第一页为封面页

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论文题目：Hodge numbers of the surface of the planes in a cubic 5-fold

Hodge numbers of the surface of planes in a cubic 5-fold

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Abstract

We study the geometry of the moduli space of planes in a general cubic 5-fold. We show that this moduli space is a smooth projective surface whose canonical bundle is ample. We also calculate the Betti numbers and the Hodge numbers of this surface.
Keywords: Cubic hypersurfaces, Noether's formula, Chern classes, Schubert calculus

Contents

1 Introduction

The moduli space of linear subspaces in a projective hypersurfaces has been extensively studied in the litterature [BD85, DM98, Voi04], starting from the classical result of Cayley and Salmon which states that every smooth projective cubic surface has exactly 27 lines. In the language of moduli spaces, it says that the moduli space of lines in a cubic surface is a finite set of 27 points.. Another important example is the moduli space of lines in a cubic 4-fold. As is proven in [BD85], this moduli space is a hyper-Kähler manifold, a fundamental building block in the classification of algebraic varieties.

In this article, we consider the following explicit moduli space of such type. Let $Y \subset \mathbb{P}^6$ be a general cubic 5-fold. Let Σ be the moduli space of planes in Y. As is proved in Section 2, Σ is a smooth projective surface of general type. The study of the surface Σ is motivated by a result of Iliev and Manivel [IM08, Proposition 4], which states that the surface Σ is a Lagrangian subvariety of the Fano variety of lines of a cubic 4-fold.

Our first result is the calculation of the Betti numbers of the smooth surface Σ.

Theorem 1.1 (Theorem 4.18)**.** *The Betti numbers of the surface Σ are as follows.*
(i) $b_0(\Sigma) = b_4(\Sigma) = 1$.
(ii) $b_1(\Sigma) = b_3(\Sigma) = 42$.
(iii) $b_2(\Sigma) = 13123$.

Since Σ is a smooth projective surface, its cohomology groups admits a Hodge decomposition (Section 3.2). Our second result is the calculation of the Hodge numbers.

Theorem 1.2 (Theorem 4.19)**.** *The Hodge numbers of the surface Σ are as follows.*
(i) $h^{2,0}(\Sigma) = h^{0,2}(\Sigma) = 3233$.
(ii) $h^{1,1}(\Sigma) = 6657$.
(iii) $h^{1,0}(\Sigma) = h^{0,1}(\Sigma) = 21$.

Let us briefly present the method for the above results.

As Σ parametrizes the planes in a general cubic 5-fold $Y \subset \mathbb{P}^6$, we may regard Σ as a subvariety of the Grassmannian $\mathrm{Gr}(3,7)$ defined as the zero locus of a general section of the vector bundle $\mathrm{Sym}^3\mathcal{U}^\vee$, where $\mathcal{U}$ is the tautological subbundle of $Gr(3,7)$. Since $\mathrm{Gr}(3,7)$ is of dimension 12 and $\mathrm{Sym}^3\mathcal{U}^\vee$ is of rank 10, the dimension of Σ is 2. Therefore, the tangent bundle T_Σ fits into the following short exact sequence

$$0 \to T_\Sigma \to T_{Gr(3,7)}|_\Sigma \to \mathrm{Sym}^3\mathcal{U}^\vee|_\Sigma \to 0, \tag{1}$$

which gives an expression of the canonical bundle K_Σ of Σ as follows

$$K_\Sigma = (K_{Gr(3,7)} \otimes \det \mathrm{Sym}^3\mathcal{U}^\vee)|_\Sigma.$$

The right-hand side can be easily seen to be ample (Proposition 2.4).

To calculate the Betti numbers of surface Σ, we first calculate the Euler characteristic of Σ (note that Σ is a compact manifold of *real* dimension 4):

$$e(\Sigma) = \sum_{i=0}^{4} (-1)^i b_i(\Sigma).$$

By the Hopf-Poincaré theorem, $e(\Sigma)$ is equal to the top degree Chern class $c_2(\Sigma)$ of the tangent bundle of T_Σ, viewed as an integer via the natural identification $\int_\Sigma : H^4(\Sigma,\mathbb{Z}) \cong \mathbb{Z}$. Basic theory of Chern classes is briefly recalled in Section 3.1. By the exact sequence (1), we can calculate the Chern class $c_2(\Sigma)$ as follows

$$c_2(\Sigma) = \big(c_1(\mathrm{Sym}^3U^\vee)^2 - c_1(U^\vee \otimes Q)\cdot c_1(\mathrm{Sym}^3U^\vee) + c_2(U^\vee \otimes Q) - c_2(\mathrm{Sym}^3U^\vee)\big)\cdot c_{10}(\mathrm{Sym}^3U^\vee).$$

By the Schubert calculus that we will briefly recall in Section 4.1, the right-hand-side can be calculated by combinatorial methods (see Section 4 for explicit calculations) and the result is 13041.

Now that we know the Euler characteristic, to calculate all the Betti numbers, it suffices to know either b_1 or b_2. However, a result of Collino [Col86] shows that $b_1(\Sigma) = 42$. This gives all the Betti numbers of Σ.

Finally, to calculate the Hodge numbers of Σ, we use the Noether's formula

$$\chi(\mathcal{O}_\Sigma) = \frac{1}{12}(c_1(\Sigma)^2 + c_2(\Sigma)).$$

As the calculation of $c_2(\Sigma)$ illustrated in the previous paragraph, $c_1(\Sigma)^2$ can be calculated by using Schubert calculus and the result is 25515. Hence, $\chi(\Sigma) = 3213$. By the result of Collino again, we get $h^{1,0}(\Sigma) = 21$. Thus, $h^{2,0}(\Sigma) = 3233$. Combined with the calculations of the Betti numbers, we finally obtain $h^{1,1}(\Sigma) = 6657$. All Hodge numbers are thus calculated.

2 Moduli space of planes in a general cubic 5-fold

Let $Y \subset \mathbb{P}^6$ be a *general* cubic 5-fold. The moduli space Σ of planes in Y can be viewed as a subvariety of the Grassmannian $\mathrm{Gr}(3,7)$ in the following way: Let $\mathcal{U}$ be the tautological subbundle of $\mathrm{Gr}(3,7)$. The defining polynomial f of Y induces a global section σ_f of the vector bundle $\mathrm{Sym}^3\mathcal{U}^\vee$. A plane $P \subset \mathbb{P}^6$ is contained in Y if and only if the section σ_f vanishes at the point $x_P \in \mathrm{Gr}(3,7)$ representing the plane P. Hence, the moduli space Σ is exactly the zero locus of the section σ_f, and thus it is a projective variety. Therefore, in order to understand the geometry of Σ, it is essential to study the vector bundle $\mathrm{Sym}^3\mathcal{U}^\vee$.

Proposition 2.1. *The vector bundle* $\mathrm{Sym}^3\mathcal{U}^\vee$ *is globally generated of rank* 10 *and every section has a zero.*

Recall that a vector bundle E on a variety X is called globally generated, if for any $x \in X$, the evaluation map

$$H^0(X, E) \to E|_x$$

is surjective. Now we proceed to the proof.

Proof. As the rank of $\mathcal{U}^\vee$ is 3, the rank of $\mathrm{Sym}^3\mathcal{U}^\vee$ is $\binom{3+3-1}{3} = 10$. Now consider a point $x \in \mathrm{Gr}(3,7)$, corresponding to a 3-dimensional subspace $W_x \subset \mathbb{C}^7$. The fiber $\mathrm{Sym}^3\mathcal{U}^\vee|_x$ is the space of homogeneous polynomials of degree 3 on W_x, i.e.,

$$\mathrm{Sym}^3\mathcal{U}^\vee|_x \cong \{\text{homogeneous polynomials of degree 3 on } W_x\}.$$

On the other hand, the global sections of $\mathrm{Sym}^3\mathcal{U}^\vee$, denoted by $H^0(\mathrm{Gr}(3,7), \mathrm{Sym}^3\mathcal{U}^\vee)$, are isomorphic to the space of homogeneous polynomials of degree 3 on $\mathbb{C}^7$:

$$H^0(\mathrm{Gr}(3,7), \mathrm{Sym}^3\mathcal{U}^\vee) \cong \{\text{homogeneous polynomials of degree 3 on } \mathbb{C}^7\}.$$

The evaluation map

$$H^0(\mathrm{Gr}(3,7), \mathrm{Sym}^3\mathcal{U}^\vee) \to \mathrm{Sym}^3\mathcal{U}^\vee|_x$$

is simply the restriction of these global polynomials to the subspace W_x. Since the restriction map is surjective, we conclude that $\mathrm{Sym}^3\mathcal{U}^\vee$ is globally generated.

Let $I = \{(x, \sigma) \in \mathrm{Gr}(3,7) \times H^0(\mathrm{Gr}(3,7), \mathrm{Sym}^3\mathcal{U}^\vee) : \sigma(x) = 0\}$ be the incidental variety and let $q : I \to H^0(\mathrm{Gr}(3,7), \mathrm{Sym}^3\mathcal{U}^\vee)$ be the second projection map. It is shown in [Bor90, Proposition 2.1] that the map q is surjective. The latter means exactly that any global section of $\mathrm{Sym}^3\mathcal{U}^\vee$ has a zero.

This concludes that $\mathrm{Sym}^3\mathcal{U}^\vee$ is globally generated and that every section has a zero. □

To proceed, we will use the following classical result in algebraic geometry. We write down a proof for completeness.

Proposition 2.2. *Let X be a projective variety of dimension n. Let E be a globally generated vector bundle of rank $r \geq n$ on X such that any global section on it has a zero. Let $s \in H^0(X, E)$ be a general global section of E. Then the zero locus of s is a* smooth *subvariety of dimension $n - r$.*

Proof. Let
$$I = \{(x, s) \in X \times \mathbb{P}H^0(X, E) : s(x) = 0\}.$$

Let $p : I \to X$ and $q : I \to \mathbb{P}H^0(X, E)$ be the two projections. We claim that the fibers of p are projective linear subspaces of $\mathbb{P}H^0(X, E)$ of codimension r. In fact, since E is globally generated, for any $x \in X$, the evaluation map
$$ev_x : H^0(X, E) \to E_x \simeq \mathbb{C}^r$$
is surjective. This implies that
$$\dim I = n + \dim \mathbb{P}H^0(X, E) - r.$$
by a dimension theorem [Har77, Ex.II.3.22(c)]. On the other hand, the fact that every global section has a zero implies that the projection map q is surjective. Hence, by the algebraic Sard's theorem [Har77, Corollary III.10.7], the general fiber of q is smooth of dimension $\dim I - \dim \mathbb{P}H^0(X, E) = n - r$, as desired. □

Since $\mathrm{Gr}(3,7)$ is a smooth projective variety of dimension 12, and $\mathrm{Sym}^3\mathcal{U}^\vee$ is a globally generated vector bundle of rank 10 such that every global section has a zero, by Proposition 2.1 and Proposition 2.2, we have the following result

Corollary 2.3. *The moduli space Σ of planes in a general cubic 5-fold Y is a smooth projective surface.*

Finally, we want to study the canonical bundle of the surface Σ. We have

Proposition 2.4. *The canonical bundle of the surface Σ is ample.*

Proof. As Σ is a smooth subvariety of $\mathrm{Gr}(3,7)$, by the adjunction formula, we have
$$K_\Sigma = \left(K_{\mathrm{Gr}} \otimes \det(\mathrm{Sym}^3\mathcal{U}^\vee)\right)\big|_\Sigma,$$
where K_{Gr} is the canonical bundle of $\mathrm{Gr}(3,7)$, and $\det(\mathrm{Sym}^3\mathcal{U}^\vee)$ is the determinant line bundle associated with the vector bundle $\mathrm{Sym}^3\mathcal{U}^\vee$. The canonical bundle of the Grassmannian $\mathrm{Gr}(3,7)$ is given by:
$$K_{\mathrm{Gr}} = \varphi^*\mathcal{O}_{\mathbb{P}(\wedge^3\mathbb{C}^7)}(-7)$$
where $\phi : \mathrm{Gr}(3,7) \to \mathbb{P}(\bigwedge^3\mathbb{C}^7)$ is the Plücker embedding. Since $\det\mathcal{U}^\vee = \phi^*\mathcal{O}_{\mathbb{P}}(1)$, a formal virtual root calculation shows that $\det\mathrm{Sym}^3\mathcal{U}^\vee = \mathcal{O}_{\mathbb{P}}(10)$. Hence, $K_\Sigma = (\phi^*\mathcal{O}_{\mathbb{P}}(3))|_\Sigma$, which is the restriction of an ample line bundle, and is hence ample. □

3 Relations of Hodge numbers and Chern classes

3.1 Chern classes

Theorem 3.1 (Grothendieck [Gro58]). *There exist Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$ for any vector bundle E of rank r of any smooth algebraic variety X, which are uniquely determined by the following properties*

(i) (Vanishing) If $k > r$, then $c_k(E) = 0$.

(ii) (Whitney Sum) For a short exact sequence

$$0 \to E' \to E \to E'' \to 0,$$

the total Chern classes satisfy the property

$$c(E) = c(E')c(E'').$$

In particular, $c_1(E) = c_1(E') + c_1(E'')$.

(iii) (Projective Bundle Formula) For a vector bundle E over X, the total Chern class of the projective bundle $\mathbb{P}(E)$ is given by

$$c(\mathbb{P}(E)) = \pi^* c(E) \cdot \sum_{i=0}^{r-1} \xi^i,$$

where $\pi : \mathbb{P}(E) \to X$ is the projection, $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$, and r is the rank of E.

(iv) (Pull-back) For a morphism $f : Y \to X$ and a vector bundle E over X, the Chern classes satisfy

$$c(f^*E) = f^*c(E).$$

(v) (Normalization) If L is a line bundle on X, then its first Chern class is given by

$$c_1(L) = [D],$$

where D is the divisor of a general rational section of L.

The Chern character of a vector bundle E is another invariant of E that can be defined by combinations of Chern classes of E. The Chern character has the following expansion (dropping E from the notation for simplicity):

$$\mathrm{ch}(E) = \mathrm{rk} + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

We refer the readers to [Har77, A.4] for strict definition of the Chern characters and the following useful property.

Proposition 3.2. *Let E, E' be vector bundles on X. Then we have*

$$ch(E \otimes E') = ch(E).ch(E').$$

3.2 Kähler Manifolds and Hodge Numbers

The best way to discuss Hodge theory is through Kähler geometry. In this section, we follow closely the presentation of [Voi02, Chapter 6] to present the Hodge theory

Definition 3.3. A **Kähler manifold** is a complex manifold X equipped with a Hermitian metric whose associated 2-form is closed.

Definition 3.4. For a Kähler manifold X, the **Dolbeault cohomology groups** $H^{p,q}(X)$ consist of equivalence classes of differential forms of type (p, q), modulo $\overline{\partial}$-exact forms:

$$H^{p,q}(X) = \frac{\ker \overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)}{\text{im } \overline{\partial} : A^{p,q-1}(X) \to A^{p,q}(X)},$$

where $A^{p,q}(X)$ denotes the space of smooth differential forms of type (p, q).

Theorem 3.5 (Hodge decomposition ([Voi02]))**.** *Let X be a compact Kähler manifold. Then there is a canonical decomposition of the de Rham cohomology of X as follows.*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

with the complex conjugate identification $H^{p,q}(X) = \overline{H^{q,p}(X)}$. Furthermore, we have a canonical isomorphism $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$.

Definition 3.6. The **Hodge numbers** of a compact Kähler manifold X are defined as the dimensions of the Dolbeault cohomology groups:

$$h^{p,q}(X) = \dim H^{p,q}(X).$$

The Hodge numbers $h^{p,q}(X)$ satisfy several symmetries, notably the following:

- **Hodge Symmetry**: $h^{p,q}(X) = h^{q,p}(X)$.
- **Serre Duality**: $h^{p,q}(X) = h^{n-p,n-q}(X)$ for $n = \dim_{\mathbb{C}} X$.

For a compact Kähler manifold X of complex dimension n, the Hodge diamond (which organizes the Hodge numbers $h^{p,q}(X)$) is structured as follows:

$$\begin{array}{c}
h^{0,0} \\
h^{1,0} \qquad h^{0,1} \\
h^{2,0} \qquad h^{1,1} \qquad h^{0,2} \\
\iddots \qquad \vdots \qquad \ddots \\
h^{n-1,0} \qquad h^{n-2,1} \qquad \cdots \qquad h^{1,n-2} \qquad h^{0,n-1} \\
h^{n,0} \qquad h^{n-1,1} \qquad h^{n-2,2} \qquad \cdots \qquad h^{1,n-1} \qquad h^{0,n} \\
h^{n,n-1} \qquad h^{n-1,n-2} \qquad \cdots \qquad h^{2,1} \qquad h^{n,0} \\
\ddots \qquad \vdots \qquad \iddots \\
h^{n-2,n} \qquad h^{n-1,n-1} \qquad h^{n,n-2} \\
h^{n,n-1} \qquad h^{n-1,n} \\
h^{n,n}
\end{array}$$

It is not hard to show that any smooth projective variety over $\mathbb{C}$ is a compact Kähler manifold, and hence the whole Hodge theory applies. The key takeaway here is that the Hodge numbers can be calculated as the dimension of the cohomology of the bundles of holomorphic differential forms, whereas the latter has the potential to be fitted into exact sequences. That is exactly the approach we take for the calculations of Hodge numbers of the smooth projective surface Σ.

3.3 Euler characteristic of Σ

The following version of Hopf-Poincaré theorem, which is called the Hopf index theorem as in [BT82, Theorem 11.25], relates the Euler characteristic of a complex manifold to its top degree Chern class.

Theorem 3.7 (Hopf-Poincaré theorem)**.** *Let X be a compact complex manifold of dimension n. Let $e(X) := \sum_{i=0}^{2n}(-1)^i b_i(X)$ be the Euler characteristic of X. Let $c_n(X) \in H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$ be the top degree Chern class of X viewed as an integer. Then*

$$e(X) = c_n(X).$$

By the Hopf-Poincaré theorem, the Euler class of the surface Σ is $c_2(\Sigma)$ viewed as an integer via the identification $\int_\Sigma : H^4(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$.

3.4 Euler characteristic of $\mathcal{O}_\Sigma$

The Euler characteristic of the structure sheaf of a surface is also determined by its Chern classes, according to the Noether's formula that we discuss below.

Let S be an arbitary smooth projective surface, we have

Theorem 3.8 (Noether's formula)**.** *Let S be a smooth projective surface, then we have*

$$\chi(\mathcal{O}_S) = \frac{1}{12}(c_1(S)^2 + c_2(S)).$$

Modern algebraic geometry views Noether's formula as a special case of the Grothendieck-Riemann-Roch formula whose proof is beyond the scope of this project. More elementary proof of the Noether's formula exists in the litterature (see, for example, [GH94, Section 4.6]). Here we follow the first step in [GH94, Section 4.6] that proves Noether's formula for surfaces in $\mathbb{P}^3$. We refer the readers to [GH94, Section 4.6] for the proof of Noether's formula for a general smooth projective surface (not necessarily in $\mathbb{P}^3$). To start with, we calculate the Chern numbers of $S \subset \mathbb{P}^3$.

Lemma 3.9. *For a smooth surface S of degree d in $\mathbb{P}^3$, the first Chern class and the second Chern class of the tangent bundle T_S over S are:*

$$c_1(S) = (4-d)H.$$

$$c_2(S) = d^3 - 4d^2 + 6d.$$

Moreover, we have $c_1(S)^2 = d^3 - 8d^2 + 16d$.

Proof. Consider the short exact sequence

$$0 \to T_S \to T_{\mathbb{P}^3}|_S \to N_{S|\mathbb{P}^3} \to 0.$$

Here, T_S is the tangent bundle of S, $T_{\mathbb{P}^3}|_S$ is the restriction of the tangent bundle $T_{\mathbb{P}^3}$ of $\mathbb{P}^3$ to S, and $N_{S|\mathbb{P}^3}$ is the normal bundle of S in $\mathbb{P}^3$. From the above sequence, we have

$$c(T_{\mathbb{P}^3}|_S) = c(S) \cdot c(N_{S|\mathbb{P}^3}).$$

We know that the total Chern class of $T_{\mathbb{P}^3}$ is $c(T_{\mathbb{P}^3}) = (1+h)^4$, where h is the hyperplane class in $\mathbb{P}^3$. Restricting to S, we get

$$c(T_{\mathbb{P}^3}|_S) = (1+h|_S)^4 = (1+H)^4$$

where $H := h|_S$.

Now, the normal bundle $N_{S/\mathbb{P}^3}$ is $\mathcal{O}(d)$, so its total Chern class is

$$c(N_{S|\mathbb{P}^3}) = 1 + dH.$$

Hence we have

$$(1+H)^4 = c(S) \cdot (1+dH).$$

Expanding and comparing terms, we get

$$c_1(S) = (4-d)H,$$

$$c_2(S) = (6-4d+d^2)H^2.$$

Hence

$$c_1(S)^2 = ((4-d)H)^2 = (4-d)^2H^2 = (16-8d+d^2)H^2.$$

Since H^2 represents the self-intersection of two hyperplane sections on a smooth surface of degree d in $\mathbb{P}^3$, we have $H^2 = d$. Therefore we have

$$c_1(S)^2 = d^3 - 8d^2 + 16d.$$

and

$$c_2(S) = d^3 - 4d^2 + 6d.$$

□

Now we can prove the Noether's formula in the special case where S is a surface in $\mathbb{P}^3$.

Proof of Noether's formula when $S \subset \mathbb{P}^3$. Firstly, we consider the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0.$$

According to the property of Euler characteristic, we have

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{\mathbb{P}^3}) - \chi(\mathcal{O}_{\mathbb{P}^3}(-d)).$$

According to [Har77, Theorem II.5.1], the Euler characteristic for $\mathcal{O}_{\mathbb{P}^n}$(d) is given by

$$\chi(\mathcal{O}_{\mathbb{P}^n}(d)) = \binom{d+n}{n}.$$

In particular, we have:

$$\chi(O_{\mathbb{P}^3}) = 1,$$

$$\chi(O_{\mathbb{P}^3}(-d)) = \frac{(-d+3)(-d+2)(-d+1)}{6}.$$

Hence we have

$$\chi(\mathcal{O}_S) = 1 - \frac{(-d+3)(-d+2)(-d+1)}{6} = \frac{d^3 - 6d^2 + 11d}{6}.$$

However, by Lemma 3.9, we have

$$\frac{1}{12}(c_1(S)^2 + c_2(S)) = \frac{2d^3 - 12d^2 + 22d}{12} = \frac{d^3 - 6d^2 + 11d}{6}.$$

Therefore,

$$\chi(\mathcal{O}_S) = \frac{1}{12}(c_1(S)^2 + c_2(S)).$$

□

Applying Theorem 3.8 directly to the surface Σ, the Euler characteristic $\chi(\mathcal{O}_\Sigma)$ of the surface Σ is $\frac{1}{12}(c_1(\Sigma)^2 + c_2(\Sigma))$.

3.5 The Abel-Jacobi map of the surface

Definition 3.10. The *intermediate Jacobian* $J^k(X)$ for a smooth projective variety X is defined as

$$J^k(X) = H^{2k-1}(X, \mathbb{C})/(F^k H^{2k-1}(X) + H^{2k-1}(X, \mathbb{Z}))$$

where F^k denotes the k-th level of the Hodge filtration.

Definition 3.11. The *Albanese variety* $\mathrm{Alb}(X)$ of a smooth projective variety X is defined as:

$$\mathrm{Alb}(X) = H^0(X, \Omega_X^1)^*/H_1(X, \mathbb{Z}).$$

Definition 3.12. Let X be a smooth projective variety and x_0 is a point on X, then the *Albanese map* $\mathrm{alb} : X \to \mathrm{Alb}(X)$ is a morphism defined by integrating holomorphic 1-forms on the path connecting an arbitrary point $x \in X$ and the fixed point x_0.

The Albanese map is universal among morphisms from X to abelian varieties, namely, for any abelian variety A and any morphism $\phi : X \to A$ that sends x_0 to $0 \in A$, there is a unique morphism $\mathrm{Alb}(X) \to A$ such that ϕ factorizes through this morphism. Let $J^5(Y)$ be the intermediate Jacobian of the cubic 5-fold Y, and $\mathrm{Alb}(\Sigma)$ the Albanese variety of the surface Σ. Consider the Abel-Jacobi map

$$a : \mathrm{Alb}(\Sigma) \to J^5 Y,$$

defined by the universal property of the Albanese map of Σ.

The following theorem of Collino [Col86] is crucial in our calculations of Hodge numbers of the surface Σ.

Theorem 3.13 (Collino [Col86])**.** *The Abel-Jacobi map $a : Alb(\Sigma) \to J^5(Y)$ is an isomorphism.*

3.6 Hodge numbers as Chern classes

Theorem 3.14. *The Betti numbers of the surface Σ are as follows.*
(i) $b_0(\Sigma) = b_4(\Sigma) = 1$.
(ii) $b_1(\Sigma) = b_3(\Sigma) = 42$.
(iii) $b_2(\Sigma) = c_2(\Sigma) + 82$.

Proof. (i) is obvious since the surface Σ is connected [DM98, Théorème 2.1 (c)]. For (ii), we first prove that $b_3(\Sigma) = b_5(Y)$. By Theorem 3.13, the Abel-Jacobi map $a : \mathrm{Alb}(\Sigma) \to J^5(Y)$ is an isomorphism. This implies that the dimensions of the Albanese variety $\mathrm{Alb}(\Sigma)$ and the intermediate Jacobian $J^5(Y)$ are equal. We can compute the dimension of the Albanese variety $\mathrm{Alb}(\Sigma)$ for the surface Σ as follows

$$\dim \mathrm{Alb}(\Sigma) = \dim H^{2,1}(\Sigma, \mathbb{C}) = \frac{1}{2} \dim H^3(\Sigma, \mathbb{C}).$$

On the other hand, the intermediate Jacobian $J^5(Y)$ of the cubic 5-fold Y is defined as

$$J^5(Y) = H^{3,2}(Y, \mathbb{C}) \oplus H^{4,1}(Y, \mathbb{C}) \oplus H^{5,0}(Y, \mathbb{C}),$$

so the dimension of $J^5(Y)$ is

$$\dim(J^5(Y)) = \frac{1}{2} \dim H^5(Y, \mathbb{C}).$$

Since $\mathrm{Alb}(\Sigma) \cong J^5(Y)$, we have

$$\dim H^3(\Sigma, \mathbb{C}) = \dim H^5(Y, \mathbb{C}).$$

Next, by [Huy23, Corollary 1.12], we find $b_5(Y) = 42$. Hence, $b_3(\Sigma 0 = 42$. Finally, according to the Poincaré duality, we have $b_1(\Sigma) = b_3(\Sigma) = 42$. As for (iii), the Hopf-Poincaré theorem shows that $\sum_{i=0}^{4} (-1)^i b_i(\Sigma) = c_2(\Sigma)$. By (i) and (ii), we find $b_2(\Sigma) = c_2(\Sigma) + 82$. □

Theorem 3.15. *The Hodge numbers of the surface Σ is as follows.*
(i) $h^{0,0}(\Sigma) = h^{2,2}(\Sigma) = 1$.
(ii) $h^{1,0}(\Sigma) = h^{0,1}(\Sigma) = h^{2,1}(\Sigma) = h^{1,2}(\Sigma) = 21$.
(iii) $h^{2,0}(\Sigma) = h^{0,2}(\Sigma) = \frac{1}{12}(c_1(\Sigma)^2 + c_2(\Sigma)) + 20$.
(iv) $h^{1,1}(\Sigma) = \frac{1}{6}(5c_2(\Sigma) - c_1(\Sigma)^2) + 42$.

Proof. (i) is obvious since Σ is a connected surface. For (ii), Theorem 3.13 and the table at the end of [Huy23, Section 1.4] show that $h^{3,2}(Y) = h^{2,1}(\Sigma) = 21$. The other three Hodge numbers are also 21 by the symmetries mentioned below Defintion 3.6. As for (iii), by Hodge symmetry, $h^{2,0}(\Sigma) = h^{0,2}(\Sigma)$. Hence, it suffices to compute $h^{0,2}(\Sigma)$. By Noether's formula, we have $h^{0,0}(\Sigma) - h^{0,1}(\Sigma) + h^{0,2}(\Sigma) = \chi(\mathcal{O}_\Sigma) = \frac{1}{12}(c_1(\Sigma)^2 + c_2(\Sigma))$. Hence, by (i) and (ii), we get $h^{0,2}(\Sigma) = \frac{1}{12}(c_1(\Sigma)^2 + c_2(\Sigma)) + 20$. Finally, we notice that the Betti number b_2 for a surface Σ can be decomposed in terms of Hodge numbers as: $b_2(\Sigma) = h^{2,0}(\Sigma) + h^{0,2}(\Sigma) + h^{1,1}(\Sigma)$. As we know the value of $h^{2,0}(\Sigma)$ and $h^{0,2}(\Sigma)$ in (iii), we find that the value of $h^{1,1}(\Sigma)$ is $\frac{1}{6}(5c_2(\Sigma) - c_1(\Sigma)^2) + 42$. □

In this section, we have expressed all the Betti numbers and Hodge numbers as expressions of $c_1(\Sigma)^2$ and $c_2(\Sigma)$. Therefore, it remains to calculate $c_1(\Sigma)^2$ and $c_2(\Sigma)$ as integers. This is done in the next section.

4 Calculations of $c_1(\Sigma)^2$ and $c_2(\Sigma)$

4.1 Schubert Cycles and Schubert Calculus

This section recalls the basic theory of the Schubert calculus which is the main calculation tool for the Chern classes. The presentation here follows closely [Ful96].

Let $\mathrm{Gr}(k,n)$ denote the Grassmannian parametrizing k-dimensional subspaces of an n-dimensional vector space $\mathbb{C}^n$. The Grassmannian has a complex manifold structure. The complex dimension of $\mathrm{Gr}(k,n)$ is $k(n-k)$.

Definition 4.1. A partition $\lambda = (\lambda_1, ..., \lambda_k)$ is a weakly decreasing sequence of non-negative integers. We denote the size of λ by $|\lambda| = \sum_{i=1}^{k} \lambda_i$.

Two partitions λ and μ are said to be complentary if $\lambda_i + \mu_{k+1-i} = n-k$ for all i.

Definition 4.2. A *complete flag* F in $\mathbb{C}^n$ is a sequence of nested subspaces:

$$F_\bullet : \quad \{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n,$$

where F_i is an i-dimensional subspace of $\mathbb{C}^n$ for each i.

Definition 4.3. For a complete flag F and a partition λ, the associated Schubert cell is defined as

$$\Omega^\circ_\lambda(F) = \{V \in \mathrm{Gr}(k,n) : \dim(V \cap F_r) = i, \text{ for } n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1}, \forall i\}.$$

The associated Schubert variety with respect to F and λ is defined as

$$\Omega_\lambda(F) = \{V \in \mathrm{Gr}(k,n) : \dim(V \cap F_{n-k+i-\lambda_i}) \geq i, \ \forall i\}.$$

Definition 4.4. The fundamental class $[\Omega_\lambda(F)]$ of a Schubert variety $\Omega_\lambda(F)$ is called a Schubert class and is denoted by σ_λ. If $\lambda = (\lambda_1, 0, ..., 0)$, then we say σ_λ is a special Schubert class.

The Schubert cells give a cell complex structure on the Grassmannian $\mathrm{Gr}(k,n)$.

Example 4.5. The Grassmannian $\mathrm{Gr}(2,4)$ has a cellular decomposition

$$\mathrm{Gr}(2,4) = \Omega^\circ_{(2,2)}(F) \cup \Omega^\circ_{(2,1)}(F) \cup \Omega^\circ_{(2,0)}(F) \cup \Omega^\circ_{(1,1)}(F) \cup \Omega^\circ_{(1,0)}(F) \cup \Omega^\circ_{\emptyset}(F)$$

where the zero skeleton is the point $\Omega^\circ_{(2,2)}$.

By the general theory of cell complex, we have the following proposition.

Proposition 4.6. *As a $\mathbb{Z}$-module, the cohomology ring $H^*(Gr(k,n))$ is generated by Schubert classes*

$$H^*(Gr(k,n)) = \oplus_\lambda \mathbb{Z}\sigma_\lambda.$$

The Schubert cell $\Omega^\circ_\lambda(F_\bullet)$ is isomorphic to an affine space of dimension $\sum_{i=1}^{k}(n-k+a_i-i)$.

Now we have the following results for the ring $H^*(\mathrm{Gr}(k,n))$.

Proposition 4.7 (Poincaré duality)**.** *Let λ and μ be two partitions contained in an $k \times (n-k)$ rectangle such that $|\lambda| + |\mu| = k(n-k)$, then we have*

$$\sigma_\lambda \cup \sigma_\mu = \sigma_{\mathrm{pt}}$$

if λ and μ are complementary and

$$\sigma_\lambda \cup \sigma_\mu = 0$$

otherwise.

Proposition 4.8 (Pieri rule). *For two partitions $\mu = (\mu_1, 0, ..., 0)$ and $\lambda = (\lambda_1, .., \lambda_k)$, we have*

$$\sigma_\mu \cdot \sigma_\lambda = \sum_r \sigma_r,$$

where the sum is over all partitions r such that $|r| = |\lambda| + |\mu|$ and $\lambda_i \leq r_i \leq \lambda_{i-1}$.

Proposition 4.9 (Giambelli Formula). *Any Schubert class $\sigma_\lambda \in H^*(Gr(k, n))$ can be expressed as $\sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{1 \leq i,j \leq k}$ where we set $\sigma_p = 1$ if $p = 0$ and $\sigma_p = 0$ if $p < 0$ or $p > n - k$.*

Using the Pieri rule and the Giambelli formula, we can compute any cup product $\sigma_\lambda \cup \sigma_\mu$. First, by using the Giambelli fomula, we express σ_λ in terms of special Schubert classes, then we can compute $\sigma_\lambda \cup \sigma_\mu$ by the Pieri rule.

4.2 Chern classes of Σ as Chern classes in the Grassmannian

In this section, we show that the Chern classes of Σ can be viewed as intersections of the Chern classes of the dual of the tautological subbundle $U^\vee$ and those of the tautological quotient bundle Q on $\mathrm{Gr}(3, 7)$. The explicit expression is shown in Proposition 4.13.

Proposition 4.10. *the Chern classes of Σ can be expressed as*

$$c_1(\Sigma) = \left(c_1(U^\vee \otimes Q) - c_1(\mathrm{Sym}^3 U^\vee)\right)|_\Sigma,$$

and

$$c_2(\Sigma) = \left(c_1(\mathrm{Sym}^3 U^\vee)^2 - c_1(U^\vee \otimes Q) \cdot c_1(\mathrm{Sym}^3 U^\vee) + c_2(U^\vee \otimes Q) - c_2(\mathrm{Sym}^3 U^\vee)\right)|_\Sigma.$$

Proof. The normal sequence of $\Sigma \subset \mathrm{Gr}(3, 7)$ gives

$$0 \to T_\Sigma \to (U^\vee \otimes Q)|_\Sigma \to (\mathrm{Sym}^3 U^\vee)|_\Sigma \to 0.$$

By applying the Whitney sum formula for Chern classes, we have:

$$c((U^\vee \otimes Q)|_\Sigma) = c(T_\Sigma) \cdot c((\mathrm{Sym}^3 U^\vee)|_\Sigma).$$

Expanding this product using the definition of the total Chern class

$$c(E) = 1 + c_1(E) + c_2(E) + \dots,$$

we obtain the following relations for the first and second Chern classes. For the first Chern class, we get

$$c_1((U^\vee \otimes Q)|_\Sigma) = c_1(T_\Sigma) + c_1((\mathrm{Sym}^3 U^\vee)|_\Sigma).$$

Thus, solving for $c_1(T_\Sigma)$, we have

$$c_1(T_\Sigma) = c_1((U^\vee \otimes Q)|_\Sigma) - c_1((\mathrm{Sym}^3 U^\vee)|_\Sigma).$$

At this point, we use the fact that T_Σ is the tangent bundle of Σ, which implies that its Chern classes are exactly those of Σ. In other words

$$c(T_\Sigma) = c(\Sigma).$$

Therefore, we conclude

$$c_1(\Sigma) = c_1((U^\vee \otimes Q)|_\Sigma) - c_1((\mathrm{Sym}^3 U^\vee)|_\Sigma).$$

For the second Chern class, the Whitney sum formula gives

$$c_2((U^\vee \otimes Q)|_\Sigma) = c_2(T_\Sigma) + c_1(T_\Sigma) \cdot c_1((\mathrm{Sym}^3 U^\vee)|_\Sigma) + c_2((\mathrm{Sym}^3 U^\vee)|_\Sigma).$$

Substituting the expression for $c_1(T_\Sigma)$ from the previous step, we find

$$c_2(T_\Sigma) = c_2((U^\vee \otimes Q)|_\Sigma) - (c_1((U^\vee \otimes Q)|_\Sigma) - c_1((\mathrm{Sym}^3 U^\vee)|_\Sigma)) \cdot c_1((\mathrm{Sym}^3 U^\vee)|_\Sigma) + c_2((\mathrm{Sym}^3 U^\vee)|_\Sigma).$$

Since $c_2(T_\Sigma) = c_2(\Sigma)$, we have

$$c_2(\Sigma) = \left(c_1(\mathrm{Sym}^3 U^\vee)^2 - c_1(U^\vee \otimes Q) \cdot c_1(\mathrm{Sym}^3 U^\vee) + c_2(U^\vee \otimes Q) - c_2(\mathrm{Sym}^3 U^\vee)\right)|_\Sigma.$$

This completes the proof. □

The following theorem is well-known and its proof can be found in [BT82, (5.14) p. 51].

Theorem 4.11. *[BT82] Let X be a differential manifold of real dimension N. Let $Y \subset X$ be a closed submanifold of real codimension c. Let $\omega \in H^{N-c}(X, \mathbb{Z})$. Then*

$$\int_Y \omega|_Y = \int_X \omega \cdot [Y].$$

Theorem 4.12. *Let $Z \subset X$ be the zero locus of a vector bundle E of rank r. Then:*

$$[Z] = c_r(E) \in H^{2r}(X, \mathbb{Z}),$$

where $[Z]$ is the cohomology class represented by the zero locus Z.

Proof. By the *splitting principle*, we can assume that the bundle E can be pulled back to a space X', where it splits as a direct sum of line bundles:

$$\pi^* E = L_1 \oplus L_2 \oplus \cdots \oplus L_r \quad \text{on} \quad X',$$

where $\pi : X' \to X$ is a projection map and $L_1, L_2, \ldots, L_r$ are line bundles.

Let $s = (s_1, s_2, \ldots, s_r)$ be a section of the bundle $\pi^* E$, where each component s_i is a section of the corresponding line bundle L_i. The zero locus of s is

$$\pi^*(Z) = \mathrm{Zero}(s) = \mathrm{Zero}(s_1) \cap \mathrm{Zero}(s_2) \cap \cdots \cap \mathrm{Zero}(s_r).$$

The class of $\pi^*(Z)$ in cohomology is the product of the classes of the zero loci of the individual sections s_i

$$[\pi^*(Z)] = [\mathrm{Zero}(s_1)] \cdot [\mathrm{Zero}(s_2)] \cdot \cdots \cdot [\mathrm{Zero}(s_r)].$$

By the splitting principle, each of these classes corresponds to the first Chern class of the respective line bundle L_i

$$[\pi^*(Z)] = c_1(L_1) \cdot c_1(L_2) \cdot \cdots \cdot c_1(L_r).$$

The total Chern class of the bundle $\pi^*E = L_1 \oplus L_2 \oplus \cdots \oplus L_r$ is

$$c(\pi^*E) = c(L_1) \cdot c(L_2) \cdot \cdots \cdot c(L_r),$$

where each $c(L_i) = 1 + c_1(L_i)$ is the total Chern class of the line bundle L_i. Applying the Whitney sum formula, we have

$$c(\pi^*E) = (1 + c_1(L_1)) \cdot (1 + c_1(L_2)) \cdot \cdots \cdot (1 + c_1(L_r)).$$

The degree-r part of this product is the r-th Chern class of the bundle

$$c_r(\pi^*E) = c_1(L_1) \cdot c_1(L_2) \cdot \cdots \cdot c_1(L_r).$$

Therefore, we see that

$$[\pi^*(Z)] = c_r(\pi^*E).$$

Since $\pi^* : H^{2r}(X, \mathbb{Z}) \to H^{2r}(X', \mathbb{Z})$ is injective, we have

$$[Z] = c_r(E) \in H^{2r}(X, \mathbb{Z}).$$

This completes the proof. □

Proposition 4.13. *Viewing $c_1(\Sigma)^2$ and $c_2(\Sigma)$ as integers, we have*

$$c_1(\Sigma)^2 = \big(c_1(U^\vee \otimes Q) - c_1(\mathrm{Sym}^3 U^\vee)\big)^2 \cdot c_{10}(\mathrm{Sym}^3 U^\vee),$$

and

$$c_2(\Sigma) = \big(c_1(\mathrm{Sym}^3 U^\vee)^2 - c_1(U^\vee \otimes Q) \cdot c_1(\mathrm{Sym}^3 U^\vee) + c_2(U^\vee \otimes Q) - c_2(\mathrm{Sym}^3 U^\vee)\big) \cdot c_{10}(\mathrm{Sym}^3 U^\vee).$$

Proof. By the Theorem 4.11, we have:

$$\int_\Sigma c_1(\Sigma)^2|_\Sigma = \int_{\mathrm{Gr}(3,7)} c_1(\Sigma)^2 \cdot [\Sigma].$$

From the Theorem 4.12, since the rank of $\mathrm{Sym}^3 U^\vee$ equals to 10, we have $[\Sigma] = c_{10}(\mathrm{Sym}^3 U^\vee)$. Thus, we can substitute into the integral:

$$\int_\Sigma c_1(\Sigma)^2|_\Sigma = \int_{\mathrm{Gr}(3,7)} \big(c_1(U^\vee \otimes Q) - c_1(\mathrm{Sym}^3 U^\vee)\big)^2 \cdot c_{10}(\mathrm{Sym}^3 U^\vee).$$

This directly proves that:

$$c_1(\Sigma)^2 = \big(c_1(U^\vee \otimes Q) - c_1(\mathrm{Sym}^3 U^\vee)\big)^2 \cdot c_{10}(\mathrm{Sym}^3 U^\vee).$$

Similarly, for $c_2(\Sigma)$, we apply the Theorem 4.11

$$\int_\Sigma c_2(\Sigma)|_\Sigma = \int_{\mathrm{Gr}(3,7)} c_2(\Sigma) \cdot [\Sigma].$$

Then we have

$$\int_\Sigma c_2(\Sigma)|_\Sigma = \int_{\mathrm{Gr}(3,7)} \big(c_1(\mathrm{Sym}^3 U^\vee)^2 - c_1(U^\vee \otimes Q) \cdot c_1(\mathrm{Sym}^3 U^\vee) + c_2(U^\vee \otimes Q) - c_2(\mathrm{Sym}^3 U^\vee)\big) \cdot c_{10}(\mathrm{Sym}^3 U^\vee).$$

This proves that

$$c_2(\Sigma) = \big(c_1(\mathrm{Sym}^3 U^\vee)^2 - c_1(U^\vee \otimes Q) \cdot c_1(\mathrm{Sym}^3 U^\vee) + c_2(U^\vee \otimes Q) - c_2(\mathrm{Sym}^3 U^\vee)\big) \cdot c_{10}(\mathrm{Sym}^3 U^\vee).$$

□

4.3 Chern classes as Schubert classes on the Grassmannian

Since the Schubert classes σ_λ form a basis for $H^*(Gr, \mathbb{Z})$, we are able to represent the Chern classes of $U^\vee$ and Q as Schubert classes. In fact, it is well-known [EJ16, Section 5.6.2] that

$$c(U^\vee) = 1 + \sigma_1 + \sigma_{1,1} + \ldots + \sigma_{1,1,\ldots,1},$$

and that

$$c(Q) = 1 + \sigma_1 + \sigma_2 + \ldots + \sigma_{n-k}.$$

Proposition 4.14. *We have* $c_1(U^\vee \otimes Q) = 7\sigma_1$ *and* $c_2(U^\vee \otimes Q) = 24\sigma_{11} + 23\sigma_2$.

Proof. Using Proposition 3.2, we get

$$\mathrm{ch}(U^\vee \otimes Q) = \left(4 + \sigma_1 + \frac{1}{2}\left(\sigma_{11} + \sigma_{20} - 2\sigma_2 + \cdots\right)\right)\left(3 + \sigma_1 + \frac{1}{2}\left(\sigma_{11} + \sigma_{20} + 2\sigma_{11} + \cdots\right)\right).$$

By collecting the degree 1 factors, we have

$$c_1(U^\vee \otimes Q) = 4\sigma_1 + 3\sigma_1 = 7\sigma_1.$$

By collecting the degree 2 factors, we have

$$\frac{1}{2}c_1(U^\vee \otimes Q)^2 - 2c_2(U^\vee \otimes Q) = \frac{1}{2}(\sigma_2 - \sigma_{11}) + 2\sigma_1^2.$$

Simplifying the equation, we get

$$c_2(U^\vee \otimes Q) = 24\sigma_{11} + 23\sigma_2.$$

□

Proposition 4.15. *We have* $c_1(\mathrm{Sym}^3 U^\vee) = 10\sigma_1$, $c_2(\mathrm{Sym}^3 U^\vee) = 40\sigma_1^2 + 15\sigma_{11}$, *and* $c_{10}(\mathrm{Sym}^3 U^\vee) = 216\sigma_1^3\sigma_{11}^2\sigma_{111} + 108\sigma_1\sigma_{11}^3\sigma_{111} + 108\sigma_1^4\sigma_{111}^2 - 486\sigma_1^2\sigma_{11}\sigma_{111}^2 + 729\sigma_1\sigma_{111}^3$.

Proof. Assume the virtual roots of $U^\vee$ are l_1, l_2, l_3. The virtual roots of $\mathrm{Sym}^3 U^\vee$ are given by $l_{i_1} + l_{i_2} + l_{i_3}$, where the indices satisfy $1 \leq i_1 \leq i_2 \leq i_3 \leq 3$. The number of terms is

$$\binom{3+2}{3} = \binom{5}{3} = 10.$$

Thus, the total Chern class of $\mathrm{Sym}^3 U^\vee$ is

$$c(\mathrm{Sym}^3 U^\vee) = \prod_{1 \leq i_1 \leq i_2 \leq i_3 \leq 3} (1 + l_{i_1} + l_{i_2} + l_{i_3}),$$

leading to the expansion into individual Chern classes

$$c(\mathrm{Sym}^3 U^\vee) = 1 + c_1(\mathrm{Sym}^3 U^\vee) + c_2(\mathrm{Sym}^3 U^\vee) + \cdots + c_{10}(\mathrm{Sym}^3 U^\vee).$$

(i) The first Chern class is sum of the symmetric polynomials of degree 1

$$c_1(\mathrm{Sym}^3 U^\vee) = \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq 3} (1 + l_{i_1} + l_{i_2} + l_{i_3}).$$

This simplifies to

$$c_1(\mathrm{Sym}^3 U^\vee) = 10(l_1 + l_2 + l_3).$$

Since $c_1(U^\vee) = l_1 + l_2 + l_3 = \sigma_1$, we have

$$c_1(\mathrm{Sym}^3 U^\vee) = 10\sigma_1.$$

(ii) The second Chern class is sum of the symmetric polynomials of degree 2 in l_1, l_2, l_3, which takes the form

$$c_2(\mathrm{Sym}^3 U^\vee) = a \cdot (l_1 + l_2 + l_3)^2 + b \cdot (l_1 l_2 + l_2 l_3 + l_3 l_1),$$

where a and b are constants. This simplifies to

$$c_2(\mathrm{Sym}^3 U^\vee) = a \cdot \sigma_1^2 + b \cdot \sigma_{11}.$$

The coefficients a and b can be determined as follows. The coefficient a is the term of l_1^2. Among the 10 terms $l_{i_1} + l_{i_2} + l_{i_3}$ in the product, there are 4 terms with no l_1, 3 terms with 1 l_1, 2 terms with 2 l_1's and 1 term with 3 l_1's. Hence, the coefficient of l_1 in the expansion is $\binom{3}{2} \times 1 \times 1 + \binom{2}{2} \times 2 \times 2 + 3 \times 2 \times 1 \times 2 + 3 \times 1 \times 1 \times 3 + 2 \times 1 \times 2 \times 3 = 40$. Therefore, $a = 40$. By similar counting methods, the coefficient of the term $l_1 l_2$ in the expansion is 95. Hence, $b = 95 - 2a = 15$. Therefore, we get

$$c_2(\mathrm{Sym}^3 U^\vee) = 40\sigma_1^2 + 15\sigma_{11}.$$

(iii) The degree 10 part of the expansion of $\prod_{1 \le i_1 \le i_2 \le i_3 \le 3}(1 + l_{i_1} + l_{i_2} + l_{i_3})$ is simply

$$\prod_{1 \le i_1 \le i_2 \le i_3 \le 3} (l_{i_1} + l_{i_2} + l_{i_3}).$$

To express $c_{10}(\mathrm{Sym}^3 U^\vee)$ as Chern classes of $U^\vee$, it suffices to express the symmetric polynomial $\prod_{1 \le i_1 \le i_2 \le i_3 \le 3}(l_{i_1} + l_{i_2} + l_{i_3})$ as elementary symmetric polynomials. We do so using Mathematica with the following code

```
eu = (3 l1) (3 l2) (3 l3) (2 l1 + l2) (2 l1 + l3) (2 l2 + l3) (2 l2 +
 l1) (2 l3 + l1) (2 l3 + l2) (l1 + l2 + l3);
SymmetricReduction[eu, Variables[eu], {c1, c2, c3}]
```

The output is

```
{216 c1^3 c2^2 c3 + 108 c1 c2^3 c3 + 108 c1^4 c3^2 - 486 c1^2 c2 c3^2 +
729 c1 c3^3, 0}
```

This terminates the calculation. □

4.4 Explicit Schubert calculus on $\mathrm{Gr}(3, 7)$

Lemma 4.16. *We have the following results*

(i) $\sigma_1^6 \cdot \sigma_{111}^2 = 5$*;*
(ii) $\sigma_1^5 \cdot \sigma_{11}^2 \cdot \sigma_{111} = 11$*;*
(iii) $\sigma_1^3 \cdot \sigma_{111} \cdot \sigma_{11}^3 = 6$*;*
(iv) $\sigma_1^4 \cdot \sigma_{11} \cdot \sigma_{111}^2 = 3$*;*
(v) $\sigma_1^3 \cdot \sigma_{111}^3 = 1$*;*

(vi) $\sigma_1 \cdot \sigma_{11}^4 \cdot \sigma_{111} = 3$*;*
(vii) $\sigma_1^2 \cdot \sigma_{11}^2 \cdot \sigma_{111}^2 = 2$*;*
(viii) $\sigma_1 \cdot \sigma_{11} \cdot \sigma_{111}^3 = 1$.

Proof. (i) We start with

$$\sigma_1^6 \cdot \sigma_{111} = \sigma_1^5 \cdot \sigma_1 \cdot \sigma_{111}$$

Using Pieri's rule, we proceed step by step

$$\begin{aligned}
\sigma_1^5 \cdot \sigma_{211} &= \sigma_1^4 \cdot (\sigma_{311} + \sigma_{221}) \\
&= \sigma_1^3 \cdot (\sigma_{411} + 2\sigma_{321} + \sigma_{222}) \\
&= \sigma_1^2 \cdot (3\sigma_{421} + 3\sigma_{322} + 2\sigma_{331}) \\
&= \sigma_1 \cdot (5\sigma_{431} + 6\sigma_{422} + 5\sigma_{332}) \\
&= \sigma_1 \cdot (5\sigma_{441} + 16\sigma_{432} + 5\sigma_{333}).
\end{aligned}$$

Finally, applying Poincaré duality, we calculate

$$\sigma_1^6 \cdot \sigma_{111} \cdot \sigma_{111} = (5\sigma_{441} + 16\sigma_{432} + 5\sigma_{333}) \cdot \sigma_{111} = 5 \cdot \sigma_{333} \cdot \sigma_{111} = 5.$$

(ii) We start with

$$\sigma_1^6 \cdot \sigma_{111}.$$

Using Pieri's rule, we proceed step by step

$$\begin{aligned}
\sigma_1^6 \cdot \sigma_{111} &= \sigma_1^5 \cdot \sigma_1 \cdot \sigma_{111} \\
&= \sigma_1^5 \cdot \sigma_{211} = \sigma_1^4 \cdot \sigma_1 \cdot \sigma_{211} \\
&= \sigma_1^3 \cdot \sigma_1 \cdot (\sigma_{311} + \sigma_{221}) \\
&= \sigma_1^3 \cdot (\sigma_{411} + 2\sigma_{321} + \sigma_{222}) \\
&= \sigma_1^2 \cdot \sigma_1 \cdot (\sigma_{411} + 2\sigma_{321} + \sigma_{222}) \\
&= \sigma_1^2 \cdot (\sigma_{421} + 3\sigma_{421} + 2\sigma_{331} + 2\sigma_{322} + \sigma_{322}) \\
&= \sigma_1 \cdot \sigma_1 \cdot (3\sigma_{431} + 3\sigma_{422} + 3\sigma_{332} + 2\sigma_{331}) \\
&= \sigma_1 \cdot (3\sigma_{431} + 3\sigma_{422} + 3\sigma_{332} + 2\sigma_{331}) \\
&= \sigma_1 \cdot (5\sigma_{431} + 6\sigma_{422} + 5\sigma_{332}).
\end{aligned}$$

Finally, applying Poincaré duality, we calculate

$$\sigma_1^5(\sigma_{22} + \sigma_{211}) \cdot \sigma_{111} = (15\sigma_{441} + 36\sigma_{432} + 11\sigma_{333}) \cdot \sigma_{111}.$$

This results in:

$$\sigma_1^5(\sigma_{22} + \sigma_{211}) \cdot \sigma_{111} = 11.$$

(iii) By the Giambelli formula, we have

$$\sigma_{11} = \det\left(\sigma_{\lambda_i+j-i}\right) = \det \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_0 & \sigma_1 & \sigma_2 \\ \sigma_{-2} & \sigma_{-1} & \sigma_0 \end{pmatrix} = \det \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & \sigma_1 & \sigma_2 \\ 0 & 0 & 1 \end{pmatrix} = \sigma_1^2 - \sigma_2.$$

Then we have

$$\begin{aligned}\sigma_{11}^2 &= \sigma_{11} \cdot (\sigma_1^2 - \sigma_2) \\ &= \sigma_{11} \cdot \sigma_1 \cdot \sigma_1 - \sigma_{11} \cdot \sigma_2 \\ &= (\sigma_{21} + \sigma_{111}) \cdot \sigma_1 - (\sigma_{31} + \sigma_{211}) \\ &= \sigma_{31} + \sigma_{22} + \sigma_{211} + \sigma_{211} - \sigma_{31} - \sigma_{211} \\ &= \sigma_{22} + \sigma_{211}.\end{aligned}$$

Now, for the cube of σ_{11}, we have

$$\begin{aligned}\sigma_{11}^3 &= \sigma_{11}^2 \cdot \sigma_{11} \\ &= (\sigma_{22} + \sigma_{211}) \cdot (\sigma_1^2 - \sigma_2) \\ &= (\sigma_{22} + \sigma_{211}) \cdot \sigma_1^2 - (\sigma_{22} + \sigma_{211}) \cdot \sigma_2 \\ &= \sigma_1 \cdot \sigma_1 \cdot (\sigma_{22} + \sigma_{211}) - \sigma_2 \cdot (\sigma_{22} + \sigma_{211}) \\ &= \sigma_1 \cdot (\sigma_{32} + \sigma_{311} + \sigma_{221}) - (\sigma_{42} + \sigma_{321} + \sigma_{222} + \sigma_{411} + \sigma_{321}) \\ &= (\sigma_{42} + \sigma_{33} + \sigma_{321} + \sigma_{222} + \sigma_{411} + \sigma_{321}) - (\sigma_{42} + \sigma_{321} + \sigma_{222} + \sigma_{411} + \sigma_{321}) \\ &= \sigma_{33} + 2\sigma_{321} + \sigma_{222}.\end{aligned}$$

Next, we compute

$$\begin{aligned}\sigma_1^3 \cdot \sigma_{111} &= \sigma_1 \cdot \sigma_1 \cdot \sigma_1 \cdot \sigma_{111} \\ &= \sigma_1 \cdot \sigma_1 \cdot \sigma_{211} \\ &= \sigma_1 \cdot (\sigma_{311} + \sigma_{221}) \\ &= \sigma_1 \cdot \sigma_{311} + \sigma_1 \cdot \sigma_{221} \\ &= \sigma_{411} + \sigma_{321} + \sigma_{321} + \sigma_{222} \\ &= \sigma_{411} + 2\sigma_{321} + \sigma_{222}.\end{aligned}$$

Then, by applying Poincaré duality, we calculate

$$\begin{aligned}\sigma_1^3 \cdot \sigma_{111} \cdot \sigma_{11}^3 &= (\sigma_{411} + 2\sigma_{321} + \sigma_{222}) \cdot (\sigma_{33} + 2\sigma_{321} + \sigma_{222}) \\ &= \sigma_{411} \cdot \sigma_{33} + 2\sigma_{321} \cdot 2\sigma_{321} + \sigma_{222} \cdot \sigma_{222} \\ &= 1 + 4 + 1 = 6.\end{aligned}$$

(iv) We start with

$$\sigma_1^4 \cdot \sigma_{11} \cdot \sigma_{111}^2 = (\sigma_1^2 \cdot \sigma_{11} \cdot \sigma_{111}) \times (\sigma_1^2 \cdot \sigma_{111})$$

Note that we can't use the Pieri rule for $\sigma_{11} \cdot \sigma_{11}$. We know

$$\sigma_{11} = \sigma_1^2 - \sigma_2$$

Now,

$$(\sigma_1^2 - \sigma_2) \cdot \sigma_{111} = \sigma_1 \cdot \sigma_1 \cdot \sigma_{111} - \sigma_2 \cdot \sigma_{111} = \sigma_1 \cdot \sigma_{211} - \sigma_{311} = \sigma_{221}$$

Thus, we have

$$\sigma_1^2\left(\sigma_1^2-\sigma_2\right)\cdot\sigma_{111}=\sigma_1\cdot(\sigma_{321}+\sigma_{222})=\sigma_{421}+\sigma_{331}+\sigma_{322}=\sigma_{421}+\sigma_{331}+2\sigma_{322}$$

Next, we compute

$$\sigma_1^2\cdot\sigma_{111}=\sigma_1\cdot\sigma_{211}=\sigma_{311}+\sigma_{221}$$

Now, expanding the expression

$$\begin{aligned}(\sigma_1^2\cdot\sigma_{11}\cdot\sigma_{111})\times(\sigma_1^2\cdot\sigma_{111})&=\left[\sigma_1^2(\sigma_1^2-\sigma_2)\cdot\sigma_{111}\right]\cdot(\sigma_1^2\cdot\sigma_{111})\\&=(\sigma_{421}+\sigma_{331}+2\sigma_{322})\cdot(\sigma_{221}+\sigma_{311}).\end{aligned}$$

Finally, by applying Poincaré duality, we get

$$\sigma_1^4\cdot\sigma_{11}\cdot\sigma_{111}^2=3.$$

(v) We start with the expression

$$\sigma_{111}=\det\left(\sigma_{\lambda_i+j-i}\right)_{1\le i,j\le 3}=\det\begin{pmatrix}\sigma_1&\sigma_2&\sigma_3\\\sigma_0&\sigma_1&\sigma_2\\\sigma_{-1}&\sigma_0&\sigma_1\end{pmatrix}=\sigma_1^3+\sigma_3-2\sigma_1\sigma_2.$$

Next, we compute

$$\begin{aligned}\sigma_{111}\cdot\sigma_{111}&=\left(\sigma_1^3+\sigma_3-2\sigma_1\sigma_2\right)\cdot\sigma_{111}\\&=\sigma_1^3\cdot\sigma_{111}+\sigma_3\cdot\sigma_{111}-2\sigma_1\sigma_2\cdot\sigma_{111}\\&=\sigma_1\cdot\sigma_1\cdot\sigma_{211}+\sigma_{411}-2\sigma_{321}-2\sigma_{411}\\&=\sigma_1\left(\sigma_{311}+\sigma_{221}\right)-2\sigma_{321}-2\sigma_{411}\\&=\sigma_{421}+\sigma_{331}+\sigma_{222}-2\sigma_{321}-\sigma_{411}\\&=\sigma_{411}+\sigma_{321}+\sigma_{321}+\sigma_{222}-2\sigma_{321}-\sigma_{411}\\&=\sigma_{222}.\end{aligned}$$

Thus, we conclude that

$$\sigma_{111}^2=\sigma_{222}.$$

Finally, by applying Poincaré duality, we have

$$\sigma_1^3\cdot\sigma_{111}^3=\sigma_1^3\cdot\sigma_{111}\cdot\sigma_{111}^2=(\sigma_{411}+2\sigma_{321}+\sigma_{222})\cdot\sigma_{222}=1.$$

(vi) We start with

$$\begin{aligned}\sigma_1\sigma_{11}^4\sigma_{111}&=\sigma_1^4\sigma_{211}\\&=\sigma_{11}^2(\sigma_1^2-\sigma_2)\sigma_{11}\sigma_{211}\\&=\sigma_{11}^2(\sigma_{22}+\sigma_{211})\sigma_{211}\\&=\sigma_{11}\left((\sigma_1^2-\sigma_2)(\sigma_{22}+\sigma_{211})\sigma_{221}\right)\\&=\sigma_{11}(\sigma_{321}+\sigma_{321}+\sigma_{33}+\sigma_{222})\sigma_{222}\\&=(\sigma_{332}+\sigma_{332}+\sigma_{332})\sigma_{211},\end{aligned}$$

By using Poincaré duality, we obtain

$$\sigma_1\sigma_{11}^4\sigma_{11} = 3.$$

(vii) We start with

$$\begin{aligned}\sigma_1^2\sigma_{11}^2\sigma_{111}^2 &= \sigma_1^2\sigma_{11}^2\sigma_{222}\\ &= \sigma_1^2(\sigma_1^2 - \sigma_2)\sigma_{11}\sigma_{222}\\ &= \sigma_1^2(\sigma_{22} + \sigma_{221})\sigma_{222}\\ &= (\sigma_{321} + \sigma_{222} + \sigma_{321} + \sigma_{222})\sigma_{222}.\end{aligned}$$

By applying Poincaré duality, we have

$$\sigma_1^2\sigma_{11}^2\sigma_{111}^2 = 2.$$

(viii) We start with

$$\sigma_1\sigma_{11}\sigma_{111}^3 = \sigma_1(\sigma_1^2 - \sigma_2)\sigma_{111}\sigma_{222} = \sigma_1 \cdot \sigma_{221} \cdot \sigma_{222} = (\sigma_{321} + \sigma_{222})\sigma_{222}.$$

By applying Poincaré duality, we conclude

$$\sigma_1\sigma_{11}\sigma_{111}^3 = 1.$$

□

4.5 Final results

Theorem 4.17. *Under the canonical identification $\int : H^4(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$, we have*

$$c_1(\Sigma)^2 = 25515$$

$$c_2(\Sigma) = 13041$$

Proof. By Proposition 4.13, Proposition 4.14, Proposition 4.15 and Lemma 4.16, we get

$$\begin{aligned}c_1(\Sigma)^2 &= (7\sigma_1 - 10\sigma_1)^2 \cdot \left(216\sigma_1^3\sigma_{11}^2\sigma_{111} + 108\sigma_1\sigma_{11}^3\sigma_{111} + 108\sigma_1^4\sigma_{111}^2 - 486\sigma_1^2\sigma_{11}\sigma_{111}^2 + 729\sigma_1\sigma_{111}^3\right)\\ &= 1944\sigma_1^3\sigma_{11}^2\sigma_{111} + 972\sigma_1\sigma_{11}^3\sigma_{111} + 972\sigma_1^4\sigma_{111}^2 - 4374\sigma_1^2\sigma_{11}\sigma_{111}^2 + 6561\sigma_1\sigma_{111}^3\\ &= 25515,\end{aligned}$$

and

$$\begin{aligned}c_2(\Sigma) =&(100\sigma_1^2 - 70\sigma_1^2 + (23\sigma_1^2 + \sigma_{11}) - (40\sigma_1^2 + 15\sigma_{11}))\\ &\times \left(216\sigma_1^3\sigma_{11}^2\sigma_{111} + 108\sigma_1\sigma_{11}^3\sigma_{111} + 108\sigma_1^4\sigma_{111}^2 - 486\sigma_1^2\sigma_{11}\sigma_{111}^2 + 729\sigma_1\sigma_{111}^3\right)\\ =&2808\sigma_1^5\sigma_{11}^2\sigma_{111} + 1404\sigma_1^3\sigma_{11}^3\sigma_{111} + 1404\sigma_1^6\sigma_{111}^2 - 6318\sigma_1^4\sigma_{11}\sigma_{111}^2 + 9477\sigma_1^3\sigma_{111}^3\\ &- 3024\sigma_1^3\sigma_{11}^3\sigma_{111} - 1512\sigma_1\sigma_{11}^4\sigma_{111} - 1512\sigma_1^4\sigma_{11}\sigma_{111}^2 + 6804\sigma_1^2\sigma_{11}^2\sigma_{111}^2 - 10206\sigma_1\sigma_{11}\sigma_{111}^3\\ =&13041.\end{aligned}$$

□

Combining Theorem 4.17 and Theorem 3.14, we finally get

Theorem 4.18. *The Betti numbers of the surface* Σ *are as follows*
(i) $b_0(\Sigma) = b_4(\Sigma) = 1$.
(ii) $b_1(\Sigma) = b_3(\Sigma) = 42$.
(iii) $b_2(\Sigma) = 13123$.

Similarly, combining Theorem 4.17 and Theorem 3.15, we get

Theorem 4.19. $h^{2,0}(\Sigma) = 3233$. $h^{1,1}(\Sigma) = 6657$.

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Acknowledgements

I would like to express my sincere gratitude to my mentors, Meng Zhang and Chenyu Bai, who have profoundly influenced my appreciation of mathematics. Throughout my mathematical journey, they have provided invaluable suggestions that have significantly enhanced the presentation of this work. I am particularly indebted to Meng Zhang for instilling in me a deep passion for mathematics. I was also fortunate to have met Chenyu Bai, who enriched my understanding of algebraic geometry and recommended extensive reading materials. Their selfless and unwavering support and guidance were instrumental in enabling me to complete the intricate computations in this article. Additionally, I would like to thank the Yau Competition Committee for providing me with the opportunity to delve into pure mathematics and to explore this new and fascinating world.