2024 S.T. Yau High School Science Award (Asia)

Research Report

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Title of Research Report: The Maximum Area of N-gons within the Intersection Region of Two Congruent Circles

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The Maximum Area of N-gons within the Intersection Region of Two Congruent Circles

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Abstract

At the 61st National High School Science Fair of Taiwan, the first-rank paper "The Study of the Largest Area of Inscribed Triangle within the Intersection of two circles" was presented. The authors discussed several properties of maximum area of inscribed triangles within intersection regions of two congruent circles. They only claim their results but without providing a rigorous proof. However, we give a proof by showing the convergence of the iteration of finding the largest height.

Subsequently, we offer new methods to approach the problems such as the trigonometric identities, Jensen's Inequality to prove the maximum area of triangles and quadrangles within the intersection region of two congruent circles.

Finally, we determined the maximum area for the case of n-gons. We conducted further research and discussion on this issue. In the future, we hope to prove why the maximum area of n-gons within the intersection region of two congruent circles occurs when there are two points on the intersection points of the two circles. We aim similar problems in the three-dimensional space, namely the maximum volume of tetrahedron within the intersection of two unit balls.

Keywords: Trigonometric identities, Jensen's Inequality, Maximum area, Iteration

Acknowledgement

We greatly appreciate Chin-Hsuan Liu at Wu-Ling high school for the useful discussion. We also appreciate Chung-Yen Hsieh and Hsiang-Chi Hu for encouraging us.

Commitments on Academic Honesty and Integrity

We hereby declare that we

1. are fully committed to the principle of honesty, integrity and fair play throughout the competition.
2. actually perform the research work ourselves and thus truly understand the content of the work.
3. observe the common standard of academic integrity adopted by most journals and degree theses.
4. have declared all the assistance and contribution we have received from any personnel, agency, institution, etc. for the research work.
5. undertake to avoid getting in touch with assessment panel members in a way that may lead to direct or indirect conflict of interest.
6. undertake to avoid any interaction with assessment panel members that would undermine the neutrality of the panel member and fairness of the assessment process.
7. observe the safety regulations of the laboratory(ies) where the we conduct the experiment(s), if applicable.
8. observe all rules and regulations of the competition.
9. agree that the decision of YHSA(Asia) is final in all matters related to the competition.

We understand and agree that failure to honour the above commitments may lead to disqualification from the competition and/or removal of reward, if applicable; that any unethical deeds, if found, will be disclosed to the school principal of team member(s) and relevant parties if deemed necessary; and that the decision of YHSA(Asia) is final and no appeal will be accepted.

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Content 1: Introduction

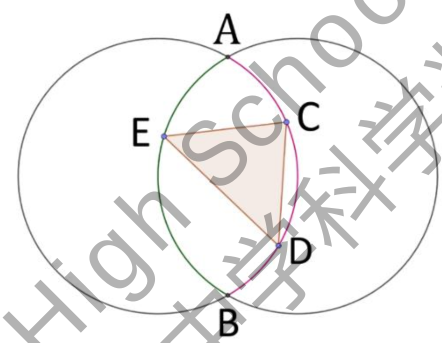
1. Motivation:

When searching for inspiration from past science exhibitions, we found several projects exploring the maximum area of the inscribed triangle in the intersecting region of two circles. Chen, Hong, Tsai, and Sun [1] considered the closed region formed by the intersection of two circles at first. Restricting the triangles to this region, investigate the maximum area and describe the figure that achieves the maximum area. Subsequently, Yang [2] considered various cases, including changing the radii and distances of two circles. Chen [1] et al. only claim their results, which after iterations of finding largest height for many times, the triangle will approach to an isosceles triangle with 70°, 70°, and 40°. But they didn't offer a rigorous proof. Therefore, we show the convergence of this iteration of finding the largest height and generalize this question to n points, ellipse and three-dimentional cases.

2. Definition:

(1) Inscribed triangle

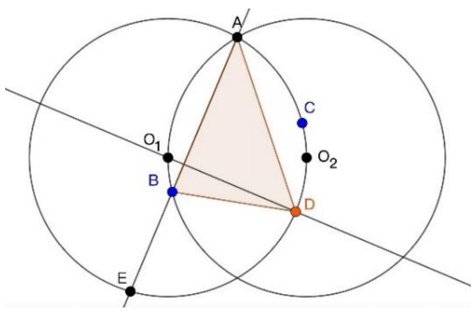
A triangle inscribed in two unit circles passing through each other's centers (hereafter referred to as intersecting circles, both unit circles, passing through each other's centers)



Pic 1 An inscribed triangle within the intersection area of two unit circles passing through each other's centers

Take three points on the two arcs in the intersection region of two intersecting circles $\widehat{AEB}$ and $\widehat{ACB}$ and make a triangle, such as ΔCDE.

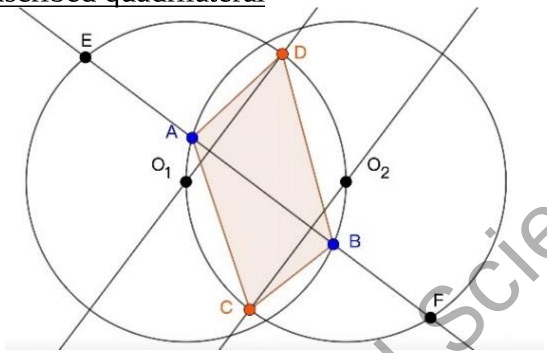
(2) **Iteration of finding largest height for triangle:** Make a perpendicular bisector of a side of a inscribed triangle



Pic 2 Take two points, point A and point B, on the two arcs in the intersection region of two circles and find the largest inscribed triangle that include those two points in the intersection region of two circles.

Take point A, one of the intersection points of the two intersecting circles, and choose any point B locating on the two arcs in the intersection region of two circles. To find the largest triangle that includes point A and point B, we extended $\overline{AB}$ to $\overline{AE}$, which is a chord of the left circle. Thus, the perpendicular bisector of $\overline{AE}$ passes the center of the left circle, intersecting two points on it. Only point D is on the arc of intersection region of two circles (Pic 2). Thus, the largest triangle that includes point A and point B is ΔABD.

(3) **Iteration of finding largest height for quadrilateral:** Make a perpendicular bisector of a diagonal of a inscribed quadrilateral



Pic 3 Take two points, point A and point B, on the two arcs in the intersection region of two circles and find the largest inscribed quadrilateral that include those two points.

To find the largest quadrilateral that includes point A and point B, we extended $\overline{AB}$ to $\overline{EF}$. Because $\overline{BE}$ is a chord of the left circle, the perpendicular bisector of it passes the center of the left circle O_1, intersecting two points on the left circle. But only point D is on the arc of intersection region of two circles. Similarly, because $\overline{AF}$ is a chord of the right circle, the perpendicular bisector of it passes the center of the right circle O_2, intersecting two points on the right circle. But only point C is on the arc of intersection region of two intersecting circles. Thus, the largest quadrilateral that includes point A and point B is $ACBD$.

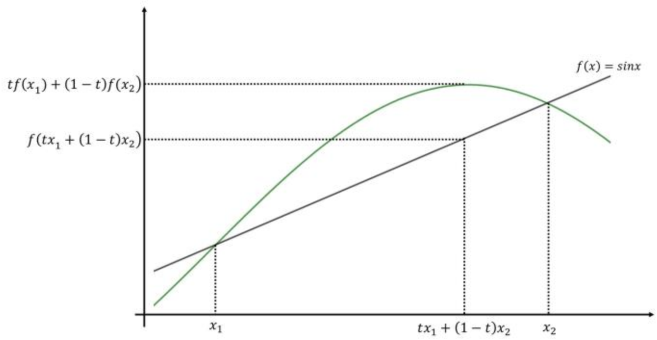
3. Theorems and Properties:

(1) Use Cauchy-Schwarz Inequality to find the maximum of a trigonometric function:
Suppose a trigonometric function is $acos\theta + bsin\theta$, we can know the inequation below by Cauchy-Schwarz Inequality.

$$(a^2 + b^2)(cos^2\theta + sin^2\theta) \geq (acos\theta + bsin\theta)^2$$

Thus, the maximum of the trigonometric function occurs at $a: cos\theta = b: sin\theta$

(2) Jensen's Inequality
If $f(x)$ is a convex function on $[a, b]$, for $x_i \in [a, b]$, $\frac{\sum f(x_i)}{n} \geq f(\frac{\sum x_i}{n})$. When $x_1 = x_2 = \cdots = x_n$, the equation holds true.
Now suppose $f(x) = sin(x)$, it is a concave function when $0 \leq x \leq \frac{2}{3}\pi$. The area we require is $\frac{1}{2}[sin(x_1) + sin(x_2) + \cdots + sin(x_n)]$. When $0 \leq x_1, x_2, \ldots, x_n \leq \frac{2}{3}\pi$, $\frac{\sum sin(x_i)}{2n} \leq sin(\frac{\sum x_i}{2n})$. When $x_1 = x_2 = \cdots = x_n$, the equation holds true, which the area is the maximum.



Pic 4 The Illustration Jensen’s Inequality by using concave function

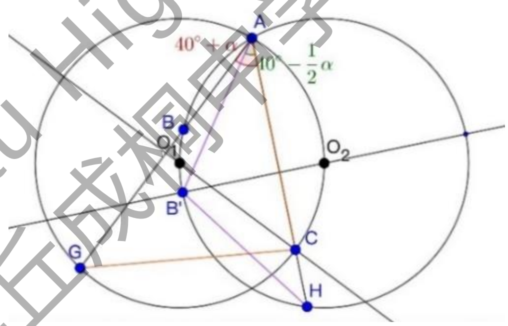
Content 2: Triangle

1. Use the geometric method to prove that by using the iteration of Chen et al. [1] many times, the apex angle of the isosceles triangle will converge to 40°.

Theorem 1
We prove that by using the iteration of Chen et al. [1] many times, the apex angle of the isosceles triangle will converge to 40°.

Proof.
We divide into two cases below and above O_1 and in both cases the angle subtracting 40 degrees will have the behavior of geometric series with ratio negative one half.

(1) <u>Point B is **above** point O_1</u>



Pic 5 The relation of two apex angles from two times of iteration when point B is above point O_1

Suppose $\angle BAC = 40^{\circ} + \alpha$, and $\overleftrightarrow{O_1C}$ is the perpendicular bisector of $\overline{AG}$, $\overleftrightarrow{O_2B'}$ is the perpendicular bisector of $\overline{AH}$.

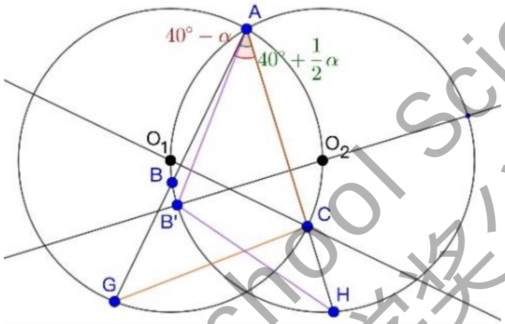
$$\because \overleftrightarrow{O_1C} \text{ is the perpendicular bisector of } \overline{AG}$$
$$\therefore \angle O_1CA = 90^{\circ} - \angle BAC = 50^{\circ} - \alpha$$
$$\because \overline{O_1A} = \overline{O_1C} = 1$$
$$\therefore \angle O_1AC = \angle O_1CA = 50^{\circ} - \alpha$$
$$\because \overline{O_1A} = \overline{O_1O_2} = \overline{O_2A} = 1$$
$$\therefore O_1AO_2 \text{ is an equilateral triangle}$$
$$\angle O_1AO_2 = 60° \Longrightarrow \angle CAO_2 = \angle O_1AO_2 - \angle O_1AC = \alpha + 10°$$

$$\because \overleftrightarrow{O_2B'} \text{ is the perpendicular bisector of } \overline{AH}$$
$$\angle AO_2B' = 90° - (\alpha + 10°) = 80° - \alpha$$
$$\widehat{AB'} = 80° - \alpha$$
$$\angle AHB' = 40° - \frac{1}{2}\alpha$$
$$\because \overline{B'A} = \overline{B'H}$$
$$\angle B'AC = 40° - \frac{1}{2}\alpha$$

Repeat the process above for n times, the apex angle is $40^{\circ} + (-\frac{1}{2})^n\alpha$.

$$\Rightarrow \lim_{n\to\infty}[40^{\circ} + \left(-\frac{1}{2}\right)^n \alpha] = 40^{\circ}$$

(2) Point B is **below** point O_1



Pic 6 The relation of two apex angles from two times of iteration when point B is below point O_1

Suppose $\angle B'AC = 40° - \alpha$, and $\overleftrightarrow{O_1C}$ is the perpendicular bisector of $\overline{AG}$, $\overleftrightarrow{O_2B'}$ is the perpendicular bisector of $\overline{AH}$.

$$\because \overleftrightarrow{O_1C} \text{ is the perpendicular bisector of } \overline{AG}$$
$$\therefore \overline{CE} \perp \overline{AG}$$
$$\angle EC'A = 90° - \angle EAC' = 50° + \alpha$$
$$\because \overline{O_1A} = \overline{O_1C}$$
$$\therefore \angle O_1AC = \angle O_1CA = 50^{\circ} + \alpha$$
$$\angle O_1AC = \angle O_1AO_2 - \angle O_1AC = 60^{\circ} - (50^{\circ} + \alpha) = 10^{\circ} - \alpha$$
$$\because \overline{BF} \perp \overline{AC}, \angle AO_2F = 90^{\circ} - (10^{\circ} - \alpha) = 80^{\circ} + \alpha$$
$$\widehat{AB'} = 80^{\circ} + \alpha, \angle AHB' = 40^{\circ} + \frac{1}{2}\alpha$$
$$\because \overline{B'A} = \overline{B'H} = 1$$
$$\therefore \angle B'AC = 40^{\circ} + \frac{1}{2}\alpha$$

Repeat the process above for n times, the apex angle is $40^{\circ} - (-\frac{1}{2})^n\alpha$

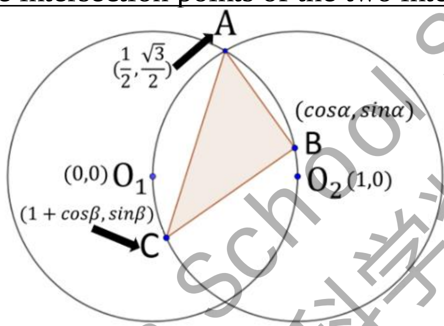
$$\Rightarrow \lim_{n\to\infty}[40^{\circ} - \left(-\frac{1}{2}\right)^n \alpha] = 40^{\circ}$$

2. Use the method of overlaying a trigonometric function to prove the angles of the largest inscribed triangle in the intersection region of two intersecting circles are 70°, 70°, and 40°

Theorem 2
The angles of the largest area of inscribed triangle of two circles are 70°, 70°, 40°.

Proof.
We divide into two cases: first we assume that there is one point on the intersection points of the two intersecting circles and second without this assumption. We prove that the area of the former case is strictly bigger than the latter.

(1) Fix a point on one of the intersection points of the two intersecting circles



Pic 7 A triangle whose one of apex is fixed on the one of the intersection points of the two intersecting circles

The equation of the left circle is $x^2+y^2=1$, and the equation of the right circle is $(x-1)^2+y^2=1$. In the following passage, the equations of the two circles remain the same.

First, the area function $f_1(\alpha,\beta)=$

$$\frac{1}{2}\left\|\begin{matrix}\frac{1}{2} & 1+\cos\beta & \cos\alpha & \frac{1}{2}\\ \frac{\sqrt{3}}{2} & \sin\beta & \sin\alpha & \frac{\sqrt{3}}{2}\end{matrix}\right\|$$

$$=\left|-\frac{1}{2}\sin\beta\cos\alpha+\frac{1}{2}\sin\alpha\cos\beta+\frac{1}{4}\sin\alpha+\frac{\sqrt{3}}{4}\cos\alpha-\frac{\sqrt{3}}{4}\cos\beta+\frac{1}{4}\sin\beta-\frac{\sqrt{3}}{4}\right|$$

where $-\frac{\pi}{3}\leq\alpha\leq\frac{\pi}{3}$ and $\frac{2\pi}{3}\leq\beta\leq\frac{4\pi}{3}$

$$f_1(\alpha,\beta)=\frac{-1}{2}(\sin\beta\cos\alpha-\sin\alpha\cos\beta-\frac{1}{2}\sin\alpha-\frac{\sqrt{3}}{2}\cos\alpha+\frac{\sqrt{3}}{2}\cos\beta-\frac{1}{2}\sin\beta+\frac{\sqrt{3}}{2})$$

View the above equation as the function of α:

$$f_1(\alpha)=\frac{-1}{2}((\sin\beta-\frac{\sqrt{3}}{2})\cos\alpha-(\cos\beta+\frac{1}{2})\sin\alpha+\underbrace{(\frac{\sqrt{3}}{2}\cos\beta-\frac{1}{2}\sin\beta+\frac{\sqrt{3}}{2})}_{\text{constant}})$$

By using Cauchy-Schwarz Inequality, the extremum occurs at

$$\left(sin\beta - \frac{\sqrt{3}}{2}\right) : -\left(cos\beta + \frac{1}{2}\right) = cos\alpha : sin\alpha$$

After organizing, we obtain the equation below.

$$-\frac{1}{2}cos\,\alpha + \frac{\sqrt{3}}{2}sin\,\alpha = cos\,\alpha\,cos\,\beta + sin\,\alpha\,sin\,\beta \;...(1)$$

Similarly, if we view equation as the function of β:

$$f_1(\beta) = \frac{-1}{2}((cos\,\alpha - \frac{1}{2})\,sin\,\beta - (sin\,\alpha - \frac{\sqrt{3}}{2})\,cos\,\beta \underbrace{- \frac{\sqrt{3}}{2}cos\,\alpha - \frac{1}{2}sin\,\alpha + \frac{\sqrt{3}}{2}}_{\text{constant}})$$

By using Cauchy-Schwarz Inequality, the extremum occurs at

$$\left(cos\,\alpha - \frac{1}{2}\right) : -\left(sin\,\alpha - \frac{\sqrt{3}}{2}\right) = sin\,\beta : cos\,\beta$$

After organizing, we obtain the equation below.

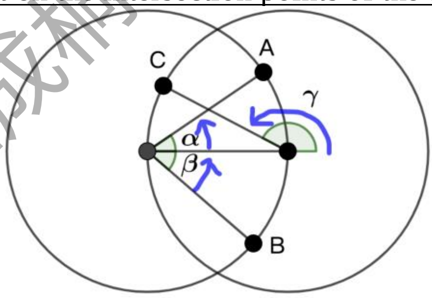
$$\frac{1}{2}cos\,\beta + \frac{\sqrt{3}}{2}sin\,\beta = cos\,\alpha\,cos\,\beta + sin\,\alpha\,sin\,\beta \;...(2)$$

By (1) and (2), we can obtain the equations below.

$$cos(\beta - \frac{\pi}{3}) = cos(\frac{2\pi}{3} - \alpha) = cos(\beta - \alpha)$$

In the equation above, consider the range of α and β, we can find that $\frac{2\pi}{3} - \alpha = \beta - \frac{\pi}{3}$. Thus $\pi = \alpha + \beta$. After putting the relation above in the second equation, we obtain that $cos(\frac{2\pi}{3} - \alpha) = cos(\pi - 2\alpha)$. Considering $-\frac{\pi}{3} \le \alpha \le \frac{\pi}{3}$, the relation of the angle can only be $\frac{2\pi}{3} - \alpha = -(\pi - 2\alpha)$ Therfore, $\alpha = -\frac{\pi}{9} = -20^{\circ}$, $\beta = \pi - \alpha = 200^{\circ}$, and the angles of the largest inscribed triangle in the intersection region of two intersecting circles are 70°,70°,40°.

(2) Without fixing any point on the intersection points of the two intersecting circles.



Pic 8 Define α, β, γ three degree by A, B, C three points

$$-\frac{\pi}{3} \le \alpha, \beta \le \frac{\pi}{3}, \alpha > \beta, \frac{2\pi}{3} \le \gamma \le \frac{4\pi}{3}$$

$$2f_2(\alpha,\beta,\gamma) = \begin{vmatrix} 1 + cos\gamma - cos\alpha & sin\gamma - sin\alpha \\ cos\beta - cos\alpha & sin\beta - sin\alpha \end{vmatrix}$$

$$= -(sin\gamma - sin\alpha)cos\beta + (1 + cos\gamma - cos\alpha)sin\beta + (sin\gamma - sin\alpha)cos\alpha - (1 + cos\gamma - cos\alpha)sin\alpha$$

We view $2f_2(\alpha,\beta,\gamma)$ as the function of α. Thus, we can regard this function as

$$f_2(\alpha) = -(cos\beta - cos\gamma - 1)sin\alpha - (sin\gamma - sin\beta)cos\alpha + k$$

The maximum of $f_2(\gamma)$ occurs when the equation below is held.

$$\frac{cos\beta - cos\gamma - 1}{sin\gamma - sin\beta} = \frac{sin\alpha}{cos\alpha}$$
$$\Rightarrow cos\alpha cos\beta - cos\gamma cos\alpha - cos\alpha = sin\alpha sin\gamma - sin\alpha sin\beta$$
$$\Rightarrow cos\alpha + cos(\gamma - \alpha) = cos(\alpha - \beta) \ \dots \ (3)$$

We view $2f_2(\alpha, \beta, \gamma)$ as the function of β. Thus, we can regard this function as
$$f_2(\beta) = -(sin\gamma - sin\alpha)cos\beta + (1 + cos\gamma - cos\alpha)sin\beta + k$$
The maximum of $f_2(\gamma)$ occurs when the equation below is held.
$$\frac{-1 - cos\gamma + cos\alpha}{sin\gamma - sin\alpha} = \frac{sin\beta}{cos\beta}$$
$$cos\alpha cos\beta - cos\gamma cos\beta - cos\beta = sin\gamma sin\beta - sin\alpha sin\beta$$
$$cos\beta + cos(\gamma - \beta) = cos(\alpha - \beta) \ \dots \ (4)$$

Due to (3) and (4), we can obtain the equation below.
$$cos\alpha + cos(\gamma - \alpha) = cos\beta + cos(\gamma - \beta) \ \dots \ (5)$$

We view $2f_2(\alpha, \beta, \gamma)$ as the function of γ. Thus, we can regard this function as
$$f_2(\gamma) = -(cos\alpha - cos\beta)sin\gamma - (sin\beta - sin\alpha)cos\gamma + k$$
The maximum of $f_2(\gamma)$ occurs when the equation below is held.
$$\frac{cos\alpha - cos\beta}{sin\beta - sin\alpha} = \frac{sin\gamma}{cos\gamma}$$
$$cos(\gamma - \alpha) = cos(\gamma - \beta) \ \dots \ (6)$$

We substitute (5) in (6), we obtain the equation below.
$$cos\alpha = cos\beta$$

In this case, the middle point of $\overline{AB}$ is on x-axis. Therefore, when the maximum area of the triangle occurs, point C and point O_1 will overlay, and the area is: $sin\,\alpha\, cos\,\alpha = \frac{1}{2} sin\,2\alpha \leq 0.5$.

However, since the number is smaller than $f_1(-\frac{\pi}{9}, \frac{10\pi}{9}) \approx 0.53$, we can confirm that at least one of point A, point B, and point C will locate at the intersection points of the two intersecting circles.

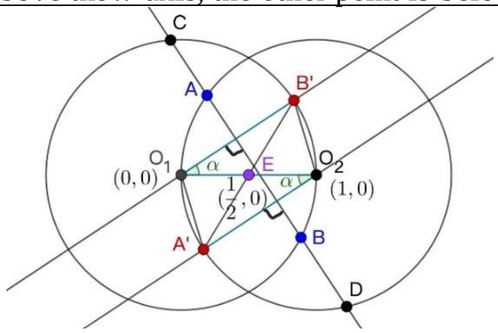
Content 3: Quadrilateral

1. **Prove the relative lemma of inscribed quadrilateral in the intersection region of two intersecting circles**

Lemma 1
By doing an iteration of two points, which are on the different intersecting arcs respectively, we can obtain point A' and B'. Then, $(\frac{1}{2}, 0)$ is on $\overline{A'B'}$.

Proof.
We divide the proof into two cases below:

(1) <u>One of the points is **above** the x-axis, the other point is **below** the x-axis</u>



Pic 9 Choose two points on the arcs in the intersection region. One of them is above the x-axis, the other one is below the x-axis

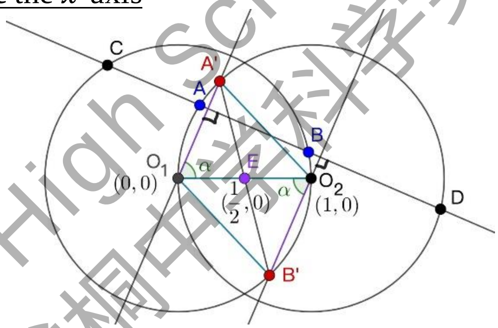
$$\because \overline{O_1B'} \perp \overline{AB} \text{ and } \overline{O_2A'} \perp \overline{AB}$$
$$\therefore \overline{O_1B'} \parallel \overline{O_2A'}$$
$$\angle B'O_1O_2 = \angle A'O_2O_1 \text{ (Alternate interior angles are equal)}$$

Furthermore, $\overline{O_1B'} = \overline{O_1O_2} = \overline{O_2A'} = 1$

$$\therefore \triangle B'O_1O_2 \cong \triangle A'O_2O_1 (SAS)$$

Thus, $O_1A'O_2B'$ is a parallelogram. Moreover, in a parallelogram, the diagonals bisect each other. Therefore, the diagonals intersect at $(\frac{1}{2}, 0)$.

(2) <u>Both points are **above** the x-axis</u>



Pic 10 Choose two points on the arcs in the intersection region. Both of them are above the x-axis.

$$\because \overline{O_1A'} \perp \overline{AB} \text{ and } \overline{O_2B'} \perp \overline{AB},$$

$$\therefore \overline{O_1A'} \parallel \overline{O_2B'} \Rightarrow \angle A'O_1O_2 = \angle B'O_2O_1 \text{ (Alternate interior angles are equal)}$$

Furthermore, $\overline{O_2A'} = \overline{O_1O_2} = \overline{O_1B'} = 1$

Thus, $\triangle A'O_1O_2$ and $\triangle B'O_2O_1$ are isosceles triangles. Moreover, the base angles of those two triangles are same, so the apex angles of those two triangles are same.

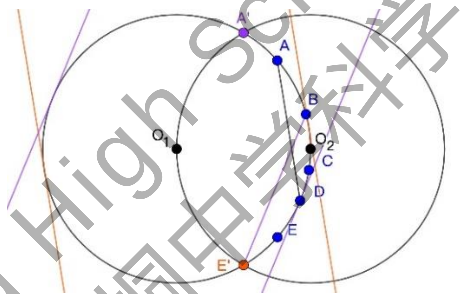
$$\therefore \triangle A'O_1O_2 \cong \triangle B'O_2O_1 (SAS)$$

Therefore, $O_1A'O_2B'$ is a parallelogram. In addition, in a parallelogram, the diagonals bisect each other. Therefore, the diagonals intersect at $(\frac{1}{2}, 0)$.

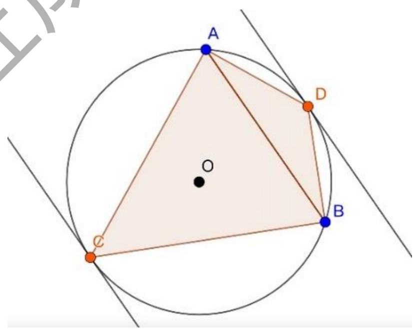
2. Prove that when n vertices are all on the same side, the maximum area occurs when both upper and lower intersection vertices are included and the other vertices are at the equal divided points of the arc.

> Theorem 3
> When n vertices are all on the same side, the maximum area occurs when both upper and lower intersection vertices are included and the other vertices are at the equal divided points of the arc.

Proof.
When all the vertices are located at the same side, we draw a line between the top one (point A) and the second one counted from the bottom (point D). Then, we do iteration. From Pic 11, we can obviously find the maximum height of ΔADE when the tangent line is on the major arc. However, it will exceed the area we consider. Therefore, the new vertex will move to the boundary which is the lower intersection point (point E'). Without loss of generality, we can do the same action to the bottom (point E') and the second one counted from the top (point B). Hence, the top will move to the upper intersection point (point A'). After that, by using Jensen's Inequality, we can arrange the left vertices to the equal divided points of the arc, which forms the maximum figure of this case.



Pic 11 When all the points are on the same arc, point A and point E will be moved to the two intersecting points via iteration

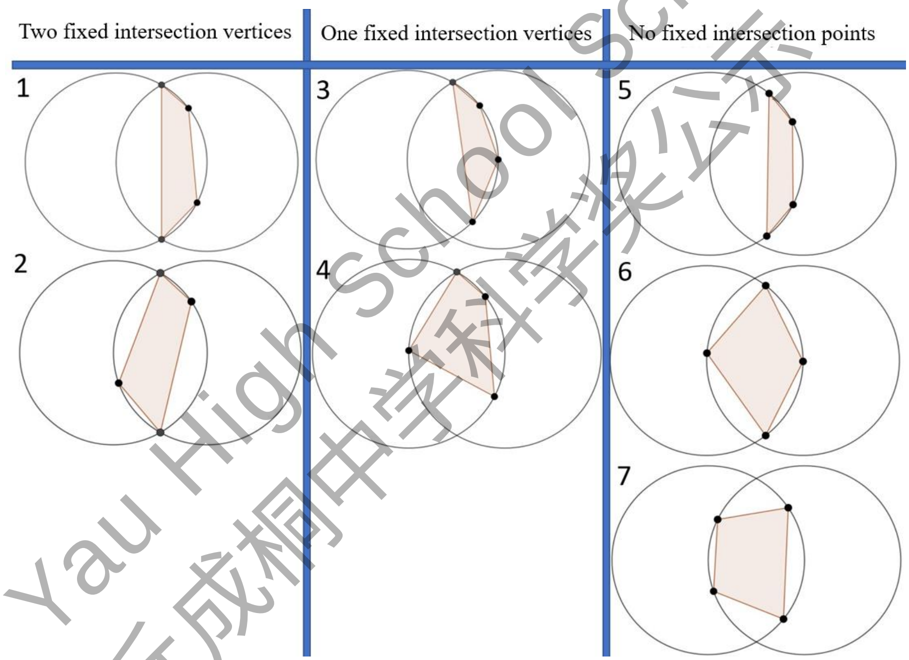


Pic 12 The iteration of $\overline{AB}$, where ΔABC is the largest inscribed triangle containing $\overline{AB}$

3. Proving the largest quadrilateral within the intersection of two circles

Theorem 4
The largest quadrilateral within two circles is a rhombus with angles of 60°, 120°, 60°, 120° and side lengths of 1. The area is $\frac{\sqrt{3}}{2}$.

Proof.
We divide into three cases. The first one is that two vertices are fixed intersection vertices of the two circles. The second one is that only one vertex is fixed at the intersection vertex of the two circles. The last one is that none of the four vertices is fixed at the intersection vertices.
After organizing, a total of seven cases can be categorized as follows:

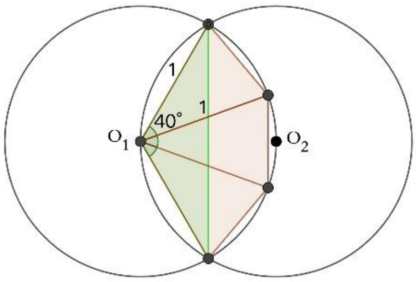


Pic 13 Seven cases of the inscribed quadrilateral within two intersection circles

<u>(1) Fixing two vertices and putting the two others on the same side</u>
This case can use Theorem 3 (p.14) to find out that the other two vertices should be on the equal divided vertices, which means that four vertices divide 120°, and the area of it is

$$\frac{1}{2}(3 \times 1 \times 1 \times sin\,40° - 1 \times 1 \times sin\,120°) \approx \frac{1}{2}(1.93 - 0.87) = 0.53$$

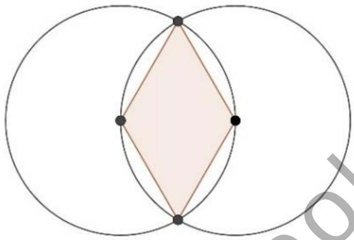
"1"in the formula means the radius of unit circle, and the formula used to calculate the triangle is $\frac{1}{2}absin\theta$



Pic 14 the maximum quadrilateral of fixing two vertices on the intersection vertices and putting the two others on the same side

(2) <u>Fixing two vertices and putting the two others on different sides</u>

This case can also use Theorem 3 (p.14) to find out that the left and right vertices should be on equal divided vertices, for example as shown in the following figure:



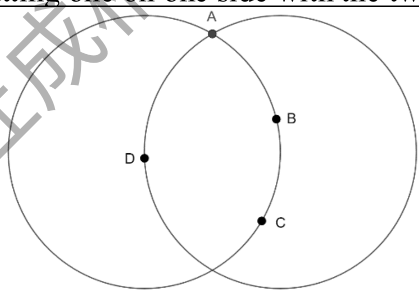
Pic 15 the maximum quadrilateral of fixing two vertices and putting the two others on different sides

Its area is $\frac{\sqrt{3}}{2} \approx 0.87$

(3) <u>Fixing only one vertex and putting the three others on same sides</u>

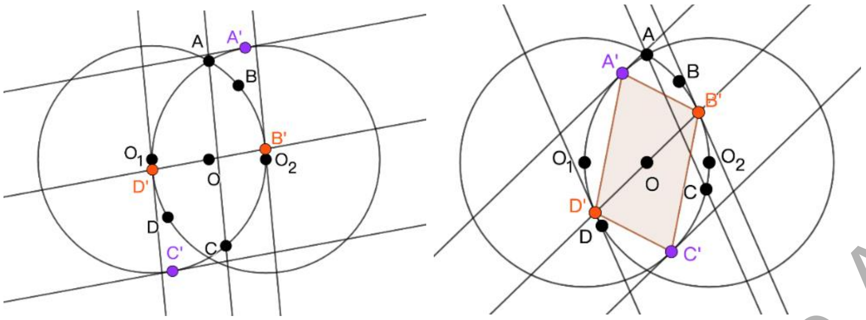
Using Theorem 3 (p.14) can know that three vertices should be on equal diversion vertices, and the figure will be same as the first case, and its area is approximately 0.53.

(4) <u>Fixing one vertex and putting one on one side with the two others on the other side</u>



Pic 16 The maximum quadrilateral where one point is fixed at the intersection and the remaining three points have two points on the same side

First, draw a line between point A and point C, and do iteration of $\overline{AC}$. Since the slope is between the range of $(-\infty, -\frac{1}{\sqrt{3}})$, the tangent lines will touch the left circle at the bottom part, which is point D', and touch the right circle at the topper part, which is point B'. The two tangent points will pass $(\frac{1}{2}, 0)$. Then, do iteration of $\overline{B'D'}$, which makes point A' and point C'. Because the two points A' and C' are point symmetric to $(\frac{1}{2}, 0)$, or exceed the borden, the case will turn to “fixing two points on intersection points or two points on one side and the other two on the different side”, which is case 2 or case 7.



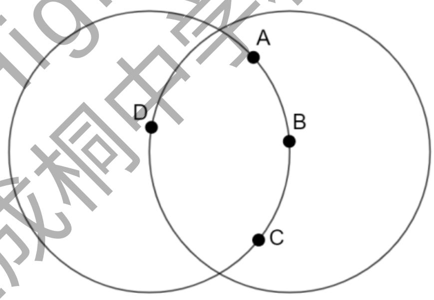
Pic 17 Find the maximum area using parallel lines in the case where one point is fixed and two points lie on the same side.

(5) No vertices are fixed and put four vertices on the same side

Using Theorem 3 (p.14), the four points are on equal divided vertices, causing the topmost point of the four to approach the upper intersection point infinitely, and the bottommost point to approach the lower intersection point infinitely. This is the same as the case where two points are at the intersection points and the remaining two points are on the same side, with an area of approximately 0.53.

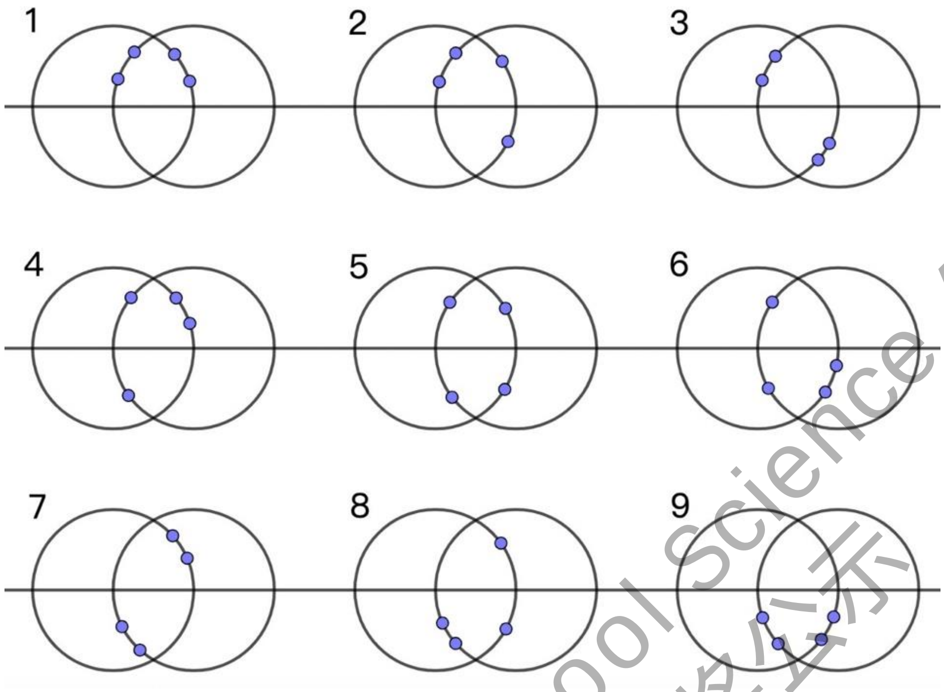
(6) No vertices are fixed and put three vertices on one side with one other on the other side\

First, connect points A and C. Then, by doing iteration of $\overline{AC}$, point B and point D will move to point B' and point D', respectively. Next, do iteration of $\overline{B'D'}$. The case will eventually become case 2 or 7.



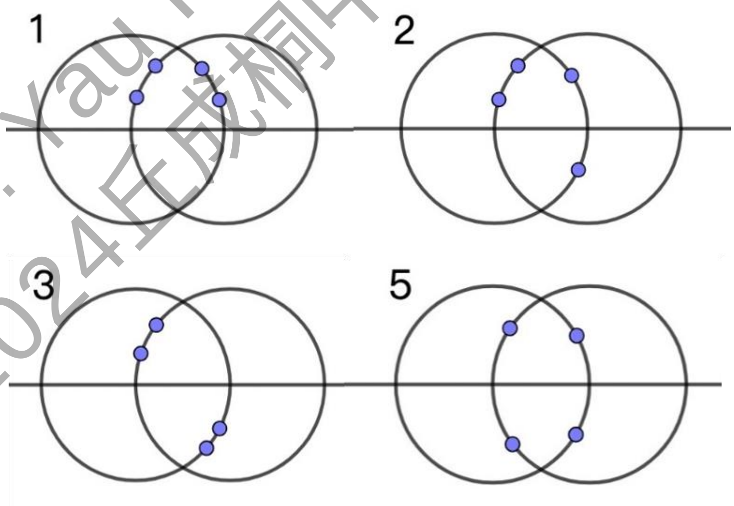
Pic 18 None of the four vertices are at the intersection points and put three vertices on the same side

(7) <u>No vertices are fixed and put two vertices on one side with two others on the other side</u>



Pic 19 Nine cases where none of the four points are fixed and two points lie on the same side

The first case can turn into the ninth case by flipping the graph upside down. The second, third, sixth are equal to the fourth, seventh, eighth, respectively. Therefore, the nine cases above can be further simplified into the following four cases:



Pic 20 Four cases remain after simplification

Suppose that the line connecting the intersection points of the two circles is the y-axis, we divide the left and right arcs into two equal parts with a straight line as the x-axis. This divides the space into four quadrants.

In the first case, select two points from the left and right arcs for one each. The slope of the line formed by the two points is between $\left(-\sqrt{3},\sqrt{3}\right)$. We do the iteration of the sector between them. If the slope is in the range of $[\frac{1}{\sqrt{3}},\infty)$, it intersects the right arc in the fourth quadrant; if the slope is in the range of $(-\infty,-\frac{1}{\sqrt{3}}]$, it intersects the left arc in the third quadrant, resulting in a case the same as the second case. If the slope is in the range of $\left(-\infty,-\frac{1}{\sqrt{3}}\right]$, there will be no intersection after an iteration. Therefore, the maximum area occurs when two of the points are at the intersection points of the two circles.

In the second case, select one point from the fourth quadrant and another point from the second quadrant. The slope of the line formed by the two points is in the range of $(-\infty,0)$. If the slope is greater than $\frac{-1}{\sqrt{3}}$, after an iteration, it will intersect the left arc in the third quadrant, forming the fifth case. With the similar method, if the slope is between the range of $(\frac{-1}{\sqrt{3}},0)$, then there will be no intersection after an iteration. Therefore, the maximum area occurs when two of the points are at the intersection points of the two circles.

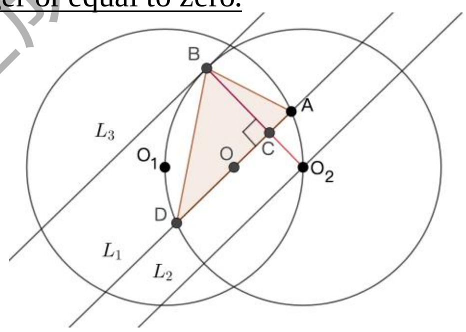
In the third case, select one point from the fourth quadrant and another point from the second quadrant, which will lead to the same conclusion as the second case. From the above, it can be seen that all cases eventually lead to the fifth case. Therefore, we only need to discuss the fifth case.

Discussing the fifth case

In the fifth case, a bigger quadrilateral can be found by using quadrilateral iteration. According to Lemma 1, both diagonals will pass through the point $(\frac{1}{2},0)$. It will transfer to the figure of parallelogram.

Using Theorem 3 (p.14) can know that three vertices should be on equal diversion vertices, and the figure will be same as the first case, and its area is approximately 0.53.

a. <u>When the slope is bigger or equal to zero:</u>



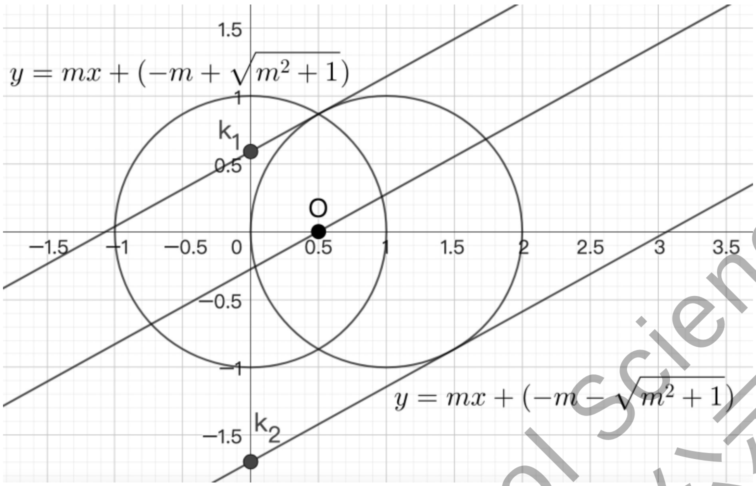
Pic 21 Draw parallel lines with the same slope to find the maximum area

The equation of the left circle is $x^2+y^2=1$, and the equation of the right circle is $(x-1)^2+y^2=1$. Assume $m\geq 0$,

$$L_1: y=m\left(x-\frac{1}{2}\right)\Rightarrow mx-y-\frac{m}{2}=0$$
$$L_2: y=m(x-1)\Rightarrow mx-y-m=0$$
$$L_3: y=mx+k\Rightarrow mx-y+k=0$$

$$d(L_2, L_3) = 1 = \frac{|-m-k|}{\sqrt{m^2+1}}$$
$$k = -m \pm \sqrt{m^2+1}$$

Because we need the positive y-intercept of L_3, so we choose $k = -m + \sqrt{m^2+1}$



Pic 22 The maximum quadrilateral where one point is fixed at the intersection and the remaining three points have two points on the same side

$d(L_1, L_3) = \frac{\sqrt{m^2+1}-m+\frac{m}{2}}{\sqrt{m^2+1}} = \frac{\sqrt{m^2+1}-\frac{m}{2}}{\sqrt{m^2+1}} = 1 - \frac{m}{2\sqrt{m^2+1}}$ is the height of the triangle

Find the intersection of L_1 and the left circle

$$x^2 + m^2x^2 - m^2x + \frac{m^2}{4} = 1$$
$$(m+1)x^2 - m^2x + \frac{m^2-4}{4} = 0$$
$$x = \frac{m^2 \pm \sqrt{3m^2+4}}{2m^2+2}$$

By Pic 22, we choose the positive one.

$$x - \frac{1}{2} = \frac{m^2 + \sqrt{3m^2+4}}{2m^2+2} - \frac{1}{2} = \frac{\sqrt{3m^2+4}-1}{2m^2+2}$$

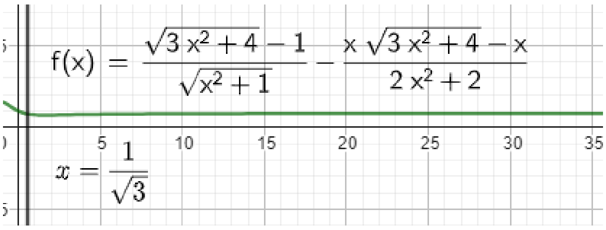
$\frac{\sqrt{3m^2+4}-1}{2m^2+2} \times \sqrt{m^2+1} \times 2$, is the base of triangle height

The sum of two triangle is base times height

$$= \frac{\sqrt{3m^2+4}-1}{\sqrt{m^2+1}} - \frac{m\sqrt{3m^2+4}-m}{2m^2+2}$$

At the same time, m must be bigger than $\frac{1}{\sqrt{3}}$, so from the picture, the maximum occurs when $m \to \infty$

$$\Rightarrow \lim_{m\to\infty}\left(\frac{\sqrt{3m^2+4}-1}{\sqrt{m^2+1}} - \frac{m\sqrt{3m^2+4}-m}{2m^2+2}\right) = \frac{\sqrt{3}}{2}$$



Pic 23 use m to show the maximum quadrilateral area ($m > 0$)

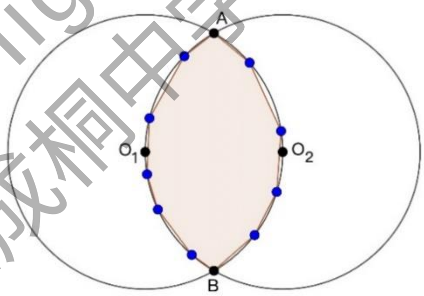
b. <u>When the slope is smaller or equal to zero:</u> without loss of generality, it will lead to the same consequence.

Content 4: Polygon

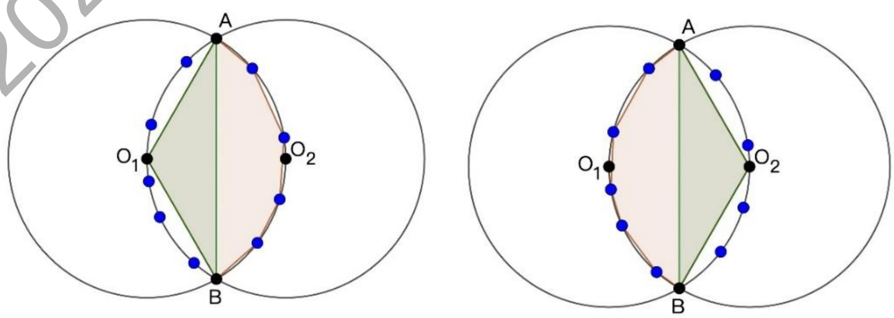
1. Prove the maximum figure of n-sided polygon within the intersection of two circles, including both of intersection points

> Theorem 5
> When two points are fixed on both upper and lower intersection points, the maximum shape of n-gons occurs when x points on the left arc and $k - x$ points on the right arc are at the points of bisection. If k is an even number, $x = \frac{k}{2}$. If k is an odd number, $x = \frac{k+1}{2}$ or $\frac{k-1}{2}$.
> $(n = k + 2)$

Proof.



Pic 24 the inscribed polygon within the two intersecting circles when two points are fixed at the intersection points



Pic 25 Divide the above diagram into left and right halves

Suppose there are $(k+2)$ points in total consisting of a polygon, with two points located at the intersection points of the two circles. Let there be $(k-x)$ points on the left arc and x points on the right arc, where $k \geq 2$ and $0 < x < k$. Let $f_3(x)$ be the area function of the inscribed polygon within the intersection region of the two circles. By Jensen's Inequality, we can deduce that the $(k-x)$ points on the left and the x points on the right locate at the point of bisection of $\widehat{AO_1B}$ and $\widehat{AO_2B}$, respectively. The figure can be divided into two parts, left and right. For one side, the area of the right half can be explained as the orange area minus the green area, with the orange area being composed of multiple triangles. Similarly, the area of the left half can be obtained. Therefore, $f_3(x)$ can be written as:

$$f_3(x) = \frac{1}{2}[(x+1)\,sin\left(\frac{2\pi}{3(x+1)}\right) + (k-x+1)\,sin\left(\frac{2\pi}{3(k-x+1)}\right)] - \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2}f_4(x) - \frac{\sqrt{3}}{2}$$

$$0 \leq x \leq k, k \geq 2, x \in \mathbb{N}$$

$$\Rightarrow f_4'(x) = sin\left(\frac{2\pi}{3(x+1)}\right) - \frac{2\pi}{3(x+1)}cos\left(\frac{2\pi}{3(x+1)}\right) - sin\left(\frac{2\pi}{3(k-x+1)}\right) + \frac{2\pi}{3(k-x+1)}cos\left(\frac{2\pi}{3(k-x+1)}\right) = 0$$

$\Rightarrow x = \frac{k}{2}$ is an apparent solution

$$f_4''(x) = \frac{1}{2}[-\frac{4\pi^2}{9(x+1)^3}sin\left(\frac{2\pi}{3(x+1)}\right) - \frac{4\pi^2}{9(k-x+1)^3}sin\left(\frac{2\pi}{3(k-x+1)}\right)]$$

When $0 < x < k$, $f_4''(x) < 0 \Rightarrow f_3'(x)$ is a strictly decreasing function

$\Rightarrow x = \frac{k}{2}$ is the only solution

Assume $f_4(x) = f_5(x) + f_5(k-x)$, $f_5(x) = (x+1)\,sin\left(\frac{2\pi}{3(x+1)}\right)$

$\Rightarrow f_5'(x) = sin\left(\frac{2\pi}{3(x+1)}\right) - \frac{2\pi}{3(x+1)}cos\left(\frac{2\pi}{3(x+1)}\right)$, $f_5''(x) = -\frac{4\pi^2}{9(x+1)^3}sin\left(\frac{2\pi}{3(x+1)}\right)$

$$\because f_5''(x) = -\frac{4\pi^2}{9(x+1)^3}sin\left(\frac{2\pi}{3(x+1)}\right) < 0$$

$$\therefore f_5\left(\frac{k}{2}\right) > \frac{f_5(0) + f_5(k)}{2} \Rightarrow 2f_5\left(\frac{k}{2}\right) - \left(f_5(0) + f_5(k)\right) > 0$$

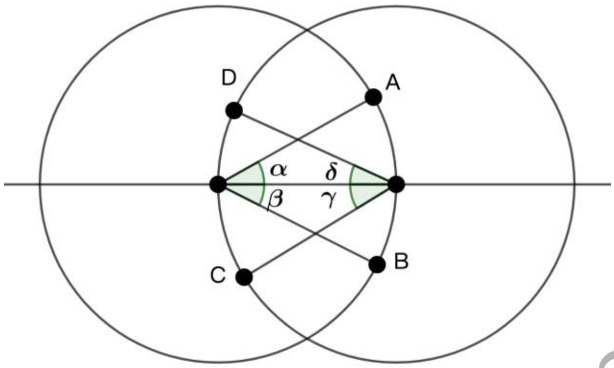
$$\Rightarrow f_4\left(\frac{k}{2}\right) - f_4(0) > 0$$

$$\Rightarrow f_4\left(\frac{k}{2}\right) > f_4(0)$$

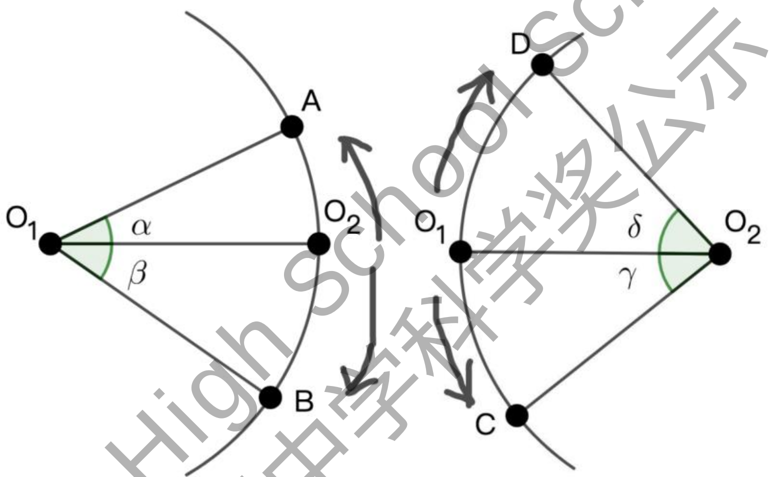
If k is an even number, $x = \frac{k}{2}$ will have the maximum area. If k is an odd number, $x = \frac{k+1}{2}$ or $\frac{k-1}{2}$ will have the maximum area. $(k \geq 2)$

2. Prove the case that we can use iteration to remove the most bottom point to the lower intersection point

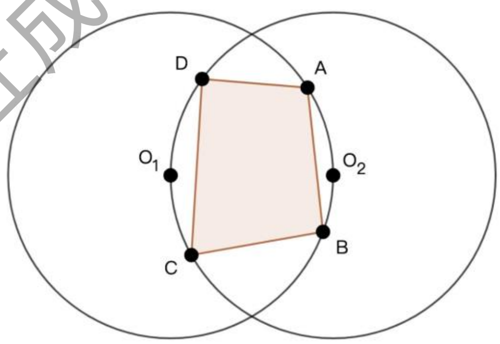
Proof.



Pic 26 The definitions of the angles α, β, γ and δ correspond to four points on the circle.



Pic 27 Define that all four angles are greater than or equal to zero



Pic 28 An example of a quadrilateral derived from the given angle definitions

$$f_6(\alpha, \beta, x) = \frac{1}{2}\left(xsin(\frac{\alpha+\beta}{x}) - sin(\alpha+\beta)\right) \dots (7)$$
$$f_6(\gamma, \delta, n-x) = \frac{1}{2}\left((n-x)sin\,(\frac{\gamma+\delta}{n-x}) - sin(\gamma+\delta)\right) \dots (8)$$
$$x \geq 2$$

$$0 \le \alpha, \beta, \gamma, \delta \le \frac{\pi}{3}$$

$$f_7(\alpha, \beta, \gamma, \delta) = \frac{1}{2}\begin{vmatrix} cos\alpha & 1 - cos\delta & 1 - cos\gamma & cos\beta & cos\alpha \\ sin\alpha & sin\delta & -sin\gamma & -sin\beta & sin\alpha \end{vmatrix}$$

$$= \frac{1}{2}(cos\alpha sin\delta - sin\gamma + sin\gamma cos\delta - sin\beta + sin\delta + sin\beta cos\gamma + sin\alpha cos\beta$$
$$-sin\alpha + sin\alpha cos\delta - sin\delta + cos\gamma sin\delta + cos\beta sin\gamma + cos\alpha sin\beta) \ldots (9)$$

Suppose $f_8(\alpha, \beta, \gamma, \delta, x) = (7) + (8) + (9)$

$$\Rightarrow 2\frac{\partial f_8(\alpha, \beta, \gamma, \delta, x)}{\partial x}$$
$$= sin\left(\frac{\alpha+\beta}{x}\right) - \frac{\alpha+\beta}{x} cos\left(\frac{\alpha+\beta}{x}\right) - sin\left(\frac{\gamma+\delta}{n-x}\right) + \frac{\gamma+\delta}{n-x} cos\left(\frac{\gamma+\delta}{n-x}\right)$$
$$2\frac{\partial^2 f_8(\alpha, \beta, \gamma, \delta, x)}{\partial x^2} = -\frac{(\alpha+\beta)^2}{x^3} sin\left(\frac{\alpha+\beta}{x}\right) - \frac{(\gamma+\delta)^2}{(n-x)^3} sin\left(\frac{\gamma+\delta}{n-x}\right)$$

Since $\alpha, \beta, \gamma, \delta, x > 0$ and $0 \le \alpha, \beta, \gamma, \delta \le \frac{\pi}{3}$, we can know that $\frac{\partial^2 f_8(\alpha,\beta,\gamma,\delta,x)}{\partial x^2} \le 0$. We Can also find that if $\alpha, \beta, \gamma, \delta$ are constant, $2\frac{\partial f_8(\alpha,\beta,\gamma,\delta,x)}{\partial x}$ strictly decreases in the range. Because $\frac{\partial f_8(\alpha,\beta,\gamma,\delta,x)}{\partial x}$ strictly decreases in the range and find that there must be a point that $\frac{\partial f_8(\alpha,\beta,\gamma,\delta,x)}{\partial x} = 0$ by observing. Therefore, there must be a maximum in the range, and when $\alpha + \beta = \gamma + \delta$, the maximum occurs when $x = \frac{n}{2}$

After that, we use partial differentiation to find the maximum of the range.

$$\frac{\partial 2f_8(\alpha, \beta, \gamma, \delta, x)}{\partial \alpha} = cos\left(\frac{\alpha+\beta}{x}\right) + \cos(\alpha + \delta) - cos\alpha = 0 \ldots (10)$$
$$\frac{\partial 2f_8(\alpha, \beta, \gamma, \delta, x)}{\partial \beta} = cos\left(\frac{\alpha+\beta}{x}\right) + \cos(\beta + \gamma) - cos\beta = 0 \ldots (11)$$
$$\frac{\partial 2f_8(\alpha, \beta, \gamma, \delta, x)}{\partial \gamma} = cos\left(\frac{\gamma+\delta}{n-x}\right) + \cos(\beta + \gamma) - cos\gamma = 0 \ldots (12)$$
$$\frac{\partial 2f_8(\alpha, \beta, \gamma, \delta, x)}{\partial \delta} = cos\left(\frac{\gamma+\delta}{n-x}\right) + \cos(\alpha + \delta) - cos\delta = 0 \ldots (13)$$

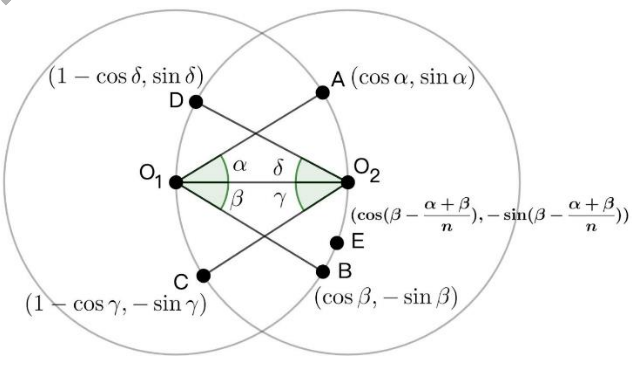
$(10) - (11) + (12) - (13)$, we can find the following equation:

$$cos\alpha + cos\gamma = cos\beta + cos\delta$$

Because the formula is sophisticates, we choose several possible extremum values.

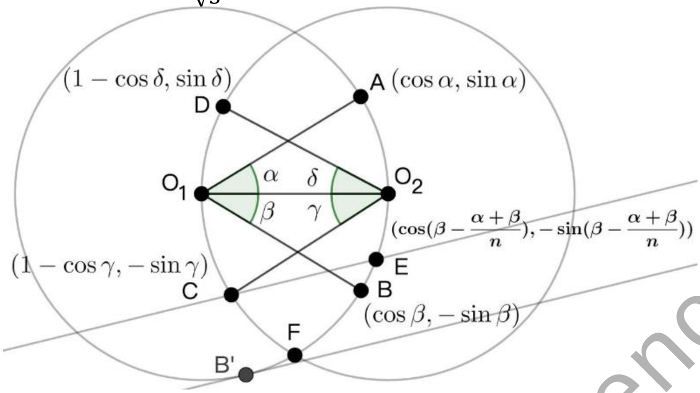
3. Prove the case that we can use adjustment to remove the most bottom point to the lower intersection point

Proof.



Pic 29 To move the lowest point on one of the arcs to the lower intersection point of the two circles

Because the lower intersection point F has a slope of $\frac{1}{\sqrt{3}}$ on the tangent of the left circle, so when $0 < m_{CE} \leq \frac{1}{\sqrt{3}}$, B' will be located at F after iteration because of exceeding the range. Hence, when $0 < m_{CE} \leq \frac{1}{\sqrt{3}}$, B will become F after iteration.



Pic 30 The case that can move B' to the lower intersecting point F

$$m = \frac{-\sin\left(\beta - \frac{\alpha+\beta}{n}\right) + \sin\gamma}{\cos(\beta - \frac{\alpha+\beta}{n}) + \cos\gamma - 1}$$

When $n \to \infty, m = \frac{-\sin\beta + \sin\gamma}{\cos\beta + \cos\gamma - 1}$

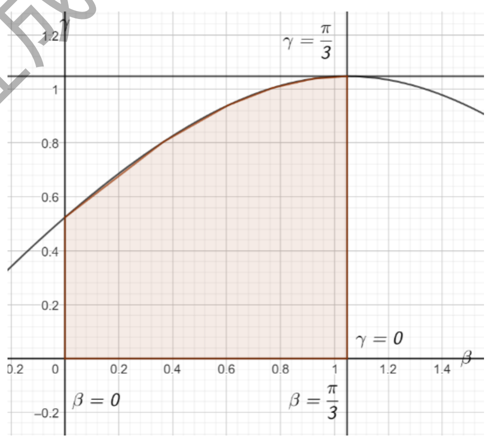
$$m = \frac{-\sin\beta + \sin\gamma}{\cos\beta + \cos\gamma - 1} \leq \frac{1}{\sqrt{3}}$$

$$\frac{-\sin\beta + \sin\gamma}{\cos\beta + \cos\gamma - 1} \leq \frac{1}{\sqrt{3}}$$

$$-\sqrt{3}\sin\beta + \sqrt{3}\sin\gamma \leq \cos\beta + \cos\gamma - 1$$

$$\frac{1}{2} \leq \sin\left(\beta + \frac{\pi}{6}\right) - \sin\left(\gamma - \frac{\pi}{6}\right)$$

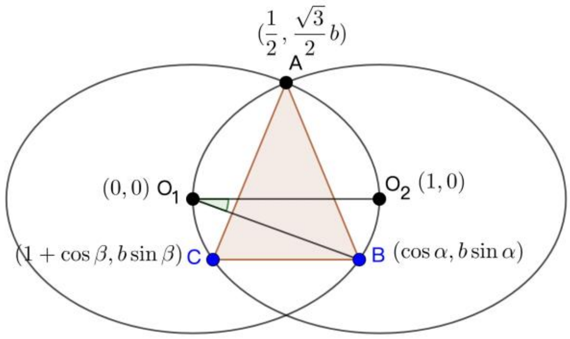
$$\gamma \leq \frac{\pi}{6} + \sin^{-1}\left(\sin\left(\beta + \frac{\pi}{6}\right) - \frac{1}{2}\right)$$



Pic 31 The case where it can be directly corrected to the intersection point

Content 5: Ellipse

In addition to using Cauchy-Schwarz Inequality and differential calculus to discuss the cases of circles, we also consider the cases of ellipses. In fact, when we switch to ellipses, their properties remain quite similar.

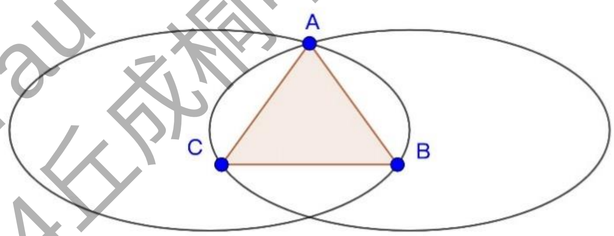


Pic 32 The largest inscribed triangle formed by two ellipses with equal major and minor axes intersecting at the center.

The reason is that an ellipse can be viewed as a circle stretched along the x-axis or y-axis. Consequently, the triangle of maximum area within the ellipse must also be a result of stretching an isosceles triangle with three angles of 70°, 70°, and 40°. According to Shoelace formula, the area can be described as

$$\frac{1}{2}\left\|\begin{matrix} \frac{1}{2} & \cos\alpha & 1+\cos\beta & \frac{1}{2} \\ \frac{\sqrt{3}}{2}b & b\sin\alpha & b\sin\beta & \frac{\sqrt{3}}{2}b \end{matrix}\right\| = b \times f_1(\alpha, \beta)$$

Thus, we obtain a similar equation as previously discussed, where the extremum occurs at $\alpha = -\frac{\pi}{9}, \beta = \frac{10\pi}{9}$.



Pic 33 The largest inscribed triangle in two ellipses of the same size (major axis: minor axis = 2:1) intersecting at the center.

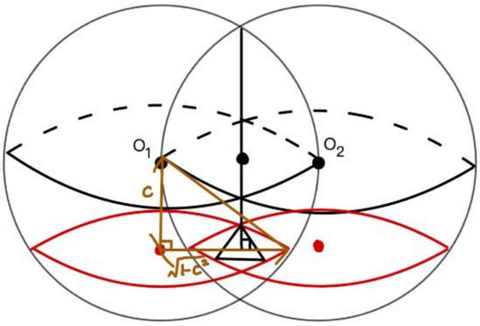
Given the equation of the polygon (f_9), we find that it only differs from the original f_1 by a factor of b. Therefore, the area can be easily calculated from the results of the circle.

$$\frac{1}{2}\left|\begin{matrix} 1+\cos\beta_1 & 1+\cos\beta_2 & \cdots & \cos\alpha_2 & \cos\alpha_1 \\ b\sin\beta_1 & b\sin\beta_2 & & b\sin\alpha_2 & b\sin\alpha_1 \end{matrix}\right| = b\times f_9(\alpha_1, \dots, \beta_{n-x})$$

Besides ellipses, we can also extend our discussion to other conic sections. However, we have decided to focus on three-dimensional cases. Therefore, the next chapter will explain our preliminary ideas and the solutions we provide.

Content 6: Three-dimensional case

What's more, we extend this question to three-dimentional cases. Differently, the intersecting circles become intersecting sphere. What we are going to find is the biggest tetrahedron. The restriction that we set is that the base of the tetrahedron will be parallel to the xy-plane.



Pic 34 The perspective view of the intersection of two spheres

Assume the equations of two spheres is $x^2+y^2+z^2=1$, $(x-1)^2+y^2+z^2=1$.
Assume c is the distance between the base of the tetrahedron and xy-plane.
To find the maximum tetrahedron, we can divide into two steps:

1. <u>Find the base area</u>

From Pic 34, the base will be two intersecting circles which both radii are $\sqrt{1-c^2}$ and the length of O_1 and O_2 is 1. By Yang [2], the maximum area is $2r^2 sin^3\theta$ when $\theta = \frac{2}{3}cos^{-1}(\frac{s}{2r})$, which r is the radius, s is the length of O_1 and O_2, and θ is the vertex angle of the triangle.
Therefore, we can obtain the area function:

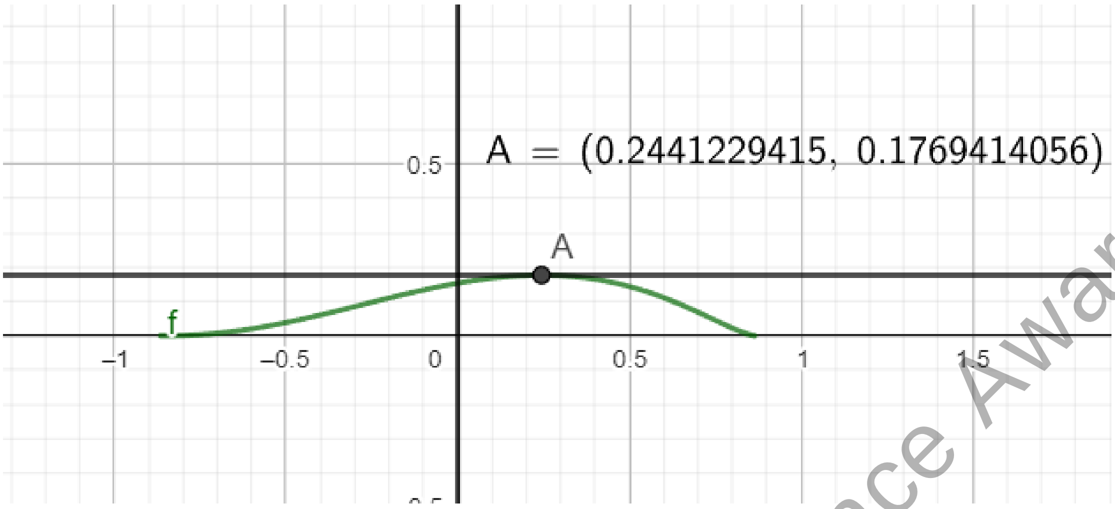
$$2(1-c^2)sin^3(\frac{2}{3}cos^{-1}\left(\frac{1}{2\sqrt{1-c^2}}\right))$$

2. <u>Find the height</u>

Because the base will be parallel to xy-plane, the maximum height occurs at the point with the largest z-coordinate within the intersection of the two spheres. Therefore, we can obtain the height is $\frac{\sqrt{3}}{2}+c$.
Combine the area and the height, we can obtain the volume function is

$$\frac{1}{3}\cdot 2(1-c^2)sin^3(\frac{2}{3}cos^{-1}\left(\frac{1}{2\sqrt{1-c^2}}\right))\cdot(\frac{\sqrt{3}}{2}+\text{c})$$

By Pic 35, we can find that the maximum volume is approximately 0.177. At this point, $c \approx\ 0.244$ and $\theta = 0.688 = 39.419°$.



Pic 35 The graph of the volume equation plotted with c as the x-axis

3. Remark:

We also know how to solve the maximum volume tetrahedron in two congruent ellipsoid. Since we know the case for two ellipses, the calculation will be similar to the case of two spheres. Therefore, we plan to skip the details here.

References

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