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论文题目：“颗粒壶铃”现象的产生机制及其影响因素研究

“颗粒壶铃”现象的产生机制及其影响因素研究

周可欣、刘福凤

摘要：为探究装有颗粒物质的容器，即“颗粒壶铃”被浸入物体提起的原理及条件，查阅已有研究成果并推导了颗粒物质和浸入物体作用方式，即摩擦。设计了实验装置及探究方法使颗粒物质与浸入物体间的摩擦力能转化为重力求出。理论推导了颗粒物质与浸入物体间摩擦力随颗粒物质半径、容器直径、浸入物体底面半径、颗粒物质堆积高度的影响，进而探究“颗粒壶铃”能被提起的条件。结果表明：颗粒物质半径与摩擦力呈负相关，容器直径与摩擦力呈开口向上的二次函数关系、浸入物体底面半径与摩擦力呈正相关、颗粒物质堆积高度影响系统重力。

关键词：颗粒物质、摩擦力、Janssen 模型

目录

第一章 引言

基于 2023 年 IYPT 12 题 Rice Kettlebells 大米壶铃赛题，引申出“颗粒壶铃”概念，即在一个容器内倒入一些颗粒状物质，将该容器称之为“颗粒壶铃”。其中，颗粒物质是指直径大于 1μm 的大量离散的固体颗粒相互作用而组成的复杂体系，是常见的物态形式，也是地球上广泛存在的、为人们所熟悉的物质类型之一。颗粒物质有许多不同于固体、流体、气体的性质，由于颗粒体系通常是远离平衡态的非平衡态，即使在静态堆积时其状态结构也处于亚稳态，具有典型的非线性、自组织、能量耗散等特征。

而当将筷子等物体浸入颗粒物质中时，在一定的浸入深度下可以通过提起筷子来提起容器及其中的颗粒。该现象被称为“颗粒壶铃”现象。针对这个现象，已有相关研究对容器底部所承受的压强进行了分析。如 1884 年，英国科学家 Roberts 首次观察到：粮仓底部所承受的压力在粮食堆积高度大于两倍底面直径后趋于饱和而不再随质量增加而增加，这就是粮仓效应。粮仓效应展现了颗粒物质在静力学方面不同于流体的性质。而 1895 年，德国工程师 Janssen 所提出的连续介质模型简化了颗粒物质内部复杂的相互作用，巧妙地解释了粮仓效应的产生：由于颗粒间的相互作用力，重力方向的力被分解到水平方向，粮仓边壁承担了颗粒的部分重量，使粮仓底部压强趋于饱和。

根据已有研究可以发现，“颗粒壶铃”现象的本质是颗粒物质与浸入物体的相互作用与重力的平衡，而该现象的产生也受相关参数影响。

大自然中的许多灾害也与颗粒物质性质紧密联系，如雪崩、泥石流、山体滑坡、沙尘暴等。同时，日常生活中有许多生产生活活动都与颗粒物质相关，如粮食的堆积、铁路路基、建筑及斜拉式大桥地基与土地的相互作用等。

目前，人们对颗粒物质的产生，运输，储存等方面进行了一定研究，但处理、控制颗粒物质、在颗粒物质与其他物质相互作用机理仍未得到系统性充分的发展。而在今天，随着人们对建筑物安全性要求的增高与对自然灾害防护治理的需要，研究颗粒物质静力学及其与其他物质相互作用原理有着重要意义。本研究不仅有助于深化对颗粒物质基本性质的理解，还可能为颗粒材料的设计、加工和应用提供新的思路和方法。

基于前人的研究，我们对“颗粒壶铃”现象在颗粒物质半径、容器直径、浸入物体底面半径、颗粒物质堆积高度等方面进行延伸探究。

第二章 “颗粒壶铃”理论模型

2.1 Janssen 模型

Janssen 模型首先提出以下假设：

1.圆筒内竖直压强在同一水平面上处处相等。

2.水平方向压强和竖直方向压强成正比。

即：

$$p_x = kp_z \quad \text{(1.1)}$$

其中，k 为应力转向系数，随颗粒物质堆积高度增加而增加。

3.颗粒与容器壁的摩擦力达到最大静摩擦力。

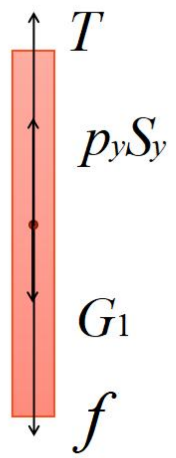
经过实验与理论研究，得出粮仓高度为 z 处的压强 p 的表达式为：

$$p_z = \frac{\rho g D}{4\mu k}\left[1 - \exp\left(-\frac{4\mu k H}{D}\right)\right] \quad \text{(1.2)}$$

其中，ρ为颗粒物质密度、g 为重力加速度、D 为容器内径、μ为筷子与容器壁之间的动摩擦系数、k 为应力转向系数、H 为颗粒物质堆积高度。

2.2 浸入物体与整体的受力分析模型

为研究“颗粒壶铃”现象的产生及相关参数的影响，现对该系统建立理论分析与受力模型。将颗粒物质抽象为一粒粒球形颗粒，容器为直筒圆柱型容器，浸入物质为一细长圆柱体。对浸入物体受力分析有：



(图 2.1)

T 为浸入物体所受拉力，p_y为浸入物体底部所受压强、S 为浸入物体底面积、G_1为浸入物体重力、f 为颗粒物质与浸入物体之间摩擦力

由浸入物体受力平衡有：

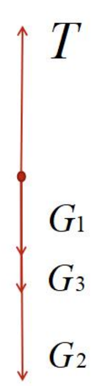
$$T + p_z S = f + G_1 \cdots\cdots (1.3)$$

若颗粒物质与浸入物体之间摩擦力达到最大静摩擦力，最大静摩擦力约等于滑动摩擦力，则：

$$f = \mu\sigma_x \cdots\cdots (1.4)$$

μ 为颗粒物质与浸入物体之间的动摩擦系数、σ_x 为浸入物体所受颗粒物质水平方向应力。

对整个系统受力分析有：



(图 2.2)

G_2 为颗粒物质重力、G_3 为容器重力

由系统受力平衡有：

$$T = G_1 + G_2 + G_3 \quad \cdots\cdots(1.5)$$

2.3 摩擦力求解

我们将浸入物体抽象为圆柱形棒。对浸入物体所受水平应力积分得 0 到 H 上的水平应力为：

$$\sigma_x = \int_0^H p_x S_x dy \quad \cdots\cdots(1.6)$$

由(1.1)(1.2)(1.4)(1.6)整理可得：

$$f = 2\mu k \pi r \int_0^H p_z dy \quad \cdots\cdots(1.7)$$

其中 r 为浸入物体底面半径

带入、化简求解可得：

$$f = 2\mu k \pi r \int_0^H \frac{\rho g D}{4\mu k}(1 - e^{-\frac{4\mu k y}{D}})dy$$

$$= \frac{\pi r \rho g D}{2} H - \frac{\pi r \rho g D}{2} \int_0^H e^{-\frac{4\mu k y}{D}} dy$$

$$= \frac{\pi r \rho g D}{2} H - \frac{\pi r \rho g D}{2} \bullet \frac{D}{4\mu k} e^{-\frac{4\mu k y}{D}} \Big|_H^0$$

$$= \frac{\pi r \rho g D}{2} H - \frac{\rho g \pi r D^2}{8\mu k}(1 - e^{-\frac{4\mu k y}{D}})$$

即：

$$f=\frac{\pi r\rho gD}{2}H-\frac{\rho g\pi rD^2}{8\mu k}(1-e^{-\frac{4\mu kH}{D}})\cdots\cdots\cdots\cdots\cdots\cdots\textbf{(1.8)}$$

当 $\frac{4\mu kH}{D}>>1$ 时，可解得：

$$\begin{aligned}f&\approx\frac{\pi r\rho gDH}{2}-\frac{\rho g\pi rD^2}{8\mu k}\\&=\frac{\pi r\rho gD}{2}(H-\frac{D}{4\mu k})\end{aligned}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\textbf{(1.9)}$$

将其转化为二次函数关系式：

$$f=\frac{\pi r\rho g}{8\mu k}[-(D-2\mu k)^2+4\mu^2k^2]\cdots\cdots\cdots\cdots\cdots\textbf{（1.10）}$$

由(1.10)可得，存在一个最适容器内径 D_0，使摩擦力最大：

$$D_0=2\mu k\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\textbf{（1.11）}$$

由(1.3)可得：

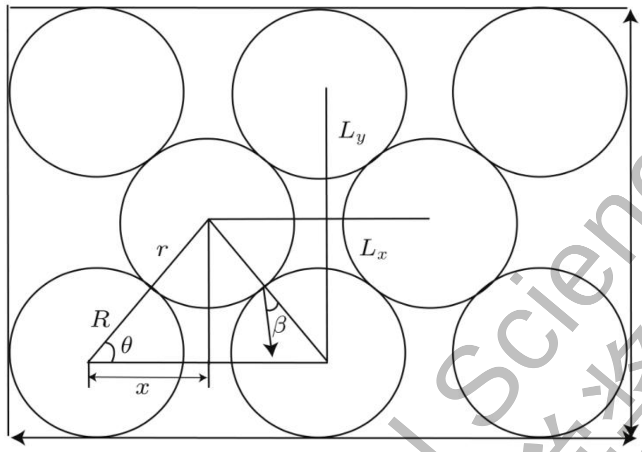
$$f=T+p_zS-G_1\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\textbf{（1.12）}$$

由(1.5)(1.10)(1.12)，要使“颗粒壶铃”刚好能被提起，需满足：

$$p_zS+G_2+G_3=\frac{\pi r\rho g}{8\mu k}[-(D-2\mu k)^2+4\mu^2k^2]\cdots\cdots\cdots\cdots\textbf{（1.13）}$$

2.4 对于应力转向系数 k 的探究

据参考文献[1]，我们将颗粒物质抽象为下图所示的颗粒元胞，其堆积方式大致如下：



（图 2.3）

在颗粒物质的堆积结构中，设颗粒之间实际接触力F与颗粒接触方向之间的夹角为β，则力 F 在水平方向上的分力为：

$$f_x = F \bullet \cos(\theta + \beta) = \sigma_x \bullet L_y \cdots\cdots\cdots\cdots\cdots \quad (1.14)$$

F 在竖直方向上的分力为：

$$f_y = F \bullet \sin(\theta + \beta) = \sigma_y \bullet L_x \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1.15)$$

其中 L_x 和 L_y 分别为颗粒元胞的长和宽。

因此：

$$\frac{L_y}{L_x} = \tan\theta \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1.16)$$

在颗粒相互挤压的过程中，设颗粒之间的摩擦力达到最大静摩擦力，由静态体系的合外力为零可列方程：

$$F \bullet \sin\beta = \mu \bullet F \bullet \cos\beta \qquad (1.17)$$

其中μ为颗粒之间的摩擦因数，得：

$$\beta = \arctan\mu \cdots\cdots\cdots\cdots\cdots\cdots \quad (1.18)$$

因此，应力转向系数为：

$$k = \frac{\sigma_x}{\sigma_y} = \frac{1}{\tan(\theta+\beta)\bullet\tan\theta} \cdots\cdots\cdots\cdots \quad (1.19)$$

同时，我们将球形颗粒物质的堆积方式看做最密堆积，此时下层紧挨的三个球形颗粒与其形成的三角形空隙上方堆积的一个球形颗粒，四者球心构成正四面体，易得:

$$tan\theta = \sqrt{2}$$

第三章 实验探究

3.1 实验装置、目的及内容

为探究“颗粒壶铃”提起的条件及影响因素，我们有如下实验器材：直径为2mm、3mm、4mm的磨砂玻璃珠、不同内径的直筒塑料杯、不同直径的圆柱形木棒、十分度的游标卡尺与电子游标卡尺、电子秤（精确度0.1g）、力传感器及数据采集器。

（图 2.4）

实验器材

实验设计为：采用多次实验，逐步找寻能提起“颗粒壶铃”所需的浸入物体最小浸入深度。此时提起系统重力与摩擦力平衡，可得出此时颗粒物质与浸入物体之间的摩擦力与侧壁压力成正比，侧壁压力与浸入侧面积成正相关，因而可通过最小浸入深度反应。改变实验变量进行多次实验验证理论的成立，计算其误差大小。

3.2 参数测量

⑴颗粒物质和木棒的动摩擦系数，由力传感器测量拉力，根据公式间接计算动摩擦系数。

$$F_{拉} = \mu mg \quad (1.20)$$

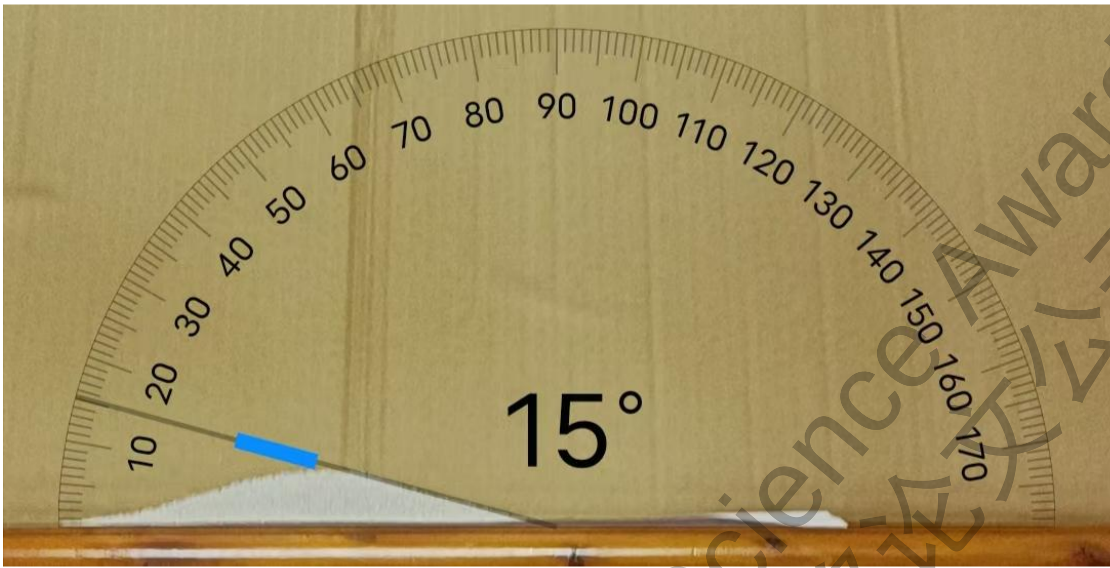
（图 2.5）

摩擦系数测量

经过多次测量取平均值后，测得颗粒物质与木棒之间动摩擦系数μ为 0.68。

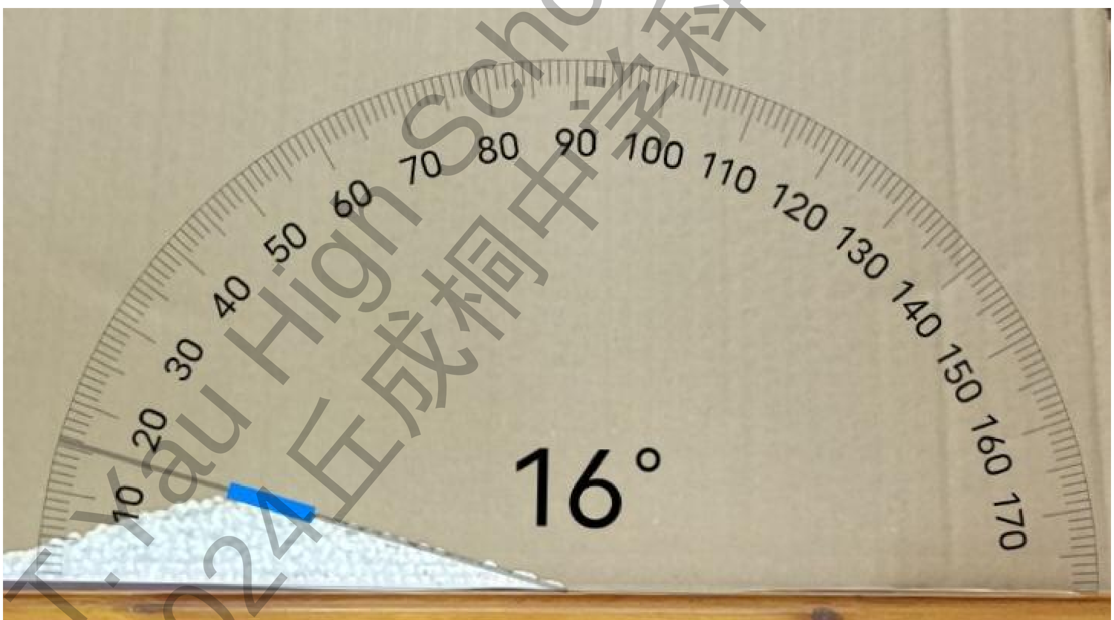
⑵颗粒物质间动摩擦系数，先测得堆积夹角θ，再根据公式$\mu = \tan\theta$间接计算动摩擦系数。

颗粒物质直径 4mm：



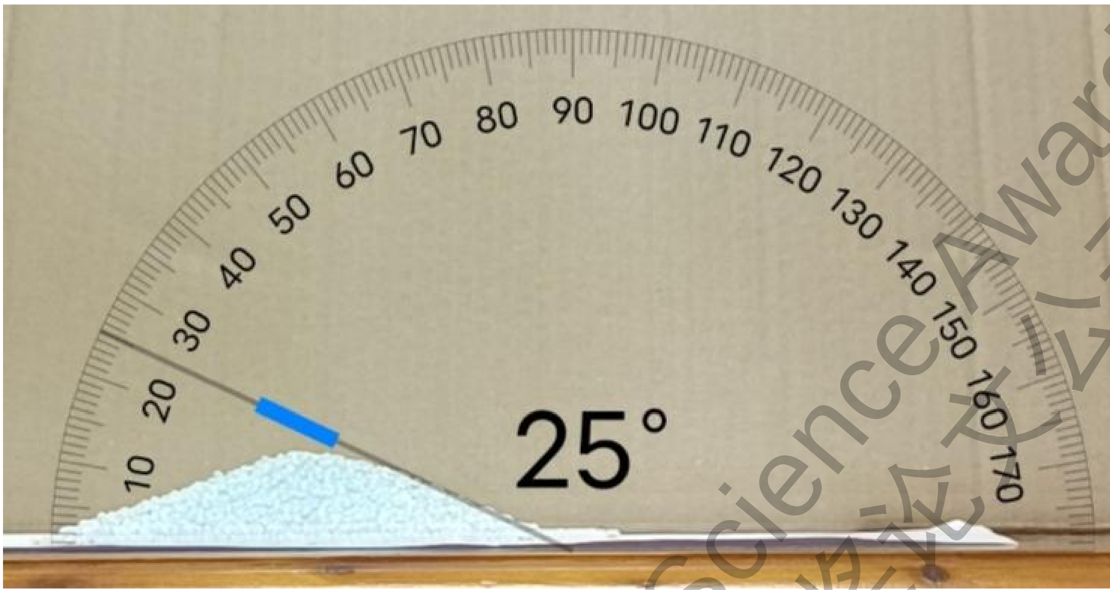
（图 2.6）

颗粒物质直径 3mm：



（图 2.7）

颗粒物质直径 2mm：



（图 2.8）

经过多次测量取平均值后，测得颗粒物质间动摩擦系数为：颗粒物质直径4mm:0.27;颗粒物质直径 3mm：0.29；颗粒物质直径 2mm：0.47。

⑶直筒塑料杯内径和木棒平均直径及其质量的测量

测量各颗粒物质直径、直筒塑料杯内径和木棒平均直径及其质量，采用游标卡尺和电子秤。

杯子	内径 /cm	质量/g
杯 1	2.20	21.4
杯 2	3.35	42.3
杯 3	4.67	39.8
杯 4	6.15	52.3

（表 2.9）

木棒	直径 /mm	质量 /g
棒 1	2.50	5.5

棒 2	**2.55**	**6.7**
棒 3	**2.75**	**8.1**
棒 4	**2.9**	**9.3**

（表 2.10）

颗粒物质	**直径/mm**
颗粒 1	**2**
颗粒 2	**3**
颗粒 3	**4**

（表 2.11）

3.3 实验

取以下几个自变量进行实验：

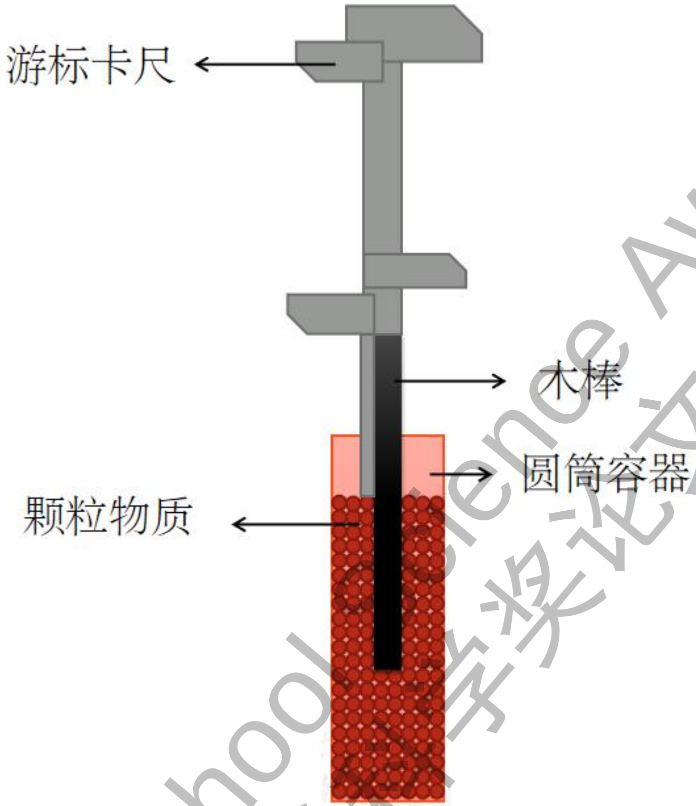
1.容器内径

2.木棒直径

3.颗粒物质堆积高度

4.颗粒物质直径

我们用游标卡尺进行测量，不断调整，将其限制在一定区间后，不断缩小，以求得最小浸入深度。



（图 2.12）

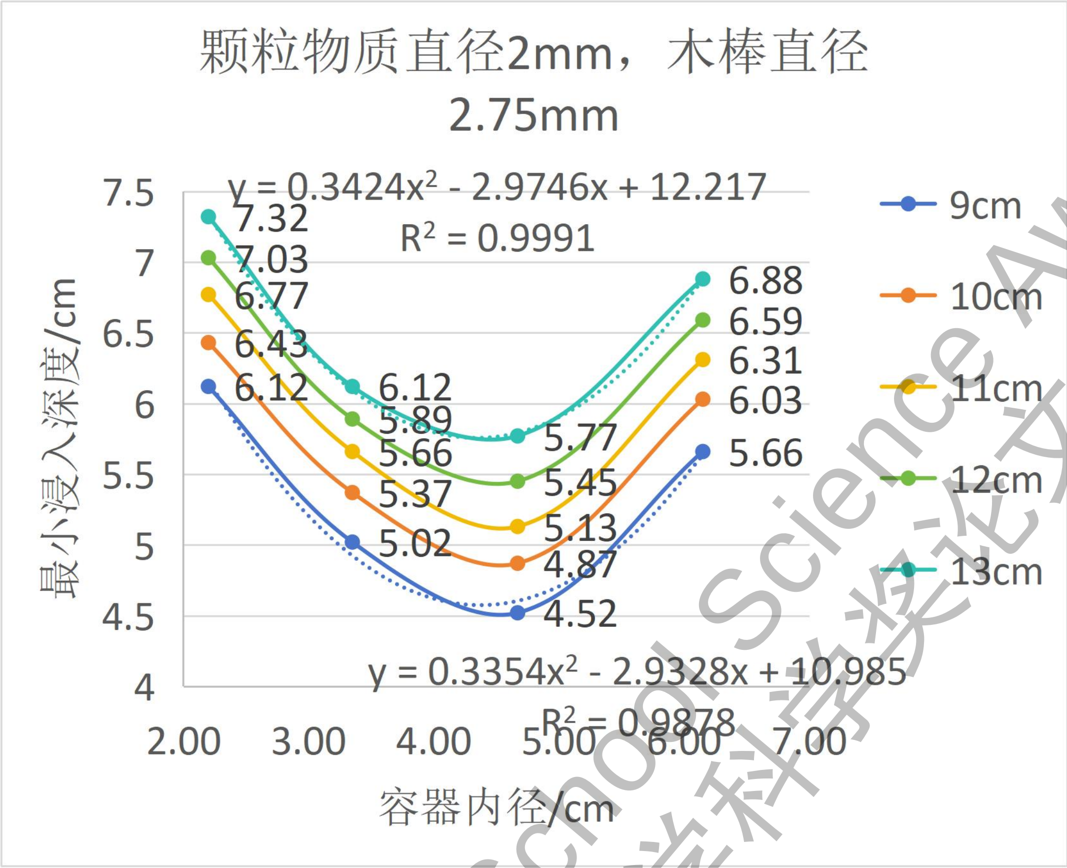
实验示意图

接下来，对各自变量进行定量实验。通过改变自变量，测得各组数据下使系统提起所需的木棒最小浸入深度。

3.3.1 容器内径

改变多组无关变量的值后，测得多组最小浸入深度与容器内径的关系。

以直径为 2mm 的颗粒物质、直径为 2.75mm 的木棒为例，在各堆积高度下测得不同塑料杯内径下提起系统所需的最小浸入深度如下：



（图 2.13）

根据图表可以看出，在其他无关变量相同时，最小浸入深度与容器内径大致成二次函数关系，并且存在最适容器内径使最小浸入深度最小。

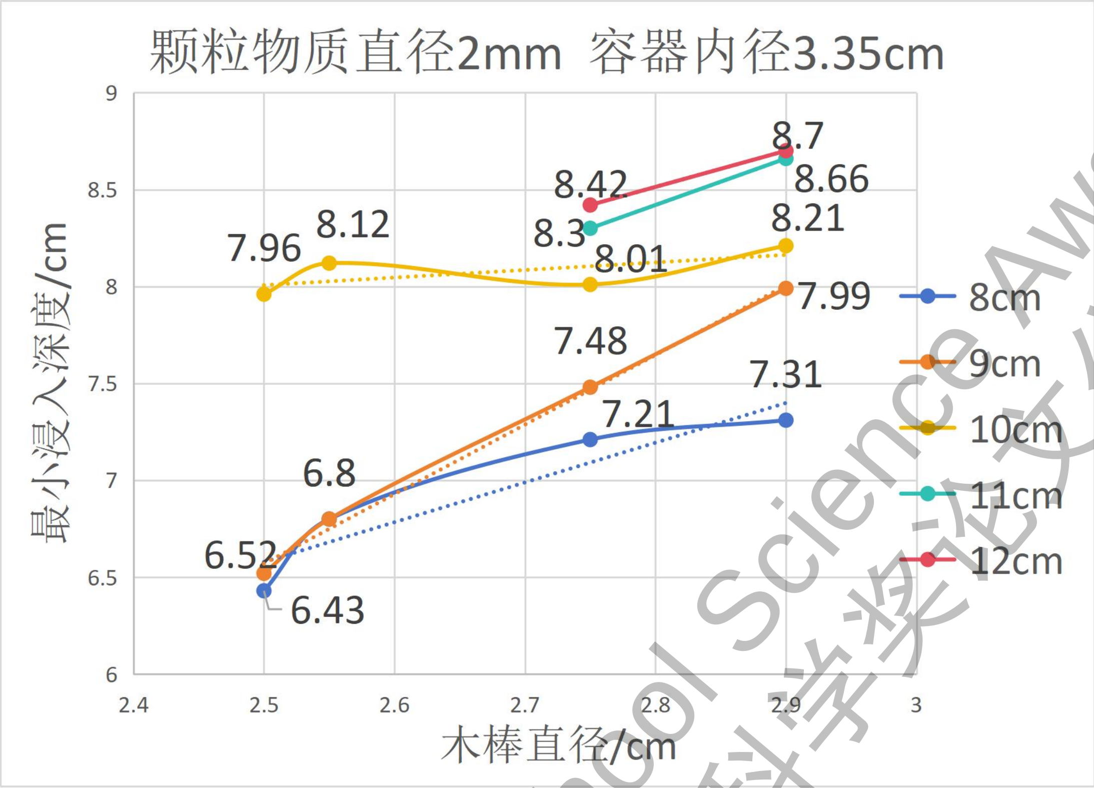
统计得到此情况下最适容器内径约为 4.36cm。

据(1.19)式，计算得理论应力转向系数约为 0.378。根据颗粒物质与木棒的动摩擦因数为 0.68、颗粒间动摩擦因数为 0.61 可根据(1.11)式计算得到理论上最适容器内径为 4.56cm，与实际值比较接近。

3.3.2 木棒直径

改变多组无关变量的值后，测得多组最小浸入深度与木棒直径的关系。

以直径为 2mm 的颗粒物质、内径为 3.35cm 的直筒塑料杯为例，在各堆积高度下测得不同木棒直径下提起系统所需的最小浸入深度如下：



（图 2.14）

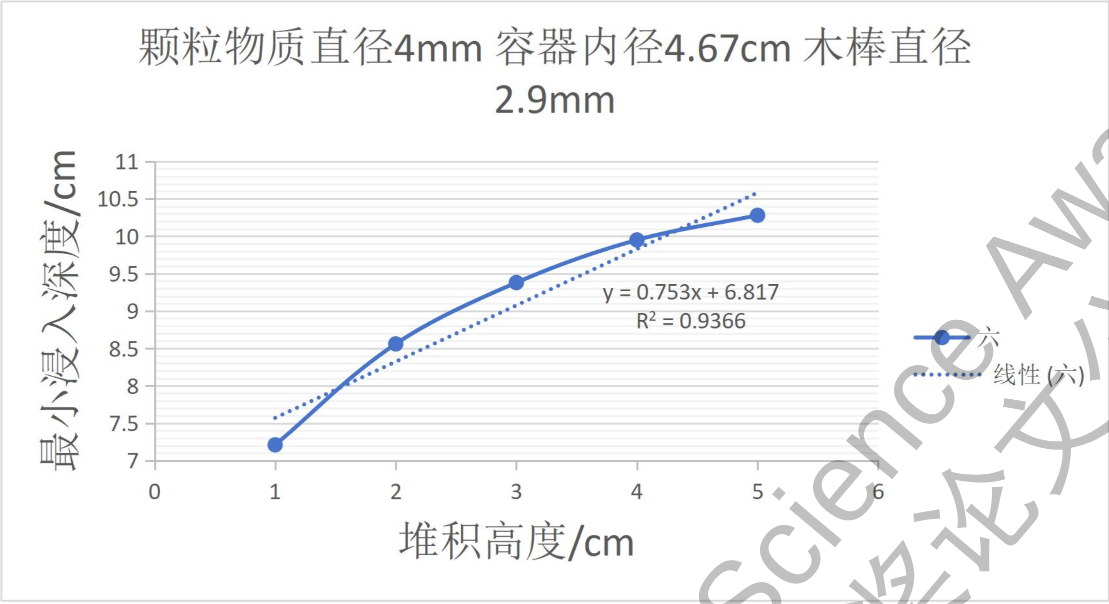
由图表可以看出，实验测得的最小浸入深度与木棒直径呈正相关关系，与理论相匹配。

3.3.3 颗粒物质堆积高度

随着颗粒物质堆积高度增加，系统重力增加，理论上最小浸入深度增加。

改变多组无关变量的值后，测得多组最小浸入深度与颗粒物质堆积高度的关系。

以直径为 2mm 的颗粒物质、内径为 4.67cm 的直筒塑料杯与 2.75mm 的木棒直径为例，测得不同颗粒物质堆积高度下提起系统所需的最小浸入深度如下：



（图 2.15）

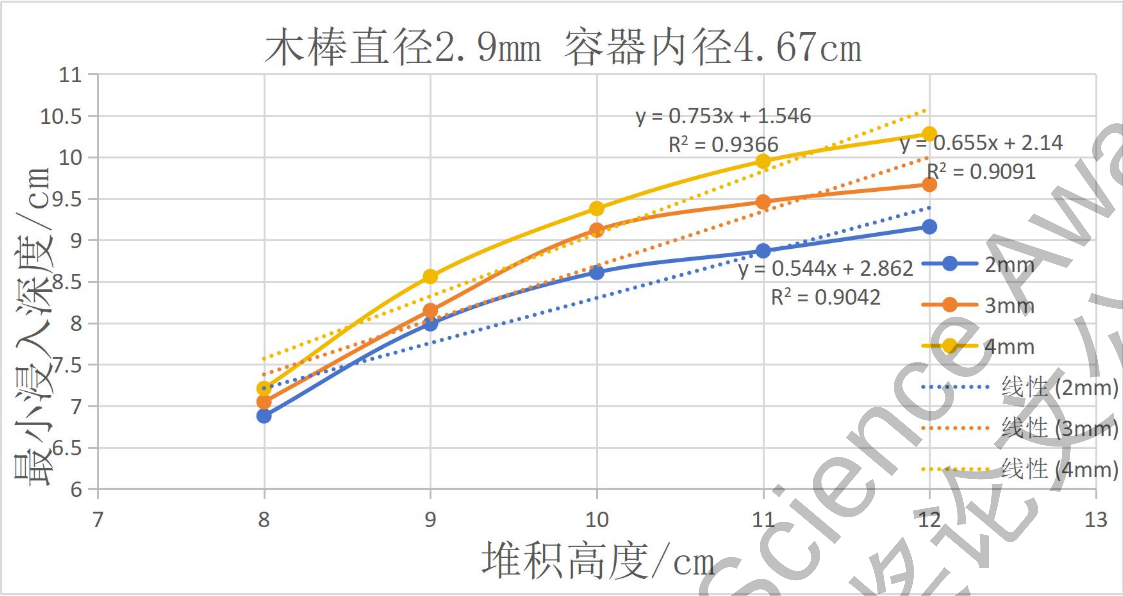
由图表可以看出，实验测得的最小浸入深度与颗粒物质堆积高度呈正相关关系，与理论相匹配。

3.3.4 颗粒物质直径

颗粒物质直径越小，堆积密度越大，颗粒物质平均密度越大，产生的摩擦力越大，理论上所需最小浸入深度减小。

改变多组无关变量的值后，测得多组最小浸入深度与颗粒物质直径的关系。

以内径为 4.67cm 的直筒塑料杯、2.9mm 的木棒直径为例，在各堆积高度下测得不同颗粒物质直径提起系统所需的最小浸入深度如下：



（图 2.16）

由图表可以看出，最小浸入深度与颗粒物质直径成正相关关系，与理论相匹配。

3.4 本章小结

本章通过实验，证明了各个变量与摩擦力的关系。首先，我们通过摩擦力与其他易测量的数据的正相关关系，将摩擦力的大小转化为最小浸入深度的大小来表示。接着，我们通过控制变量法对容器内径、木棒直径、颗粒物质堆积高度、颗粒物质直径等四个自变量对摩擦力的影响进行探究，得到的实验结果为：浸入物体与颗粒物质之间的摩擦力与容器内径大致成二次函数关系，并且存在最适容器内径使摩擦力最大；与浸入物体底面半径成正相关；与颗粒物质堆积高度成正相关；与颗粒物质直径成负相关。

第四章 总结与展望

4.1 总结

通过理论与实验分析，我们得到了“颗粒壶铃”现象产生的原因，即由浸入物体与颗粒物质之间的摩擦、颗粒物质之间的摩擦、颗粒物质与器壁之间的摩擦相互作用，使物体达到稳定的平衡状态。其中，最重要的是颗粒物质与浸入物体之间的摩擦，该摩擦力与颗粒物质与容器的总重力相互平衡，使系统被提起。

进而我们对颗粒物质与浸入物体之间的摩擦力大小进行深入探究，运用 Janssen 模型与颗粒元胞模型得出了该摩擦力的表达式。在实验中，我们用提起系统所需的最小浸入深度来表现该摩擦力的大小。理论与实验相互印证，我们得出以下结论：

1.摩擦力与各个参数之间有近似于下述函数关系：

$$f \approx \frac{\pi r \rho g D}{2}\left(H - \frac{D}{4\mu k}\right)$$

2.浸入物体与颗粒物质之间的摩擦力与容器内径大致成二次函数关系，并且存在最适容器内径使摩擦力最大，且最适容器内径为：

$$D_0 = 2\mu k$$

其中 μ 为颗粒物质与浸入物体之间的动摩擦系数，k 为应力转向系数。

3.浸入物体与颗粒物质之间的摩擦力与浸入物体底面半径成正相关。

4.浸入物体与颗粒物质之间的摩擦力与颗粒物质直径成负相关。

5.颗粒物质堆积高度越高，所需浸入深度越大。

4.2 展望

本文在探讨“颗粒壶铃”现象的产生原因与影响因素时对物理模型有诸多简

化。首先我们采用的 Janssen 模型为连续介质模型，而颗粒物质本身为大量颗粒组成的离散体系，因而模型所得的结果与实际稍有误差，并且理论经过近似处理，所得到的关系式仍有误差和局限性。其次，实际实验中磨砂玻璃珠的堆积方式并非都是最密堆积，因而据此计算得来的角度有偏差。并且，由于颗粒物质每次堆积的随机性，实验所得数据即使经过多次测量仍然存在偏差。

基于以上问题，在后续研究中，我们还会在理论研究中加入对颗粒物质中力链的分析考虑，并结合他人对颗粒物质堆积随机性的探究，以使理论模型更加精确。同时，在实验中，我们也可使用更丰富的器材，丰富数据，同时依据理论计算摩擦力与各个物理量的正比系数。

然而如果要使研究更接近与实际，我们还需探究椭圆形颗粒的“颗粒壶铃”效应，浸入物体也不应局限于直棒状物体。

参考文献

[1] 1001-246X(2006)06-0642-05 粮仓效应的圆盘堆积模型　张兴刚，胡林，隆正文

[2] DIO:10.7498/aps.64.134502 二维晶格颗粒堆积中侧壁的压力分布与转向系数　杨林，胡林，张兴刚

[3]　DOI:10.19655/j.cnki.1005-4642.2021.01.009.韩志强,沈仲驰,唐明君.粮仓效应的底部与侧壁压力的实验验证[J].物理实验.2021,(1).

[4]彭政,李湘群,蒋礼,等.应力未饱和粮仓系统中器壁与颗粒的摩擦阻力[J].物理学报.2009,(3).2090-2096.

[5]N. P. Kruyt,L. Rothenburg.On micromechanical characteristics of the critical state of two-dimensional granular materials[J].Acta Mechanica.2014,225(8).

[6]孙其诚, 王光谦著. 颗粒物质力学导论 [M].科学出版社,2009.

致谢

本文为学生周可欣与刘福凤参加 2023 年 CYPT 所研究的成果。

在此十分感谢本校的赵博涵、陈雪、刘成礼老师，在高一就开始引导我们参与物理实验的课题研究，并且在我们研究“颗粒壶铃”时无偿给予学术和部分器材上的帮助。

同时感谢北京师范大学马宇翰教授在我们理论推导遇到瓶颈时给予大学知识上的支持与帮助。

在 2023 年 CYPT 的选题中，我们两人有缘共同选到 12 题 Rice Kettlebells 大米壶铃，并一起进行接下来的研究。

在开始阶段，我们一起进行预实验：使用简单的器具，先用筷子插进大米中，将系统提起来，由此共同确定了影响“颗粒壶铃”现象产生的几个因素，同时在网上查找颗粒物质的相关论文。

我们很容易地找到了关于颗粒物质性质的经典模型 Janssen 模型，但却不知道如何应用。然而根据预实验中筷子插的越深越易提起，我们受到从侧壁摩擦去考虑的启发。

接下来，由周可欣同学主要研究理论方面，同时刘福凤同学先就细砂、米、白沙这三种颗粒物质进行实验，同时，赵博涵、陈雪老师对实验的精确度改进做出指导。

在实验过程中，据预实验现象我们首先认为最小浸入深度随容器内径的增大而增大，然而实验结果并不如此，由于最初我们的容器内径有限，实验结果甚至出现了负相关的关系，这与预实验的结论大相径庭。

然而随着理论研究的进一步深入，周可欣同学在马宇翰教授的指导下对压力进行了积分，化简整理的结果表明侧压力与容器内径成二次函数关系。这将我们的第一个难关打通。同时，我们得出了颗粒物质与浸入物体之间的摩擦力的表达式，使实验方向更加明确。

当正式实验基本完成时，我们发现一个致命的问题，即我们最初使用的细砂、米、白沙这三种材料的材质不同，密度等性质也相去甚远，因而我们决定将颗粒物质改用磨砂玻璃珠，再次进行实验。

在论文撰写方面，由周可欣同学主要负责理论模型部分，刘福凤同学主要负责实验探究部分，剩下章节由两人一起写作、修改、润色。

同时感谢本校其他 CYPT 参赛队员在论文查找、颗粒物质与浸入物体动摩擦力测量当中给予的技术与体力帮助。

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Thesis Title:Research on the Mechanism and Influencing Factors of the ″Granular Kettlebell″ Phenomenon

Research on the Generation Mechanism and Influencing Factors of the "Granular Kettlebell" Phenomenon

By Zhou Kexin & Liu Fufeng

Abstract:To investigate the principles and conditions under which a container filled with granular materials, termed the "Granular Kettlebell," can be lifted by immersing an object within it, we reviewed existing research and derived the interaction mode between granular materials and the immersed object, specifically friction. An experimental setup and methodology were devised to convert the friction force between the granular materials and the immersed object into a measurable gravitational force. Theoretical derivations explored how the friction force between the granular materials and the immersed object varies with the radius of the granular materials, the diameter of the container, the radius of the base of the immersed object, and the height of the granular material accumulation, thereby elucidating the conditions under which the "Granular Kettlebell" can be lifted. The results indicated that the radius of the granular materials negatively correlates with friction, the container diameter exhibits a quadratic function relationship with friction with an opening upwards, the radius of the base of the immersed object positively correlates with friction, and the height of the granular material accumulation affects the system's gravitational force.

Keywords: Granular Materials, Friction, Janssen Model

Table of Contents

Chapter 1: Introduction

Based on the 2023 IYPT Problem 12, Rice Kettlebells, we introduce the concept of the "Granular Kettlebell," which refers to a container filled with granular materials. Granular materials are complex systems composed of a large number of discrete solid particles with diameters greater than $1\ \mu m$, interacting with each other. They are a common state of matter and one of the most widely existing and familiar material types on Earth. Granular materials exhibit properties distinct from solids, fluids, and gases, as they are typically in a non-equilibrium state far from equilibrium, with their state structures remaining metastable even during static accumulation, characterized by nonlinearity, self-organization, and energy dissipation.

When an object such as a chopstick is immersed into granular materials, the container and its contents can be lifted by pulling the object to a certain depth. This phenomenon is termed the "Granular Kettlebell" effect. Existing research has analyzed the pressure at the bottom of the container. For instance, in 1884, British scientist Roberts first observed that the pressure at the bottom of a granary tends to saturate after the height of the grain accumulation exceeds twice the diameter of the base, known as the granary effect. This demonstrates the unique static properties of granular materials compared to fluids. In 1895, German engineer Janssen proposed a continuous medium model that simplifies the complex internal interactions of granular materials, elegantly explaining the granary effect: due to interparticle forces, the force in the direction of gravity is redirected horizontally, with the walls of the granary supporting part of the weight of the grains, causing the pressure at the bottom to saturate.

Based on previous studies, it is evident that the essence of the "Granular Kettlebell" phenomenon lies in the balance between the interaction between granular materials and the immersed object and gravity, and its occurrence is influenced by relevant parameters.

Numerous natural disasters are intimately related to the properties of granular materials, such as avalanches, mudslides, landslides, and sandstorms. Meanwhile, many daily production and living activities involve granular materials, such as grain storage, railway subgrades, and the interaction between buildings, cable-stayed bridges' foundations, and the soil.

While some research has been conducted on the generation, transportation, and storage of granular materials, the handling, control, and interaction mechanisms between granular materials and other substances have not been systematically and fully developed. Today, with increasing demands for building safety and the need for natural disaster prevention and mitigation, studying the statics of granular materials and their interaction principles with other substances is of great significance. This research not only deepens the understanding of the fundamental properties ofgranular materials but also potentially provides new ideas and methods for the design, processing, and application of granular materials.

Drawing on the work of previous scholars, we extend our investigation of the "Granular Kettlebell" phenomenon to explore its sensitivity to factors such as the radius of the granular materials, the diameter of the container, the radius of the base of the immersed object, and the height of the granular material accumulation.

Chapter 2: Theoretical Model of the "Granular Kettlebell"

2.1 Janssen Model

The vertical pressure within the cylinder is equal everywhere on the same horizontal plane.

1.The horizontal pressure is proportional to the vertical pressure.

2.Mathematically, this can be expressed as:

$$p_x = kp_z \quad \text{………………………………（1.1）}$$

where k is the stress redirection coefficient, which increases with the height of the particulate material.

3.The friction between the particles and the container wall reaches the maximum static friction.

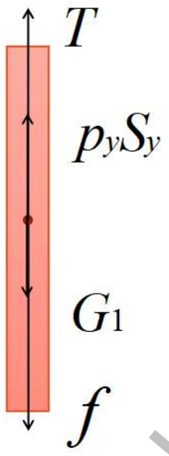
Through experimental and theoretical research, the expression for pressure $p_z$at height z in the silo is derived as:

$$p_z = \frac{\rho g D}{4\mu k}[1 - \exp(-\frac{4\mu k H}{D})] \quad \text{……………………（1.2）}$$

where ρ is the base of the natural logarithm, μ is the coefficient of kinetic friction between the particles and the container wall, D is the inner diameter of the container, and H is the height of the particulate material accumulation,g is the gravitational acceleration

2.2 Force Analysis Model for the Immersed Object and the Enti re System

To investigate the "granular kettlebell" phenomenon and the influence of r elated parameters, a theoretical analysis and force model for the system are est ablished. The particulate material is abstracted as spherical particles, the container is a straight cylindrical container, and the immersed object is a slender cylinder. The force analysis on the immersed object yields:



(Picture2.1)

From the force balance of the immersed object:

$$T + p_zS = f + G_1 \cdots\cdots\cdots\cdots\cdots\cdots \textbf{(1.3)}$$

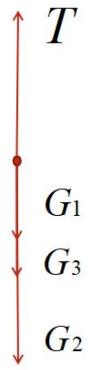
where T is the tension force acting on the immersed object, p_y is the pressure at the bottom of the immersed object, S is the base area of the immersed object, G_1 is the weight of the immersed object, and f is the friction force between the particulate material and the immersed object.

If the friction force between the particulate material and the immersed object reaches the maximum static friction force, which is approximatel equal to he sliding friction force, then:

$$f = \mu\sigma_x \cdots\cdots\cdots\cdots\cdots\cdots \textbf{(1.4)}$$

μ It is the coefficient of kinetic friction between the particulate material and the immersed object, σ_x is the horizontal stress acting on the immersed object from the particulate materia.

A force analysis for the entire system involves:



(Picture2.2)

G_2 is the gravity of the particulate material、G_3 is the gravity of the contain

From the force balance of the immersed object:

$$T = G_1 + G_2 + G_3 \quad \text{(1.5)}$$

2.3 Solving for Friction Force

The immersed object is abstracted as a cylindrical rod. Integrating the horizontal stress acting on the immersed object from 0 to H yields:

$$\sigma_x = \int_0^H p_x S_x dy \quad \text{(1.6)}$$

From(1.1)(1.2)(1.4)(1.6),by organizing and summarizing, we can obtain：

$$f = 2\mu k \pi r \int_0^H p_z dy \quad \text{(1.7)}$$

Where r is the radius of the base of the immersed object.

By substituting and simplifying the equation, we obtain：

$$
\begin{aligned}
f &= 2\mu k\pi r\int_0^H \frac{\rho g D}{4\mu k}(1-e^{-\frac{4\mu k y}{D}})dy \\
&= \frac{\pi r\rho g D}{2}H - \frac{\pi r\rho g D}{2}\int_0^H e^{-\frac{4\mu k y}{D}}dy \\
&= \frac{\pi r\rho g D}{2}H - \frac{\pi r\rho g D}{2}\bullet\frac{D}{4\mu k}e^{-\frac{4\mu k y}{D}}\Big|_H^0 \\
&= \frac{\pi r\rho g D}{2}H - \frac{\rho g\pi r D^2}{8\mu k}(1-e^{-\frac{4\mu k y}{D}})
\end{aligned}
$$

thus：

$$f = \frac{\pi r\rho g D}{2}H - \frac{\rho g\pi r D^2}{8\mu k}(1-e^{-\frac{4\mu k H}{D}}) \quad \text{(1.8)}$$

When $\frac{4\mu kH}{D} >> 1$，we can obtain：

$$
\begin{aligned}
f &\approx \frac{\pi r\rho g D H}{2} - \frac{\rho g\pi r D^2}{8\mu k} \\
&= \frac{\pi r\rho g D}{2}(H - \frac{D}{4\mu k})
\end{aligned}
\quad \text{(1.9)}
$$

By transforming it, we obtain a quadratic function relationship：

$$f = \frac{\pi r\rho g}{8\mu k}[-(D-2\mu k)^2 + 4\mu^2 k^2] \quad \text{（1.10）}$$

From(1.10)，there exists an optimal inner diameter of the container. D_0， to maximize the friction force:

$$D_0 = 2\mu k \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad \textbf{(1.11)}$$

By（3）：

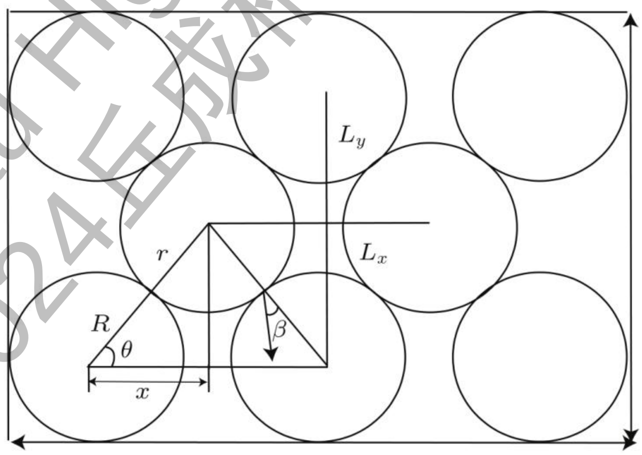
$$f = T + p_z S - G_1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad \textbf{(1.12)}$$

By(1.5)(1.10)(1.12)，for the "granular kettlebell" to be lifted just barely, it is necessary to satisfy:

$$p_z S + G_2 + G_3 = \frac{\pi r \rho g}{8\mu k}[-(D-2\mu k)^2 + 4\mu^2 k^2] \cdots\cdots\cdots \quad \textbf{(1.13)}$$

2.4 Exploration of the Stress Redirection Coefficient k

Based on reference [1], the particulate material is abstracted into particle cells as shown in the figure, with a stacking pattern approximately as described.



（Picture2.3）

In the stacking structure of particulate material, let the angle between the actual contact force Fbetween particles and the direction of particle contact be β， Then, the component of force F in the horizontal direction is:

$$f_x = F \bullet \cos(\theta+\beta) = \sigma_x \bullet L_y \cdots\cdots\cdots\cdots\cdots\cdots \quad \textbf{(1.14)}$$

The component force of F in the vertical direction is:

$$f_y = F \bullet \sin(\theta+\beta) = \sigma_y \bullet L_x \cdots\cdots\cdots\cdots\cdots\cdots \quad \textbf{(1.15)}$$

Where L_x and L_y are the length and width of the particle cell, respectively.

Thus：

$$\frac{L_y}{L_x} = \tan\theta \cdots\cdots\cdots\cdots\cdots\cdots \quad \textbf{(1.16)}$$

During the process of particles pulling against each other, assuming that the friction force between particles reaches the maximum static friction force, an equation can be derived from the fact that the resultant external force on a static system is zero:

$$F \bullet \sin\beta = \mu \bullet F \bullet \cos\beta \quad \textbf{(1.17)}$$

Where μ is the coefficient of friction between particles, we can obtain:

$$\beta = \arctan\mu \cdots\cdots\cdots\cdots\cdots\cdots \quad \textbf{(1.18)}$$

Therefore, the stress redirection coefficient is given by:

$$k = \frac{\sigma_x}{\sigma_y} = \frac{1}{\tan(\theta+\beta) \bullet \tan\theta} \cdots\cdots\cdots\cdots \quad \textbf{(1.19)}$$

Simultaneously, we consider the packing arrangement of spherical particulate material as the closest packing, where three spherical particles in the lower layer are closely arranged, and above the triangular void formed by these particles, another spherical particle is stacked. The centers of these four particles form a regular tetrahedron, and it can be easily derived that:

$$tan\theta = \sqrt{2}$$

Chapter Three: Experimental Exploration

3.1 Experimental Setup, Objectives, and Content

To explore the conditions and influencing factors for lifting the "granular kettlebell," we have the following experimental equipment: frosted glass beads with diameters of 2mm, 3mm, and 4mm, straight plastic cups with different inner diameters, cylindrical wooden sticks with different diameters, vernier calipers and electronic vernier calipers with tenth-degree accuracy, electronic scales (with an accuracy of 0.1g), force sensors, and data collectors.

（Picture2.4）

Experimental equipment

The experimental design is as follows: Multiple experiments are conducted to gradually identify the minimum immersion depth required for lifting the "granular kettlebell." At this point, the gravitational force and frictional force of the lifting system are balanced, indicating that the frictional force between the granular material and the immersed object is proportional to the lateral wall pressure, which in turn is positively correlated with the immersed lateral area. Therefore, this relationship can be reflected through the minimum immersion depth. To validate the theory, multiple experiments are conducted by varying the experimental variables, and the magnitude of the error is calculated.

3.2 Parameter Measurement

(1) The coefficient of kinetic friction between the granular material and the wooden stick is measured by a force sensor, which measures the tensile force, and the coefficient of kinetic friction is indirectly calculated based on the formula.

$$F_{拉} = \mu mg \quad \text{(1.20)}$$

（Picture2.5）

Measurement of friction coefficient

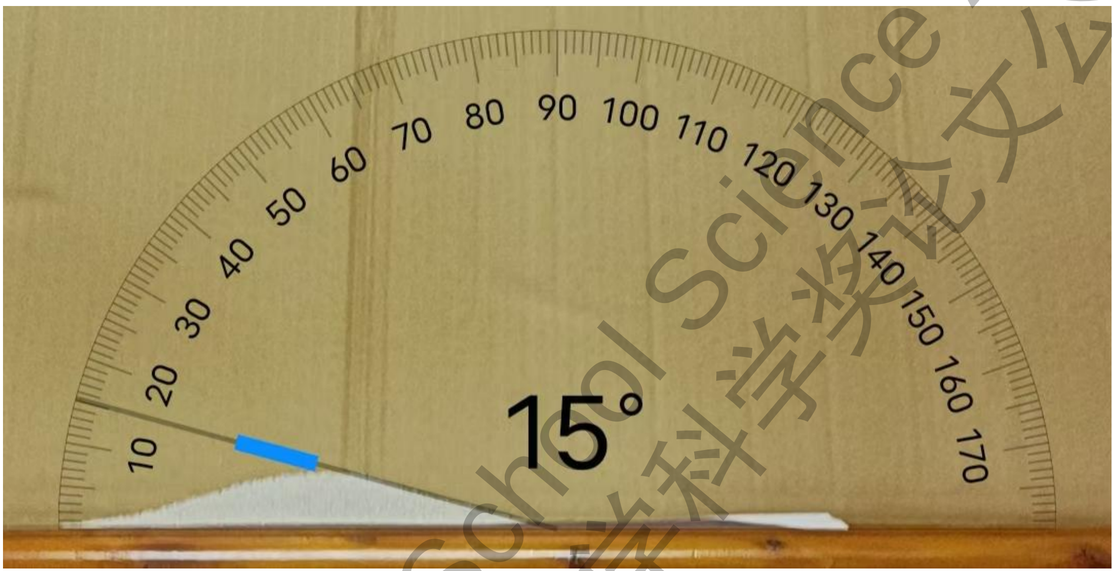
After taking the average of multiple measurements, the coefficient of kinetic friction μ between the granular material and the wooden stick was found to be 0.68.

(2) To measure the coefficient of kinetic friction between the granular materials,

the angle of repose is first measured, and then the coefficient of kinetic friction is indirectly calculated based on the formula:

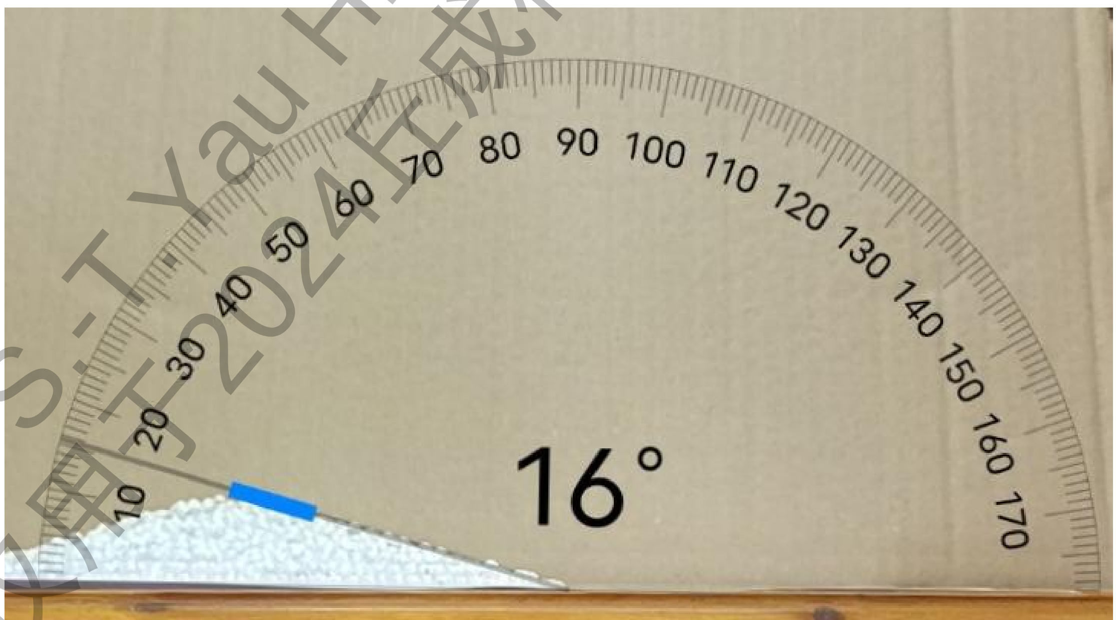
$$\mu = \tan\theta$$

For granular material with a diameter of 4mm:



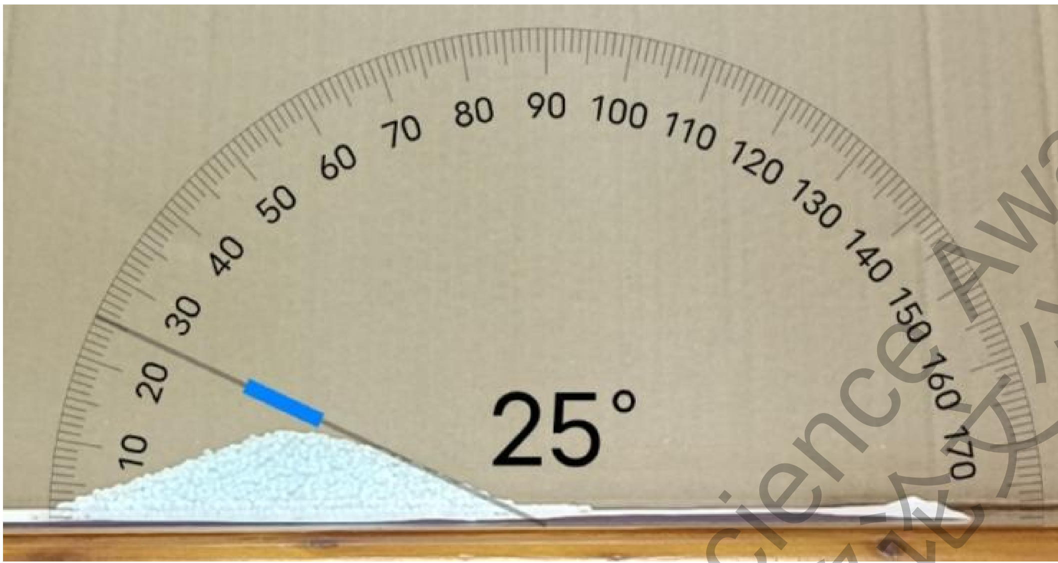
（**Picture2.6**）

For granular material with a diameter of 3mm:



（**Picture2.7**）

For granular material with a diameter of 2mm:



（Picture2.8）

After taking the average of multiple measurements, the coefficients of kinetic friction between the granular materials were found to be: for granular materials with a diameter of 4mm: 0.27; for granular materials with a diameter of 3mm: 0.29; and for granular materials with a diameter of 2mm: 0.47.

(3) Measurement of the inner diameter of the straight plastic cup, the average diameter of the wooden stick, and their masses

The diameters of the granular materials, the inner diameter of the straight plastic cup, the average diameter of the wooden stick, and their masses were measured using vernier calipers and an electronic scale.

Cup	Inner diameter/cm	Mass/g
Cup1	2.20	21.4
Cup2	3.35	42.3
Cup3	4.67	39.8
Cup4	6.15	52.3

（Table2.9）

Wooden	Diameter/mm	Mass/g

stick		
Stick1	2.50	5.5
Stick2	2.55	6.7
Stick3	2.75	8.1
Stick4	2.9	9.3

（Table2.10）

Granular material	Diameter/mm
Granular 1	2
Granular 2	3
Granular 3	4

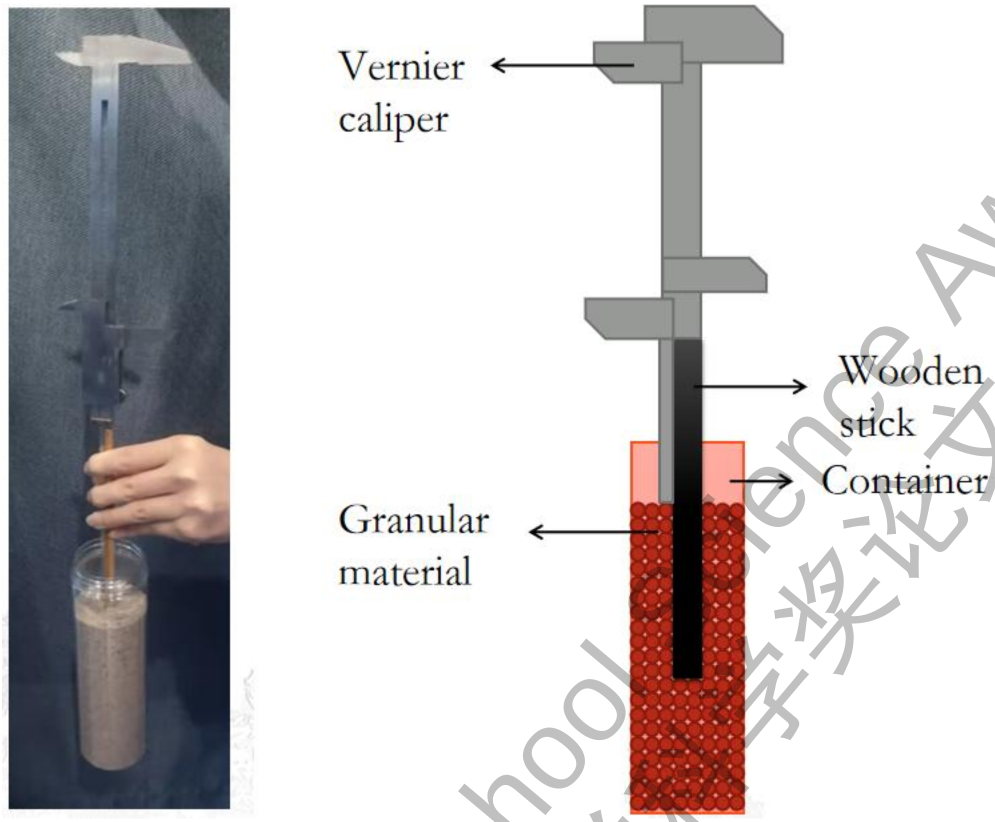
（Table2.11）

3.3 Experiment

The following independent variables were selected for experimentation:

1. Container inner diameter
2. Wooden stick diameter
3. Pile height of granular material
4. Diameter of granular material

We used vernier calipers to measure and continuously adjust these variables, restricting them within a certain range and then gradually narrowing it down in order to find the minimum immersion depth.



（Picture2.12）

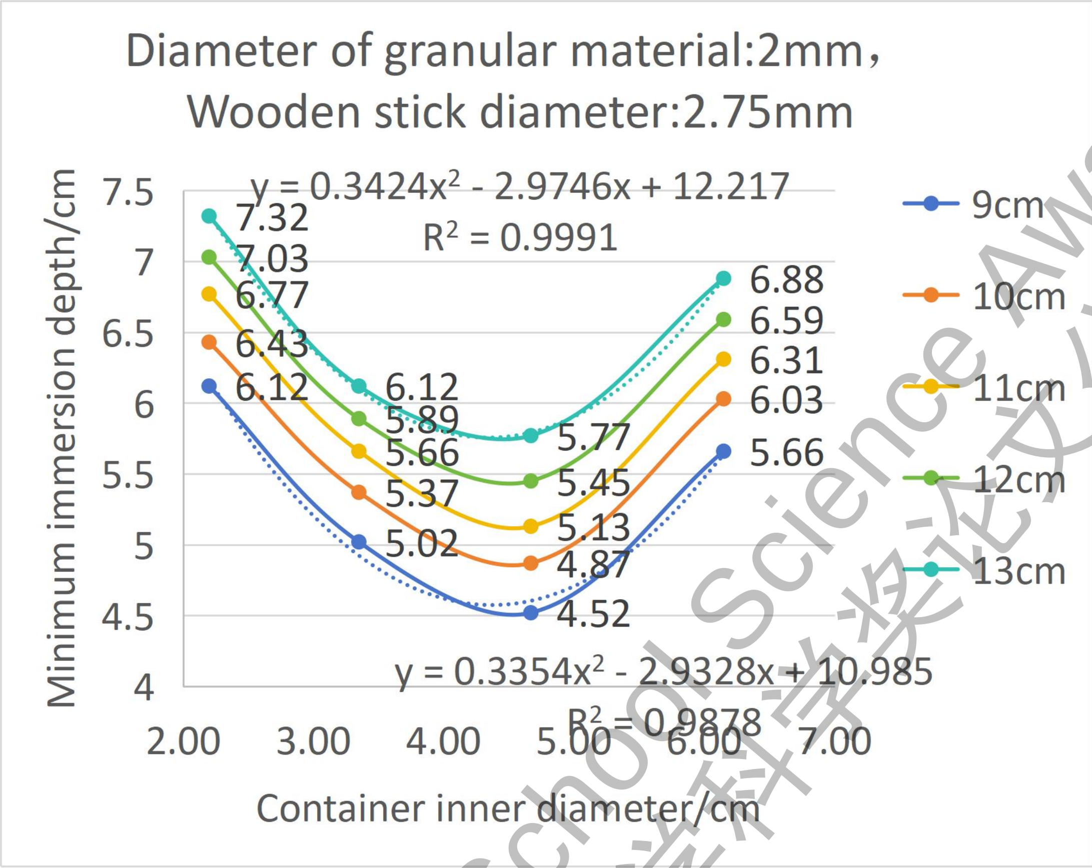
Schematic diagram of the experiment

Next, quantitative experiments were conducted on each independent variable. By varying the independent variables, the minimum immersion depth of the wooden stick required to lift the system was measured for each set of data.

3.3.1 Container Inner Diameter

After changing the values of multiple irrelevant variables, the relationship between the minimum immersion depth and the inner diameter of the container was measured for multiple sets of data.

Taking granular material with a diameter of 2mm and a wooden stick with a diameter of 2.75mm as an example, the minimum immersion depth required to lift the system at different inner diameters of plastic cups was measured at various pile heights as follows:



（**Picture2.13**）

As can be seen from the graph, when other irrelevant variables are the same, the minimum immersion depth has a roughly quadratic relationship with the inner diameter of the container, and there is an optimal container inner diameter that minimizes the minimum immersion depth.

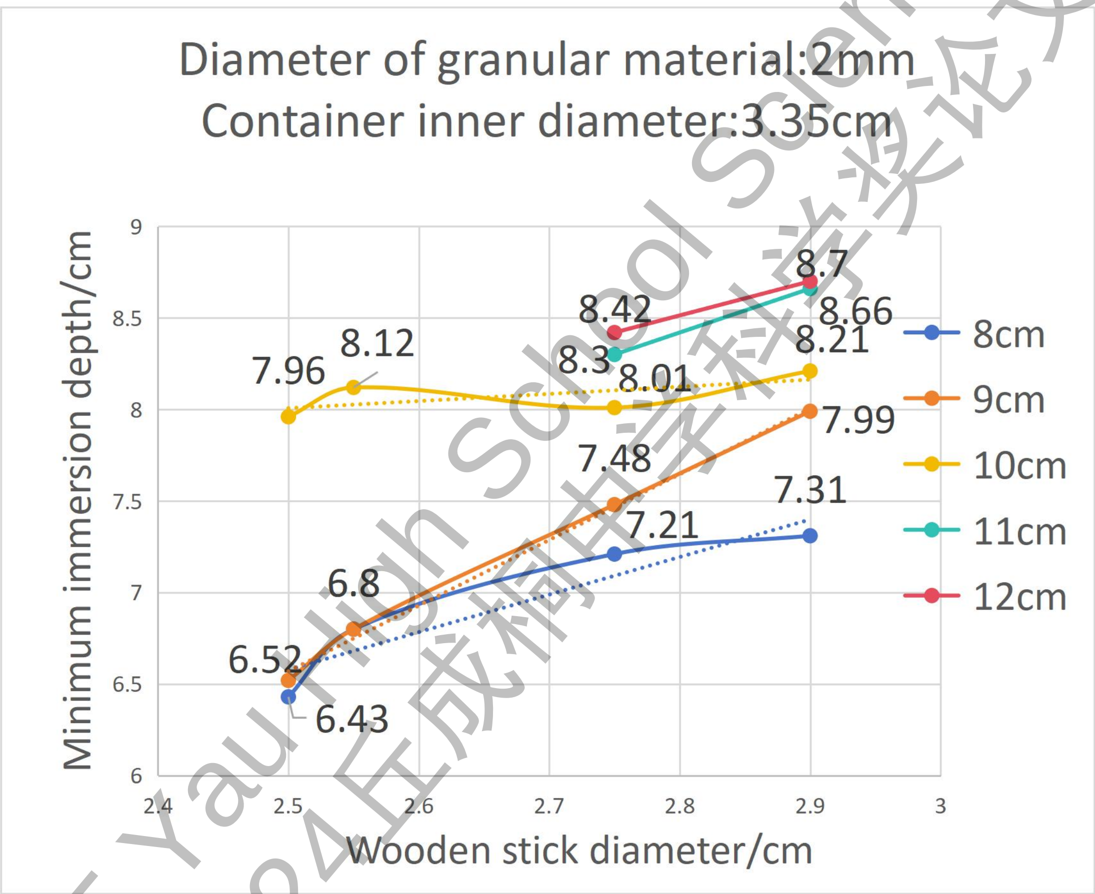
Statistical analysis revealed that the optimal container inner diameter in this case is approximately 4.36cm.

Using Equation (1.19), the theoretical stress redirection factor is calculated to be approximately 0.378. Based on the coefficient of kinetic friction between the granular material and the wooden stick of 0.68, and the coefficient of kinetic friction between the granular particles of 0.61, the theoretically optimal container inner diameter is calculated to be 4.56cm using Equation (1.11), which is close to the actual value.

3.3.2 Wooden Stick Diameter

After changing the values of multiple irrelevant variables, the relationship between the minimum immersion depth and the diameter of the wooden stick was measured for multiple sets of data.

Taking granular material with a diameter of 2mm and a straight plastic cup with an inner diameter of 3.35cm as an example, the minimum immersion depth required to lift the system at different diameters of the wooden stick was measured at various pile heights as follows:



（**Picture2.14**）

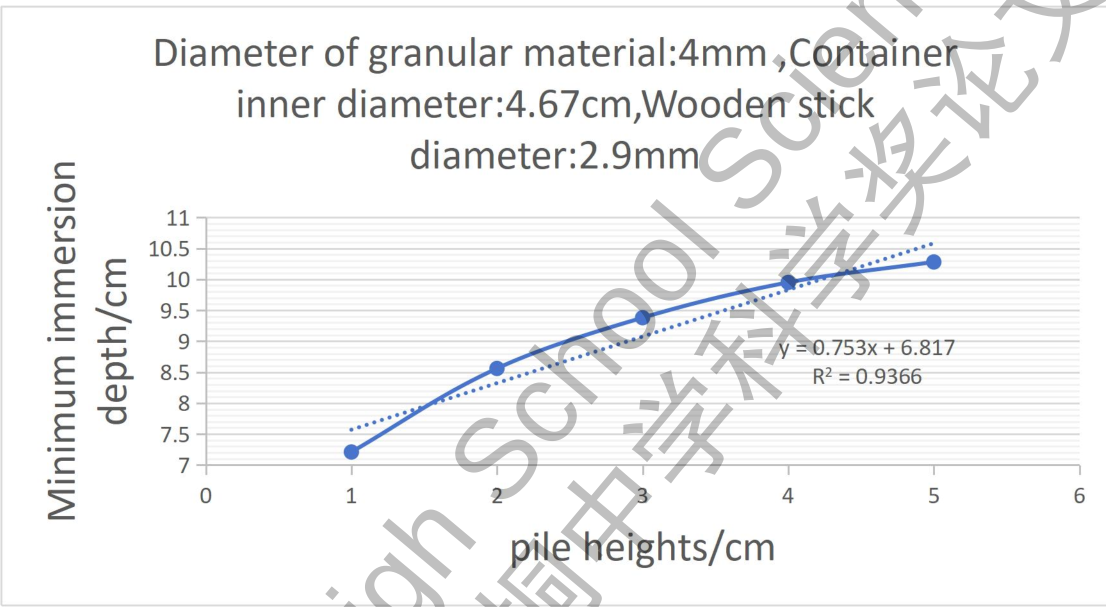
As can be seen from the graph, the experimentally measured minimum immersion depth has a positive correlation with the diameter of the wooden stick, which is consistent with the theory.

3.3.3 Pile Height of granular material

As the pile height ofgranular material increases, the gravity of the system increases, theoretically leading to an increase in the minimum immersion depth.

After altering the values of multiple sets of irrelevant variables, the relationship between the minimum immersion depth and the pile height of granular material was measured for multiple sets.

Taking granular material with a diameter of 2mm, a straight plastic cup with an inner diameter of 4.67cm, and a wooden stick with a diameter of 2.75mm as examples, the minimum immersion depth required to lift the system under different pile heights of granular material was measured as follows:



（Picture2.15）

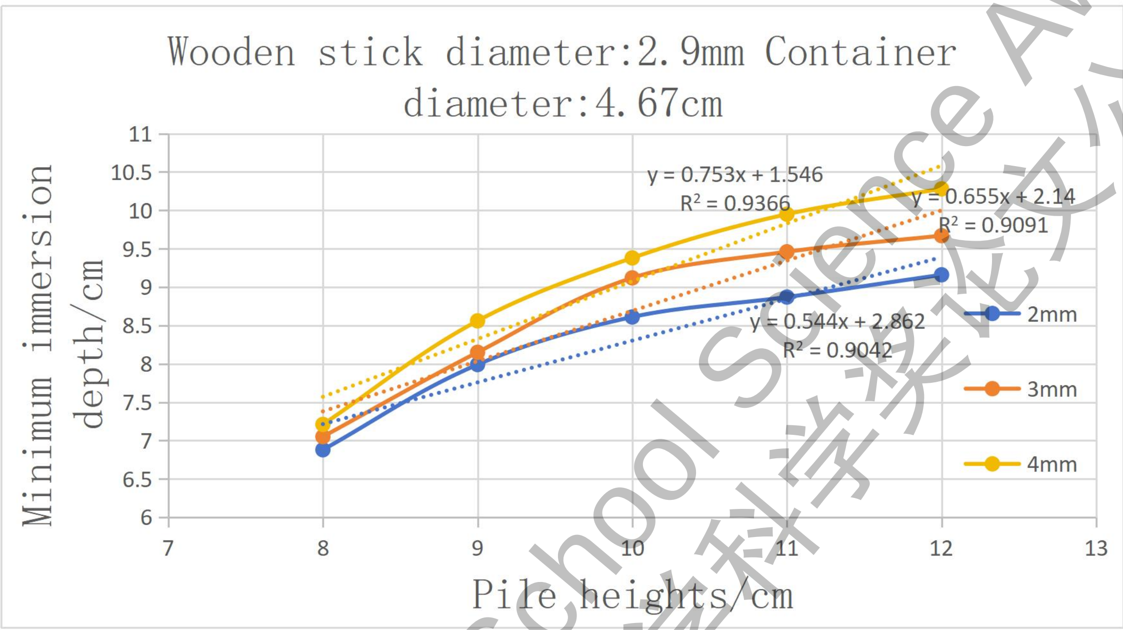
As can be seen from the chart, the experimentally measured minimum immersion depth shows a positive correlation with the pile height of granular material, which is consistent with the theory.

3.3.4 Diameter of Granular Material

The smaller the diameter of granular material, the greater the packing density and average density of the granular material, resulting in greater friction and theoretically requiring a smaller minimum immersion depth.

After altering the values of multiple sets of irrelevant variables, the relationship between the minimum immersion depth and the diameter of granular material was measured for multiple sets.

Taking a straight plastic cup with an inner diameter of 4.67cm and a wooden stick with a diameter of 2.9mm as examples, the minimum immersion depth required to lift the system with different diameters of granular material under various pile heights was measured as follows:



（Picture2.16）

As can be seen from the chart, there is a positive correlation between the minimum immersion depth and the diameter of granular material, which is consistent with the theory.

3.4 Summary of This Chapter

This chapter experimentally demonstrates the relationship between various variables and friction. Firstly, we transformed the magnitude of friction into the minimum immersion depth by leveraging the positive correlation between friction and other easily measurable data. Subsequently, we employed the controlled variable method to investigate the influence of four independent variables—the inner diameter of the container, the diameter of the wooden stick, the pile height of granular materials, and the diameter of granular materials—on friction. The experimental results show that the friction between the immersed object and granular materials exhibits an approximately quadratic function relationship with the inner diameter of the container, and there exists an optimal inner diameter that maximizes friction. Additionally, friction is positively correlated with the radius of the immersed object's bottom surface and the pile height of granular materials, while it is negatively correlated with the diameter of granular materials.

Chapter 4: Summary and Future Prospects

4.1 Summary

Through theoretical and experimental analyses, we have elucidated the underlying causes of the "Granular Kettlebell" phenomenon, which arises from the interplay of friction between the immersed object and the granular materials, friction among the granular materials themselves, and friction between the granular materials and the container walls, leading to a stable equilibrium state. Notably, the most significant factor is the friction between the granular materials and the immersed object, which balances against the total gravitational force of the granular materials and the container, enabling the system to be lifted.

Further delving into the magnitude of this friction force, we derived an expression using the Janssen model and the granular cellular model. In experiments, we represented the magnitude of this friction force by the minimum immersion depth required to lift the system. By corroborating theoretical and experimental results, we reached the following conclusions:

1.The friction force exhibits an approximate functional relationship with various parameters:

$$f \approx \frac{\pi r \rho g D}{2}(H - \frac{D}{4\mu k})$$

2.The friction force between the immersed object and the granular materials follows a quadratic function with respect to the inner diameter of the container, with an optimal container diameter maximizing the friction force. This optimal diameter is given by:

$$D_0 = 2\mu k$$

where μ is the coefficient of kinetic frictioan between the granular materials and the immersed object, and k is the stress redirection coefficient.

3.The friction force positively correlates with the radius of the base of the immersed object.

4.The friction force negatively correlates with the diameter of the granular materials.

5.As the height of the granular material accumulation increases, the required immersion depth also increases.

4.2 Future Prospects

In our exploration of the causes and influencing factors of the "Granular Kettlebell" phenomenon, numerous simplifications were made in the physical models. Firstly, the Janssen model, a continuous medium model, was adopted, whereas granular materials inherently constitute a discrete system of numerous particles. Consequently, the results derived from the model exhibit slight deviations from reality, and the theoretically approximated relationships still have errors and limitations. Secondly, the packing of frosted glass beads in actual experiments does not always achieve the densest packing, leading to deviations in the calculated angles. Furthermore, due to the randomness in the packing of granular materials, experimental data, even after multiple measurements, still exhibits deviations.

To address these issues, in subsequent research, we will incorporate the analysis of force chains within granular materials into our theoretical studies and integrate others' investigations into the randomness of granular packing to refine our theoretical models. Additionally, in experiments, we will utilize a broader range of equipment to enrich our data and determine the proportional coefficients between the friction force and various physical quantities based on theoretical calculations.

However, to bring our research closer to practical applications, we must also explore the "Granular Kettlebell" effect in elliptical-shaped particles, and the immersed objects should not be limited to straight sticks.

References

[1] Zhang Xinggang, Hu Lin, Long Zhengwen. Disk Stacking Model for the Granary Effect. 1001-246X(2006)06-0642-05.

[2] Yang Lin, Hu Lin, Zhang Xinggang. Pressure Distribution and Stress Diversion Coefficient Along Sidewalls in Two-Dimensional Lattice Granular Packings. DOI: 10.7498/aps.64.134502.

[3] Han Zhiqiang, Shen Zhongchi, Tang Mingjun. Experimental Verification of Bottom and Sidewall Pressures in the Granary Effect. Journal of Physics Experiments. 2021, (1). DOI: 10.19655/j.cnki.1005-4642.2021.01.009.

[4] Peng Zheng, Li Xiangqun, Jiang Li, et al. Frictional Resistance Between the Wall and Granular Particles in an Unsaturated Granary System. Acta Physica Sinica. 2009, (3). 2090-2096.

[5] N. P. Kruyt, L. Rothenburg. On Micromechanical Characteristics of the Critical State of Two-Dimensional Granular Materials. Acta Mechanica. 2014, 225(8).

[6] Sun Qicheng, Wang Guangqian. Introduction to Mechanics of Granular Materials. Science Press, 2009.

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We also extend our thanks to Professor Ma Yuhan from Beijing Normal University for his invaluable university-level knowledge support and assistance when we encountered difficulties in our theoretical derivations.

In the 2023 CYPT competition, we were fortunate to have both chosen Problem 12, Rice Kettlebells, and embarked on this research journey together. At the initial stage, we conducted preliminary experiments using simple tools, inserting chopsticks into rice and lifting the system, thereby identifying several factors influencing the "Granular Kettlebell" phenomenon. We also searched for relevant papers on granular materials online.

While we easily found the classic Janssen model regarding the properties of granular materials, we struggled with its application. However, inspired by the observation that deeper insertion of chopsticks facilitated lifting, we began to consider the role of friction along the container walls.

Subsequently, Zhou Kexin focused primarily on the theoretical aspects, while Liu Fufeng conducted initial experiments using fine sand, rice, and white sand. Our teachers, Mr. Zhao Bohan and Ms. Chen Xue, provided invaluable guidance to improve the accuracy of our experiments.

During the experimental process, our initial hypothesis was that the minimum immersion depth would increase with the inner diameter of the container. However, the experimental results contradicted this assumption, initially showing a negative

correlation due to the limited range of container diameters we tested.

As our theoretical research progressed, Zhou Kexin, under the guidance of Professor Ma Yuhan, integrated the pressure distribution, revealing a quadratic relationship between the lateral pressure and the container diameter. This breakthrough paved the way for us to derive an expression for the friction force between the granular materials and the immersed object, clarifying our experimental direction.

Upon completing the formal experiments, we encountered a crucial issue: the different materials (fine sand, rice, and white sand) varied significantly in texture and density. Consequently, we decided to switch to frosted glass beads for further experiments.

In terms of paper writing, Zhou Kexin primarily handled the theoretical model section, while Liu Fufeng focused on the experimental investigation. We collaborated on writing, revising, and polishing the remaining chapters.

Additionally, we are grateful to our fellow CYPT participants for their technical and physical assistance in literature searches, as well as in measuring the kinetic friction between granular materials and immersed objects.