \le n$, let $V_{i_1,\cdots,i_k} \in Gr_k(\mathbf{C}^n)$ denote the subspace spanned by $e_{i_1}, \cdots, e_{i_k}$. In the next three subsections we will prove that

Theorem 4.5. f is a Morse-Smale function with $\binom{n}{k}$ critical points:

$$\mathcal{C}(f) = \{V_{i_1,\cdots,i_k} | 1 \le i_1 < \cdots < i_k \le n\}. \tag{4.5}$$

The index of $V_{i_1,\cdots,i_k}$ is $2\sum_{j=1}^k (i_j - j)$.

4.2. **Step two: compute the tangent map.** Recall (proposition 2.3) $\varphi_V : U_V \xrightarrow{\cong} Hom_{\mathbf{C}}(V, V^\perp)$ defines the local coordinate. For $\omega \in Hom_{\mathbf{C}}(V, V^\perp)$, consider the curve $\gamma_\omega(t) := \varphi_V^{-1}(t\omega)$ in U_V, $\gamma_\omega(0) = V$. We will determine $\nabla_V f$ by computing $f_*(\gamma_\omega)$ for every ω. We will write $\frac{\partial f}{\partial \omega} := f_*(\gamma_\omega)$, understood as the directional derivative along ω. The goal of this subsection is to prove

$$\frac{\partial f}{\partial \omega} = Tr(C(\omega\pi_V + \omega^*\pi_{V^\perp})). \tag{4.6}$$

Remark 4.6. In the notation $\frac{\partial f}{\partial \omega}$, we have identified $Hom_{\mathbf{C}}(V, V^\perp)$ with $T_V U_V = T_V Gr_k(\mathbf{C}^n)$. This identification will often show up implicitly in the later works.

First, for $W \in U_V$ we need an explicit formula expressing π_W in π_V, $\pi_{V^\perp}$ and $\omega := \varphi_V(W)$.

Lemma 4.7. We have the composition

$$\pi_W = (id + \omega) \circ (id + \omega^*\omega)^{-1} \circ (\pi_V + \omega^*\pi_{V^\perp}) : \mathbf{C}^n \xrightarrow{\pi_V + \omega^*\pi_{V^\perp}} V \xrightarrow{(id+\omega^*\omega)^{-1}} V \xrightarrow{id+\omega} W. \tag{4.7}$$

Proof: For any $x \in \mathbf{C}^n$, denote $a := \pi_V(x)$, $b := \pi_{V^\perp}(x)$, then $x = a + b$. Recall in lemma 2.4 we have seen that

$$W = \{v + \omega(v) | v \in V\}. \tag{4.8}$$

$$W^\perp = \{v' - \omega^*(v') | v' \in V^\perp\}. \tag{4.9}$$

In order to find $\pi_W(x)$, we need to find $v \in V$ and $v' \in V^\perp$ such that

$$x = (v + \omega(v)) + (v' - \omega^*(v')) \tag{4.10}$$

then $\pi_W(x) = v + \omega(v)$. We rewrite the equation as

$$a + b = (v + \omega(v)) + (v' - \omega^*(v')) = (v - \omega^*(v')) + (v' + \omega(v)) \tag{4.11}$$

which is then equivalent to the system of equations

$$a = v - \omega^*(v'). \tag{4.12}$$

$$b = v' + \omega(v). \tag{4.13}$$

Eliminating v' from the equations:

$$a + \omega^* b = v + \omega^* \omega(v) \tag{4.14}$$

thus $v = (id + \omega^*\omega)^{-1} \circ (\pi_V + \omega^* \pi_{V^\perp})(x)$, and $\pi_W(x) = (id + \omega)(v)$ as desired. □

We formally denote $tW := \varphi_V^{-1}(t\omega)$, then by 4.3

$$\frac{\partial f}{\partial \omega} = \frac{df(tW)}{dt}|_{t=0} = \sum_{i=1}^{n} (C \frac{d\pi_{tW}}{dt}|_{t=0} e_i, e_i). \tag{4.15}$$

Substituting tW for W in lemma 4.7 yields

$$\pi_{tW} = (id + t\omega) \circ (id + t^2 w^* w)^{-1} \circ (\pi_V + t\omega^* \pi_{V^\perp}) := X(t)Y(t)Z(t). \tag{4.16}$$

Then take the derivative, and notice that $Y(0) = id$, $Y'(0) = 0$:

$$\frac{d\pi_{tW}}{dt}|_{t=0} = X'(0)Z(0) + X(0)Z'(0) = \omega\pi_V + \omega^* \pi_{V^\perp}. \tag{4.17}$$

Finally, substitute this into 4.15, we get the equation 4.6.

4.3. **Step three: compute the gradient flow.** To speak of the gradient as a vector field, we need an inner product structure on $T_V Gr_k(\mathbf{C}^n)$. Since we have identified it with $Hom_{\mathbf{C}}(V, V^\perp) = \mathbf{C}^{k(n-k)}$ via φ_V, we get an inner product on $T_V Gr_k(\mathbf{C}^n)$ transported from the standard Hermitian inner product on $\mathbf{C}^{k(n-k)}$:

$$(\omega, \tau) := Tr(\omega \tau^*),\ \omega, \tau \in Hom_{\mathbf{C}}(V, V^\perp). \tag{4.18}$$

Hence we can view $\nabla_V f$ as in $T_V Gr_k(\mathbf{C}^n)$ by 3.6. The goal of this subsection is to prove

Theorem 4.8. The gradient flow of f generated by V is

$$\gamma(t) := A(t) \cdot V := \mathrm{diag}(e^{2c_1 t}, \cdots, e^{2c_n t}) \cdot V. \tag{4.19}$$

Remark 4.9. Here the diagonal matrix $A(t)$ is understood as a linear automorphism of $\mathbf{C}^n$, that takes V to another k-dimensional subspace.

We need to show that $\gamma'(t) = \nabla_{\gamma(t)} f$ everywhere. Notice that $A(t)$ is a one-parameter group, meaning $A(t_1)A(t_2) = A(t_1 + t_2)$. It suffices to show that at $t = 0$, $\gamma'(0) = \nabla_V f$, then the equation holds for all t. We use the simple fact deduced from equation 3.6 that, when $|v|$ is fixed, $\frac{\partial f}{\partial v}$ attains its maximum at the direction of gradient. Hence our strategy to determine the gradient, is to maximize the expression 4.6.

Fix an orthonormal basis $\{e'_i\}$ of $\mathbf{C}^n$ with $e'_1, \cdots, e'_k \in V$, $e'_{k+1}, \cdots, e'_n \in V^\perp$. After the change of basis $\{e_i\} \to \{e'_i\}$, suppose that C is represented by the block matrix $\begin{pmatrix} A & B^* \\ B & D \end{pmatrix}$ where $A \in Hom(V,V)$, $B \in Hom(V, V^\perp)$, $D \in Hom(V^\perp, V^\perp)$. In the same form of block matrix, the other operator in 4.6 is expressed as

$$\omega\pi_V + \omega^*\pi_{V^\perp} = \begin{pmatrix} 0 & \omega^* \\ \omega & 0 \end{pmatrix} \Rightarrow C(\omega\pi_V + \omega^*\pi_{V^\perp}) = \begin{pmatrix} B^*\omega & \cdots \\ \cdots & B\omega^* \end{pmatrix}. \tag{4.20}$$

It follows that

$$\frac{\partial f}{\partial \omega} = Tr(B^*\omega) + Tr(B\omega^*) = (\omega, B) + (B, \omega) = |B+\omega|^2 - |\omega|^2 - |B|^2. \tag{4.21}$$

Proposition 4.10. $\nabla_V f = 2B$ (under the identification $T_V Gr_k(\mathbf{C}^n) = Hom(V, V^\perp)$).

Proof: If $B = 0$, then $\frac{\partial f}{\partial \omega} \equiv 0$, so $\nabla_V f = 0$. If $B \neq 0$, we maximize $\frac{\partial f}{\partial \omega}$, under the restriction $|\omega| = |B|$. Then it is to maximize $|B+\omega|$. By $|B+\omega| \leq |B| + |\omega| = 2|B|$, $\frac{\partial f}{\partial \omega}$ attains its maximum $2|B|^2$, at $\omega = B$. Hence it follows from 3.6 that $\nabla_V f = 2B$. □

Corollary 4.11. The critical points of f are $V_{i_1,\cdots,i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$.

Proof: $V = V_{i_1,\cdots,i_k} \iff V$ is an invariant subspace of $C \iff B = 0 \iff \frac{\partial f}{\partial \omega} \equiv 0 \iff \nabla_V f = 0$. In the first equivalence, we have used the assumption that the eigenvalues $c_1, \cdots, c_n$ of C are distinct to each other. □

Lemma 4.12. For all t, $\gamma(t) = A(t) \cdot V \in U_V$.

Proof: We need to show $(A(t) \cdot V) \cap V^\perp = \{0\}$. Suppose $v \in V$ such that $A(t)(v) \in V^\perp$, then $(A(t)(v), v) = 0$. But $A(t)$ is by definition positive definite, so $v = 0$. □

Now we also write $A(t)$ in the block matrix form $\begin{pmatrix} \alpha(t) & \beta^*(t) \\ \beta(t) & \delta(t) \end{pmatrix}$.

Lemma 4.13. $\varphi_V(\gamma(t)) = \beta(t) \circ \alpha^{-1}(t)$.

Proof: For $v \in V$, $A(t)(v) = \alpha(t)(v) + \beta(t)(v)$ is the decomposition in $V \oplus V^\perp$. Recall that $\varphi_V(W) : V \xrightarrow{\pi_V|_W^{-1}} W \xrightarrow{\pi_{V^\perp}|_W} V^\perp$. When $W = \gamma(t) = A(t) \cdot V$, we have $\varphi_V(W) : \alpha(t)(v) \mapsto \beta(t)(v)$, hence $\varphi_V(W) = \beta(t)\alpha^{-1}(t)$. □

Finally, theorem 4.8 will follow from proposition 4.10 and the next result.

Proposition 4.14. $\gamma'(0) = 2B$.

Proof: Via the identification φ_V, $\gamma(t)$ becomes the curve $\beta(t)\alpha^{-1}(t)$ in $Hom(V, V^\perp)$, and the derivative is taken in $Hom(V, V^\perp)$. Since $A(0) = id$, $\alpha(0) = id$ and $\beta(0) = 0$. Since $A'(0) = 2C = \begin{pmatrix} 2A & 2B^* \\ 2B & 2D \end{pmatrix}$, $\beta'(0) = 2B$. It follows that $\gamma'(0) = \beta'(0)\alpha^{-1}(0) = 2B$. □

4.4. **Step four: determine the unstable manifolds.** To find the unstable manifold decomposition, we let $t \to +\infty$ in theorem 4.8. The key observation is that $A(t)$ scales the i-th coordinate by $e^{2c_i t}$, when $t \to +\infty$, $e^{2c_i t} = o(e^{2c_j t})$ for $i < j$.

For all $1 \le i \le n$, let F_i denote the subspace spanned by $e_1, \cdots, e_i$. Consider $0 := F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbf{C}^n$, called a *complete flag* in linear algebra. The previous observation implies that $\forall$ 1-dimensional subspace $V_1 \subset F_i - F_{i-1}$,

$$\lim_{t\to+\infty} A(t)\cdot V_1 = \mathbf{C}e_i. \tag{4.22}$$

For each $1 \le i_1 < \cdots < i_k \le n$, let $e_{i_1,\cdots,i_k} \subset Gr_k(\mathbf{C}^n)$ be the subspace consisting of V such that

$$\{i|\dim(V\cap F_i) > \dim(V\cap F_{i-1})\} = \{i_1,\cdots,i_k\} \tag{4.23}$$

Since $\forall V \in Gr_k(\mathbf{C}^n)$, the dimensions from $V\cap F_0$ to $V \cap F_n$ are from 0 to k, increase at most 1 each step, we deduce that V must be in some $e_{i_1,\cdots,i_k}$ (precisely, $i_j = min\{i|\dim(V\cap F_i) = j\}$), and $\{e_{i_1,\cdots,i_k}\}$ forms a disjoint decomposition of $Gr_k(\mathbf{C}^n)$.

Proposition 4.15. $\forall V \in e_{i_1,\cdots,i_k}$, $\lim_{t\to+\infty} A(t)\cdot V = V_{i_1,\cdots,i_k}$. As an immediate corollary:

$$U(V_{i_1,\cdots,i_k}) = e_{i_1,\cdots,i_k}. \tag{4.24}$$

Proof: For each $1 \le j \le k$, take $v_j \in V$ such that $v_j \in F_{i_j} - F_{i_j-1}$. It is easy to see that $\{v_1,\cdots,v_k\}$ forms a basis of V. From

$$\lim_{t\to+\infty} A(t)\cdot(\mathbf{C}v_j) = \mathbf{C}e_{i_j} \subset \lim_{t\to+\infty} A(t)\cdot V \tag{4.25}$$

we deduce that $e_{i_j} \in \lim_{t\to+\infty} A(t)\cdot V$, $\forall j$. Hence $\lim_{t\to+\infty} A(t)\cdot V = V_{i_1,\cdots,i_k}$. □

We give $U(V_{i_1,\cdots,i_k})$ a more concrete description. Let $u_{i_1,\cdots,i_k} \subset Hom(V_{i_1,\cdots,i_k}, V^{\perp}_{i_1,\cdots,i_k})$ be the linear subspace consisting of ω such that

$$\omega(e_{i_j}) \subset F_{i_j-1},\ \forall 1\le j\le k. \tag{4.26}$$

Lemma 4.16. $\dim_{\mathbf{C}}(u_{i_1,\cdots,i_k}) = \sum_{j=1}^k (i_j - j)$.

Proof: Since $\dim(F_{i_j-1}\cap V^{\perp}_{i_1,\cdots,i_k}) = i_j - j$, as a matrix of $(n-k)\times k$, $\omega \in u_{i_1,\cdots,i_k} \iff$ only the first $i_j - j$ entries in the j-th column are non zero, $\forall j$. Hence such matrices form a linear subspace of $\mathbf{C}^{k(n-k)}$ of dimension $\sum_{j=1}^k (i_j - j)$. □

Proposition 4.17. $e_{i_1,\cdots,i_k} = \varphi^{-1}_{V_{i_1,\cdots,i_k}}(u_{i_1,\cdots,i_k})$.

Remark 4.18. So $U(V_{i_1,\cdots,i_k})$ consists of $(id+\omega)\cdot V_{i_1,\cdots,i_k}$, where $\omega \in u_{i_1,\cdots,i_k}$. In terms of matrices, they are exactly represented by the row canonical forms as in theorem 4.1.

Proof: We use that fact that $V \in e_{i_1,\cdots,i_k} \iff \exists v_j \in V$ $(j = 1,\cdots,k)$, $v_j \in F_{i_j} - F_{i_j-1}$.

For $(id+\omega)\cdot V_{i_1,\cdots,i_k}$ where $\omega \in u_{i_1,\cdots,i_k}$, we simply take $v_j = (id+\omega)(e_{i_j})$, the condition is satisfied. Conversely, if such $\{v_j\}$ exists, without loss of generality, we can assume for v_j that its i_l-th coordinate is 0 when $l < j$, and 1 when $l = j$. Then ω determined by $\omega(e_{i_j}) = v_j - e_{i_j}$ satisfies $\omega \in u_{i_1,\cdots,i_k}$ and $v_j = (id + \omega)(e_{i_j})$, so $(id + \omega) \cdot V_{i_1,\cdots,i_k} = V$. □

For the stable manifolds, we consider the complete flag $F_n^\perp \subset F_{n-1}^\perp \subset \cdots \subset F_0^\perp$ with respect to the reversed basis $e_n, \cdots, e_1$. Then it is completely analogous that

Proposition 4.19. $S(V_{i_1,\cdots,i_k})$ is $\varphi^{-1}_{V_{i_1,\cdots,i_k}}$ of a linear subspace of $Hom(V_{i_1,\cdots,i_k}, V^\perp_{i_1,\cdots,i_k})$ defined by

$$\omega(e_{i_j}) \subset F^\perp_{i_j},\ \forall 1 \le j \le k. \tag{4.27}$$

Since $F_i^\perp$ is spanned by $e_{i+1}, \cdots, e_n$, $\dim(F^\perp_{i_j} \cap V^\perp_{i_1,\cdots,i_k}) = n - k - (i_j - j)$. Hence

Lemma 4.20. $\dim_{\mathbf{C}}(S(V_{i_1,\cdots,i_k})) = k(n-k) - \sum_{j=1}^k (i_j - j) = \text{codim}_{\mathbf{C}}(U(V_{i_1,\cdots,i_k}))$.

Thus $V_{i_1,\cdots,i_k}$ is non degenerate of index $\dim_{\mathbf{R}}(U(V_{i_1,\cdots,i_k})) = 2\sum_{j=1}^k (i_j - j)$. To complete the proof of theorem 4.5, it remains to verify that f satisfies the Smale condition. That is, for any $V \in U(V_{i_1,\cdots,i_k})) \cap S(V_{i'_1,\cdots,i'_k})$, we need to show that $U(V_{i_1,\cdots,i_k})$ and $S(V_{i'_1,\cdots,i'_k})$ intersect transversally at V.

Lemma 4.21. $i'_j \le i_j$, for all j.

Proof: This essentially boils down to the simple fact in linear algebra that

$$\dim(V \cap F_i) + \dim(V \cap F_i^\perp) \le \dim(V) = k. \tag{4.28}$$

Thus $\dim(V \cap F_i) \le \text{codim}(V \cap F_i^\perp)$, $\forall i$. Since they both are non decreasing in i, we deduce that

$$i_j = min\{i | \dim(V \cap F_i) = j\} \ge min\{i | \text{codim}(V \cap F_i^\perp) = j\} = i'_j,\ \forall j. \tag{4.29}$$

□

Proposition 4.22. $T_V U(V_{i_1,\cdots,i_k}) + T_V S(V_{i'_1,\cdots,i'_k}) = T_V Gr_k(\mathbf{C}^n)$, so $U(V_{i_1,\cdots,i_k}) \pitchfork S(V_{i'_1,\cdots,i'_k})$.

Proof: Recall that $U(V_{i_1,\cdots,i_k}) \subset U_{V_{i_1,\cdots,i_k}} = Hom(V_{i_1,\cdots,i_k}, V^\perp_{i_1,\cdots,i_k})$ is identified via $\varphi^{-1}_{V_{i_1,\cdots,i_k}}$ with the linear subspace $u_{i_1,\cdots,i_k}$. A complement of $u_{i_1,\cdots,i_k}$ is given by

$$u'_{i_1,\cdots,i_k} = \{\omega \in Hom(V_{i_1,\cdots,i_k}, V^\perp_{i_1,\cdots,i_k}) | \omega(e_{i_j}) \subset F^\perp_{i_j},\ \forall 1 \le j \le k\}. \tag{4.30}$$

Since $V \in S(V_{i'_1,\cdots,i'_k})$ and $i'_j \le i_j$, for $\omega \in u'_{i_1,\cdots,i_k}$, the limit at $t \to -\infty$ of $V + \omega$ does not depend on ω. Hence $V + \omega \in S(V_{i'_1,\cdots,i'_k})$, $\forall \omega \in u'_{i_1,\cdots,i_k}$. It follows that $T_V U(V_{i_1,\cdots,i_k}) + T_V S(V_{i'_1,\cdots,i'_k})$ is the full space. □

Now we can finally reap the rewards: theorem 4.1 immediately follows from specifying for the case $k = 2$, $n = 4$. The Schubert cells are $e^0 = e_{1,2}$, $e^2 = e_{1,3}$, $e^4_1 = e_{1,4}$, $e^4_2 = e_{2,3}$, $e^6 = e_{2,4}$, $e^8 = e_{3,4}$. The superscript indicates the index or (real) dimension.

5. Schubert's calculus

In this section we solve Schubert's problem, and clarify all the equivocal aspects in Schubert's original proof in section 1. We showed in the previous section that $Gr_2(\mathbf{C}^4)$ is a CW complex with one 0-cell, one 2-cell, two 4-cells, one 6-cell and one 8-cell. Since they are concentrated in even dimensions, the cohomology groups are easily determined:

$$H^0(Gr_2(\mathbf{C}^4)) = \mathbf{Z},\ H^2(Gr_2(\mathbf{C}^4)) = \mathbf{Z},\ H^4(Gr_2(\mathbf{C}^4)) = \mathbf{Z}^2,\ H^6(Gr_2(\mathbf{C}^4)) = \mathbf{Z},\ H^8(Gr_2(\mathbf{C}^4)) = \mathbf{Z} \tag{5.1}$$

where the generators correspond to the cells with canonical orientation. We will see that these 6 cells are the incarnations of the 6 logical propositions introduced in section 1.

5.1. **The Schubert cells.** We first specify the given data of the propositions $*, i, p, e, r$, after the projectivization $\mathcal{P}$. These are very simple:

- The point (in p and r) is the 1-dimensional space $(*, 0, 0, 0)$, i.e. the origin $(0, 0, 0) \in \mathbf{R}^3$.
- The line (in $*$ and i) is the 2-dimensional space $(*, *, 0, 0)$, i.e. the x-axis in $\mathbf{R}^3$.
- The plane (in e and r) is the 3-dimensional space $(*, *, *, 0)$, i.e. the xy-plane in $\mathbf{R}^3$.

Let $*, I, P, E, R \subset Gr_2(\mathbf{C}^4)$ respectively denote the subspace of the elements satisfying the propositions $*, i, p, e, r$ (reformulated in section 2). Clearly, from the given data:

$$* \ \subset \ R \ \subset \ P,\ E \ \subset \ I \ \subset \ Gr_2(\mathbf{C}^4). \tag{5.2}$$

Proposition 5.1. We have

$$* = e^0,\ e^2 \subset R,\ e^4_1 \subset P,\ e^4_2 \subset E,\ e^6 \subset I. \tag{5.3}$$

Proof: All are straightforward, except the last one. The fact that every $V = \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix} \in e^6$ satisfies i can be seen from

$$det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix} = 0 \tag{5.4}$$

□

Proposition 5.2. We have

$$e^8 \cap I = \varnothing,\ e^6 \cap (P \cup E) = \varnothing,\ e^4_1 \cap E = \varnothing,\ e^4_2 \cap P = \varnothing,\ e^2 \cap * = \varnothing. \tag{5.5}$$

Proof: All are straightforward, except the first one. The fact that no $V = \begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix} \in e^8$ satisfies i can be seen from

$$det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix} = 1 \neq 0 \tag{5.6}$$

□

Proposition 5.3. Let the overline indicate the closure of a space, then

(1) $R = \overline{e^2} = e^0 \cup e^2$.
(2) $P = \overline{e_1^4} = e^0 \cup e^2 \cup e_1^4$.
(3) $E = \overline{e_2^4} = e^0 \cup e^2 \cup e_2^4$.
(4) $I = \overline{e^6} = e^0 \cup e^2 \cup e_1^4 \cup e_2^4 \cup e^6$.
(5) $Gr_2(\mathbf{C}^4) = \overline{e^8} = e^0 \cup e^2 \cup e_1^4 \cup e_2^4 \cup e^6 \cup e^8$.

Proof: We just do (4), the others are completely analogous. The fact $I = e^0 \cup e^2 \cup e_1^4 \cup e_2^4 \cup e^6$ is deduced from the previous two propositions, together with the inclusion relations 5.2. By the statement of i, I is closed. Hence $\overline{e^6} \subset I$. Since e^6 consists of the elements that "intersect the x-axis, but do not contain 0 nor be contained in the xy-plane", obviously $\overline{e^6} = I$. □

Remark 5.4. Alternatively, $\overline{e^6} = e^0 \cup e^2 \cup e_1^4 \cup e_2^4 \cup e^6$ can be seen from letting the free variables in $\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix} \in e^6$ tend to ∞.

Lemma 5.5. $*, I, P, E, R$ are closed submanifolds.

Proof: The normal method, is to express the conditions of $*, i, p, e, r$ locally as equations in the coordinate chart $Hom_{\mathbf{C}}(V, V^{\perp})$, then use the constant-rank theorem to show that they are submanifolds. This is doable but tedious. We prefer to sketch an alternative proof that is inspiring. We just proved that $*, I, P, E, R$ are closed cells. Take $I = \overline{e^6}$ for example, clearly the interior points $x \in e^6$ (i.e. not in the plane, not contain the point) have local coordinates. For the boundary points, there is no essential difference: we can simply "forget" the given plane and point, or move them elsewhere (while keeping the given line, of course), then those points also have local coordinates for the same reason as the interior points do. □

. *The upshot. The cohomology classes $[I], [P], [E], [R], [*]$ associated to $I, P, E, R, *$ (as defined in section 3) are the generators. Precisely, denote them again by $i, p, e, r, *$ by slightly abusing the notation, then they are the generators of H^2, H^4, H^4, H^6, H^8, respectively.*

. *Important observation. Any given datum will induce the same cohomology classes $i, p, e, r, *$.*

This is because the propositions $i, p, e, r, *$ are stated essentially without reference to the coordinate, so any datum will give the generator class as above. For example, no matter what line is given in i, we can always make it the "x-axis", then the class $[I]$ will always be the generator of $H^2(Gr_2(\mathbf{C}^4))$. For p and e, there might be some ambiguity since H^4 has two generators, but it is due to the *homotopy invariance* of homology and cohomology classes (as we will explain in the next subsection), that any two given data will induce the same class p and e.

Recall the formulation of Schubert's problem in section 2, where there are four submanifolds I_1, I_2, I_3, I_4 of codimension 2. In the transverse (i.e. generic) case, by theorem 3.1:

$$[I_1 \cap I_2 \cap I_3 \cap I_4] = [I_1] \cup [I_2] \cup [I_3] \cup [I_4] = i^4 \in H^8(Gr_2(\mathbf{C}^4)) = \mathbf{Z}. \quad (5.7)$$

The resulting number, is exactly the cardinality of $I_1 \cap I_2 \cap I_3 \cap I_4$.

Remark 5.6. Recall that strictly speaking, the intersection number is a signed counting. But for complex manifolds the canonical orientation ensures that every intersection is counted as $+1$, not -1, so we get a genuine number of intersections.

Now we determine the cohomology ring. Since it is concentrated in even degrees, the graded commutativity becomes the genuine commutativity.

Proposition 5.7. In the following product relations, we simply omit the symbol $\cup$:

(1) $p^2 = e^2 = *$, $pe = 0$.
(2) $ir = *$, $ip = ie = r$.
(3) $i^2 = p + e$.

Proof: The strategy is to find a transverse pair, then determine the intersection. Recall that transversality always holds up to small perturbation. While in section 1, the relation $i^2 = p + e$ is very hard to justify, here we can derive it purely algebraically.

(1) Take two distinct points A, B and denote by p_A, p_B the corresponding propositions. Clearly $P_A \cap P_B = \{AB\}$ which is the generator of $H_0 = H^8$, so

$$p^2 = p_A p_B = [P_A \cap P_B] = *. \tag{5.8}$$

The relation $e^2 = *$ is completely analogous. If the point in p is not in the plane in e, then $P \cap E = \varnothing$ and by theorem 3.1, $pe = 0$.

(2) Since $R, P, E \subset I$, the intersections are not transverse. We use a trick: the stable manifolds of f are exactly the unstable manifolds of $-f$. For the generator $i \in H^2$, we can represent it by (the closure of) $U_{-f}(V_{1,3}) = S_f(V_{1,3})$, which amounts to change the given datum $(*, *, 0, 0)$ in i into $(0, 0, *, *)$. Since f is Smale, the intersections are now transverse. We can compute the intersections purely in linear algebra, but the better way is to use the (projective) geometry: $(0, 0, *, *)$ is the ∞ line at the yz-plane, so $I \cap R$ is the y-axis, $I \cap P$ is the pencil in the yz-plane through 0, $I \cap E$ is the pencil in the xy-plane through the ∞ point at the y-axis. Thus $ir = *$, $ip = ie = r$.

(3) Since i^2 is in H^4 generated by p and e, there are integers a, b such that $i^2 = ap + be$. Using (1) and (2):

$$a* = i^2 p = ir = *,\ b* = i^2 e = ir = *. \tag{5.9}$$

So $a = b = 1$, $i^2 = p + e$.

$\square$

Finally, we can use these product relations as input, to compute i^4 purely algebraically. We emphasize, thanks to the properties of cup product listed in section 3, that this computation can be understood in a precise and rigorous sense:

$$
\begin{aligned}
i^4 &= i^2(p+e) && \textit{associativity} && (5.10)\\
&= i(r+r) && \textit{associativity and distributivity} && (5.11)\\
&= 2(ir) && \textit{distributivity} && (5.12)\\
&= 2*. && && (5.13)
\end{aligned}
$$

Recall that $*$ is the generator of H^8, so we deduce that $|I_1 \cap I_2 \cap I_3 \cap I_4| = 2$.

Remark 5.8. In the non transverse case, we would have an element $V \in \cap_{k=1}^4 I_k$ such that

$$\operatorname{codim}(\cap_{k=1}^4 T_V I_k) < \sum_{k=1}^{4} \operatorname{codim}(T_V I_k) = 8. \tag{5.14}$$

That is, $\dim(\cap_{k=1}^4 T_V I_k) \geq 1$. This holds if there is a 1-dimensional family of intersections near V. This could also occur, when V is an "unstable" point, in the sense that a slight perturbation annihilates the intersection V.

5.2. **Schubert's proof revisited.** We just presented a proof of theorem 1.1 based on rigorous mathematics. Comparing what we have done with what Schubert did, in this subsection we can give very precise answers to the four questions raised in section 1.

Question 1.5. The genericity exactly means the transversality, and the answer number in the generic cases is always 2. This is a precise notion, otherwise one would have to conduct an exhaustive enumeration of degenerate positions to define genericity. Note that Thom's transversality theorem well gets across what the word "generic" wants to convey: one can always restore the genericity infinitesimally locally.

Question 1.6. The "principle of continuity" is a consequence of the homotopy invariance of homology / cohomology. Recall that to define homology classes, we modulo the boundary relation. When we move a line to another position, we get two closed subspaces of $Gr_2(\mathbf{C}^4)$ that are coboundary, hence they induce the same homology class. More precisely, a deforming process of the given line can be viewed as a homotopy between two maps f, g from $\overline{e^6}$ to $Gr_2(\mathbf{C}^4)$, so $f_* = g_*$ on homology. Passing via the Poincare duality, we get a well defined cohomology class $i \in H^2(Gr_2(\mathbf{C}^4))$. Since no matter what data are given, they all represent the same cohomology class, once we can compute for one specific representative, the result applies to all generic cases.

Question 1.7. The inclusion relations exhibit a cell structure of $Gr_2(\mathbf{C}^4)$, and the "degree of freedom" is the dimension of cell. While in section 1, the symbol $\subset$ is understood as implication under some special conditions, here $\subset$ genuinely means inclusion of cells at their boundaries. That is, $\subset$ encodes the gluing information in the CW complex $Gr_2(\mathbf{C}^4)$.

Question 1.8. The product relations are the cup product formulas in the cohomology ring $H^*(Gr_2(\mathbf{C}^4))$. Schubert meant the product as taking $\wedge$ ("and") of logical propositions, or equivalently, taking intersection of those elements that satisfy the propositions. From theorem 3.1 we know that this is just secretly computing the cup product, though Schubert himself would not be aware of it. The formal calculus of Schubert turns out to give the correct answer, because the cup product really behaves like a product as in the usual sense.

Let's recap what we have done: we convert the problem of finding the intersection number, to the algebraic problem of computing the cup product, and we use transverse pairs in geometry to represent the cohomology classes. Along the way, we have seen that many important terminologies can only be made rigorous after the introduction of some more advanced and abstract mathematical concepts. However, the Schubert problem itself is stated in a completely elementary way, that literally anyone can understand. In this project we have witnessed the intriguing process of unraveling the mysterious original arguments of Schubert, and we feel very much inspired, to see that these abstract theories we have learnt in geometry and topology, could eventually lead to an elegant solution to such a concrete problem.

6. The octahedron

The paper could just end here, but we learned from [Gue01] an interesting visualizable reinterpretation of the results that we would like to present. Although we have not managed to understand the general theory there, we are able to prove the theorem for the particular Grassmannian example of ours, using no more than linear algebra and basic topology.

Now we consider the Grassmannian $Gr_2(\mathbf{R}^4)$, not $Gr_2(\mathbf{C}^4)$. First recall again $\pi_V : \mathbf{R}^4 \to V \subset \mathbf{R}^4$ is the orthogonal projection for $V \in Gr_2(\mathbf{R}^4)$, and $\{e_i\}_{1\leq i\leq 4}$ denotes the standard orthonormal basis. Define a map μ from $Gr_2(\mathbf{R}^4)$ to $\mathbf{R}^4$ by

$$\mu(V) = ((\pi_V e_i, e_i))_{1\leq i\leq 4} \tag{6.1}$$

called the *moment map*, for reasons that are beyond the scope of the paper. Equivalently, $\mu(V)$ is formed by the diagonal entries of the matrix π_V in the basis $\{e_i\}$. It is the image of μ that reveals interesting patterns.

Lemma 6.1. The image of μ is contained in

$$O := \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 | x_1 + x_2 + x_3 + x_4 = 2,\ x_i \in [0, 1]\}. \tag{6.2}$$

Proof: As a projection map of rank 2, π_V has trace 2 so the image of μ is in the 3-dimensional hyperplane $x_1+x_2+x_3+x_4 = 2$. As a projection map, π_V is positive semidefinite, so $(\pi_V e_i, e_i) \geq 0$, and $(\pi_V e_i, e_i) \leq |e_i|^2 = 1$. □

For $V = V_{i,j}$ (using the notation in section 4), $1 \leq i < j \leq 4$, obviously π_V is a diagonal matrix and $\mu(V) = e_i + e_j$. Explicitly, we denote

$$A := \mu(V_{1,2}) = (1, 1, 0, 0),\ B := \mu(V_{1,3}) = (1, 0, 1, 0),\ C := \mu(V_{1,4}) = (1, 0, 0, 1) \tag{6.3}$$

$$D := \mu(V_{2,3}) = (0, 1, 1, 0),\ E := \mu(V_{2,4}) = (0, 1, 0, 1),\ F := \mu(V_{3,4}) = (0, 0, 1, 1). \tag{6.4}$$

We deduce from the distance relations that $ABCDEF$ forms a regular octahedron in the hyperplane $x_1 + x_2 + x_3 + x_4 = 2$ of edge length $\sqrt{2}$, with AF, BE, CD being the opposite vertices. From now on, everything is implicitly in the hyperplane.

Lemma 6.2. The regular octahedron $ABCDEF$ is O.

Proof: It is easy to see that O is convex. Then as the convex hull of $A, B, C, D, E, F \in O$, the octahedron is contained in O. Notice that the eight faces have the underlying planes $x_i = 0, 1$ $(1 \le i \le 4)$ respectively. The other sides of the planes, are the regions $x_i < 0, x_i > 1$ $(1 \le i \le 4)$. So no point in O can be outside the octahedron, hence $O = ABCDEF$. □

It is a much more general theorem of Atiyah-Guillemin-Sternberg ([Gue01], theorem 4.1.3) that for our case states that

Theorem 6.3. $\mu(Gr_2(\mathbf{R}^4)) = O$. In more detail, the images of the closed Schubert cells are

(1) $\mu(\overline{e^2}) = AB = O \cap \{x_1 = 1, x_4 = 0\}$.
(2) $\mu(\overline{e_1^4}) = ABC = O \cap \{x_1 = 1\}$.
(3) $\mu(\overline{e_2^4}) = ABD = O \cap \{x_4 = 0\}$.
(4) $\mu(\overline{e^6}) = ABCDE = O \cap \{x_1 + x_2 \ge x_3 + x_4\} = O \cap \{x_1 + x_2 \ge 1\}$.

In all the cases above, the image is the convex hull of the images of critical points in the region.

Proof: We start from the simple cases to the hard ones.

(1) Since $\forall V \in \overline{e^2} = e^0 \cup e^2$ is spanned by $(1, 0, 0, 0)$ and another vector in $(*, *, *, 0)$ (see theorem 4.1), $(\pi_V e_1, e_1) = 1$ and $(\pi_V e_4, e_4) = 0$, so $\mu(\overline{e^2}) \subset AB$. Since $\overline{e^2}$ is connected containing $V_{1,2}$, $V_{1,3}$, and $A = \mu(V_{1,2})$, $B = \mu(V_{1,3})$, we must have $\mu(\overline{e^2}) = AB$.

(2) Since $\forall V \in \overline{e_1^4} = e^0 \cup e^2 \cup e_1^4$ contains $(1, 0, 0, 0)$, $(\pi_V e_1, e_1) = 1$ so $\mu(\overline{e_1^4}) \subset ABC$. For any element $(1, x_2, x_3, x_4)$ in ABC (so $x_2 + x_3 + x_4 = 1$, $x_i \in [0, 1]$), consider $V \in \overline{e_1^4}$ spanned by the orthonormal basis $e_1 = (1, 0, 0, 0)$, $v := (0, \sqrt{x_2}, \sqrt{x_3}, \sqrt{x_4})$. Then for $i = 2, 3, 4$, since $e_i \perp e_1$,

$$(\pi_V e_i, e_i) = |\pi_V e_i|^2 = (e_i, v)^2 = x_i. \tag{6.5}$$

So $(1, x_2, x_3, x_4) = \mu(V)$, hence $\mu(\overline{e_1^4}) = ABC$.

(3) Since $\forall V \in \overline{e_2^4} = e^0 \cup e^2 \cup e_2^4$ is contained in $(*, *, *, 0)$, $(\pi_V e_4, e_4) = 0$ so $\mu(\overline{e_2^4}) \subset ABD$. For any element $(x_1, x_2, x_3, 0)$ in ABD (so $x_1 + x_2 + x_3 = 2$, $x_i \in [0, 1]$), consider $V \in \overline{e_2^4}$ to be the normal plane of the unit vector $v := (\sqrt{1 - x_1}, \sqrt{1 - x_2}, \sqrt{1 - x_3}, 0)$ in $(*, *, *, 0)$. Then for $i = 1, 2, 3$, since $(e_i, v) = \sqrt{1 - x_i}$,

$$(\pi_V e_i, e_i) = |\pi_V e_i|^2 = 1 - (e_i, v)^2 = x_i. \tag{6.6}$$

So $(x_1, x_2, x_3, 0) = \mu(V)$, hence $\mu(\overline{e_2^4}) = ABD$.

(4) Since $\forall V \in \overline{e^6}$ has non zero intersection with the 2-dimensional space $(*, *, 0, 0)$, $\exists$ an element $v := (\sqrt{a}, \sqrt{1 - a}, 0, 0) \in V$, so

$$(\pi_V e_1, e_1) + (\pi_V e_2, e_2) = |\pi_V e_1|^2 + |\pi_V e_2|^2 \ge (e_1, v)^2 + (e_2, v)^2 = 1. \tag{6.7}$$

Hence $\mu(\overline{e^6}) \subset ABCDE$. Conversely, for any $(x_1, x_2, x_3, x_4) \in O$ with $x_1 + x_2 \geq 1$, we show that $\exists x$ such that $V(x) := \begin{pmatrix} \sqrt{x} & \sqrt{1-x} & 0 & 0 \\ -\sqrt{x_1 - x} & \sqrt{x_2 + x - 1} & \sqrt{x_3} & \sqrt{x_4} \end{pmatrix} \in \overline{e^6}$ and $\mu(V(x)) = (x_1, x_2, x_3, x_4)$, so $\mu(\overline{e^6}) = ABCDE$. Note that the two basis vectors $v_1(x)$ and $v_2(x)$ are unit vectors, if we have $v_1 \perp v_2$, then

$$(\pi_V e_i, e_i) = |\pi_V e_i|^2 = (e_i, v_1)^2 + (e_i, v_2)^2 = x_i, \ i = 1, 2, 3, 4 \tag{6.8}$$

so that $\mu(V(x)) = (x_1, x_2, x_3, x_4)$. It remains to show $\exists x$, $v_1(x) \perp v_2(x)$. Consider the continuous function

$$g(x) := v_1(x) \cdot v_2(x), \ x \in [1 - x_2, x_1]. \tag{6.9}$$

Since $g(1 - x_2) \leq 0 \leq g(x_1)$, $\exists x$ such that $g(x) = 0$, i.e. $v_1(x) \perp v_2(x)$.

Finally, $\mu(Gr_2(\mathbf{R}^4)) = O$ can be proved in an exactly the same manner as in (4). Alternatively, we remark the following facts. Recall that $\overline{e^2} = R = \{V \in Gr_2(\mathbf{R}^4) | (*, 0, 0, 0) \subset V \subset (*, *, *, 0)\}$. For any $i \neq j$, the subset $R_{ij} := \{V \in Gr_2(\mathbf{R}^4) | e_i \in V \subset e_j^\perp\}$ has the edge connecting $\mu(V_{i,k})$ $(k \neq i, j)$ as its image, for the same reason as $R_{14} = \overline{e^2}$ does. So all the 12 edges of the octahedron can be $\mu(R)$ for some R. Similarly, for $P = \overline{e_1^4}$, let the given datum in p vary from $(*, 0, 0, 0)$ to $(0, 0, 0, *)$, then $\mu(P) = ABC, ADE, BDF, CEF$ the 4 alternate faces, and $\mu(E)$ can be the other 4 alternate faces ABD, ACE, BCF, DEF. Lastly, the original given datum in i is $V_{1,2}$, and the image of $I = \overline{e^6}$ is the regular square pyramid with the apex $A = \mu(V_{1,2})$. Any $V_{i,j}$ can be the given datum in i, together they give all the 6 regular square pyramids with the apex $\mu(V_{i,j})$ as $\mu(I)$ respectively. It is then clear that $\mu(Gr_2(\mathbf{R}^4))$ is the whole octahedron. □

After taking $Gr_2(\mathbf{R}^4)$ to the regular octahedron via μ, we can "visualize" the cup product of cohomology classes in the picture as taking intersection among vertices, edges, faces and regular pyramids, and this looks quite interesting for us. For example, if we take two regular pyramids with adjacent apexes, then their intersection is two adjacent faces. This can be viewed as a visualization of the relation $i^2 = p + e$. Similarly, consider given four lines: the x-axis, the y-axis, the ∞ line at the xz-plane, and the ∞ line at the yz-plane. Then the lines intersecting all of them are the z-axis, or the ∞ line at the xy-plane. This can be seen from the picture that the four pyramids with apexes A, B, E, F have the intersection $\{C, D\}$, which reflects the fact that $i^4 = 2$. In fact, example 4.1.10 of [Gue01] tells us that just by contemplating the regular octahedron, we find all the structures in the cohomology ring $H^*(Gr_2(\mathbf{C}^4))$.

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