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论文题目: Spectral coskewness and the valuation of stocks

Spectral coskewness and the valuation of stocks

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Abstract

The investors in China are primarily retail investors, and retail investors' information sources rely on channels such as financial reports or news. They are prone to overreaction to such information in a short period of time, exacerbating the asymmetry of stock returns, thus affecting coskewness risk. This phenomenon is usually corrected by the market in a relatively short period of time. Therefore, we speculate that coskewness risk is an important risk in China's A-share market, and its impact on stock returns is mainly reflected in the short-cycle effect. This article explores the impact of co-skewness risk on asset pricing in China's A-share market, especially analyzing the pricing ability of co-skewness on stock cross-sectional returns under different frequencies/cycles from a frequency domain perspective. By combining the skewness asset pricing model and the extended Wold representation theorem, we calculate the coskewness at different frequencies and study its impact on stock returns. The results show that short-period coskewness plays a more significant role in stock return pricing. In particular, coskewness with a period of 1-4 months (consistent with the earnings release cycle and the news impact cycle on stocks) still has a strong ability to explain stock returns even after considering the impact of other factors. The research in this article not only provides a new perspective for understanding the role of coskewness in asset pricing, but also reveals the unique impact of retail investor behavior on China's A-share market. This finding has practical implications for short-term investment decisions and risk management, while also offering theoretical support for improving risk regulation in China's capital markets.

Keywords: coskewness, frequency analysis, asset pricing, extended Wold representation, China A-share stock market

Contents

1 Introduction

As one of the largest emerging markets in the world, China's A-share stock market has a unique investor structure. Unlike developed markets, which are dominated by institutional investors, China A-share stock market is primarily driven by retail investors. According to the 2019 Shanghai Stock Exchange Annual Report, the total number of investors in China's A-share market was approximately 214.5 million, of which 213.8 million were retail investors, while only 700,000 were institutional investors. Retail investors held 99.8% of the stock accounts[1]. Compared with institutional investors, retail investors have weaker information collection capabilities and face challenges in accessing comprehensive and timely market information. This information asymmetry makes them more prone to overreaction, contributing to the asymmetry in the distribution of stock returns.

When new information, such as news or earnings reports, emerges, retail investors, due to their lack of professional expertise, often exhibit herding behavior. This leads to a shift of skewness risk from individual stocks to the broader market, amplifying systemic risk. For example, when a company releases an excellent quarterly financial report, retail investors usually buy a large number of the company's shares, causing the stock price to rise sharply in the short term, thereby affecting the distribution of stock returns. However, this overbought behavior caused by retail investors' overreaction is usually corrected by the market when the next quarterly earnings report is released. This phenomenon shows that retail investors' overreaction to earnings reports may have a cyclical characteristic of about one quarter. The same is true for news information in the market. A good (bad) piece of information may cause investors to be excited (panic) in a short period of time, causing stock prices to rise (fall), but it does not last long. Prices will return to their proper levels in the short term. These observations indicate that the overreaction and information asymmetry among retail investors exacerbate the asymmetry in the distribution of stock returns, and this effect generally exhibits a short-

[1] [See](Leippold et al., 2022)

term cyclical pattern. Therefore, analyzing the cyclical nature of coskewness can help us better understand stock pricing in China's A-share market.

This paper combines the skewness asset pricing model with frequency domain analysis by employing the extended Wold representation to investigate the pricing ability of coskewness at different frequencies (cycles). Using the extended Wold representation proposed by Ortu et al. (2020), we could decompose stock returns and the market factor into time series representing different frequencies. This allows us to calculate the coskewness between stock returns and the market factor at various frequencies, thereby exploring the explanatory power of coskewness across different frequencies/cycles in explaining the cross-section of stock returns. It is worth noting that the representation of the extended Wold representation is derived in the time domain. Therefore, although it has similar insights with many frequency domain methods, this method is particularly suitable for time series forecasting. Its adaptability of time and localization in both frequency and time domain provide significant advantages in analytical applications.

By utilizing the extended Wold representation and combining it with traditional financial models, we can examine the influence of different factors on stock returns across different frequencies. In recent years, several studies have begun focusing on this area (Chaudhuri and Lo, 2015,0; Bandi et al., 2019,0). For instance, Bandi et al. (2021) apply the extended Wold representation to decompose the market factor into different frequencies based on the Capital Asset Pricing Model (CAPM), thereby deriving market betas corresponding to specific frequencies, and they refer the new model as the spectral factor model. They find that a market beta with a cycle of 32- to 64-month horizon effectively explain the cross-sectional variation in stock returns. The spectral factor model, containing only this frequency component, exhibits the same explanatory power as other multi-factor models. However, Bandi et al. (2021) build their spectral factor research on CAPM, which is widely established in the literature that CAPM has several shortcomings in explaining stock returns.

CAPM posits that the expected return and risk of an asset can be described by its mean and variance, and assume that asset return distributions follow a normal distribution. However, many scholars have found in subsequent studies that the return distribution of an asset does not necessarily follow a normal distribution, but instead exhibits characteristics of "peaked and thick-tailed" and "skewed" distribution (Gray and French, 1990; Peiro, 1999). As a result, it has been argued that incorporating skewness into CAPM is necessary (Levy, 1969; Alderfer and Bierman, 1970; Kraus and Litzenberger, 1976; Harvey and Siddique, 2000; Baruník and Křehlík, 2018). Notably, Kraus and Litzenberger (1976) firstly introduce the skewness asset pricing model (three-moment CAPM) by utilizing the stochastic discount factor and investors' utility function. The skewness asset pricing model builds on CAPM by including the skewness of the return distribution, allowing it to better capture asymmetry of return distribution and addressing CAPM's limitation of assuming normally distributed returns. In this model, expected returns are influenced not only by the market factor but also by the systematic skewness factor, with coskewness measured by the covariance between stock returns and the systematic skewness factor. Their results indicate that the skewness factor could significantly explain the cross-sectional differences in stock returns. Harvey and Siddique (2000) propose a direct measurement for coskewness from a conditional expectation perspective based on the work of Kraus and Litzenberger (1976) and develope a new coskewness factor. They find that this new coskewness factor helps explain variations in expected returns across different assets. Therefore, we believe that the study of spectral factors should not only be based solely on CAPM but also consider the impact of skewness.

This article uses all stock data from China's A-share stock market from January 1993 to December 2022 for research, and uses the direct measurement method of coskewness proposed by Harvey and Siddique (2000) to calculate the coskewness between assets and the market factor. By employing the extended Wold representation, we decompose the market factor and stock returns into different frequencies and then use the direct coskewness measurement to compute the coskewness at each frequency. Finally, we test the conditional skewness asset pricing model by utilizing the cross-sectional regression from Fama and MacBeth (1973).

Following the approach of Bandi et al. (2021), we first decompose individual stock returns and market returns into high-frequency ($j \leq 4$) and low-frequency ($j \geq 4$) components, and apply Harvey and Siddique (2000)'s method to calculate the coskewness for both high- and low-frequency components. Here, j describes different frequency components, with larger values corresponding to lower frequencies. We subsequently perform Fama-MacBeth cross-sectional regressions on total coskewness as well as the high- and low-frequency coskewness components, finding that Both the coskewness and its high-frequency component exhibit negative coefficients, which are statistically significant at the 5% confidence level. This aligns with previous research, indicating that higher coskewness is associated with lower expected stock returns.

Furthermore, we decompose the high-frequency component into four frequency components (from j=1 to j=4) and find that the coskewness of frequency components at j=1 and j=2 is more significant and has a greater impact on expected stock returns than the ones of j=3 and j=4. These results demonstrate that the impact of coskewness risk on stock returns primarily manifests over short time period. This aligns with previous observations that short-term market information, such as earnings reports and news releases, may lead to emotional investment decisions and overreactions by retail investors, causing skewed stock return distributions. Moreover, the effects of these behaviors are typically corrected by the market within a short time.

However, when we add other control variables into the regression, the coefficient of overall coskewness becomes insignificant. This result indicates that overall coskewness is not stable and has weak robustness, failing to remain significant under the interference of other factors. Nevertheless, the coefficients of high-frequency coskewness (i.e., at j=1 and j=2) remain significant despite the influence of the multi-factor model, demonstrating strong robustness. This suggests that even after controlling for other risk factors, high-frequency coskewness is still significantly negatively correlated with stock returns.

Furthermore, after applying the portfolio sorting method proposed by Fama and French (1992), we find that a long-short strategy based on overall coskewness fails to generate significant positive average returns or positive alpha. However, a long-short strategy using high-frequency coskewness can produce significant positive average returns and alpha. This indicates that by utilizing the extended Wold representation theorem, we can extract the effective information from the factors, thereby enhancing their pricing power. Additionally, we performed portfolio sorting analysis using $\beta^{(1)}_{SKD}$ and $\beta^{(2)}_{SKD}$ separately, and found that only the long-short strategy based on $\beta^{(1)}_{SKD}$ yields significant positive average returns and alpha. This shows that the pricing source of coskewness mainly comes from the high-frequency component of individual stock and market portfolio returns, especially the frequency component with a period of 1-2 months.

Distinguished from existing literature, the main contributions of this paper are as follows: First, this study demonstrates the critical role of coskewness risk in the pricing of stock returns. Second, by combining the extended Wold representation with the skewness asset pricing model, this paper offers a new perspective to the literature. Unlike previous studies that primarily explore cross-sectional differences in stock returns using various factor models, this research delves into the impact of coskewness at different frequencies on stock returns in the cross-section. Third, the findings of this paper not only help investors better understand the risk characteristics of the Chinese stock market but also provide valuable insights for Chinese regulators to improve their risk supervision systems.

The structure of this paper is as follows: in Section 2 we review the relevant literature, in Section 3 we introduce the data and how to build the model, in Section 4 we report the main empirical analysis results and robustness tests, and finally, in Section 5 we summarize the paper.

2 Literature Review

2.1 Skewness and asset pricing

In traditional methods, investors describe an asset's return and risk using its mean and variance. Since the introduction of the mean-variance model by Markowitz (1952), it has been widely applied in portfolio selection and has become the foundation of modern portfolio theory. This model uses variance and mean to depict an asset's risk and return, assuming that asset returns follow a normal distribution. Based on this theory, Sharpe (1964), Treynor (1961a,9), Lintner (1969), and Mossin (1966) jointly proposed the Capital Asset Pricing Model (CAPM). CAPM believes that there are two main parts determine the assets' expected return, the systematic risk related to the market and the non-systematic risk unrelated to market factors. The sensitivity of an asset to systematic risk is described by the coefficient,beta, of the market factor. The more sensitive the asset is to the market factor, the higher the beta, and the higher its expected return should be. Although the CAPM model first realized the quantification of risk and derived the relationship between expected return and risk, subsequent studies on CAPM found that it cannot explain many asset returns.

Many studies argue that one of the reasons for the failure of CAPM is its omission of skewness. Kraus and Litzenberger (1976) firstly suggest that risk-averse investors have preference for assets with positive skewness, which is similar to how they dislike variance in their portfolios, and are willing to pay a premium for purchasing assets with positive skewness. Friend and Westerfield (1980) tests Kraus and Litzenberger (1976)'s theory by using various stock and bond return indices. Their findings support the argument made by Kraus and Litzenberger (1976), showing that in addition to covariance, co-skewness should also be considered to capture the variation of asset return, though this conclusion may depend on the market index and estimation methods used. Harvey and Siddique (2000) formalized the insights of Kraus and Litzenberger (1976) by emphasizing conditional skewness over unconditional skewness. Their results demonstrate that conditional skewness significantly captures the variation in cross-sectional expected returns, even when controlling for size and value factors.

The literature on skewness primarily focuses on two types: one stream of research examines the role of systematic skewness in asset pricing, while another investigates the effects of idiosyncratic skewness. Systematic skewness mainly refers to the skewness of the market portfolio. Studies in this area focus on the skewness and co-skewness between stock and market returns. Kraus and Litzenberger (1976) extend the CAPM by including the systematic skewness factor, which is called the skewness asset pricing model. In this model, they incorporate an quadratic form of themarket factor and name it systematic skewness. Coskewness is measured by the covariance between the asset excess return and systematic skewness, which is the numerator of the coefficient of systematic skewness. It captures the contribution of the asset to the market portfolio skewness. Their result confirm a significant risk price of systematic skewness. Yang et al. (2010) employ a bivariate regime-switching model under the framework of the skewness asset pricing model to describe the coskewness between stock and bond returns. Their findings demonstrate that both stock and bond coskewness can generate significant and robust risk premiums. Engle (2011) shows that the negative long-term skewness could rise the default rate and high correlated defaults reinforce the possibility of another systematic risk. Chabi-Yo et al. (2014) study the relation between demand in equilibrium and risk premium incorporating the skewness risk. They show that an asset's yield risk premium of market skewness under the condition that both the co-skewness of it with market portfolio and market skew-tolerance exist. Langlois (2020) investigate the effect of systematic skewness on expected stock returns. Their results reveal that the estimated systematic skewness risk factor is linked to a substantial and robust risk premium, ranging from 6% to 12% per year.

Idiosyncratic skewness refers to the firm-level skewness. Some literature believe idiosyncratic skewness is generated by under-diversified portfolios and investors may remain under-diversified to obtain the likelihood of extreme positive large return. In this case, idiosyncratic skewness should be priced (Mitton and Vorkink, 2007; Barberis and Huang, 2008; Boyer et al., 2010; Conrad et al., 2013; Amaya et al., 2015). For example, Mitton and Vorkink (2007) document that portfolios owned by under-diversified investors exhibit more positive skewed than those owned by diversified investors and a strong negative relationship exists between idiosyncratic skewness and the Sharpe ratio. Amaya et al. (2015) report a negative correlation between future returns and individual realized skewness, demonstrating that a long-short portfolio ranked by realized skewness produces a significant average return.

Although both downside risk and coskewness risk describe some perspective of downside variation, they are two different risk in the literature (Ang et al., 2006; Huang et al., 2012; Kelly and Jiang, 2014; Van Oordt and Zhou, 2016).For instance, Ang et al. (2006) contend that downside beta specifically accounts for market downturns, whereas coskewness does not explicitly highlight market asymmetries. Their result confirm that past co-skewness could predict future return, but its predictive power is not due to the capture of exposure to downside risk indicating they are two different risk loading. Kelly and Jiang (2014) show the tail risk premium is robust and significant after controlling downside beta and coskewness.

2.2 Time-varying skewness

With the further study of skewness, some find that the persistence of skewness is not consistent over time. Some studies focus on time-varying skewness by using different time periods. Singleton and Wingender (1986) estimate skewness over 5 year time horizon and find that positive skewness occurs at constant frequency over time. In contrast, the skewness of individual stock and portfolio does not show persistence in different time periods. Farago and Hjalmarsson (2023) derive a theoretical model for skewness in compounded return in long-horizon to investigate the skew-inducing effect. They find that compounded returns have positive skewness and their skewness increase with the time horizon. Aretz and Arisoy (2023) construct the realized return skewness over different horizons suing block bootstrap estimator. They provide evidence that the pricing ability of skewness is mainly driven by short-term instead of long-term return horizons.

Several studies employ time-series analysis to model time-varying skewness. Harvey and Siddique (1999) estimate time-varying conditional skewness by extending the traditional GARCH model within a maximum likelihood framework, showing that autoregressive conditional skewness is significant and influences variance persistence. Smith (2007) apply the GMM method to estimate conditional two- and three-moment CAPM, rejecting the two-moment CAPM but supporting the three-moment version. Their findings indicate that both coskewness and investors' preferences for skewness vary over time, with greater concern for coskewness risk when the market exhibits positive skewness. Yang et al. (2010) explore time-varying coskewness using a bivariate regime-switching model, demonstrating that both stock and bond coskewness yield statistically and economically significant risk premiums.

The literature mentioned above illustrate that there is an improving performance after considering the time-varying skewness. Some literature provide the evidence that skewness is related to earnings announcement events and return horizon. For example, Albuquerque (2012) propose periodicity in earnings announcement events could lead to positive skewness in firm return. It is natural to raise question whether skewness should change with frequency. In recent two decades, a large amount of literature consider the variation of the factor risk across different frequencies (Chaudhuri and Lo, 2015; Dew-Becker and Giglio, 2016; Baruník and Křehlík, 2018; Neuhierl and Varneskov, 2021; Bandi et al., 2021). For example, Dew-Becker and Giglio (2016) estimate frequency -specific risk prices by a novel frequency domain decomposition and find that long-run risks that is longer than business cycle are priced significant in the stock market. Neuhierl and Varneskov (2021) provide a model-free framework to decompose the state vector into different frequency and allow its risk prices change with frequency. They find evidence that the risk of low and high-frequency state vector are priced differently in the stock market.

In China, there is no consensus on whether co-skewness risk is priced in the Chinese A-share market. Wang (2006) explore the improvement of the traditional CAPM model by incorporating higher moments, finding that higher-moment information significantly enhances the predictive accuracy of models in the Chinese stock market. Gang and Lin (2021) explore the role of skewness in asset pricing within the Chinese A-share market and demonstrate that co-skewness is negatively correlated with expected excess stock returns, while also improving the explanatory power of traditional factor models. However, Yu (2017) find that co-skewness does not exhibit significant pricing power, suggesting that higher-order risks are not fully rationally priced by investors.

In China, some literature has recognized the importance of skewness. Wang (2006) studied the asset pricing role of skewness in China's A-share market, showing that coskewness is negatively related to expected excess returns and improves the explanatory power of traditional factor models. Yu (2017) found that coskewness did not exhibit significant pricing power, suggesting that higher-order risks are not fully rationally priced by investors. Wang (2006) explored improving the traditional CAPM model by incorporating higher-order moments and found that higher-order moment information significantly improved the prediction accuracy of the model in the Chinese stock market.

Although some Chinese studies have considered incorporating skewness into asset pricing models, few of them have examined the relationship between skewness and cycles from a frequency perspective. This paper uses the method of Harvey and Siddique (2000) to construct conditional co-skewness and applies the extended Wold representation to decompose asset return and the market factor into different frequencies. Then, we study the impact of coskewness across different frequencies on stock returns.

3 Data

The dataset used in this study comprises all stocks from China's A-share market, spanning the period from January 1993 to December 2022. All data are sourced from the China Stock Market & Accounting Research (CSMAR) database. Given the necessity to compute multiple indicators, the regression analysis focuses on the period from January 2000 to December 2022. There are two primary reasons for using data post-2000:

First, choosing data after 2000 could ensure the consistency of accounting data. China's financial reporting regulations were gradually refined around 1999. Although the principles of fair trade and financial disclosure were introduced in 1993, the lack of clear guidelines led to inconsistencies in the application of these standards by different firms, limiting the comparability of accounting data. It was not until 1998 and 1999 that more detailed laws and regulations were introduced to regulate transactions and financial reporting.

Second, this time period provides a sufficiently robust data sample. To apply the Fama-MacBeth regression analysis and ensure reasonable precision and statistical power, each stock must have at least 35 months of data. The filtering criteria are as follows: stocks listed for fewer than six months, stocks with fewer than 120 trading records in the previous year, or fewer than 15 trading records in the prior month. The last set of criteria is designed to mitigate the impact of stock returns after prolonged trading suspensions. It is only after 1999 that the number of stocks in the market satisfies these requirements. Additionally, data from the first six months following an IPO (including the listing month) are excluded, along with ST (Special Treatment), *ST (delisting risk warning), PT (Special Transfer) stocks, financial sector stocks, and stocks with negative book values. These measures are implemented to further enhance the robustness of the results.

[Insert Table 1 here]

Table 1 presents the descriptive statistics of individual stock returns and the five factors, along with the correlation coefficients between the factors. Panel A reports the descriptive statistics of individual stock returns and each factor. First, the average stock return is 1.112%, with standard deviation of 15.202% and skewness of 11.123%. This positive skewness suggests that stock returns in the Chinese market exhibit a right-skewed distribution. Among the five factors, the average return of market factor is 0.4948%, with standard deviation of 8.0742% and skewness of 0.1575%. The positive skewness of the market factor also reflects the "right-skewed" nature of the returns in China's A-share market. Next, the mean return of the size factor is the highest among the five factors, which is 0.8445%, while the investment factor has the largest skewness, which is 0.3263%. Table 2 shows the construction methods of the factors.

[Insert Table 2 here]

Panel B displays the correlation between the factors. The market factor (MKT) has weak correlations with other factors, with the highest correlation being 0.1573 with the size factor (SMB). The strongest negative correlation is between the profitability factor (RMW) and the size factor (SMB) at -0.6304. The investment factor(CMA) has a relatively strong negative correlation with the profitability factor (RMW), with a correlation of -0.6139.

4 Method

4.1 Why do investors prefer stocks with positive skewness?

[Insert Figure 1 here]

Skewness is measured by the third moment of stock returns. Intuitively, positive skewness (negative skewness) means that there is a greater probability of small losses (gains) and a smaller probability of large positive gains (losses). Figure 1 illustrates the positively and negatively skewed distributions of stock returns, where the horizontal axis represents stock return, and the vertical axis represents the probability of the return. In general, we assume the mean of stock returns is zero. For stocks with positive skewness, the right tail of the distribution is longer, indicating a small probability of very large positive returns, while the peak of the probability density function is on the left of the mean, implying a higher probability of small losses. For stocks with negative skewness, the situation is reversed. Avoiding potential large losses is crucial for a risk-averse investor, which explains why they tend to prefer stocks with positively skewed returns. Coskewness quantifies an asset's contribution to the overall skewness of a portfolio. If an asset exhibits positive coskewness with the market portfolio, adding it to the portfolio will increase the portfolio's total skewness. Since investors generally prefer assets with positively skewed returns, assets that increase the overall skewness of the portfolio are more likely to be favored by investors. However, merely relying on intuitive logic is not sufficiently rigorous; we need to mathematically demonstrate that investors indeed prefer positively skewed stocks.

[Insert Table 3 here]

In the next section, we will prove why investors prefer positively skewed stocks, starting from the utility function, with the explanation of symbols provided in Table 3. Let w represent the investor's wealth, and x denote the investor's income (a random variable). Assuming the investor's utility function, U, only depends on their total wealth and income, and it can be defined as $U = U(\widetilde{x} + w)$. The return of the investor's investment, w, is given by $\tilde{r} = \widetilde{x}/w$, and the utility function can be expressed as $U = U(rw + w)$. Next, let $\mu = E(w + rw)$ denote the expected return of the investment. Expand U using a Taylor series and take the expectation on both sides of the equation, then we could obtain:

$$E(U) = U(\mu) + \frac{U^2(\mu)}{2}\sigma^2 + \sum_{i=3}^{\infty} \frac{\mu_i}{i!} U^n(\mu) \tag{1}$$

where U^n denotes the n-th derivative of U, and μ_i represents the i-th moment of U. In order to analyze the expected utility of investment returns w, it is necessary to define the relevant moments of the return distribution and the sign of the coefficient for each moment. For certain specific return distributions, such as the normal distribution, uniform distribution, and binomial distribution, the mean and variance can fully describe the expected utility $E(U)$. However, in the following cases: (i) the return distribution of the portfolio is asymmetric, (ii) when the investor's utility function is of a higher order than quadratic,, and (iii) when the mean and variance alone are inadequate to fully characterize the distribution, it becomes essential to account for the moments higher than second moment, as well as the signs of their coefficients. For moments beyond the variance, two questions can be posed: (1) Can the direction of preference for each moment (i.e., the sign of U^n in equation (1)) be determined? (2) If the direction could be determined, what the direction of preference is for each moment?

According to Scott and Horvath (1980), an investor who exhibits strict consistency in the direction of preference for the n-th moment should have a utility function which can only result in the following three cases:

$$\begin{aligned} &U^n(w) > 0 \forall w, \\ &U^n(w) = 0 \forall w, \quad \text{or} \\ &U^n(w) < 0 \forall w. \end{aligned} \tag{2}$$

In financial theory, we typically assume that for a risk-averse investor, the first and second derivatives of the utility function have the following signs:

$$\begin{aligned} &U^1(w) > 0 \forall w, \quad \text{and} \\ &U^2(w) < 0 \forall w. \end{aligned} \tag{3}$$

An investor with consistently positive marginal utility of wealth, stable risk aversion, and strict preference consistency over time will favor positive skewness and exhibit an aversion to negative skewness. This suggests that $U^3(w) > 0$.

Proof: We first assume that $U^3(w) < 0$ for all w, or $U^3(w) = 0$ for all w. By the Mean Value Theorem, it follows that for any $w_2 > w_1$, there exists a $\bar{w} \in (w_1, w_2)$ such that:

$$U^1(w_2) - U^1(w_1) = U^2(\bar{w})(w_2 - w_1)$$

By rearranging the above equation, we obtain:

$$U^1(w_2) = U^1(w_1) + U^2(\bar{w})(w_2 - w_1)$$

$U^3(w) \leq 0$, therefore:

$$U^2(w_1) \geq U^2(\bar{w})$$
$$U^1(w_2) \leq U^1(w_1) + U^2(w_1)(w_2 - w_1)$$

For $w_2 \geq w^* = w_1 + \frac{U^1(w_1)}{-U^2(w_1)}$, we obtain $U^1(w_2) \leq 0$. For $w_2 > w^*$, $U^1(w_2) < 0$. This contradicts the assumption that $U^1(w) > 0$, therefore $U^3(w) > 0$. Consequently, it can be concluded that investors prefer stocks with positively skewed return distributions.

4.2 Coskewness and measurement

From the above derivation, we can conclude that, all else being equal, investors prefer assets whose return probability density function is positively skewed. For a portfolio, assets that reduce its skewness are less favored by investors. In order to remain competitive, such assets must offer higher expected returns. Similarly, assets that enhance the skewness of a portfolio are expected to generate lower returns. Coskewness quantifies the degree to which an asset contributes to the overall skewness of the portfolio. Harvey and Siddique (2000) propose a method to directly measure co-skewness from a conditional expectation perspective. This paper adopts the latter method, and the formula is as follows:

$$\hat{\beta}_{\mathrm{SKD}_i} = \frac{E\left[\epsilon_{i,t+1}\epsilon^2_{M,t+1}\right]}{\sqrt{E\left[\epsilon^2_{i,t+1}\right]}E\left[\epsilon^2_{M,t+1}\right]}, \tag{4}$$

where $\epsilon_{i,t+1} = r_{i,t+1} - \alpha - \beta_i(r_{M,t+1})$, represents the residual of corresponding CAPM for the same period. $\epsilon_{M,t+1} = r_{M,t+1} - r_{f,t} - \mu_M$ denotes the difference between the market portfolio return and its mean. β_{SKD} represents a stock's contribution to the skewness of the market portfolio. A negative β_{SKD} indicates that a stock reduces the skewness of the market portfolio. As earlier derivation, a stock exhibiting negative coskewness with the market is expected to have a higher return, implying that the coskewness risk premium is negative.

4.3 The extended Wold representation

According to Wold (1938), a non-deterministic stationary process can always be represented as the sum of a moving average of white noise and a deterministic stationary process. Suppose we have a bivariate covariance stationary non-deterministic process $\boldsymbol{X} = (y_t, x_t)^T{}_{t\in\mathbb{Z}}$, which can represent stock and factor returns. Then, it can be expressed as:

$$\boldsymbol{X}_t = \begin{pmatrix} y_t \\ x_t \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha_k^1 & \alpha_k^2 \\ \alpha_k^3 & \alpha_k^4 \end{pmatrix} \begin{pmatrix} \varepsilon_{t-k}^1 \\ \varepsilon_{t-k}^2 \end{pmatrix} = \sum_{k=0}^{\infty} \boldsymbol{\alpha_k \varepsilon_{t-k}} + \boldsymbol{V_t} \tag{5}$$

where $\boldsymbol{V_t}$ is the deterministic process and is always omitted. The equation holds under the L^2 norm, and $\sum_{k=0}^{\infty} tr^{1/2}(\boldsymbol{\alpha_k}^T\boldsymbol{\alpha_k}) < \infty$, with $\boldsymbol{\alpha_0} = I_2$ (I_2 is a 2×2 identity matrix), where $tr()$ denotes the sum of the diagonal elements.

To make the explanation more intuitive, we begin with a finite one-dimensional sequence and consider a moving average process with seven lags, which is defined as follows:

$$x_t = \sum_{k=0}^{7} \alpha_k \varepsilon_{t-k} \tag{6}$$

By calculating the means and differences between adjacent terms, x_t can be alternatively expressed as:

$$x_t = \sum_{k=0}^{3} \beta_k^{(1)} \varepsilon_{t-2k}^{(1)} + \sum_{k=0}^{3} \gamma_k^{(1)} \bar{\varepsilon}_{t-2k}^{(1)} \tag{7}$$

where $\varepsilon_{t-2k}^{(1)}$ contains high-frequency information, while $\bar{\varepsilon}_{t-2k}^{(1)}$ contains low-frequency information [2]. The coefficients $\beta_k^{(1)}$ and $\gamma_k^{(1)}$ describe the sensitivity of x_t to $\varepsilon_{t-2k}^{(1)}$ and $\bar{\varepsilon}_{t-2k}^{(1)}$, respectively. Specifically, $\beta_k^{(1)}$ and $\gamma_k^{(1)}$ can be obtained from the original coefficients α_h through the following linear transformation:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \beta_k^{(1)} \\ \gamma_k^{(1)} \end{bmatrix} = \begin{bmatrix} \alpha_{2k} \\ \alpha_{2k+1} \end{bmatrix} \tag{8}$$

Next, we concentrate on the low-frequency component $\pi^{(1)} = \sum_{k=0}^{3} \gamma_k^{(1)} \bar{\varepsilon}_{t-2k}^{(1)}$, and combine the innovation terms in the same way as before:

$$\varepsilon_{t-4k}^{(2)} = \frac{\bar{\varepsilon}_{t-4k}^{(1)} - \bar{\varepsilon}_{t-4k-2}^{(1)}}{\sqrt{2}}, \quad \bar{\varepsilon}_{t-2k}^{(2)} = \frac{\bar{\varepsilon}_{t-4k}^{(1)} + \bar{\varepsilon}_{t-4k-2}^{(1)}}{\sqrt{2}}.$$

such that:

$$\pi^{(1)} = \sum_{k=0}^{1} \beta_k^{(2)} \varepsilon_{t-4k}^{(2)} + \sum_{k=0}^{1} \gamma_k^{(2)} \bar{\varepsilon}_{t-4k}^{(2)}$$

Through recursive iteration, the original sequence can be expressed as:

$$x_t = \sum_{j=1}^{3} \sum_{k=0}^{2^{3-j}-1} \beta_k^{(j)} \varepsilon_{t-k2^j}^{(j)} + \gamma_0^{(3)} \bar{\varepsilon}_t^{(3)} \tag{9}$$

[2] $\varepsilon_{t-2k}^{(1)}$ can be seen as the difference between adjacent terms, while $\bar{\varepsilon}_{t-2k}^{(1)}$ can be seen as their mean. In signal processing, averaging acts as a low-pass filter, while differencing acts as a high-pass filter.

where each $\varepsilon t - k2^{j(j)}$ and $\bar{\varepsilon}_t^{(3)}$ can be represented in terms of the white noise of the original sequence as:

$$\varepsilon_t^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-i} - \sum_{i=0}^{2^{j-1}-1} \varepsilon_{t-2^{j-1}-i} \right), \quad \bar{\varepsilon}_t^{(3)} = \frac{1}{\sqrt{2^3}} \sum_{i=0}^{2^3-1} \varepsilon_{t-i}$$

The coefficients $\beta_k^{(j)}$ and $\gamma_k^{(3)}$ can be expressed in terms of the original coefficients α_k as:

$$\beta_k^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+i} - \sum_{i=0}^{2^{j-1}-1} \alpha_{k2^j+2^{j-1}+i} \right), \quad \gamma_k^{(3)} = \frac{1}{\sqrt{2^3}} \sum_{i=0}^{2^3-1} \alpha_{k2^3+i}$$

The parameter j describes different frequency components, with each frequency component distinguished through averaging and differencing.

Now, we extend the finite time series process to an infinite process $\boldsymbol{X_t} = (y_t, x_t)^T{}_{t\in\mathbb{Z}}$, which can be expressed as:

$$\boldsymbol{X}_t = \begin{pmatrix} y_t \\ x_t \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha_k^1 & \alpha_k^2 \\ \alpha_k^3 & \alpha_k^4 \end{pmatrix} \begin{pmatrix} \varepsilon_{t-k}^1 \\ \varepsilon_{t-k}^2 \end{pmatrix} = \sum_{k=0}^{\infty} \boldsymbol{\alpha_k \varepsilon_{t-k}} \tag{10}$$

By applying the previous procedure, a bivariate stationary non-deterministic process $\boldsymbol{X_t}$ with zero mean can be represented as:

$$\boldsymbol{X}_t = \begin{pmatrix} y_t \\ x_t \end{pmatrix} = \sum_{j=1}^{+\infty} \sum_{k=0}^{+\infty} \boldsymbol{\Psi_k^{(j)} \varepsilon_{t-k2^j}^{(j)}} \tag{11}$$

where the dependence between the rescaled $\varepsilon_t^{(j)}$ and the original innovation is as follow:

$$\boldsymbol{\varepsilon}_t^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \boldsymbol{\varepsilon}_{t-i} - \sum_{i=0}^{2^{j-1}-1} \boldsymbol{\varepsilon}_{t-2^{j-1}-i} \right) \tag{12}$$

and the coefficients are uniquely determined by:

$$\boldsymbol{\Psi}_k^{(j)} = \frac{1}{\sqrt{2^j}} \left(\sum_{i=0}^{2^{j-1}-1} \boldsymbol{\alpha}_{k2^j+i} - \sum_{i=0}^{2^{j-1}-1} \boldsymbol{\alpha}_{k2^j+2^{j-1}+i} \right) \tag{13}$$

In this way, the original process can be represented as the sum of orthogonal components at different frequencies:

$$\boldsymbol{X}_t = \begin{pmatrix} y_t \\ x_t \end{pmatrix} = \sum_{j=1}^{+\infty} \sum_{k=0}^{+\infty} \boldsymbol{\Psi_k^{(j)}} \boldsymbol{\varepsilon_{t-k2^j}^{(j)}} = \sum_{j=1}^{+\infty} \boldsymbol{X}_t^{(j)} \tag{14}$$

where $\boldsymbol{X_t^{(j)}} = \sum_{k=0}^{+\infty} \boldsymbol{\Psi_k^{(j)}} \boldsymbol{\varepsilon_{t-k2^j}^{(j)}}$. The frequency components $\boldsymbol{X_t^{(j)}}$ and $\boldsymbol{X_t^{(k)}}$ are uncorrelated:

$$\mathbb{E}\left[\mathbf{x}_{t-m2^j}^{(j)} \mathbf{x}_{t-n2^k}^{(k)}\right] = \mathbf{0} \quad \forall j \neq k, \quad \forall m, n \in \mathbb{N}_0, \quad \forall t \in \mathbb{Z}$$

In summary, the bivariate time series $\boldsymbol{X}_t$ can be expressed as the sum of infinitely many orthogonal frequency components, i.e., $\boldsymbol{X}_t = \sum_{j=1}^{\infty} \boldsymbol{X}_t^{(j)}$. Each frequency component captures periods between 2^{j-1} and 2^j. The interpretation of parameter j is shown in Table 4.

[Insert Table 4 here]

4.4 Identification

In this section, we explain how to apply the extended Wold representation to real-world data. Essentially, the Wold decomposition is a time series technique that distinguishes different frequency components within a time series. Thus, the target of the decomposition is a time series, such as returns or other macroeconomic variables. Suppose we have a series of stock returns and aim to decompose it into different frequencies using extended Wold representation. First, we need to compute the Wold representation coefficients $\boldsymbol{\alpha_k}$ of $\boldsymbol{x_t}$.

We assume that $\boldsymbol{x_t}$ follows a p-th order vector autoregressive process (VAR(p)) of the form:

$$\boldsymbol{x}_t = \boldsymbol{A}_1\boldsymbol{x}_{t-1} + ... + \boldsymbol{A}_p\boldsymbol{x}_{t-p} + \varepsilon_t \tag{15}$$

where A_i are k dimension square matrices and $\boldsymbol{\varepsilon_t}$ is a k dimension white noise vector. VAR(p) process could be transformed to VAR(1) process by stacking $\boldsymbol{x}$ into a larger vector, that is:

$$\boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{\varepsilon}_t \tag{16}$$

If all eigenvalues of the matrix $\boldsymbol{A}$ lie outside the unit circle, the VAR process is stable, and the Wold decomposition can be applied. The residuals in equation (16), also known as innovations, represent the white noise process in the traditional Wold representation described in equation (14). The above VAR(1) process can then be transformed into a moving average representation:

$$\boldsymbol{X}_t = \sum_{k=0}^{\infty} \boldsymbol{\alpha}_k \boldsymbol{\varepsilon}_t \tag{17}$$

The coefficients $\boldsymbol{\alpha_k}$ for the Wold decomposition can be computed as $\boldsymbol{\alpha_k} = A^k$.

To summarize this section: First, we assume that the stock return series follows a VAR(p) process, and we convert it into a VAR(1) process as shown in equation (16). The VAR(1) model can be estimated using unconstrained least squares. Then, we compute the Wold representation coefficients using $\boldsymbol{\alpha}_k = A^k$. Finally, we substitute the residuals of the VAR(1) process into equation (17) to obtain the frequency-specific Wold representation of the process $\boldsymbol{X}$.

4.5 Frequency-specific coskewness

To calculate the coskewness at each frequency, we perform regressions of individual stock returns on market excess returns every 60 months to estimate β_M and α, thereby obtaining the residuals. Next, we use the extended Wold representation to decompose the CAPM residuals of each stock and the market's excess returns. Then, we use the market return at each frequency to subtract its past 60-month average, which is denoted as $(R_M - \mu_{RM})$. The coskewness of stock returns with the market at frequency j is calculated as follows:

$$\hat{\beta}^{(j)}_{\mathrm{SKD}_i} = \frac{E\left[\epsilon^{(j)}_{i,t+1} \cdot (\epsilon^{(j)}_{M,t+1})^2\right]}{\sqrt{E\left[\epsilon^{(j)}_{i,t+1}{}^2\right] E\left[\epsilon^{(j)}_{M,t+1}{}^2\right]}} \tag{18}$$

where $\epsilon^{(j)}_{i,t+1}$ is the residual series at frequency j obtained from the extended Wold representation. $\epsilon^{(j)}_{M,t+1} = (r_{M,t+1} - r_{f,t})^{(j)} - \mu^{(j)}_M$ represents the difference between market excess return and its past 60-month average at frequency j. $\beta^{(j)}_{SKD}$ measures stock i's contribution to market skewness at frequency j.

5 Empirical results and discussion

In this section, we empirically examine the impact of coskewness at different frequencies/cycles on stock returns in the China A-share stock market. Since we mainly focus on individual stocks, the dependent variable is the return of individual stocks. The returns of portfolios, on the other hand, are calculated by value-weighting the returns of multiple stocks. Therefore, portfolio returns are less influenced by skewness and are not ideal tested assets.

5.1 Fama-Macbeth regression

This paper primarily employs the Fama-Macbeth cross-sectional regression method (Fama and MacBeth, 1973) for regression analysis. The main drawbacks of traditional cross-sectional regression are its neglect of time variation and its assumption of constant coefficients, which can lead to unstable estimates and biased standard errors, reducing the accuracy of significance tests and limiting its ability to capture dynamic changes in data. In contrast, the Fama-Macbeth cross-sectional regression performs regressions at each time point separately and averages the coefficients, allowing it to better account for time variation. This approach provides more robust coefficient estimates and more accurate standard errors while allowing for time-varying regression coefficients, enhancing its ability to handle heterogeneity and dynamic characteristics. As a result, the Fama-Macbeth method offers a significant advantage in validating asset pricing models and conducting financial research.

[Insert Table 5 here]

Table 5 reports the result of the Fama-Macbeth cross-sectional regression. Our sample period spans from January 2000 to December 2022. First, for each individual stock, we estimate its exposure to the market factor, which is denoted as β_{RM}, using time-series regression based on the past 60 months of historical data. Then, at each time point, we conduct a cross-sectional regression using the beta of all individual stocks and their corresponding asset returns. This yields the coefficient of the market beta, also known as the price of risk, at each time point. Finally, we calculate the average price of market risk across all cross-sections to obtain the final market factor risk price.

In regression 1, we perform a Fama-Macbeth regression on individual stock returns against β_{RM} and β_{SKD}. The result shows that the price of coskewness risk, which is the coefficient of β_{SKD}, is -0.713 and statistically significant at the 5% confidence level, with a t-value of -2.30. However, the price of market factor risk is statistically insignificant, with a t-value of only -0.81. The negative coskewness risk price is consistent with our earlier theoretical derivation, indicating that investors prefer stocks with positive skewness. If a stock increases the skewness of the market portfolio, it should be preferred by investors, and its expected return should be lower. This result demonstrates that coskewness risk can significantly explain stock returns in the Chinese A-share market.

Next, to explore the separate effects of the high- and low-frequency components of β_{SKD}, we adopt the high- and low-frequency classification proposed by Bandi et al. (2021). We sum all the frequency components less than or equal to 4 ($j <= 4$) to obtain the high-frequency component (HF). Similarly, we sum all the frequency components greater than 4 ($j > 4$) to obtain the low-frequency component (LF).

In regressions 2 and 3, we perform Fama-Macbeth regressions on individual stock returns against β_{RM}, high-frequency β^{HF}_{SKD}, and low-frequency β^{LF}_{SKD}, respectively. The results show that the coefficient of β^{HF}_{SKD} is significant at the 1% confidence level, while the coefficient of β^{LF}_{SKD} is not significant. Their respective values are -1.622 and 0.245, with t-values of -3.70 and 1.45. Only the coefficient of the high-frequency β^{HF}_{SKD} aligns with our previous theoretical derivation that stocks with higher coskewness tend to have lower expected returns. The coefficient of β^{HF}_{SKD} indicates that for every unit increase in the coskewness between the stock and the market, the expected return of the stock decreases by 1.622. In contrast, the coefficient of β^{LF}_{SKD} suggests that for every unit increase in coskewness, the expected return of the stock increases by 0.245. Meanwhile, the magnitude of the coefficient of β^{HF}_{SKD} is much larger than the one of β^{LF}_{SKD}, which means the expected return of a stock is primarily determined by the high-frequency component of coskewness.

Through the above tests, we find that individual stock returns are mainly related to the high-frequency component β^{HF}_{SKD}. However, the high-frequency component consists of four different frequency components, from $j = 1$ to $j = 4$, with their respective cycle lengths shown in Table 4. To further investigate which specific frequency component affects stock returns, we calculate the β_{SKD} corresponding to $j = 1$ to $j = 4$, and include them in the Fama-Macbeth cross-sectional regression.

[Insert Table 6 here]

Table 6 presents the results of the Fama-Macbeth cross-sectional regression of individual stock returns on coskewness across different frequency components from $j = 1$ to $j = 4$. In regressions 1 and 2, we regress individual stock returns on the coskewness of frequency components $j = 1$ and $j = 2$, and find that the risk prices of coskewness for these two frequency components are statistically significant at the 5% confidence level. The coefficients are -1.113 and -0.613, and t-values are -2.22 and -2.98, respectively.

In contrast, in regressions 3 and 4, the coefficients of β_{SKD} for the frequency components $j = 3$ and $j = 4$ are not significant, with t-values of -1.02 and -0.92. This indicates that individual stock returns are primarily influenced by the coskewness of frequency components $j = 1$ and $j = 2$, which correspond to cycles of 1-2 months and 2-4 months, respectively. This is consistent with the findings of Albuquerque (2012), who discovered that the heterogeneity of earnings announcements leads to positive skewness in individual stock returns and negative skewness in market factors. In their paper, they mention that earnings announcements are concentrated in the second to eighth week of each quarter, implying that about half of each quarter is earnings season. Thus, quarterly earnings announcements, i.e., earnings reports, contribute to the skewness of individual stock returns and market returns. This aligns with our earlier hypothesis: during the release of quarterly financial reports, retail investors, due to their limited capacity to gather comprehensive and timely market information, tend to overreact or delay their responses. The overbought behavior or delayed reactions caused by earnings reports are corrected by the market when the next report is released, thereby influencing the coskewness between stock returns and the market. Therefore, the variation in stock returns is primarily driven by the coskewness of market factors with cycles of four months or less.

5.2 Robustness test

Based on the previous analysis, we have identified that high-frequency coskewness has a significant impact on stock returns, particularly for frequency components with cycles of 1 to 4 months. To ensure the robustness and reliability of this finding, we now conduct a series of robustness tests. These tests further verify whether our conclusions hold when additional control variables are included and whether the pricing ability of coskewness across different frequency components remains consistent and robust. Through these robustness tests, we aim to comprehensively confirm the critical role of high-frequency coskewness in stock return pricing.

[Insert Table 7 here]

To test the robustness of our conclusions, we incorporate five factors as control variables in the regression: the market factor (MKT), size factor (SMB), value factor (HML), profitability factor (RMW), and investment factor (CMA). We then reconstruct $\beta_{SKD}^{(HF)}$ by summing the frequency components of the CAPM residuals for the market factor and individual stocks at $j = 1$ and $j = 2$, and calculate the new high-frequency component using Equation 4.

Table 7 presents the regression results. In regression 1, the coefficient of β_{SKD} is only -0.0081, indicating that after accounting for the five additional factors, the overall β_{SKD} has a minimal effect on expected stock returns, with a t-value of -1.78, which is not significant at the 5% confidence level. However, in regressions 2 and 3, the coefficients of β_{SKD} at $j = 1$ and $j = 2$ are -1.2684 and -0.7331, respectively, which are much larger than the coefficient of the overall β_{SKD}, indicating a greater impact on expected stock returns. The t-values of the coefficients are -2.55 and -2.53, which shows that these coefficients are significant at the 5% confidence level.

Finally, in regression 4, the coefficient of β_{SKD}^{HF} is -0.6858, with a t-value of -2.27. Table 7 illustrates that the overall coskewness is not sufficiently stable in the presence of other factors and shows weak robustness. However, coskewness in the high-frequency components remains highly robust, further confirming the earlier conclusion that β_{SKD} has the most significant impact on stock returns when the cycle is between 1-4 months. Additionally, it demonstrates that by utilizing the extended Wold decomposition, we can extract the effective information from the factors, enhancing their pricing ability.

[Insert Table 8 here]

From the previous tests, we have established that the coefficient of coskewness between individual stocks and the market is negative, meaning that the larger the coskewness between an individual stock and the market, the lower the stock's expected return. Using the portfolio sorting method proposed by Fama and French (1992), we should observe that a portfolio consisting of the 10% of stocks with the smallest coskewness with the market factor should generate higher returns, while a portfolio consisting of the 10% of stocks with the largest coskewness should produce lower returns. In this scenario, we could long the portfolio of stocks with the smallest coskewness and short the portfolio of stocks with the largest coskewness to hedge risk.

We calculate the coskewness of each stock with the market portfolio using the past 60 months of individual stock returns. In the 61st month, we rank all stocks based on their calculated coskewness and divide them into 10 portfolios. Portfolio 1 consists of the 10% of stocks with the smallest coskewness relative to the market, while Portfolio 10 includes the 10% of stocks with the largest coskewness. The return of the long-short strategy is obtained by subtracting the return of Portfolio 10 from that of Portfolio 1.

We mainly focus on whether the average return of the long-short strategy is significantly different from zero, and whether the strategy's return exhibits a significantly positive alpha relative to the CAPM. A significantly positive alpha indicates that the portfolio's returns cannot be fully explained by the market factor and that the portfolio outperforms the market portfolio with a positive excess return.

Table 8 reports the summary statistics of portfolios generated using different grouping methods. For each portfolio, we report its average return, the CAPM alpha, and standard deviation, with p-values presented in parentheses. Panel A reports the statistics for portfolios grouped by overall coskewness. From Portfolio 1 to Portfolio 10, we observe a monotonic decrease in average returns, consistent with our previous results, which indicated that the greater the coskewness between an individual stock and the market, the lower the expected stock return. However, the average return of the long-short portfolio is not significantly different from zero at the 10% confidence level. Additionally, the calculated alpha of the strategy's return is not significantly different from zero at the 10% confidence level, indicating that the long-short strategy based on overall coskewness does not produce a significant alpha relative to CAPM.

Panel B reports the statistics of portfolios grouped by high-frequency coskewness. The last column shows the average return and the market factor's alpha for the long-short strategy, with values of 0.489 and 0.485, respectively, and p-values of 0.096 and 0.054, both of which are significantly different from zero at the 10% confidence level. This suggests that high-frequency coskewness, indeed could lead the cross-sectional difference of stock return. Moreover, the long-short strategy based on sorting high-frequency coskewness generates significant positive returns and positive alpha. This further indicates that, by using the extended Wold decomposition, we can extract the effective information embedded in the original factors, enhancing their pricing ability.

Next, we aim to further refine the high-frequency components and investigate the pricing ability of the $j = 1$ and $j = 2$ frequency components. Similarly, we calculate $\beta_{SKD}^{(1)}$ and $\beta_{SKD}^{(2)}$ using the past 60 months of data, and in the 61st month, we sort individual stocks based on the calculated frequency coskewness. Panels C and D present the data for portfolios grouped by $\beta_{SKD}^{(1)}$ and $\beta_{SKD}^{(2)}$, respectively. Again, we focus primarily on the returns and alpha of the long-short strategy.

The long-short strategy grouped by $\beta_{SKD}^{(1)}$ produces highly significant positive average returns and positive alpha, with values of 1.026 and 1.027, and p-values of 0.001 and 0.000, respectively. Compared to the overall coskewness and high-frequency coskewness used earlier, the average positive return generated by $\beta_{SKD}^{(1)}$ is approximately three times larger. However, the long-short strategy grouped by $\beta_{SKD}^{(2)}$ fails to generate significant positive returns or positive alpha, with values of 0.378 and 0.400, and p-values of 0.133 and 0.140, respectively. Among these four panels, the long-short strategy consistently shows the lowest standard deviation, demonstrating its ability to hedge risk effectively.

In this study, the empirical analysis of China's A-share market reveals the critical role of coskewness in asset pricing. The research finds a significant negative correlation between coskewness and stock returns, particularly in the short term (cycles of 1-4 months), indicating that stock returns are mainly associated with high-frequency coskewness. Further analysis shows that the explanatory power of overall coskewness weakens when considering other factors, but high-frequency coskewness remains robust, suggesting that high-frequency coskewness could significantly explain the stock returns. The analysis of portfolios demonstrates that investment strategies based on high-frequency coskewness can generate significant positive returns and market alpha, validating the pricing efficiency of high-frequency coskewness. These findings provide new direction for risk management and investment strategies in the Chinese A-share market.

6 Conclusion

This paper investigates the role of coskewness in asset pricing through an empirical analysis of the Chinese A-share market, particularly examining the impact of coskewness on cross-sectional stock returns from a frequency/cycle perspective. The study is grounded in the context of the Chinese A-share market, where retail investors dominate. Retail investors tend to overreact or delay their response to financial reports and news over short cycles, leading to a deviation of market returns from a normal distribution. To better understand this phenomenon, this paper introduces frequency-domain analysis, combining the extended Wold representation with the skewness asset pricing model to systematically analyze the effect of coskewness on stock cross-sectional returns at different frequencies.

The findings indicate that coskewness risk has significant explanatory power for stock returns, with a negative correlation between high-frequency coskewness and stock returns. This is consistent with the conclusions of Albuquerque (2012), who found that quarterly earnings announcements trigger market reactions, leading to the change of skewness in stock returns. Therefore, the impact of coskewness on stock returns typically spans a cycle of approximately 1 to 4 months.

Further empirical analysis shows that, even after decomposing the market factor and individual stock CAPM residuals into different frequency components, the high-frequency portion of coskewness retains significant pricing power. Even when additional control variables such as the market factor, size factor, and value factor are included, the high-frequency component of coskewness still exhibits significant explanatory power. This suggests that, even under the interference of other factors, high-frequency coskewness remains significant in pricing stock returns, especially for frequency components with cycles of 1-2 months. This result highlights the crucial role of high-frequency coskewness in asset pricing. Moreover, this study validates the practical application of high-frequency coskewness in investment strategies through portfolio sorting analysis. It is found that a long-short strategy based on high-frequency coskewness can generate significant positive returns and positive market alpha, further confirming the pricing efficiency of high-frequency coskewness. In contrast, strategies based on overall coskewness fail to produce significant returns or alpha, indicating that extracting information from high-frequency coskewness is more effective and offers more practical investment decision support. In contrast, strategies based on overall coskewness fail to produce significant returns or alpha, indicating that the information from high-frequency coskewness is more effective and offers more practical investment decision support.

This research has important implications for both academia and practice. It not only enriches the theoretical study of coskewness in asset pricing and reveals the uniqueness of the Chinese A-share market, but also provides investors with practical investment strategy references based on high-frequency coskewness. Additionally, the findings offer scientific support for regulatory bodies in formulating market policies and promote the application of frequency-domain analysis methods in financial market research, expanding the boundaries of asset pricing theory and enhancing the precision of market dynamic analysis.

Despite the valuable insights provided by this study, it has certain limitations. For instance, the research primarily focuses on the Chinese A-share market, and future studies could consider applying the coskewness pricing model to other markets to test its generalizability. Moreover, the analysis relies on historical data for regression, which may not fully reflect changes in future market conditions, posing challenges for the practical application of investment strategies. Therefore, future research could expand the sample scope or introduce more dynamic market models to further verify and refine the conclusions of this paper.

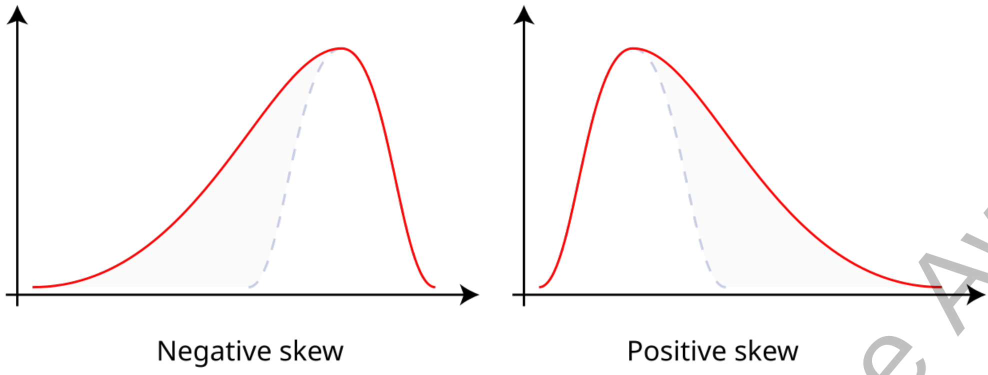

Figure 1: Representation of Positive and Negative Skewness in Stock Return Distributions

Table 1: Summary statistics

This table reports the summary statistics and correlation of individual stock return and five factors in the China A-share stock market. Panel A shows the summary statistics of individual stock returns and five factors. The numbers are showed in percentage. Panel B reports the correlation between five factors.

		Panel A Summary Statistics					
Variables	Observation	Mean	Standard deviation	Skewness	P10	Median	P90
Individual stock return	485988	1.112	15.202	11.123	-13.7017	-0.2012	17.035
Market factor(MKT)	348	0.4948	8.0742	0.1575	-8.7283	0.3795	10.5739
Size factor(SMB)	348	0.8445	4.3707	0.0474	-4.7835	0.8208	6.8337
Value factor(HML)	348	0.1629	3.2241	-0.3032	-3.6543	0.3127	4.1868
Profitability factor(RMW)	348	-0.1042	3.2613	-0.1241	-3.7304	-0.2607	4.0991
Investment factor(CMA)	348	0.1785	2.9171	0.3263	-3.3187	0.2669	3.3009

Panel B Factor correlation					
	MKT	SMB	HML	RMW	CMA
MKT	1				
SMB	0.1573	1			
HML	-0.0857	-0.2984	1		
RMW	-0.3055	-0.6304	0.0552	1	
CMA	0.1353	0.3934	0.2299	-0.6139	1

Table 2: The construction of factors

Variables	Factor Construction
Market factor(RM)	It consists of all stocks in the Chinese A-share stock market and are calculated as the value-weighted returns of all stock returns.
Size factor(SMB)	It represents the difference between the portfolio return of the top 50% largest market capitalization stocks and the portfolio return of the bottom 50% smallest market capitalization stocks. The monthly portfolio returns are calculated using a value-weighted method.
Value factor(HML)	It represents the difference between the portfolio return of the top 30% stocks with highest book-to-market ratio and the portfolio return of the bottom 30% stocks with lowest book-to-market ratio. The monthly portfolio returns are calculated using a value-weighted method.
Profitability factor(RMW)	It represents the difference between the portfolio return of the top 30% stocks with robust profitability and the portfolio return of the bottom 30% stocks with weak profitablity. The monthly portfolio returns are calculated using a value-weighted method.
Investment factor(CMA)	It represents the difference between the portfolio return of the top 30% stocks with low investment (conservation) and the portfolio return of the bottom 30% stocks with high investment (aggressive). The monthly portfolio returns are calculated using a value-weighted method.

Table 3: Symbol interpretation

Symbol	Definition
w	The wealth of investor
x	The income of investors(a random variable)
U	The utility function used to measure the investor's preferences for different outcomes of investors.
r	The rate of return on investment w.
μ	The expected return of investment.
U_n	The n-th derivative of U.
μ_i	The i-th central moment of return.
$\epsilon_{i,t+1}$	The residual of CAPM corresponding to stock i.
$\epsilon_{M,t+1}$	The difference between the excess return of market factor and its average.
$R_{M,t}$	The excess return of the market factor at month t.
$R_{i,t}$	The excess return of stock i at month t.

Table 4: Scale Interpretation

This table provide the interpretation of the scale based on monthly time interval.

Scale	Time horizon
j = 1	1 - 2 months
j = 2	2 - 4 months
j = 3	4 - 8 months
j = 4	8 - 16 months
j = 5	16 - 32 months
j = 6	32 - 64 months
j > 6	> 64 months

Table 5: Individual-level Fama-Macbeth cross sectional regressions

This table presents the results of the Fama-MacBeth cross-sectional regression analysis. We regress the one-month-ahead excess returns of individual stocks on β_{R_M} and β_{SKD}, corresponding to their high-frequency and low-frequency components. To calculate the high-frequency component (HF), we sum all frequency components with scales less than or equal to 4 ($j \leq 4$). Similarly, the low-frequency component (LF) is obtained by summing all frequency components with scales greater than 4 ($j > 4$). These regressions are conducted monthly from January 2000 to December 2022. Newey-West t-statistics are shown in parentheses, and adjusted R^2 values are listed in the final column. Bolded numbers indicate statistical significance at the 5% level or higher.

	Intercept	β_{R_M}	β_{SKD}	β^{HF}_{SKD}	β^{LF}_{SKD}	$Adj.R^2$
(1)	**1.461**	-0.306	**-0.713**			0.023
	(2.38)	(-0.81)	(-2.30)			
(2)	**1.459**	-0.242		**-1.622**		0.023
	(2.18)	-(0.06)		(-3.70)		
(3)	**1.304**	-0.492			0.245	0.022
	(2.18)	(-1.36)			(1.45)	

Table 6: Individual-level Fama-Macbeth cross sectional regressions

This table displays the outcomes of Fama-MacBeth cross-sectional regressions for frequency components $j = 1$ through $j = 4$. The regressions are conducted on a monthly basis from January 2000 to December 2022. Newey-West t-statistics are shown in parentheses, while the adjusted R^2 values are provided in the final column. Bold values represent those that are statistically significant at the 5% level or higher.

	Intercept	β_{R_M}	$\beta^{(1)}_{SKD}$	$\beta^{(2)}_{SKD}$	$\beta^{(3)}_{SKD}$	$\beta^{(4)}_{SKD}$	$Adj.R^2$
(1)	**2.071**	-0.476	**-1.113**				0.0215
	(3.05)	(-1.33)	(-2.22)				
(2)	**1.545**	-0.497		**-0.613**			0.0192
	(2.65)	(-1.75)		(-2.98)			
(3)	**2.204**	-0.543			-0.353		0.0192
	(3.09)	(-1.47)			(-1.02)		
(4)	**1.926**	-0.517				-0.177	0.0192
	(2.66)	(-1.38)				(-0.92)	

Table 7: Individual-level Fama-Macbeth cross sectional regressions: Robustness test

This table presents the results of Fama-MacBeth cross-sectional regressions for frequency components $j = 1$ and $j = 2$, incorporating size, value, investment, and profitability factors as control variables. The regressions are conducted monthly from January 2000 to December 2022. Newey-West t-statistics are shown in parentheses, while the adjusted R^2 values are displayed in the final column. We use bolded value to represent statistical significance at the 5% level or higher.

	Intercept	β_{SKD}	$\beta_{SKD}^{(1)}$	$\beta_{SKD}^{(2)}$	β_{SKD}^{HF}	β_{mkt}	β_{smb}	β_{hml}	β_{rmw}	β_{cma}	$Adj.R^2$
(1)	1.1069	-0.0081				-0.9083	0.1733	0.0210	-0.0554	-0.0302	0.0706
	(1.87)	(-1.78)				(-0.26)	(0.83)	(0.15)	(-0.39)	(-0.31)	
(2)	**1.5455**		**-1.2684**			-0.1498	0.1843	0.0039	-0.0330	-0.0840	0.0708
	(2.13)		(-2.55)			(-0.43)	(0.80)	(0.03)	(-0.22)	(-0.83)	
(3)	**1.6121**			**-0.7331**		-0.2319	0.2278	0.0004	-0.0504	-0.0857	0.069
	(2.47)			(-2.53)		(-0.68)	(0.99)	(0.00)	(-0.35)	(-0.87)	
(4)	**1.5490**				**-0.6858**	-0.1989	0.2083	-0.1478	-0.0643	-0.6277	0.0686
	(2.42)				(-2.27)	(-0.57)	(0.92)	(-0.11)	(-0.44)	(-0.64)	

Table 8: Sorted portfolio statistics

This table presents the statistics for portfolios sorted by both original coskewness and frequency-specific coskewness. The table includes the average return, single-factor stock market alphas (Alpha), and standard deviations for various quantile portfolios. At the start of each month, individual stocks are ranked based on coskewness, estimated using their past 60-month returns. The stocks with the highest coskewness are assigned into portfolio10, while those with the lowest coskewness are included in the portfolio 1. Portfolio returns are calculated using a value-weighted method. The sample period spans from January 2000 to December 2022. P-values are reported in parentheses, and values in bold are significant at the 5% confidence level.

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P1-P10
Panel A: Sorted by β_{SKD}											
Mean	**1.377**	**1.412**	1.047	1.110	1.132	1.104	1.162	0.858	0.860	0.903	0.474
p-value	(0.031)	(0.031)	(0.072)	(0.059)	(0.059)	(0.060)	(0.054)	(0.146)	(0.132)	(0.115)	(0.181)
Alpha	**0.691**	**0.721**	0.393	**0.441**	**0.458**	**0.434**	**0.485**	0.216	0.214	0.255	0.436
p-values	(0.003)	(0.004)	(0.073)	(0.040)	(0.043)	(0.042)	(0.025)	(0.292)	(0.285)	(0.248)	(0.164)
Standard Deviation	9.269	9.460	8.813	8.930	9.108	8.946	9.041	8.568	8.536	8.721	5.509
Panel B: Sorted by β_{SKD}^{HF}											
Mean	**1.025**	**1.161**	**0.922**	0.783	**1.211**	0.763	0.841	0.835	0.650	0.537	**0.489**
p-values	(0.095)	(0.073)	(0.096)	(0.201)	(0.037)	(0.187)	(0.152)	(0.175)	(0.285)	(0.383)	(0.096)
Alpha	**0.508**	**0.617**	**0.489**	0.276	**0.722**	0.270	0.342	0.333	0.152	0.023	**0.485**
p-values	(0.026)	(0.007)	(0.029)	(0.212)	(0.002)	(0.234)	(0.151)	(0.120)	(0.512)	(0.927)	(0.054)
Standard Deviation	8.808	9.196	8.579	8.632	8.516	8.487	8.655	8.517	8.581	8.964	4.177
Panel C: Sorted by $\beta_{SKD}^{(1)}$											
Mean	**1.358**	0.836	1.079	1.171	0.816	0.810	0.816	0.681	0.584	0.333	**1.026**
p-values	(0.035)	(0.154)	(0.082)	(0.062)	(0.161)	(0.190)	(0.179)	(0.271)	(0.305)	(0.570)	(0.001)
Alpha	**0.865**	0.357	**0.594**	**0.682**	0.339	0.300	0.323	0.184	0.128	-0.162	**1.027**
p-values	(0.000)	(0.144)	(0.013)	(0.004)	(0.160)	(0.215)	(0.143)	(0.399)	(0.593)	(0.461)	(0.000)
Standard Deviation	8.930	8.071	8.768	8.747	8.674	9.143	8.754	8.772	8.352	8.741	4.621
Panel D: Sorted by $\beta_{SKD}^{(2)}$											
Mean	1.016	0.857	0.801	**1.128**	0.790	0.746	0.909	0.782	0.817	0.638	0.378
p-values	(0.093)	(0.176)	(0.180)	(0.058)	(0.199)	(0.221)	(0.145)	(0.187)	(0.166)	(0.300)	(0.133)
Alpha	**0.539**	0.339	0.303	**0.637**	0.277	0.258	0.408	0.298	0.331	0.129	0.400
p-values	(0.023)	(0.127)	(0.171)	(0.010)	(0.213)	(0.295)	(0.074)	(0.227)	(0.158)	(0.614)	(0.140)
Standard Deviation	8.540	8.909	8.612	8.718	8.842	8.659	8.752	8.613	8.541	9.028	4.459

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7 Acknowledgement

7.1 Motivation

The initial inspiration for choosing this research topic came from my father. He is an experienced investor who often discusses stock returns and predicts market trends with his friends. While they have had their moments of success in the stock market, they have also faced disappointing results, most recently suffering a loss of over 50%. When discussing stocks, my father and his friends frequently talk about a company's financial reports. He told me that he references these reports and recent news to decide whether to invest in a stock. Inspired by my father, I began to think that whether the periodic release of financial reports have a significant impact on stock returns.

After contacting Professor Wang Yuan, I shared my thoughts on this matter with him. Professor Wang supported my ideas and suggested using frequency domain methods to explore this phenomenon. Additionally, he introduced me to the work of Albuquerque (2012), which connects the frequency domain with skewness. Initially, I hoped that my research would give me a clearer understanding of the Chinese A-share market, enabling me to help my father. As my research progressed, I became even more motivated to contribute to the broader community of investors and institutions. This is the reason I decided to systematically study coskewness and the Chinese A-share market.

7.2 Research process

Throughout this research journey, I would first like to thank my family for their unwavering support. They helped me connect with professors in the financial field, arranged visits, internships, and learning opportunities at universities, and provided me with a positive environment to focus on my research.

Secondly, I owe a great deal of thanks to my teachers. The first is Mr. Nan Hu from Shenzhen Middle School. Mr. Hu followed my progress from the inception of this idea to the completion of my paper, helping me refine the language and providing invaluable guidance. My other teacher, Professor Yuan Wang, is a top expert in the field, and I consider him my academic guide who led me into the world of research. Professor Wang not only recognized and praised my ideas but also offered support for my research. He guided me out of confusion when I encountered difficulties, corrected my direction when I veered off track, and encouraged me to persist during moments of frustration.

This project has been a process of learning, practice, and personal growth. The most challenging part was understanding the extended Wold representation theorem. I had numerous discussions with Professor Wang, progressing from partial understanding to eventually mastering the concept, which was achieved through repeated discussions and derivations. Professor Wang also suggested I use MATLAB for programming, which greatly facilitated the visualization and regression analysis. Additionally, I would like to extend special thanks to Mr. Bojie Sun. He provided numerous suggestions and assistance in formula derivation, coding, and writing the paper.

Moreover, the internet has been an invaluable resource in my learning process. From online courses to generous contributors and countless community articles and blog posts, I found immense help. I would also like to thank all the authors whose works are cited in my references. Their papers provided me with cutting-edge insights, knowledge, and techniques, allowing me to quickly grasp the latest financial theories.

Finally, I extend my sincere thanks and gratitude once again to all the teachers, family members, and friends who have accompanied and supported me throughout this journey!