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Research Report

The Team

Registration Number: Math-153

Name of Team Member: YING Liqian
School: NUS High School of Mathematics and Science
Country: Singapore

Name of Supervising Teacher: CHAI Ming Huang Royce
Job Title: Mathematics Teacher
School: NUS High School of Mathematics and Science
Country: Singapore

Title of Research Report

Make24: Bounding the Generalised Form of a Numbers Game

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Make24: Bounding the Generalised Form of a Numbers Game

YING Liqian[1]

NUS High School of Mathematics and Science[1]

Research Abstract

Make24 is a multiplayer brain teaser that you can play anywhere, anytime! The classic version of the game involves picking 4 (not necessarily distinct) numbers between 1 and 10 inclusive at random, and racing against the other players to be the fastest player to successfully form the number 24 by using each number exactly once, using +, −, ×, ÷ and ( ).

These 4 numbers can come from car plates, poker cards, or even simply making up numbers in your mind, making it a fun way to pass time with family and friends!

From here, let the word "set" and {} denote a *Multiset*, which is a set that allows duplicate entries.

Given a set $D = \{d_1, d_2, d_3, \ldots, d_k\}$, where d_i are (not necessarily distinct) integers such that $1 \leq d_i \leq 10$ for all $1 \leq i \leq k$, we say that we can Make n with D, if we can create an expression,

- using each $d_1, d_2, \ldots, d_k$ *exactly once* each,
- with the operations +, −, ×, ÷, or (),
- such that each operation yields an *integer*, and
- such that it evaluates to n.

For example, with the set $\{4, 4, 10, 10\}$ we can Make 24 because $(10 \times 10 - 4) \div 4$ is one such expression that satisfies all the given requirements. We say that this expression is a *valid solution.*

I defined the function $\eta : \mathbb{Z} \to \mathbb{N}$ such that $\eta(n)$ is the minimum $k \in \mathbb{N}$ such that given an integer n, we can Make n with every set $D = \{d_1, d_2, d_3, \ldots, d_k\}$ such that $1 \leq d_i \leq 10$ for all $1 \leq i \leq k$.

That is, $\eta(n)$ is the *minimum* size of a set D with each entry between 1 and 10 inclusive, such that we can *guarantee* to *always* Make n. In this project, I am interested to study the properties of this function.

In my research, I will first study the properties of $\eta(n)$. For example, I show that given not necessarily distinct $a, b \in \mathbb{Z}$, we have the identities $\eta(-a) = \eta(a)$, $\eta(a+b) \leq \eta(a) + \eta(b)$ and $\eta(ab) \leq \eta(a) + \eta(b)$.

I will then study the values of $\eta(n)$ for small integers n, by using a *Complete Search* algorithm to obtain exact values for $0 \leq n \leq 12$. From here, I will use the properties above to produce tight bounds for the values of $\eta(n)$ for $13 \leq n \leq 30$. In particular, I fully solve the original case of Make24 to obtain $\eta(24) = 9$.

Next, I shift my focus onto the *Asymptotic Growth Rate* of η. In particular, I show that the bounds

$$\lceil 3 \log_3 n \rceil \leq \eta(n) \leq \lfloor 3 \log_2 n \rfloor + 1$$

holds for all integers $n > 2$ using the idea of *Strong Induction*. On the way, we study other functions like $\zeta(k)$, defined to be the maximum integer we can Make with k ones, through the analysis of *Greedy Algorithms*, to aid us in producing our final result.

Applications of this project involve gaining insight into the study of similar *Combinatorics Games*, as the techniques of proof, such as *reduction into a simpler form*, *induction* and *pattern recognition* are all very common themes in modern combinatorics, especially in olympiads. Results from this paper are potentially also applicable in attempting to prove unsolved conjectures in relation to a number's *integer complexity*, defined as the minimum number of ones needed to make that number with + and ×.

Furthermore, this project is applicable in *Computer Science*, specifically in the study of *Complexity Analysis*. Since values of η can be searched through an *exponential-time complete search*, this project can be a way to test the effectiveness of a program or a programming language at implementing *recursion*, *memoization* (i.e. to store and recover previously computed results) and *pruning* (i.e. to eliminate the need to evaluate identical expressions multiple times).

Keywords: 24 Game, Make24, Combinatorics, Integer Complexity, Mahler-Popken Complexity

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3. observe the common standard of academic integrity adopted by most journals and degree theses.
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(Signatures of full team below)

X ______________________

Name of team member: YING Liqian

X ______________________

Name of team member: Nil

X ______________________

Name of team member: Nil

X ______________________

Name of supervising teacher: CHAI Ming Huang Royce

Noted and endorsed by

(signature)

Mr Goh Hock Leong
Deputy Principal (Academic)
NUS High School of Math & Science

Name of school principal: SOH Lai Leng Magdalen

Contents

This game holds very special significance to me as it was a game introduced to me by my parents and a game I played with them growing up. Hence, I dedicate this research project to them.

1 Introduction

Make24 is a game where you take k integers, $\{d_1, d_2, \ldots, d_k\}$, where $1 \leq d_i \leq 10$ as well as the four basic operations, $+$, $-$, $\times$, $\div$, and brackets, in any order, to make the number 24.

For example, for the integers $\{1, 2, 3, 5\}$, we can perform $(5 - 1) \times 2 \times 3 = 24$. We say this is a valid solution. There may be more than 1 solution for every set of integers.

The other rules are:

- Each digit must be used **exactly once**.
- The final result must be **exactly** 24 (i.e. not 23, 25 or 24.1).
- Every operation must yield an **integer**, so $6 \div 3$ is allowed, but $5 \div 3$ isn't.

1.1 Problem Statement and Definitions

What is the minimum number of integers d_i $(1 \leq d_i \leq 10)$ required such that we can guarantee to Make 24 with every such set of integers with the basic mathematical operations $+, -, \times, \div$, using each d_i **exactly once**?

Definition 1.1. *Throughout this paper, let "set" and "$\{\ldots\}$" refer to a Multiset, which is a modified version of a set that allows for duplicate entries, and all other normal normal set notations hold.*

Definition 1.2. *Let $\mathbb{N}$ denote the infinite set of naturals, $\mathbb{N} := \{1, 2, 3, \ldots\}$. That is, $0 \notin \mathbb{N}$.*

We also make these 4 following definitions that will be used:

Definition 1.3. *Let "Make n from D" refer to using the elements of D* ***exactly once*** *each, with the operations $+, -, \times$ or $\div$, to make n.*

Definition 1.4. *Let D_n $(n \in \mathbb{N})$ be the set of all sets $\{d_1, d_2, \ldots, d_n\}$ such that $1 \leq d_i \leq 10$ for each $i = 1, 2, \ldots, n$.*

Definition 1.5. *Let $\eta : \mathbb{Z} \to \mathbb{N}$ be the function such that that $\eta(n)$ is the minimum number of integers d_i required to guarantee you can Make n with* ***any*** *$D \in D_{\eta(n)}$.*

Definition 1.6. *Let I_n $(n \in \mathbb{N})$ be the set $\underbrace{\{1, 1, \ldots, 1\}}_{n \text{ ones}}$. We call this the n-th unital set.*

1.2 The 4 Numbers Game

The case $k = 4$ (where we choose exactly 4 numbers) is a well-explored case, and is sometimes referred to as the "4 Numbers Game". (4nums.com [1])

Existing datasets (4nums.com [1]) tell us there are 566 unique unordered (that is, $\{1, 2, 3, 4\}$ is the same as $\{4, 3, 2, 1\}$) solutions for a version of the 4 Numbers Game without the rule where every operation must yield an integer is removed. This is cross-checked with Python and C++ Code avaliable at (RosettaCode, 2023 [10]), which all give the result 566.

But how many sets of 4 unordered numbers are there? If we choose 4 unique numbers from 1 to 10, we have ${}^{10}C_4$ choices. If we choose 3 unique numbers and have 1 duplicate, then we have ${}^{10}C_3$ ways to choose the unique numbers, and we can add the duplicate in 3 ways, for a total of ${}^{10}C_3 \times 3$ ways. We obtain the other two values similarly.

There are then ${}^{10}C_4 + {}^{10}C_3 \times 3 + {}^{10}C_2 \times 3 + {}^{10}C_1 = 715$ total ways to choose 4 unordered sets of numbers, so $\frac{566}{715} \approx 79.2\%$ of such unordered numbers have valid solution(s).

1.3 Thinking Process and Rough Plan

In a large part of this research, many problems are approached using the idea of *bounding*. In this paper, I will derive upper and lower bounds for $\eta(24)$, which eventually work towards determining the true value of $\eta(24)$.

It is relatively simple to come up with a lower bound. We simply have to present a set $D \in D_n$ such that it is impossible to make 24. In that case, we have shown that the lower bound must be greater than n.

It is harder to come up with an upper bound, since we must show that for **any** $D \in D_n$ we can make 24, in order to show that the upper bound is at most n. We will approach this issue carefully, using the ideas of *reduction* and *induction*.

We will justify why the above is sufficient and rigorous.

2 Lower Bound for $\eta(24)$

We first establish some lemmas.

Lemma 2.1. *If you cannot Make n with some $D \in D_i$, then there must be some $S \in D_j$ such that we cannot Make n with S, for all $j < i$.*

Proof. We prove by Induction. Let $D = \{d_1, d_2, \ldots, d_i\} \in D_i$ be a set for which you can never Make n. Now, we let P_m be the assertion that you can find a set $S = \{d_1, d_2, d_3, \ldots, d_m\} \in D_m$ such that it is impossible to Make n. By definition, this is true for the Base Case $i = k$.

Now suppose P_i is true for some $i \leq k$. Then suppose by Way of Contradiction that all you can Make n for all sets $R \in D_{i-1}$. If $d_1 = d_2 = \cdots = d_i$, we consider the set $R = \{\frac{d_1}{d_2}, d_3, d_4, \ldots, d_i\}$. Otherwise, without Loss of Generality let $d_1 > d_2$ and consider the set $R = \{d_1 - d_2, d_3, d_4, \ldots, d_i\}$. Clearly, $R \in D_{i-1}$. By our assumption, we can Make n with R, and since R is formed from a valid operation of S, we can also Make n with S. But by our Induction Hypothesis, it is impossible to Make n from S. *Contradiction.* Hence there must be some set $R \in D_{i-1}$ such that you can never Make n. Hence $P_i \implies P_{i-1}$ and this completes our Inductive Step, and we are done by the **Principle of Mathematical Induction**. ■

Lemma 2.2. ***(Lower Bound Lemma)*** *If we cannot Make n with $D \in D_k$, $\eta(n) > k$.*

Proof. It is clear that $\eta(n) \neq k$, since we cannot Make n with D. Now suppose by Way of Contradiction $\eta(n) < k$. By Lemma 2.1, since $\eta(n) < k$ and we cannot make n with $D \in D_k$, we have that there must exist some $S \in D_{\eta(n)}$ such that we can never Make n. But the definition of η implies that we can Make n with all $S \in D_{\eta(n)}$. *Contradiction.* The result follows. ■

We return to the original problem. We consider $D = I_n$. That is, $D = \underbrace{\{1, 1, \ldots, 1\}}_{n \text{ ones}}$.

A trend among past papers (de Renya, 2000 [3], de Renya *et al.*, 2009 [4]), is that they focus on the *integer complexity* of a number, denoted $||n||$, which is defined to be the minimum number of 1's needed to make n.

(de Renya *et al.*, 2009 [4]) writes "*This formula is very time consuming to use for large n, but we know no other method to compute $||n||$*", referring to a result in the paper. Hence we shall not use this idea, and instead propose to focus on a more intuitive way of thinking, which is similar to the inverse of the classical method.

We ask, instead, given I_n, that is, n ones, what is the largest number we can make? If it is less than 24, then we have shown that I_n cannot Make 24.

It is quite clear that to create the largest number, we should not use $-$ or $\div$ in this case. We then want to find the largest number that can be made with n 1's, with only $+$ and $\times$. We follow previous authors (de Renya *et al.*, 2009 [4]) in terming this the *maximal integral sum-product* of length n.

Definition 2.3. *Let $\zeta : \mathbb{N} \to \mathbb{N}$ be the function such that $\zeta(n)$ is the maximal integral sum-product of length n. That is, the maximum number you can make with I_n.*

Definition 2.4. *For $k, n \in \mathbb{N}$, we say k is n-conservative if there exists $a_1, a_2, \ldots, a_m \in \mathbb{N}$ such that $a_1 + a_2 + \cdots + a_m = n$ and $a_1 \times a_2 \times \cdots \times a_m = k$.*

Intuitively we guess the maximum value attained from such a set should be n-conservative, that is, in the form of $(1+1+\cdots+1)(1+\cdots+1)\ldots(1+\cdots+1)$ where the terms sum to n. We let each of the sums be $a_1, a_2, \ldots a_k$ respectively. This is what we try to prove with the following lemma.

Lemma 2.5. *$\zeta(n)$ is n-conservative. In particular, all $a_i \in \{2, 3\}$.*

Proof. Suppose that $\mathcal{O}$ is an expression to make $\zeta(n)$ with D, and suppose we used an arbitrary addition $A + B$. Without Loss of Generality let $A \leq B$. Now suppose by Way of Contradiction $A \geq 2$, we have $A + B \leq 2B \leq AB$. We can then replace $A + B$ with $A \times B$. If the new value of $\zeta(n)$ after replacing $A + B$ with AB is the same, nothing has changed. If the value increases, this challenges the maximality of $\zeta(n)$. *Contradiction.*

In either case, we can only consider additions in the form $B + 1$. If $B = 1$, the addition is $1 + 1 = 2$ and cannot be replaced. We only consider $B > 1$ from now on.

If the last operation in $\mathcal{O}$ used to make B is $+$, then by the logic above that operation must have been $C + 1 = B$. But then the addition we consider is $(C+1)+1 = C+2$. If $C > 1$, then the logic above implies we can replace $(C+1)+1$ with $(1+1) \times C$ in $\mathcal{O}$ while not decreasing its value. If $C = 1$, the addition yields $(1+1)+1 = 3$ and cannot be replaced.

Else the last operation in $\mathcal{O}$ used to make B must be $\times$. Suppose in $\mathcal{O}$, $n \times m$ is used to make B. Since $B > 1$, without Loss of Generality we can let $n > 1$. Now the addition we are considering is $n \times m + 1$. But consider $n \times (m+1)$ instead. We have $n \times (m+1) = n \times m + n > n \times m + 1$.

This challenges the maximality of $\zeta(n)$. *Contradiction.*

Hence, all additions result in 2 or 3. This implies that the rest of the operations are all multiplication, and hence $\zeta(n)$ is n-conservative, with all terms being 1 or 2 or 3.

Finally, suppose by Way of Contradiction we have the operation $1 \times D = \zeta(n)$ in $\mathcal{O}$, noting that multiplication is associative. But then we may consider $1 + D > D = 1 \times D = \zeta(n)$, challenging the maximality of $\zeta(n)$. *Contradiction.* This implies all terms of the product must be 2 or 3. ■

Remark. This result is particularly interesting, since this does imply that by slightly modifying our algorithm, if we are given I_n, then we can start with any arbitrary *integral sum-product* of length n and iteratively apply this *greedy algorithm*, we will always end up at the optimal value, $\zeta(n)$. See **Appendix** for details.

We introduce standard modular arithmetic notation from *Number Theory*:

Definition 2.6. *We say $a \mid b$ if there exists $k \in \mathbb{Z}$ such that $ak = b$. Otherwise, $a \nmid b$.*

Definition 2.7. *We say $a \equiv b \pmod{c}$ if there exists $k \in \mathbb{Z}$ such that $a = b + ck$.*

Through experimentation, it seems like the largest products always come from products of 2 and 3. For example, $\zeta(5) = (1+1)(1+1+1) = 6$, $\zeta(11) = (1+1)(1+1+1)(1+1+1)(1+1+1)$. In fact, this observation seems to always hold. Hence, this motivates the following lemma.

Lemma 2.8. *Let $n \in \mathbb{N}$. We then have*

$$\zeta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 3^{\frac{n}{3}} & \text{if } n \equiv 0 \pmod{3} \\ 2 \cdot 3^{\frac{n-2}{3}} & \text{if } n \equiv 2 \pmod{3} \\ 4 \cdot 3^{\frac{n-4}{3}} & \text{otherwise} \end{cases}$$

Proof. Clearly, $\zeta(1) = 1$. We only consider $n > 1$ from now. By **Lemma 2.5**, we know that $\zeta(n)$ is n-conservative, with all terms in the product being 2 or 3. Hence we let $\zeta(n) = 2^a 3^b$, where a, b are non-negative integers such that $2a + 3b = n$. This is just the defining condition of being n-conservative.

Suppose by Way of Contradiction $2^3 \mid \zeta(n)$. This implies that $\zeta(n) = 8N = (1+1)(1+1)(1+1)N$ for some N. But we may replace $(1+1)(1+1)(1+1)$ with $(1+1+1)(1+1+1)$, making $(1+1+1)(1+1+1)N = 9N > 8N = \zeta(n)$. This challenges the maximality of $\zeta(n)$. *Contradiction.* Hence $2^3 \nmid \zeta(n)$, and $0 \leq a \leq 2$.

Now, if $n \equiv 0 \pmod 3$, we have $2a + 3b \equiv 0 \pmod 3 \implies 2a \equiv 0 \pmod 3$. Hence $a = 0$. Then $2(0) + 3b = n \implies b = \frac{n}{3}$ and we must have $\zeta(n) = 3^{\frac{n}{3}}$. Similarly, if $n \equiv 1 \pmod 3$, we must have $a = 2$, $b = \frac{n-4}{3}$ and $\zeta(n) = 4 \times 3^{\frac{n-4}{3}}$. If $n \equiv 2 \pmod 3$, we must have $a = 1$, $b = \frac{n-2}{3}$ and $\zeta(n) = 2 \times 3^{\frac{n-2}{3}}$. ■

In simple terms, the above lemma just gives a general expression for the maximum number that you can make with n 1's. $\zeta(n)$ is actually on the OEIS as A000792 (OEIS, 2023 [11]).

An interesting result is that if we extend this to the reals, that is, trying to find the maximum product of $a_1, a_2, \ldots, a_k$ given their sum is n, where each a_i is allowed to be any real, we will have that the largest number we can make is $e^{\frac{n}{e}}$. See **Appendix** for details.

2.1 Useful Properties of ζ

We show 3 useful properties of ζ.

Corollary 2.9. *For all $n \in \mathbb{N}$ where $n > 1$, $\zeta(n+3) = 3\,\zeta(n)$.*

Proof. If $n \equiv 0 \pmod 3$, by **Lemma 2.8**, $\zeta(n+3) = 3^{\frac{n+3}{3}} = 3 \times 3^{\frac{n}{3}} = 3\,\zeta(n)$. Indeed, similar arguments hold as direct consequences of **Lemma 2.8** for $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$. ■

Remark. **Corollary 2.9** also proves that $3 \mid \zeta(n)$ for all $n \geq 5$.

Theorem 2.10. ***(Zinc Theorem)*** *ζ is strictly monotone increasing. That is, for all $n, m \in \mathbb{N}$, we have $\zeta(n) > \zeta(m)$ if and only if $n > m$.*

Proof. It suffices to show that $\zeta(n+1) > \zeta(n)$ for all $n \in \mathbb{N}$. By definition, we can Make $\zeta(n)$ with I_n. We can then perform $\zeta(n) + 1$ with the remaining 1 from I_{n+1}, so we can Make $\zeta(n) + 1$

with I_{n+1}, which implies $\zeta(n+1) \geq \zeta(n) + 1$. Thus, for all $n \in \mathbb{N}$, we have $\zeta(n+1) > \zeta(n)$ as desired, and ζ is strictly monotone increasing. ■

Theorem 2.11. ***(Zeta Bounding Theorem)*** *For all $n \in \mathbb{N}$, $\zeta(n) \leq 3^{\frac{n}{3}}$.*

Proof. We check that $2 < 3^{\frac{2}{3}} \approx 2.08008$, $4 < 3^{\frac{4}{3}} \approx 4.32675$, so $2 \cdot 3^{\frac{n-2}{3}} < 3^{\frac{n}{3}}$, $4 \cdot 3^{\frac{n-4}{3}} < 3^{\frac{n}{3}}$. Furthermore, $1 < 3^{\frac{1}{3}} \approx 1.44225$. The result follows by **Lemma 2.8**. ■

Figure 1: *$\zeta(n)$ against n*

Figure 1 above shows some values of $\zeta(n)$ against n, denoted by blue points. The green curve is the upper bound provided by the **Zeta Bounding Theorem**. Notice equality holds if and only if $3 \mid n$.

Coming back to our problem, $\zeta(8) = 18$, which corresponds to $(1+1)(1+1+1)(1+1+1) = 18$. Since $18 < 24$, this implies that we cannot Make 24 with $I_8 = \{1, 1, 1, 1, 1, 1, 1, 1\} \in D_8$.

Then by the **Lower Bound Lemma**, 9 is a valid lower bound for $\eta(24)$, and $\eta(24) \geq 9$.

3 Upper Bound for $\eta(24)$

We first assert two highly important lemmas.

Lemma 3.1. ***(Monotonicity Lemma)*** *If it is possible to Make n with any $D \in D_i$, then it is possible to Make n with any $S \in D_j$, if $j > i$.*

Proof. We prove by Induction. Let P_n be the assertion that it is possible to Make 24 with any $D \in D_n$. By definition, P_i is true. Now assume P_n is true for some $n \in \mathbb{N}, n \geq i$.

Then now consider some set $\{d_1, d_2, \ldots, d_n, d_{n+1}\} \in D_{n+1}$. If $d_1 = d_2 = \cdots = d_{n+1}$, we consider the set $\{\frac{d_1}{d_2}, d_3, d_4, \ldots, d_{n+1}\}$. Otherwise, without Loss of Generality let $d_1 > d_2$ and consider the set $\{d_1 - d_2, d_3, d_4, \ldots, d_n, d_{n+1}\}$. Notice that in either case the set must be an element of D_n.

By our Induction Hypothesis, it is possible to Make 24 with this set. But we made this new set from an arbitrary set in D_{n+1} with a valid operation, so it is also possible to Make 24 with any set in D_{n+1}. Hence, $P_i \implies P_{i+1}$ and this completes our Inductive Step, and we are done by the **Principle of Mathematical Induction**. ■

Lemma 3.2. ***(Upper Bound Lemma)*** *If we can Make n with all $D \in D_k$, $\eta(n) \leq k$.*

Proof. By the definition of η, $\eta(n)$ is the minimum k such that we can Make n with all $D \in D_k$. Since we can Make n with all $D \in D_k$, we cannot have $\eta(n) > k$. Hence $\eta(n) \leq k$. ■

At this point, we turn to a little bit of assistance from our computing technology, to assist in determining some small cases of $\eta(n)$.

3.1 The Power of Computing

We take inspiration from an algorithm from (RosettaCode, 2023 [10]), and adapt the algorithm to fit the needs of this project by writing a Python program. This Python program runs a *complete search*, where it checks through all possible $D \in D_k$ and tries to Make n, looking for sets D such that we cannot Make n. With this program, we generate results for small values of n and tabulate them in **Table 1** below.

From this point on, we will state results from **Table 1** without proof. See **Appendix** for details with regards to the Python Program.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\eta(n)$	5	5	5	5	6	6	6	6	6	6	7	8	7

Table 1: *Table of $\eta(n)$ against n for $0 \leq n \leq 12$*

3.2 Useful Properties of η

In this chapter, we will establish some useful, simple, yet elegant properties of η.

Corollary 3.3. *η is an even function. That is, $\eta(-n) = \eta(n)$.*

Proof. The proof is obvious. Assume by Way of Contradiction $\eta(n) < \eta(-n)$. Let $D \in D_{\eta(n)}$. Since we are able to form n with D by definition, we must also be able to form $-n$ with D, simply by adding a negative sign at the start. This means that we are able to form $-n$ with any $D \in D_{\eta(n)}$, which challenges the minimality of $\eta(-n)$. *Contradiction.* By similar reasoning, it is also impossible that $\eta(n) > \eta(-n)$. But this implies $\eta(n) = \eta(-n)$. ■

Theorem 3.4. ***(Additive Bounding Theorem)*** *$\eta(a + b) \leq \eta(a) + \eta(b)$.*

Proof. We let D be an arbitrary set $D \in D_{\eta(a)+\eta(b)}$. Now consider an arbitrary subset $A \subset D$ such that $|A| = \eta(a)$. The leftover elements form another set B where $|B| = \eta(b)$. This means $A \in D_{\eta(a)}$ and $B \in D_{\eta(b)}$ and by the definition of η, we are able to form a with A and b with B. We then perform $a + b$ to Make $(a + b)$, and this means that we can form $(a + b)$ with any $D \in D_{\eta(a)+\eta(b)}$. The result follows by the **Upper Bound Lemma**. ■

Theorem 3.5. ***(Multiplicative Bounding Theorem)*** *$\eta(ab) \leq \eta(a) + \eta(b)$.*

Proof. The proof of this theorem is the exact same as the **Additive Bounding Theorem**, with the exception of replacing $a + b$ with $a \times b$ respectively where necessary. ■

Remark. More generally, given any set A of finite cardinality, we may generalise the **Additive Bounding Theorem** and **Multiplicative Bounding Theorem** respectively, to yield

$$\eta\left(\sum_{a\in A} a\right) \leq \sum_{a\in A} \eta(a) \qquad \text{and} \qquad \eta\left(\prod_{a\in A} a\right) \leq \sum_{a\in A} \eta(a).$$

Corollary 3.6. *For all $n \in \mathbb{N}$, $\eta(n)$ exists and is finite.*

Proof. This is a direct consequence of the **Additive Bounding Theorem**. For $n \in \mathbb{N}$,

$$\begin{aligned} \eta(n) &= \eta\left(\sum_{i=1}^{n} 1\right) \\ &\leq \sum_{i=1}^{n} \eta(1) \\ &= \sum_{i=1}^{n} 5 \\ &= 5n \end{aligned}$$

Appealing to the **Evenness of Eta**, η is bounded below by 0 and above by $\max\{5, 5|n|\}$, where the 5 comes from $\eta(0) = 5$. Hence, it must exist and be finite for all n. ■

3.3 Deriving an Upper Bound

With these properties, we are able to determine an upper bound for $\eta(24)$.

I claim that we can Make 24 with all $D \in D_9$.

Proof. Consider an arbitrary $D \in D_9$.

Suppose by way of contradiction, $2 \in D$. Since $\eta(12) = 7 < 8$, we can Make 12 with any $S \in D_7$. By the **Monotonicity Lemma**, we can Make 12 with any $S \in D_8$. Let $D = \{2\} \cup D'$. Notice $D' \in D_8$, so we can Make 12 with D'. Thus we Make 12 with D', and perform $2 \times 12 = 24$. *Contradiction*, and $2 \notin D$.

Suppose by way of contradiction, $3 \in D$. Since $\eta(8) = 6 < 8$, by the **Monotonicity Lemma**, we can Make 12 with any $S \in D_8$. Let $D = \{3\} \cup D'$. Notice $D' \in D_8$, so we can Make 8 with D'. Thus we Make 8 with D', and perform $3 \times 8 = 24$. *Contradiction*, and $3 \notin D$.

We repeat this argument 3 more times to see that $4 \notin D$, $6 \notin D$, $8 \notin D$, using $4 \times 6 = 24$ and $3 \times 8 = 24$.

Hence, D consists entirely of 1, 5, 7, 9, 10. To narrow our search space, we wish to show that if two numbers in D are either 5, 7, 9 or 10, then we can derive a contradiction. Suppose by way of contradiction we have $n, m \in \{5, 7, 9, 10\}$ with $\{n, m\} \subset D$. Let $D = \{n, m\} \cup D'$. Notice $D' \in D_7$.

If $n = m = 5$, since $\eta(1) = 5 < 7$, by the **Monotonicity Lemma**, we can Make 1 with D'. Thus, we can Make 1 with D' and perform $n \times m - 1 = 24$ to Make 24. *Contradiction.*

Else, we take $x = n + m$. Notice $12 = 5 + 7 \leq x \leq 10 + 10 = 20$. Thus, $4 \leq 24 - x \leq 12$. From our computations we know $\eta(24 - x) \leq 7$. Hence, by the **Monotonicity Lemma**, we can Make $(24 - x)$ with any $S \in D_7$. Since $D' \in D_7$, we can Make $(24 - x)$ with D'. Now, take $x + (24 - x) = 24$ to Make 24. Check that we have used n, m and every element of D' exactly once each. *Contradiction.*

This implies that D contains at most one number from $\{5, 7, 9, 10\}$, and 8 ones, or $D = I_9$. However,

$$(1 + 1) \times (1 + 1) \times (1 + 1) \times (1 + 1 + 1) = 24$$
$$1 + 1 + 1 + 1 + (1 + 1 + 1 + 1) \times 5 = 24$$
$$1 \times 1 \times 1 + 1 + 1 + (1 + 1 + 1) \times 7 = 24$$
$$1 - 1 - 1 - 1 - 1 + (1 + 1 + 1) \times 9 = 24$$
$$1 \times 1 \times 1 + 1 + 1 + 1 + (1 + 1) \times 10 = 24$$

Contradiction. Hence, we have exhausted all cases and we can Make 24 with all $D \in D_9$. Finally, by the **Upper Bound Lemma**, we have $\eta(24) \leq 9$ as desired. ■

3.4 Conclusion for Make24

When we place the lower and upper bounds together, we have shown that $9 \leq \eta(24) \leq 9$, which suffices in forcing $\eta(24) = 9$, and we have solved the case for Make24.

4 Generalisation

After deriving $\eta(24) = 9$, we begin to futher generalise beyond the case $n = 24$. Our generalised problem statement is:

Given **any** $n \in \mathbb{Z}$, what is the minimum number of integers d_i $(1 \leq d_i \leq 10)$ required such that we can guarantee to Make n with every such set of integers with the basic mathematical operations $+, -, \times, \div$, using each d_i **exactly once**?

From here, we are interested to analyse the *Asymptotic Growth Rate* of η, that is, how $\eta(n)$ behaves as n tends to infinity. We must first introduce standard notation from *Number Theory.*

Definition 4.1. ***(Floor and Ceiling)*** *For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the maximum $n \in \mathbb{Z}$ such that $n \leq x$, and let $\lceil x \rceil$ be the minimum $m \in \mathbb{Z}$ such that $m \geq x$. These are the floor and ceiling functions respsectively.*

Now, looking at the **Multiplicative Bounding Theorem**, one observes that this identity looks similar to the identity $\ln(ab) = \ln(a) + \ln(b)$. Hence, we guess that η should be bounded above logarithmically. This motivates the following conjecture:

Conjecture 4.2. *$\eta(n)$ scales as $\ln(n)$. That is, there exists lower and upper bounds in terms of linear expressions of logarithms for $\eta(n)$.*

We show why this result is true in two parts, by constructing a lower bound, followed by an upper bound. First, we take a look at past literature.

4.1 Mahler-Popken Complexity and m-ary Complexity

In fact, this problem is a generalised version of an interesting problem in relation to the *Mahler-Popken complexity* (also known as *Integer Complexity* in literature) of an integer, which seems to have originated from (Mahler *et al.* [8], 1953).

Since then, this function has been studied comprehensively in literature, with notable papers such as (de Reyna [3], 2000), (de Reyna *et al.* [4], 2009), (Campbell [5], 2024), (Zelinsky [6], 2022), (Cordwell *et al.*, [7], 2019).

Definition 4.3. ***(Mahler-Popken Complexity)*** *For an arbitrary $n \in \mathbb{N}$, let $||n||$ denote its Mahler-Popken Complexity, the minimum number of ones needed to Make n, only with the operations $+$ and $\times$.*

This is the classical definition of the Mahler-Popken Complexity, hereafter written "Integer Complexity" in accordance with literature for convenience. Note that only + and × is allowed in the classical version, which makes it different, but closely related to, our research.

Consider the following standard definitions from *Real Analysis*.

Definition 4.4. ***(Asymptotic Equivalence)*** *(Hardy et al., 1960 [2]) Consider two real-valued functions f and g. We say f and g are asymptotically equivalent, or $f(n) \sim g(n)$, if*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

where the limit exists.

Definition 4.5. ***(Asymptotic Density)*** *Given a set $S \subset \mathbb{N}$, define $S_n = \{1, 2, \ldots, n\} \cap S$. The asymptotic density of S over $\mathbb{N}$, or "density of S", denoted as ρ below, is defined as*

$$\rho = \lim_{n\to\infty} \frac{|S_n|}{n}$$

if the limit taken over integer values of n exists.

Definition 4.6. *We say that a set A contains "almost all" natural numbers if the set $\mathbb{N} \setminus A$ has asymptotic density 0 over $\mathbb{N}$.*

It is well known that $||n||$ is bounded logarithmically, and that for *almost all* $n \in \mathbb{N}$,

$$||n|| \leq \frac{5}{2\ln 2} \ln n \approx 3.60674 \ln n$$

More recently, (Cordwell *et al.*, [7]) proved in 2019 the stronger bound that for *almost all* $n \in \mathbb{N}$,

$$||n|| \leq \frac{2326006662}{2^{11}3^9 \ln(2^{11}3^9)} \ln n \approx 3.29496 \ln n$$

Even more recently, (Zelinsky, [6]) proved in 2022 that for *all* integers $n \in \mathbb{N}$,

$$||n|| \leq \frac{41}{\ln 55296} \ln n \approx 3.75442 \ln n$$

However, it is in fact an open problem to determine whether

Conjecture 4.7. *For all $\varepsilon > 0$, almost all integers $n \in \mathbb{N}$ satisfy*

$$||n|| \leq \left(\frac{3}{\ln 3} + \varepsilon\right) \ln n \approx (2.73072 + \varepsilon) \ln n$$

or if there are infinite, perhaps possibly a set of positive asymptotic density of integers that do not satisfy the above inequality for every ε. If almost all integers do satisfy this inequality, this immediately implies

$$||n|| \sim 3 \log_3 n$$

which is another open problem.

A generalised form of the Integer Complexity is the m-ary Complexity of a number.

Definition 4.8. ***(m-ary Complexity)*** *(Campbell, 2024 [5]) For $n, m \in \mathbb{N}$ let $||n||_m$ denote its m-ary Complexity, defined as the minimum $k \in \mathbb{N}$ such that we can always Make n with any set $D = \{d_1, d_2, \ldots, d_k\}$ with $d_i \in \{1, 2, \ldots, m\}$ for all $1 \leq i \leq k$, only with the operations $+$ and $\times$.*

Two special cases of the m-ary Complexity are $||n||_1 = ||n||$, which is the standard *Mahler-Popken Complexity*, and $||n||_2$, the *Binary Complexity*, being studied extensively by (Campbell, 2024 [5]).

$\eta(n)$ in our research is an altered form of the *Decimal Complexity* $||n||_{10}$, where the operations $\div$ and $-$ are attached. We will briefly revisit the relationships between the closely-related functions $\eta(n)$, $||n||$ and $||n||_m$ in **Chapter 7 - Possible Extensions**.

4.2 General Lower Bound

We make the following claim, for a general lower bound of η.

Lemma 4.9. *For all $n \in \mathbb{N}$, $\eta(n)$ is bounded below logarithmically by $\lceil 3 \log_3 n \rceil$.*

Proof. We obtain the following chain inequality. Notice that

$$\begin{aligned} \zeta(\lceil 3\log_3(n)\rceil - 1) &\leq 3^{\frac{\lceil 3\log_3(n)\rceil - 1}{3}} && \text{[Zeta Bounding Theorem]} \\ &< 3^{\log_3(n)} \\ &= n \end{aligned}$$

So we cannot Make n with the set with $(3\lceil \log_3(n)\rceil - 1)$ 1's. Thus, we have $\eta(n) \geq \lceil 3\log_3 n \rceil$ by the **Lower Bound Lemma**, as desired. ■

4.3 Our Strategy

Let's recall what our strategy for Make24 was. When we supposed that $2, 3, 4, 6$ or $8 \in D$, we abused the identities $2 \times 12 = 24$, $3 \times 8 = 24$, and $4 \times 6 = 24$ to narrow our search space of possible sets D by a large amount. Thus, we wish to generalise this idea.

Our approach to tackling this problem involves supposing that some set of numbers, S, was a subset of D. With S, we make some value d, and consider long division. That is, we let $n = d \times q + r$, where the remainder term must satisfy $0 \leq r < d$. This is a *Reduction Step* we take to reduce the problem of "Make n" into two smaller sub-problems, namely "Make q" and "Make r", where q and r are both strictly less than n.

Finally, we can support this form of *Reduction* by considering *Strong Induction*, that is, supposing all previous cases are already proven.

We will justify why the above is sufficient and rigorous.

4.4 Two More Important Results

The following lemma is inspired by the idea of *Dynamic Programming*, where we are able to produce a bound $\eta(n)$ by considering the past 10 values of η.

Lemma 4.10. ***(Reduction Lemma)*** *For all $n \in \mathbb{Z}$, we have $\eta(n) \leq 1 + \max_{n-10 \leq m < n} \eta(m)$.*

Proof. Define $M = \max_{n-10 \leq m < n} \eta(m)$, to ensure that M satisfies $M \geq \eta(m)$ for all $n - 10 \leq m < n$.

By the **Monotonicity Lemma**, we can Make $(n-10)$, $(n-9)$, $\dots$, or $(n-1)$ with any $D \in D_M$. For an arbitrary set $D \in D_{M+1}$, choose an arbitrary $d_1 \in D$. Now, we suppose $D = \{d_1\} \cup D'$, such that $|D'| = M$. Since $1 \leq d_1 \leq 10$, we are guaranteed to be able to Make $(n - d_1)$ with D'. Then, perform $(n - d_1) + d_1 = n$ to Make n. By the **Upper Bound Lemma**, we have $\eta(n) \leq M + 1$ as desired. ∎

Lastly, we introduce the following result, will become extremely important later.

Lemma 4.11. *For any $n \geq 8$ and $k \geq 3\lfloor \log_2 n \rfloor$, we can Make n with I_k and $\{2\} \cup I_{k-2}$.*

Proof. We show that we can Make n with $I_k = \{\underbrace{1, 1, \dots, 1}_{k \text{ ones}}\}$ first.

Indeed, we express n in Base-2 as $n = d_0 + 2d_1 + 2^2 d_2 + 2^3 d_3 + \cdots + 2^p d_p$, where $p = \lfloor \log_2 n \rfloor$, and $d_i \in \{0, 1\}$ for all $0 \leq i \leq p$. Notice $p \geq \lfloor \log_2 8 \rfloor = 3$. Then,

$$\begin{aligned} n &= d_0 + 2d_1 + 2^2 d_2 + 2^3 d_3 + \cdots + 2^{p-1} d_{p-1} + 2^p d_p \\ &= d_0 + 2(d_1 + 2d_2 + 2^2 d_3 + \cdots + 2^{p-2} d_{p-1} + 2^{p-1} d_p) \\ &= d_0 + 2(d_1 + 2(d_2 + 2(d_3 + \cdots + 2(d_{p-1} + 2d_p))) \cdots) \end{aligned}$$

We can make each of the 2's with $(1+1)$. There are p 2's and $(p+1)$ d_i values, so there are at most $2p + p + 1 = 3p + 1$ ones. But if $(3p+1)$ ones were used, this implies $d_0 = d_1 = \cdots = d_p = 1$. That is, $n = 1 + 2 + 2^2 + \cdots + 2^p = 2^{p+1} - 1$.

Then we can write n as $\underbrace{(1+1)(1+1)\dots(1+1)}_{(p+1) \text{ terms}} - 1$ with $2(p+1) + 1 = 2p + 3 \leq 3p$ ones, as $p \geq 3$.

This means in either way, we can make n with at most $3p$ ones. If there are leftover ones, we keep multiplying n by 1 to get back 1 until no leftover ones remain. The result follows.

For the sets in the form $\{2\} \cup I_{k-2}$, we may simply replace a $(1+1)$ in the previous proof by the 2 which is given. That is, instead of using two 1's, use the 2 given. The result follows. ∎

4.5 General Upper Bound

The casework begins! We will state and prove the following claim.

Lemma 4.12. *For all integers $n > 2$, we have $\eta(n) \leq \lfloor 3 \log_2 n \rfloor + 1$.*

Proof. We first show this result for all $2 < n \leq 30$.

Refer to **Table 1** on the following page.

We can reach an upper bound for each $\eta(n)$, using the method stated in the table. Hence, we may verify this upper bound holds for $2 < n \leq 30$.

Specifically, if an equation like $16 = 8 + 8$ is stated, we are implicitly applying the **Additive Bounding Theorem** or **Multiplicative Bounding Theorem**. For $n = 16$, for example, the result derives from $\eta(16) = \eta(8 + 8) \leq \eta(8) + \eta(8) = 12$.

n	$\lfloor 3\log_2 n\rfloor + 1$	$\eta(n)$	Outline for Proof
3	5	5	-
4	7	6	-
5	7	6	-
6	8	6	-
7	9	6	-
8	10	6	-
9	10	6	-
10	10	7	-
11	11	8	-
12	11	7	-
13	12	≤ 9	**Reduction Lemma**
14	12	≤ 10	**Reduction Lemma**
15	12	≤ 11	**Reduction Lemma**
16	13	≤ 12	$16 = 8 + 8$
17	13	≤ 12	$17 = 8 + 9$
18	13	≤ 12	$18 = 9 + 9$
19	13	≤ 13	$19 = 12 + 7$
20	13	≤ 12	$20 = 2 \times 10$
21	14	≤ 11	$21 = 3 \times 7$
22	14	≤ 13	$22 = 2 \times 11$
23	14	≤ 14	**Reduction Lemma**
24	14	9	-
25	14	≤ 12	$25 = 5 \times 5$
26	15	≤ 14	$26 = 24 + 2$
27	15	$\le 11^{\dagger}$	$27 = 3 \times 9$
28	15	≤ 12	$28 = 4 \times 7$
29	15	$\le 15^{\dagger}$	$29 = 24 + 5$
30	15	≤ 12	$30 = 5 \times 6$

Table 2: *Upper Bound of* $\eta(n)$ *for* $2 < n \le 30$

†In fact, $\eta(27) = 9$ and thus $\eta(29) = \eta(27 + 2) \le \eta(27) + \eta(2) = 14$. See **Appendix** for details.

We continue by disregarding the floor function and show the weaker $\eta(n) \le 3\log_2 n + 1$.

We proceed by means of Strong Induction. Let P_k be the assertion that $\eta(l) \le 3\log_2 l + 1$ holds for all $2 < l \le k$. We have shown above that P_{30} holds. Let $n > 30$ be an integer such that P_{n-1} holds. We claim P_n holds consequently.

Suppose by way of contradiction P_n doesn't hold. This implies $\eta(n) > 3\log_2 n + 1 \ge \lfloor 3\log_2 n \rfloor + 1$. For convenience define $m = \lfloor 3\log_2 n \rfloor + 1$, and by the converse of the **Upper Bound Lemma**, this implies we cannot Make n with some set in D_m. Let $D = \{d_1, d_2, \ldots, d_m\}$ be one such set, and without loss of generality suppose $d_1 \le d_2 \le \cdots \le d_m$. One may consider seven cases based on the structure of D.

Firstly suppose by way of contradiction $d_m \ge 6$. Then let $n = d_m q + r$, where $q, r \in \mathbb{Z}$ and $0 \le r < d_m$. Since $n > 30$ and $d_m \le 10$, we have $\lfloor \frac{n}{d_m} \rfloor > 2$. Then,

$$\begin{aligned} \eta(q) &= \eta\left(\left\lfloor \frac{n}{d_m} \right\rfloor\right) \\ &\le 3\log_2 \left\lfloor \frac{n}{d_m} \right\rfloor + 1 && \textbf{(Inductive Hypothesis)} \\ &\le 3\log_2 \left(\frac{n}{d_m}\right) + 1 \\ &= 3\log_2 n - 3\log_2 d_m + 1 \\ &\le 3\log_2 n - 3\log_2 6 + 1 \\ &< 3\log_2 n - 6 \end{aligned}$$

However, recall that since $\eta(q) \in \mathbb{N}$, we must have $\eta(q) \le \lfloor 3\log_2 n \rfloor - 6$. We also have $0 \le r < d_m \le 10$, so $\eta(r) \le 6$. By the **Monotonicity Lemma**, we can Make q with any $S \in D_{\lfloor 3\log_2 n \rfloor - 6}$ and we can Make r with any $S \in D_6$.

Let $A = \{d_1, d_2, \ldots, d_6\}$ and $B = \{d_7, d_8, \ldots, d_{m-1}\}$. We may check that $|A| = 6$ and $|B| = \lfloor 3\log_2 n \rfloor - 6$. By the definition of η we can Make r with A and q with B. We can take $q \times d_m + r = n$ to Make n, and we may check that we have used $d_1, d_2, \ldots, d_m$ exactly once each. *Contradiction.*

Secondly, suppose by way of contradiction $d_m = 5$ and $d_{m-1} \ge 3$. Then let $n = (d_{m-1} + d_m)q + r$ where $q, r \in \mathbb{Z}$ and $0 \le r < d_{m-1} + d_m$. Since $n > 30$ and $d_{m-1} + d_m \le 2d_m = 10$, we have

$\lfloor \frac{n}{d_{m-1}+d_m} \rfloor > 2$. So,

$$\begin{aligned}\eta(q) &= \eta\left(\left\lfloor \frac{n}{d_{m-1}+d_m} \right\rfloor\right)\\ &\le 3\log_2 \left\lfloor \frac{n}{d_{m-1}+d_m} \right\rfloor + 1 \qquad \textbf{(Inductive Hypothesis)}\\ &\le 3\log_2 \left(\frac{n}{d_{m-1}+d_m} \right) + 1\\ &= 3\log_2 n - 3\log_2(d_{m-1}+d_m) + 1\\ &\le 3\log_2 n - 3\log_2(3+5) + 1\\ &= 3\log_2 n - 8\end{aligned}$$

Again, since $\eta(q) \in \mathbb{N}$, we must have $\eta(q) \le \lfloor 3\log_2 n \rfloor - 8$. We also have $0 \le r < d_{m-1}+d_m \le 10$, so $\eta(r) \le 6$. By the **Monotonicity Lemma**, we can Make q with any $S \in D_{\lfloor 3\log_2 n \rfloor - 7}$ and we can Make r with any $S \in D_6$.

Let $A = \{d_1, d_2, \ldots, d_6\}$ and $B = \{d_7, d_8, \ldots, d_{m-2}\}$. We may check that $|A| = 6$ and $|B| = \lfloor 3\log_2 n \rfloor - 7$. By the definition of η we can Make r with A and q with B. We can take $q \times (d_{m-1}+d_m) + r = n$ to Make n, and we may check that we have used $d_1, d_2, \ldots, d_m$ exactly once each. *Contradiction.*

Thirdly, suppose by way of contradiction $d_m = 5$ and $d_{m-1} = 2$. Then let $n = d_{m-1}d_m q + r$ where $q, r \in \mathbb{Z}$ and $0 \le r < d_{m-1}d_m$. Since $n > 30$ and $d_{m-1}d_m = 10$, we have $\lfloor \frac{n}{d_{m-1}d_m} \rfloor > 2$. So,

$$\begin{aligned}\eta(q) &= \eta\left(\left\lfloor \frac{n}{d_{m-1}d_m} \right\rfloor\right)\\ &\le 3\log_2 \left\lfloor \frac{n}{d_{m-1}d_m} \right\rfloor + 1 \qquad \textbf{(Inductive Hypothesis)}\\ &\le 3\log_2 \left(\frac{n}{d_{m-1}d_m} \right) + 1\\ &= 3\log_2 n - 3\log_2(d_{m-1}d_m) + 1\\ &\le 3\log_2 n - 3\log_2(10) + 1\\ &< 3\log_2 n - 8\end{aligned}$$

Again, since $\eta(q) \in \mathbb{N}$, we must have $\eta(q) \le \lfloor 3\log_2 n \rfloor - 8$. We also have $0 \le r < d_{m-1}d_m = 10$, so $\eta(r) \le 6$. By **Monotonicity Lemma**, we can Make q with any $S \in D_{\lfloor 3\log_2 n \rfloor - 7}$ and we can Make r with any $S \in D_6$.

Let $A = \{d_1, d_2, \ldots, d_6\}$ and $B = \{d_7, d_8, \ldots, d_{m-2}\}$. We may check that $|A| = 6$ and $|B| = \lfloor 3\log_2 n \rfloor - 7$. By the definition of η we can Make r with A and q with B. We can take

$q \times d_{m-1} \times d_m + r = n$ to Make n, and we may check that we have used $d_1, d_2, \ldots, d_m$ exactly once each. *Contradiction.*

Fourthly, suppose by way of contradiction $d_m = 4$. Then let $n = 4q + r$, where $q, r \in \mathbb{Z}$ and $0 \le r < 4$. Since $n > 30$, notice $\lfloor \frac{n}{4} \rfloor > 2$. Since $0 \le r < 4$, $\eta(r) = 5$, so we can Make r with any $S \in D_5$. Hence,

$$\begin{aligned}
\eta(q) &= \eta\left(\left\lfloor \frac{n}{4} \right\rfloor\right) \\
&\le 3\log_2 \left\lfloor \frac{n}{4} \right\rfloor + 1 \qquad \textbf{(Inductive Hypothesis)} \\
&\le 3\log_2 \left(\frac{n}{4}\right) + 1 \\
&= 3\log_2 n - 3\log_2 4 + 1 \\
&= 3\log_2 n - 5
\end{aligned}$$

Again, since $\eta(q) \in \mathbb{N}$, we must have $\eta(q) \le \lfloor 3\log_2 n \rfloor - 5$. Recall $\eta(r) = 5$, so by the **Monotonicity Lemma**, we can Make q with any $S \in D_{\lfloor 3\log_2 n \rfloor - 5}$ and we can Make r with any $S \in D_5$.

Let $A = \{d_1, d_2, \ldots, d_5\}$ and $B = \{d_6, d_7, \ldots, d_{m-1}\}$. We may check that $|A| = 5$ and $|B| = \lfloor 3\log_2 n \rfloor - 5$. By the definition of η we can Make r with A and q with B. We can take $q \times d_m + r = n$ to Make n, and we may check that we have used $d_1, d_2, \ldots, d_m$ exactly once each. *Contradiction.*

Fifthly suppose by way of contradiction $d_m = d_{m-1} = 3$. Let $n = d_{m-1} d_m q + r = 9q + r$, where $q, r \in \mathbb{Z}$ and $0 \le r < 9$. Since $n > 30$, we have $\lfloor \frac{n}{9} \rfloor > 2$. Then,

$$\begin{aligned}
\eta(q) &= \eta\left(\left\lfloor \frac{n}{9} \right\rfloor\right) \\
&\le 3\log_2 \left\lfloor \frac{n}{9} \right\rfloor + 1 \qquad \textbf{(Inductive Hypothesis)} \\
&\le 3\log_2 \left(\frac{n}{9}\right) + 1 \\
&= 3\log_2 n - 3\log_2(9) + 1 \\
&< 3\log_2 n - 8
\end{aligned}$$

Again, since $\eta(q) \in \mathbb{N}$, we must have $\eta(q) \le \lfloor 3\log_2 n \rfloor - 8$. Since $0 \le r < 9$, so $\eta(r) \le 6$. so by the **Monotonicity Lemma**, we can Make q with any $S \in D_{\lfloor 3\log_2 n \rfloor - 7}$ and we can Make r with any $S \in D_6$.

Let $A = \{d_1, d_2, \ldots, d_6\}$ and $B = \{d_7, d_8, \ldots, d_{m-2}\}$. We may check that $|A| = 6$ and $|B| = \lfloor 3\log_2 n \rfloor - 7$. By the definition of η we can Make r with A and q with B. We can take

$q \times d_{m-1} \times d_m + r = n$ to Make n, and we may check that we have used $d_1, d_2, \ldots, d_m$ exactly once each. *Contradiction.*

Sixthly suppose by way of contradiction $d_m = 3$ and $d_{m-1} = d_{m-2} = 2$. We make the peculiar choice to let $n = d_{m-2}(d_{m-1} + d_m)q + r = 10q + r$, where $q, r \in \mathbb{Z}$ and $0 \le r < 10$. Since $n > 30$, we have $\lfloor \frac{n}{10} \rfloor > 2$. Then,

$$\begin{aligned}
\eta(q) &= \eta\left(\left\lfloor \frac{n}{10} \right\rfloor\right) \\
&\le 3\log_2 \left\lfloor \frac{n}{10} \right\rfloor + 1 \qquad \textbf{(Inductive Hypothesis)} \\
&\le 3\log_2 \left(\frac{n}{10}\right) + 1 \\
&= 3\log_2 n - 3\log_2(10) + 1 \\
&< 3\log_2 n - 8
\end{aligned}$$

Again, since $\eta(q) \in \mathbb{N}$, we must have $\eta(q) \le \lfloor 3\log_2 n \rfloor - 8$. Since $0 \le r < 10$, so $\eta(r) \le 6$. so by the **Monotonicity Lemma**, we can Make q with any $S \in D_{\lfloor 3\log_2 n \rfloor - 8}$ and we can Make r with any $S \in D_6$.

Let $A = \{d_1, d_2, \ldots, d_6\}$ and $B = \{d_7, d_8, \ldots, d_{m-3}\}$. We may check that $|A| = 6$ and $|B| = \lfloor 3\log_2 n \rfloor - 8$. By the definition of η we can Make r with A and q with B. We can take $q \times d_{m-2} \times (d_{m-1} + d_m) + r = n$ to Make n, and we may check that we have used $d_1, d_2, \ldots, d_m$ exactly once each. *Contradiction.*

Seventhly suppose by way of contradiction $d_m = d_{m-1} = d_{m-2} = 2$. Let $n = d_{m-2}d_{m-1}d_m q + r = 8q + r$ where $q, r \in \mathbb{Z}$ and $0 \le r < 8$. Since $n > 30$, we have $\lfloor \frac{n}{8} \rfloor > 2$. Then,

$$\begin{aligned}
\eta(q) &= \eta\left(\left\lfloor \frac{n}{8} \right\rfloor\right) \\
&\le 3\log_2 \left\lfloor \frac{n}{8} \right\rfloor + 1 \qquad \textbf{(Inductive Hypothesis)} \\
&\le 3\log_2 \left(\frac{n}{8}\right) + 1 \\
&= 3\log_2 n - 3\log_2(8) + 1 \\
&= 3\log_2 n - 8
\end{aligned}$$

Again, since $\eta(q) \in \mathbb{N}$, we must have $\eta(q) \le \lfloor 3\log_2 n \rfloor - 8$. Since $0 \le r < 8$, so $\eta(r) \le 6$. so by the **Monotonicity Lemma**, we can Make q with any $S \in D_{\lfloor 3\log_2 n \rfloor - 8}$ and we can Make r with any $S \in D_6$.

Let $A = \{d_1, d_2, \ldots, d_6\}$ and $B = \{d_7, d_8, \ldots, d_{m-3}\}$. We may check that $|A| = 6$ and $|B| =$

$\lfloor 3\log_2 n \rfloor - 8$. By the definition of η we can Make r with A and q with B. We can take $q \times d_{m-2} \times d_{m-1} \times d_m + r = n$ to Make n, and we may check that we have used $d_1, d_2, \ldots, d_m$ exactly once each. *Contradiction.*

From the seven cases above, we conclude that there are only the following six possibilities for D.

$$D = \{\underbrace{1, 1, 1, 1, \ldots, 1}_{m \text{ ones}}\} = I_{m-1} \cup \{1\}$$

$$D = \{\underbrace{1, 1, 1, \ldots, 1}_{(m-1) \text{ ones}}, 2\} = I_{m-1} \cup \{2\}$$

$$D = \{\underbrace{1, 1, 1, \ldots, 1}_{(m-1) \text{ ones}}, 3\} = I_{m-1} \cup \{3\}$$

$$D = \{\underbrace{1, 1, 1, \ldots, 1}_{(m-1) \text{ ones}}, 5\} = I_{m-1} \cup \{5\}$$

$$D = \{\underbrace{1, 1, \ldots, 1}_{(m-2) \text{ ones}}, 2, 2\} = (\{2\} \cup I_{m-2}) \cup \{2\}$$

$$D = \{\underbrace{1, 1, \ldots, 1}_{(m-2) \text{ ones}}, 2, 3\} = (\{2\} \cup I_{m-2}) \cup \{3\}$$

For the first 4 sets, let $D = I_{m-1} \cup \{a\}$. Notice that $1 \leq a \leq 5$, so $n - a \geq 8$ indeed holds. Furthermore, we have

$$\begin{aligned} 3\lfloor \log_2 (n-a) \rfloor &\leq 3\lfloor \log_2 n \rfloor \\ &\leq \lfloor 3\log_2 n \rfloor \\ &= m - 1 \end{aligned}$$

Hence, by **Lemma 4.11**, we can Make $(n-a)$ with I_{m-1}. We can then perform $(n-a)+a = n$ to Make n, and we may check that we used each element of D exactly once each. *Contradiction.*

For the last 2 sets, let $D = (\{2\} \cup I_{m-2}) \cup \{a\}$. Notice that $2 \leq a \leq 3$, so $n - a \geq 8$ indeed holds. Furthermore, we have

$$\begin{aligned} 3\lfloor \log_2 (n-a) \rfloor - 2 &\leq 3\lfloor \log_2 n \rfloor - 2 \\ &\leq \lfloor 3\log_2 n \rfloor - 2 \\ &< m - 2 \end{aligned}$$

Hence, by **Lemma 4.11**, we can Make $(n-a)$ with $\{2\} \cup I_{m-2}$. We can then perform $(n-a)+a = n$ to Make n, and we may check that we used each element of D exactly once each. *Contradiction.*

We have exhausted all cases. By contradiction, $\eta(n) \leq 3\log_2 n + 1$, so $P_{n-1} \implies P_n$ and the inductive step indeed holds. The stated inequality thus holds for all $n > 2$.

Finally, recalling $\eta(n) \in \mathbb{N}$, this weaker inequality allows us to deduce $\eta(n) \leq \lfloor 3\log_2 n \rfloor + 1$ indeed holds true for all $n > 2$. ■

4.6 Conclusion for General Bound

By considering the lower and upper bounds provided by **Lemma 4.9** and **Lemma 4.12** respectively, and considering the **Evenness of Eta**, we can summarise our findings into one theorem.

Theorem 4.13. ***(Make-n Theorem)*** *For all $n \in \mathbb{Z}$ such that $|n| > 2$, the inequality*

$$\lceil 3\log_3 |n| \rceil \leq \eta(n) \leq \lfloor 3\log_2 |n| \rfloor + 1$$

holds, and for $|n| \leq 2$, $\eta(n) = 5$.

Remark. A slightly weaker version of this result that $\eta(n) \leq \lfloor 3\log_2 n \rfloor + 2$ for all $n > 1$, that is not dependent on the result of $\eta(24) = 9$ can also be shown.

Remark. Furthermore, it appears a slightly stronger version of this theorem can show that if the inequality $\eta(n) \leq \lfloor 3\log_2 n \rfloor$ holds for all $3 < n < 40$, then it also holds for all $n > 3$, but the base cases for $3 < n < 40$ seem overly tedious and computationally intensive to prove or disprove, and therefore this has not been explored here.

In the next chapter, we shall show some other notable results, by introducing some intuition justifying why certain results are the way they are, as well as analysing the strength of our upper and lower bounds.

5 Other Notable Results

We introduce some standard definitions from *Real Analysis*,

Definition 5.1. ***(Infimum)*** *Given f defined on $A \subset \mathbb{R}$, let $\inf_{n\in A} f(n)$ denote the infimum of f over A. That is, for some $I \in \mathbb{R}$, if for all $x \in A$, we have $f(x) \geq I$, and for all $\varepsilon > 0$, there exists some $a \in A$ such that $f(a) < I + \varepsilon$, then we say*

$$\inf_{n\in A} f(n) = I.$$

Definition 5.2. ***(Limit Inferior)*** *Given $f : \mathbb{N} \to \mathbb{R}$, let $\liminf_{n\to\infty} f(n)$ denote the limit inferior of f as n tends to infinity. That is, we define*

$$\liminf_{n\to\infty} f(n) = \lim_{n\to\infty} \inf_{m\geq n} f(m)$$

where m is assumed to be taken over only integer values at least n in the infimum.

Definition 5.3. ***(Supremum)*** *Given f defined on $A \subset \mathbb{R}$, let $\sup_{n\in A} f(n)$ denote the supremum of f over A. That is, for some $S \in \mathbb{R}$, if for all $x \in A$, we have $f(x) \leq S$, and for all $\varepsilon > 0$, there exists some $a \in A$ such that $f(a) > S - \varepsilon$, then we say*

$$\sup_{n\in A} f(n) = S.$$

Definition 5.4. ***(Limit Superior)*** *Given $f : \mathbb{N} \to \mathbb{R}$, let $\limsup_{n\to\infty} f(n)$ denote the limit superior of f as n tends to infinity. That is, we define*

$$\limsup_{n\to\infty} f(n) = \lim_{n\to\infty} \sup_{m\geq n} f(m)$$

where m is assumed to be taken over only integer values at least n in the supremum.

We then make the following peculiar claim, inspired by $\eta(9) = 6$.

Theorem 5.5. *For all $n \in \mathbb{N}$, we have the equality*

$$\eta(9^n) = 6n$$

and in particular, we can conclude

$$\liminf_{n\to\infty} \frac{\eta(n)}{\ln n} = \frac{3}{\ln 3}.$$

Proof. For the first claim, bounding yields the chain equality

$$\begin{aligned} 6n &= \lceil 3\log_3(9^n)\rceil \\ &\leq \eta(9^n) && (\textbf{Make-n Theorem}) \\ &\leq n\cdot\eta(9) && (\textbf{Multiplicative Bounding Theorem}) \\ &= 6n \end{aligned}$$

which forces $\eta(9^n) = 6n$.

We may then consider the sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{N}$ given by $x_n = 9^n$. This sequence is clearly strictly monotone and unbounded, and we notice that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{\eta(x_n)}{\ln x_n} &= \frac{\eta(9^n)}{(\ln 3)(\log_3 9^n)} \\ &= \frac{6n}{(\ln 3)(2n)} \\ &= \frac{3}{\ln 3} \end{aligned}$$

It is well-known that this implies that

$$\liminf_{n\to\infty} \frac{\eta(n)}{\ln n} \leq \frac{3}{\ln 3}$$

is indeed a valid upper bound.

However, by the **Make-n Theorem**, we have the lower bound

$$\frac{\eta(n)}{\ln n} \geq \frac{\lceil 3\log_3 n \rceil}{\ln n} \geq \frac{3\log_3 n}{\ln n} = \frac{3}{\ln 3}$$

holds for all $n \in \mathbb{N}$, and it is well known that we then have

$$\liminf_{n\to\infty} \frac{\eta(n)}{\ln n} \geq \frac{3}{\ln 3}$$

but we then have

$$\frac{3}{\ln 3} \geq \liminf_{n\to\infty} \frac{\eta(n)}{\ln n} \geq \frac{3}{\ln 3} \implies \liminf_{n\to\infty} \frac{\eta(n)}{\ln n} = \frac{3}{\ln 3}$$

as desired. ■

Remark. In essence, this result tells us that our logarithmic lower bound is the *best possible* for this problem and cannot be improved further. In particular, in the **Make-n Theorem**, *equality holds* in the lower bound an *infinite* number of times (e.g. at values of the form 9^n).

Even though the $\liminf_{n\to\infty} \frac{\eta(n)}{\ln n} = \frac{3}{\ln 3}$ is the best value achievable, we are not able to conclusively determine a tight upper bound for the corresponding limit superior, $\limsup_{n\to\infty} \frac{\eta(n)}{\ln n}$. In particular, we are only able to deduce, from our **Make-n Theorem** that

$$\limsup_{n\to\infty} \frac{\eta(n)}{\ln n} \leq \frac{3}{\ln 2},$$

and we will analyse the tightness of this bound in a later part of this paper.

Furthermore, we introduce a slightly stronger form of **Theorem 5.5**.

Theorem 5.6. *For all $n \in \mathbb{N}$, we have $\eta(3^n) = 3n$, except for $n = 1$, where $\eta(3) = 5$.*

Proof. We already know that $\eta(3) = 5$, so we disregard this case.

If n is even, then by noticing that $\frac{n}{2} \in \mathbb{N}$ and directly applying **Theorem 5.5**, we indeed have

$$\eta(3^n) = \eta(9^{\frac{n}{2}}) = 6 \cdot \frac{n}{2} = 3n.$$

Now, we consider odd n. We first claim that $\eta(27) = 9$, which is done by listing. For brevity, we take this result as true, and leave the details to the **Appendix**. Thus, we have shown the case for $n = 3$ that that $\eta(3^3) = 9 = 3 \cdot 3$. Hence, it suffices to suppose $n > 3$ is odd and let $n = 2k + 3$, such that $k \in \mathbb{N}$. Then,

$$\begin{aligned}
3n &= \lceil 3 \log_3 3^n \rceil \\
&\leq \eta(3^n) && (\textbf{Make-n Theorem}) \\
&= \eta(3^{2k+3}) \\
&= \eta(9^k \cdot 3^3) \\
&\leq \eta(9^k) + \eta(3^3) && (\textbf{Multiplicative Bounding Theorem}) \\
&= 6k + 9 && (\textbf{Theorem 5.5}) \\
&= 3n
\end{aligned}$$

from where we conclude that $\eta(3^n) = 3n$ as desired, and the result follows, seeing as we have exhausted all cases. ∎

We introduce one more standard definition.

Definition 5.7. *Let* $\{x\}$ *denote the fractional part of* x*. That is,* $\{x\} = x - \lfloor x \rfloor$*.*

Consider the following identity involving the ceiling, floor and fractional part functions.

Lemma 5.8. *Given* $x \in \mathbb{R}$ *and* $k \in \mathbb{N}$ *we have*

$$k\lceil x \rceil - \lceil kx \rceil = \lfloor k\{-x\} \rfloor$$

Proof. It is well known that

$$\begin{aligned} \lceil x \rceil &= -\lfloor -x \rfloor \\ &= -(-x - \{-x\}) \\ &= x + \{-x\} \end{aligned}$$

applying which along with standard properties of $\{x\}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ yields

$$\begin{aligned} k\lceil x \rceil - \lceil kx \rceil &= k(x + \{-x\}) - (kx + \{-kx\}) \\ &= k\{-x\} - \{-kx\} \\ &= k\{-x\} - \{k\lfloor -x \rfloor + k\{-x\}\} \\ &= k\{-x\} - \{k\{-x\}\} \\ &= \lfloor k\{-x\} \rfloor \end{aligned}$$

as desired. ■

From here, we make the following elegant generalisation of the technique used in **Theorem 5.5**.

Theorem 5.9. ***(Lifting-the-Exponent Theorem)*** *Suppose* $\eta(n) = \lceil 3\log_3 n \rceil$ *for some* $n \in \mathbb{N}$*. Then for all* $k \in \mathbb{N}$*, we have*

$$0 \leq k \cdot \eta(n) - \eta(n^k) \leq \lfloor k\{-3\log_3 n\} \rfloor \leq k - 1$$

and in particular, if k *satisfies* $\{-3\log_3 n\} < \frac{1}{k}$*, then*

$$\eta(n^k) = k \cdot \eta(n).$$

Proof. For the first claim, we first consider the **Multiplicative Bounding Theorem**, giving

$$\eta(n^k) \leq k \cdot \eta(n),$$

which yields the first inequality. Now, we may consider the bounding strategy from before,

$$\begin{aligned}
\lceil 3k\log_3 n\rceil &= \lceil 3\log_3 n^k\rceil \\
&\leq \eta(n^k) \qquad \text{(Make-n Theorem)} \\
&\leq k\cdot\eta(n) \qquad \text{(Multiplicative Bounding Theorem)} \\
&= k\lceil 3\log_3 n\rceil \\
&= \lceil 3k\log_3 n\rceil + \lfloor k\{-3\log_3 n\}\rfloor \qquad \text{(Lemma 5.8)}
\end{aligned}$$

Notice that the bounds, $\lceil 3k\log_3 n\rceil$ and $\lceil 3k\log_3 n\rceil + \lfloor k\{-3\log_3 n\}\rfloor$, as well as the desired quantities, $\eta(n^k)$ and $k\cdot\eta(n)$ are all integers, which implies that $\eta(n^k)$ and $k\cdot\eta(n)$ are bounded by integers with difference $\lfloor k\{-3\log_3 n\}\rfloor$. Consequently, their difference must also be at most $\lfloor k\{-3\log_3 n\}\rfloor$ as desired, which yields the second inequality.

The last inequality results from the fact that $\{-3\log_3 n\} < 1$, so $k\{-3\log_3 n\} < k$, from which the last inequality $\lfloor k\{-3\log_3 n\}\rfloor \leq k-1$ follows. We have shown our first claim.

If $\{-3\log_3 n\} < \frac{1}{k}$, then $\lfloor k\{-3\log_3 n\}\rfloor = 0$. Substituting this into the first claim yields

$$0 \leq k\cdot\eta(n) - \eta(n^k) \leq 0 \implies \eta(n^k) = k\cdot\eta(n)$$

which proves the second claim as desired. ■

Remark. Using the fact that since the argument is always an integer, $\{-3\log_3(9^n)\}$ is always 0 and thus less than $\frac{1}{k}$ for all $k\in\mathbb{N}$, **Theorem 5.5** directly follows due to $\eta(9) = 6 = \lceil 3\log_3 9\rceil$.

Remark. The **Lifting-the-Exponent Theorem** yields significantly stronger results than the **Make-n Theorem**. For instance, since $\eta(10) = 7 = \lceil 3\log_3 10\rceil$, we may apply the **Lifting-the-Exponent Theorem**. If we wish to calculate η of a googol, that is, $\eta(10^{100})$, the former yields $629 \leq \eta(10^{100}) \leq 700$, while the latter only allows us to conclude $629 \leq \eta(10^{100}) \leq 997$.

Finally, we introduce one last result to relate η and ζ.

Lemma 5.10. ***(Eta-Zeta Inequality)*** *For any $n\in\mathbb{N}$, $\zeta(\eta(n)) \geq n$. Furthermore, $\eta(\zeta(n)) \geq n$.*

Proof. We first prove the first inequality by direct proof. Recall the definition of $\eta(n)$ implies we can Make n with every $D\in D_{\eta(n)}$. But $I_{\eta(n)}\in D_{\eta(n)}$, so we can Make n with $I_{\eta(n)}$, which implies that $\zeta(\eta(n)) \geq n$, since that describes the maximum number you can Make with $I_{\eta(n)}$.

We now prove the second inequality. For $n = 1$, we check that, $\eta(\zeta(1)) = \eta(1) = 5 \geq 1$.

From now we only consider $n > 1$. Suppose by way of contradiction, $\eta(\zeta(n)) \leq n-1$. By the **Monotonicity Lemma**, this implies we can Make $\zeta(n)$ with any $D \in D_{n-1}$. But notice $I_{n-1} \in D_{n-1}$. We know that $\zeta(n-1)$ is the maximum number that I_{n-1} can make by definition, and this means that $\zeta(n) \leq \zeta(n-1)$. But $n > n-1$, so by the **Zinc Theorem**, $\zeta(n) > \zeta(n-1)$. *Contradiction.* Hence $\eta(\zeta(n)) \geq n$. ■

Remark. This inequality should remind you of the definition of an inverse function. That is, for some bijective f, f^{-1} is the unique function given by $f(f^{-1}(x)) \equiv x \equiv f^{-1}(f(x))$. In particular, here, ***in some sense***, $\eta(n) \geq \zeta^{-1}(n)$. This explains why the lower bound for η is in fact, very similar to an inverse of ζ. However, of course, this is all just abuse of notation since ζ is not surjective over $\mathbb{N} \to \mathbb{N}$, which is why we instead upper bound ζ with $3^{\frac{n}{3}}$, which is bijective, and thus invertible, over $\mathbb{R} \to \mathbb{R}^+$.

6 Analysis and Visualisation

Figure 2 below shows the bounds of $\eta(n)$ presented in the **Make-n Theorem** for $0 < n \leq 50$ plotted against n, where red is the lower bound, green is the upper bound, the blue region is the range of possible values for η, and the black points represent actual values of $\eta(n)$.

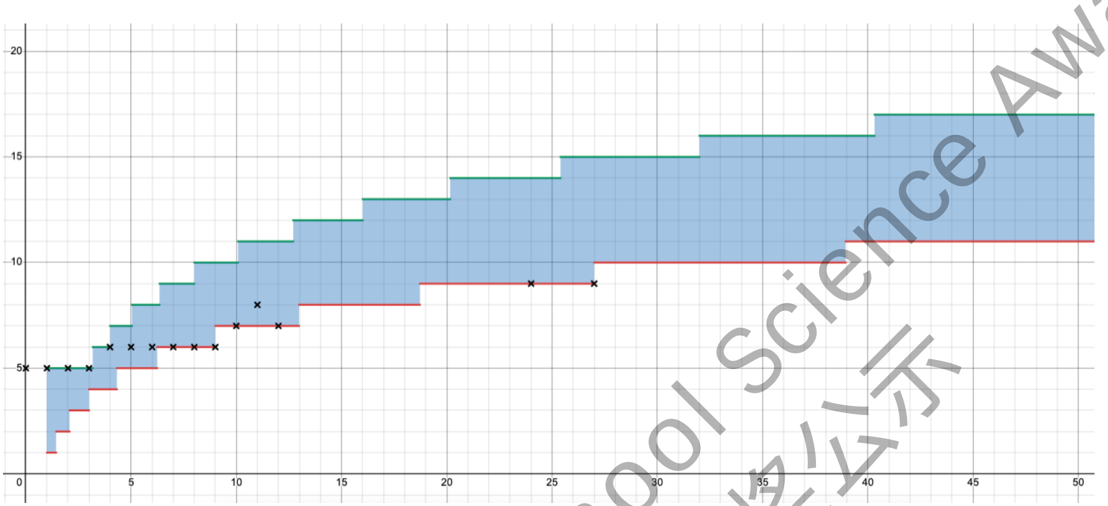

Figure 2: *Bounds for* $\eta(n)$ *against* n

Since our results are logarithmic in nature, it seems natural to plot our graphs on a logarithmic scale as well. **Figure 3** shows the bounds for $\eta(n)$ over $1 \leq n \leq 10^{50}$ plotted on a logarithmic scale against n.

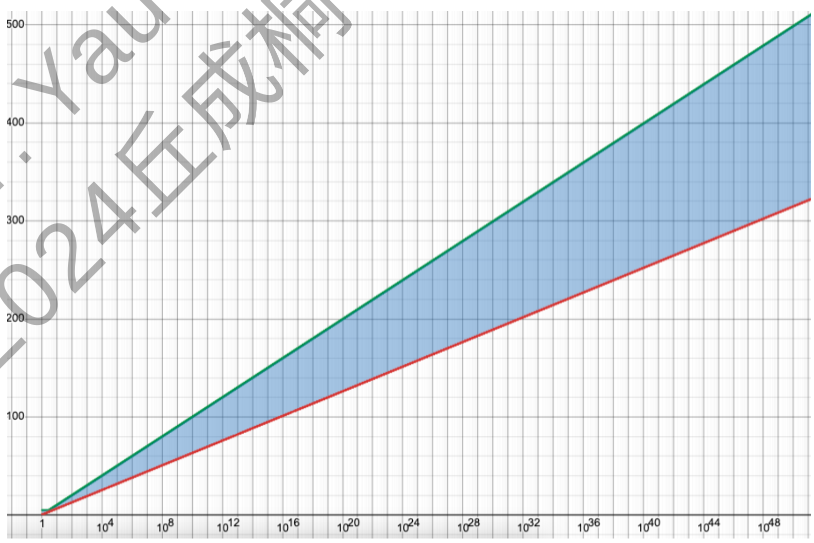

Figure 3: *Bounds for* $\eta(n)$ *against* n

We can determine interesting results from the graphs.

- With 50 digits, we can make any number up to $\lfloor 2^{\frac{50}{3}} \rfloor = 104031$.
- With 100 digits, we can make any number up to $\lfloor 2^{\frac{100}{3}} \rfloor \approx 1.082 \times 10^{10}$.
- With 1000 digits, we can make any number up to $\lfloor 2^{\frac{1000}{3}} \rfloor \approx 2.205 \times 10^{100}$.
- With 10000 digits, we can make any number up to $\lfloor 2^{\frac{10000}{3}} \rfloor \approx 2.712 \times 10^{1003}$.
- For our original $n = 24$, the **Make-n Theorem** yields $9 \leq \eta(24) \leq 14$.

where "digit" here refers to a positive integer between 1 and 10 inclusive.

7 Extensions and Conjectures

In this section, we discuss related problems, possible extensions and conjectures in relation to this problem.

7.1 Relation to Mahler-Popken Complexity

Recall that in this paper, the upper bound reached by η is given by

$$\eta(n) \leq \lfloor 3\log_2 n \rfloor + 1$$

which grows asymptotically to $\dfrac{3}{\ln 2}\ln n \approx 4.32809 \ln n$, noticeably worse than the upper bounds of $||n||$, whose upper bound for *almost all* n grows asymptotically to approximately $3.29496 \ln n$.

The difference between proving statements in relation to $||n||$ and $\eta(n)$, is that $\eta(n)$ requires a much larger search space, checking through all $D \in D_k$ in an attempt to Make n. On the other hand, $||n||$ only checks through $I_k \in D_k$, which makes the time complexity and overall casework of $||n||$ much simpler.

Since $||n||$ can be described as a subcase of $\eta(n)$, we can in fact derive the chain inequality

$$\lceil 3\log_3 n \rceil \leq ||n|| \leq \eta(n) \leq \lfloor 3\log_2 n \rfloor + 1$$

for all $n > 2$, where the lower bound for $||n||$ is derived using the exact same proof as the lower bound for η. Intuitively, this is because the counterexample we used to generate the lower bound for $\eta(n)$ are unital sets in the first place, which also applies to $||n||$.

In the following, let $||n||$ refer to a modified version of the Mahler-Popken Complexity function, where we allow the usage of $-$ and $\div$. A conjecture could result from the idea that as n gets large, "overshooting" the answer should not be too much of an issue, as bigger numbers help us reach our target n faster, and we might hypothesise that for sufficiently large n, if we can Make n with I_k, then we can Make n with any $D \in D_k$.

After all, for all known values of $\eta(n)$ with $n \geq 7$, that is, for $n \in \{7, 8, 9, 10, 11, 12, 24, 27\}$, we have $\eta(n) = ||n||$ holding true. This motivates the following conjecture.

Conjecture 7.1. *There exists a constant $c \in \mathbb{N}$ such that for all $n \geq c$, $\eta(n) = ||n||$.*

We also form the slightly weaker conjecture regarding the asymptotic growth of $\eta(n)$ and $||n||$.

Conjecture 7.2. $\limsup_{n\to\infty} \frac{\eta(n)}{\ln n} = \limsup_{n\to\infty} \frac{||n||}{\ln n}$.

7.2 Possible Extensions

There are many possible extensions to this project that I would look into if given more time.

For example, we can try to make non-integral rational numbers. Furthermore, we can include other operations, such as the *factorial* and *exponentiation*.

In fact, if the factorial is allowed, we are able to Make 24 very easily, because we are able to Make 4 and perform $4! = 24$, so we can Make 24 with $\eta(4) = 6$ numbers. However, it would be difficult to generalise this to all integers, as factorials and exponents both grow extremely quickly, and are much more unpredictable compared to $+, -, \times, \div$, making this an interesting extension to my project.

The m-ary Complexity displays many interesting properties. For instance, it is clear that by fixing $n \in \mathbb{N}$, the sequence $\{||n||_m\}_{m=0}^{\infty}$ is (non-strictly) monotone decreasing, which can bring interesting bounding arguments. Therefore, by introducing a function $\eta_S(n) : \mathbb{Z} \to \mathbb{N}$, where $S \subset \mathbb{Z}$ is the *basis set*, and defining $\eta_S(n)$ to be the minimum $k \in \mathbb{N}$ such that for all sets $D = \{d_1, d_2, \ldots, d_k\}$ with $d_i \in S$ for all $1 \leq i \leq k$, we can Make n with D, we are able to further generalise. Here,

$$||n||_m \quad \text{is closely related to} \quad \eta_{\{1,2,\ldots,m\}}(n)$$

and our original function

$$\eta(n) \equiv \eta_{\{1,2,\ldots,10\}}(n).$$

Some trivial properties about $\eta_S(n)$ can be established. For instance, if we are given two sets $R \subseteq S \subset \mathbb{Z}$, then

$$\eta_R(n) \geq \eta_S(n)$$

for all $n \in \mathbb{Z}$.

7.3 Complexity Analysis with Catalan Numbers

Let's explore the possibility of manually exhausting, or, "brute-forcing" through all possibilities to end off this paper.

Definition 7.3. *Let* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *be the n-th Catalan Number.*

The Catalan Numbers relate to many problems in Combinatorics. For example, C_n is the number of *valid bracket sequences* of length $2n$ (Davis, 2006 [12]), where a *valid bracket sequence* is a string made up of an equal number of '(' and ')', and the number of ')' never exceeds the number of '(' in any prefix of the string.

As pointed out in (Kruis *et al.*, 2020 [9]), this implies that C_{n-1} is the number of *base-patterns* given n numbers, where a *base-pattern* is the sequence, or order, in which binary operations are applied to the set of n numbers. For example, for $n = 3$, the *base-patterns* are $((a * b) * c)$ and $(a * (b * c))$. Notice here $*$ refers to **any** binary operation, not just multiplication.

We can relate this to our problem. In our case, each of the $n-1$ binary operations can be replaced by $+, -, \times$ or $\div$. Furthermore, each of the n numbers can take 10 possible values. This means that if we wanted to manually exhaust all sets with n numbers, we have to check $C_{n-1} \times 4^{n-1} \times 10^n$ possibilities, to see if they yield 24. However, for each possibility, we need to perform $(n-1)$ operations, hence that makes for a total of $(n-1)\,C_{n-1}4^{n-1}10^n = \frac{1}{4}(n-1)\,C_{n-1}\,40^n$ operations. Let this expression be $P(n)$.

In fact, $C_n \sim \frac{4^n}{n^{\frac{3}{2}}\sqrt{\pi}}$ (MathWorld, 2009 [13]), so we have:

$$P(n) \sim \frac{1}{4}\frac{4^{n-1}}{(n-1)^{\frac{3}{2}}\sqrt{\pi}}(n-1)\,40^n \sim \frac{1}{16\sqrt{\pi}}\frac{160^n}{\sqrt{n}}$$

As such, **without optimisation**, one must run a truly enormous exponential time complete search in brute forcing to determine true values of η, which is largely impractical.

7.4 Applications

For millennia, mathematicians have always been intrigued to study all sorts of games and brain teasers, from simple games like tic-tac-toe, to complex games like chess. Especially in complex decision-making games, concepts like *reduction*, *induction* and *pattern recognition* are important, to simplify a complex state into smaller, more digestible states. For instance, in chess, instead of focusing on analysing the entire board, it is equally crucial to look at smaller sub-problems such as tactics and checkmating patterns. Applications of this project include gaining insight into similar mathematical games, as well as to better understand functions closely related with $\eta(n)$, such as the Mahler-Popken Complexity $||n||$, and the m-ary Complexities $||n||_m$, perhaps even to solve certain unsolved conjectures.

Furthermore, this project is applicable in *Computer Science*, specifically in studying *Complexity Analysis*. In many complex decision-making games, there are enormous numbers of game states. In our research, the game states are the set of numbers that are available to us, and the decisions are the operations we can apply. Since values of η can be searched through an *exponential-time complete search*, this project can be a way to test the effectiveness of a program or a programming language at implementing *recursion*, *memoization* (i.e. to store and recover previously computed results) and *pruning* (i.e. to eliminate the need to evaluate identical expressions multiple times). It can also gives us insight into important *pruning* techniques to speed up searches on these complex decision-making games.

In short, there is much room for improvement and expansion in this project, which can potentially uncover plenty of insightful observations in the study of related *Combinatorics Games*.

Appendix: Greedy Algorithm

[Back to Lemma 2.5]

Consider an arbitrary expression with 8 ones. We expect to obtain $\zeta(8) = 18$ in the end. The rules for this algorithm I discovered are (the first four are new, trivial ones):

- $A - B \to A \times B$
- $A \div B \to A \times B$
- $1 \times n \to 1 + n$
- $(1+1) \times (1+1) \times (1+1) \to (1+1+1) \times (1+1+1)$
- $A + B \to A \times B$, where $A + B > 3$
- $A \times B + 1 \to A \times (B+1)$, where $A > 1$

Table 3 on the next page is a demonstration of the greedy algorithm at work, starting with an extremely suboptimal sum-product $1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 = -6$.

Notice the non-decreasing value! This is very important.

This is the *Greedy Property* of this algorithm. In fact, we can show that these rules are sufficient to guarantee that we can always reach $\zeta(n)$.

Current Expression	Value
$1-1-1-1-1-1-1-1$	-6
$1-1-1-1-1-1-1\times 1$	-5
$1-1-1-1-1-1\times 1\times 1$	-4
$1-1-1-1-1\times 1\times 1\times 1$	-3
$1-1-1-1\times 1\times 1\times 1\times 1$	-2
$1-1-1\times 1\times 1\times 1\times 1\times 1$	-1
$1-1\times 1\times 1\times 1\times 1\times 1\times 1$	0
$1\times 1\times 1\times 1\times 1\times 1\times 1\times 1$	1
$1\times 1\times 1\times 1\times 1\times 1\times (1+1)$	2
$1\times 1\times 1\times 1\times 1\times (1+1+1)$	3
$1\times 1\times 1\times 1\times (1+1+1+1)$	4
$1\times 1\times 1\times (1+1+1+1+1)$	5
$1\times 1\times (1+1+1+1+1+1)$	6
$1\times (1+1+1+1+1+1+1)$	7
$1+1+1+1+1+1+1+1$	8
$(1+1)+(1+1+1+1+1+1)$	8
$(1+1)\times (1+1+1+1+1+1)$	12
$(1+1)\times ((1+1)+(1+1+1+1))$	12
$(1+1)\times (1+1)\times (1+1+1+1)$	16
$(1+1)\times (1+1)\times ((1+1)+(1+1))$	16
$(1+1)\times (1+1)\times (1+1)\times (1+1)$	16
$(1+1+1)\times (1+1+1)\times (1+1)$	$18=\zeta(8)$

Table 3: *Greedy Algorithm at Work*

Appendix: Maximum Product Over Reals

[Back to Lemma 2.8]

Let $M : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function such that $M(n,k)$ is the maximum possible value of $a_1 \cdot a_2 \cdot a_3 \dots a_k$, over all $a_1, a_2, \dots, a_n \in \mathbb{N}$ such that $a_1 + a_2 + a_3 + \cdots + a_k = n$. Then, we have:

$$\begin{aligned} M(n,k) &= a_1 \cdot a_2 \cdot a_3 \dots a_k \\ &\leq \left(\frac{a_1 + a_2 + a_3 + \cdots + a_k}{k}\right)^k \qquad \textbf{[AM-GM Inequality]} \\ &= \left(\frac{n}{k}\right)^k \end{aligned}$$

Let $N(n,k) = \left(\frac{n}{k}\right)^k$, where $N : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. The above inequality then yields

$$M(n,k) \leq N(n,k) \tag{1}$$

for all $n, k \in \mathbb{N}$, where $\mathbb{R}^+$ is the set of *positive reals*.

However, notice that in our case, n is a constant. We want to find $\max_{k \in \mathbb{N}} M(n,k)$. We use Implicit Logarithmic Differentiation. We have $\ln(N(n,k)) = k \ln\left(\frac{n}{k}\right)$. Differentiating both sides, we have

$$\begin{aligned} \frac{\partial}{\partial k} \ln(N(n,k)) &= \frac{\partial}{\partial k} k \ln\left(\frac{n}{k}\right) \\ \frac{1}{N(n,k)} \cdot \frac{\partial N}{\partial k} &= \ln\left(\frac{n}{k}\right) - 1 \\ \implies \frac{\partial N}{\partial k} &= N(n,k)\left(\ln\left(\frac{n}{k}\right) - 1\right) \end{aligned}$$

At local extrema points, $\frac{\partial N}{\partial k} = 0 \implies N(n,k)\left(\ln\left(\frac{n}{k}\right) - 1\right) = 0.$

It is quite clear that $N(n,k) > 0$, so this implies that $\ln\left(\frac{n}{k}\right) - 1 = 0$, and an extrema exists at $k = \frac{n}{e}$. We can check that for all $0 < k < \frac{n}{e}$,

$$\begin{aligned} \frac{\partial N}{\partial k} &= N(n,k)\left(\ln\left(\frac{n}{k}\right) - 1\right) \\ &> N(n,k)(\ln e - 1) \\ &= 0 \end{aligned}$$

and for $k > \frac{n}{e}$,

$$\begin{aligned} \frac{\partial N}{\partial k} &= N(n,k)\left(\ln\left(\frac{n}{k}\right) - 1\right) \\ &< N(n,k)(\ln e - 1) \\ &= 0 \end{aligned}$$

Thus, by the **First Derivative Test**, $k = \frac{n}{e}$ is a local maximum, and also the global maximum of $N(n, k)$ over the domain $(0, \infty)$.

We have the following chain inequality.

$$\begin{aligned} \max_{k\in\mathbb{N}} M(n,k) &\leq \max_{k\in\mathbb{N}} N(n,k) \qquad \text{[Inequality 1]} \\ &\leq \max_{k\in\mathbb{R}^+} N(n,k) \\ &= N\left(n, \frac{n}{e}\right) \\ &= e^{\frac{n}{e}} \end{aligned}$$

This gives our upper bound of $e^{\frac{n}{e}}$. Since $2 < e < 3$, with $e \approx 2.718$ being closer to 3, this reassures us in that it seems to agree with **Lemma 2.8**, which only uses 2 and 3 in the product while favouring 3.

Appendix: Python Program

[Back to Chapter 3.1]

The program is as shown below.

```
from math import comb
from datetime import datetime
from operator import sub
from itertools import combinations_with_replacement

INVALID = 0.1


def div(a, b):
    return INVALID if b == 0 or a % b != 0 else a // b

def plus(a, b):
    return INVALID if a > b else a + b

def times(a, b):
    return INVALID if a > b else a * b


ops = [plus, times, sub, div]

def eta(num):
    if len(num) == 1:
        return num[0] == c

    for i, n1 in enumerate(num):
        for j, n2 in enumerate(num):
            if i == j:
                continue

            for op in ops:
                if op(n1, n2) != INVALID:
                    if eta([op(n1, n2)] + [n for k, n in enumerate(num) if k != i
                                              and k != j]):
                        return True

    return False
```

```
def etatest():
    n = 0
    s = datetime.now()
    t = comb(9 + k, k)

    print(f'Executing {t} testcases of size {k} to Make {c}.')

    for x in combinations_with_replacement(range(1, 11), k):
        if not eta(x):
            print(f'{x} failed.')
            n += 1

    print(f'All tests completed. Total: {t}, Failed: {n}, Passed: {t - n}.\n{(
                                          datetime.now() - s).total_seconds()}
                                          seconds elapsed.')


k = 5 # Number of integers in each testcase
c = 6 # Number we are trying to Make

etatest()
```

To use the program, you can modify the values of `k` and `c` at the bottom of the code. Particularly, notice the two functions below.

```
def plus(a, b):
    return INVALID if a > b else a + b

def times(a, b):
    return INVALID if a > b else a * b
```

The reason behind returning `INVALID`, which is a signal to kill the current testcase and move on to the next possibility, is so that we can reduce the number of duplicate testcases. Specifically, this is because $+$ and $\times$ are commutative (i.e. they are symmetric: $a+b=b+a$, $a\times b=b\times a$). Empirically, this cuts the running time **in half**, and is a simple example of a commonly used technique called *pruning*.

There are many potential optimisations for this code. For one, python is a very slow programming language, and I only used it because of the simplicity of `list` in python, as compared to a `vector` in `C++`, for example. Switching to `C++` should cut the running time by 2 to 3 orders of magnitude. Secondly, *memoization* can be used to check for cases that have already been evaluated. This is another example of *pruning* to save time.

Of course, in trying to speed up our computation, there will be a trade-off between computational time and computational space. In this project, we have not explored the possibility of optimising the code by any significant means. It is well-known by making a trade off between time and space with *memoization*, however, the running time can be cut down by multiple orders of magnitudes as well. In this area, there is strong potential for future work.

Appendix: Value of $\eta(27)$

[Back to Theorem 5.6]

Lemma A. *Consider an arbitrary $D \in D_9$. If there exists a subset $S \subset D$ satisfying $|S| \leq 3$ and some $1 \leq x \leq 9$ such that we can Make x with $\{27\} \cup S$, then we can Make 27 with D.*

Proof. Let $R = D \setminus S$. Then $|R| \geq 6$. By the **Monotonicity Lemma**, we can Make x with R, since $1 \leq x \leq 9$ implies $\eta(x) \leq 6$. Suppose the equation used to Make x with $\{27\} \cup S$ is $\mathcal{O}$. Simply rewrite the expression $\mathcal{O}$ in terms of the constant 27. For example, if $\mathcal{O}$ was $(27 - 1 - 1) \div 5 = 5$, we can rewrite it into $27 = 1 + 1 + 5 \times 5$. Such a rearrangement is clearly guaranteed to exist, and we have an equation to Make 27 with D, and the result follows. ■

Using **Lemma A**, we wish to prove $\eta(27) = 9$.

Proof. Firstly, by the **Make-n Theorem**, $\eta(27) \geq \lceil 3 \log_3(27) \rceil = 9$.

Suppose we have an arbitrary $D \in D_9$. Consider **Table 4** on the next page, which is exhaustive of all cases. That is, we are always able to choose a subset $S \subset D$ with $|S| \leq 3$ corresponding to one of the rows below. Using the corresponding expression $\mathcal{O}$, by **Lemma A**, we are able to Make 27 with D. Thus, we can Make 27 with all $D \in D_9$, and by the **Upper Bound Lemma**, we have $\eta(27) \leq 9$.

Finally, $9 \leq \eta(27) \leq 9$ suffices to force $\eta(27) = 9$. ■

Remark. By "exhaustive", in fact, any arbitrary choice of $S \subset D$ will suffice, and either S or a subset of S is guaranteed to be found in **Table 4**, so this result indeed holds.

Remark. This is a proof that proves a result of equal caliber as the proof for $\eta(24) \leq 9$, but admittedly this is a much more brutal approach. While the same approach will certainly have worked for $\eta(24)$, we wish to pursue a more elegant result with less listing. Because of the highly-composite nature of 24, the proof for $\eta(24)$ is much more elegant, and unfortunately I have not discovered such an elegant proof for bounding $\eta(27)$.

s_1	s_2	s_3	$\mathcal{O}$
3			$27 \div 3 = 9$
9			$27 \div 9 = 3$
1	2		$27 \div (1 + 2) = 9$
1	4		$(27 + 1) \div 4 = 7$
1	7		$(27 + 1) \div 7 = 4$
1	8		$27 \div (1 + 8) = 3$
1	10		$27 \div (10 - 1) = 9$
2	5		$(27 - 2) \div 5 = 5$
2	10		$27 - 2 \times 10 = 7$
4	5		$27 - 4 \times 5 = 7$
4	6		$27 - 4 \times 6 = 3$
4	7		$4 \times 7 - 27 = 1$
4	8		$4 \times 8 - 27 = 5$
5	5		$27 - 5 \times 5 = 2$
5	6		$5 \times 6 - 27 = 3$
5	7		$5 \times 7 - 27 = 8$
6	6		$6 \times 6 - 27 = 9$
7	10		$(27 - 7) \div 10 = 2$
10	10		$27 - 10 - 10 = 7$
1	1	1	$27 \div (1 + 1 + 1) = 9$
1	1	5	$(27 - 1 - 1) \div 5 = 5$
1	1	6	$27 \times (1 + 1) \div 6 = 9$
2	2	2	$27 \div (2 + 2 \div 2) = 9$
2	2	4	$27 \times 2 \div (2 + 4) = 9$
2	2	6	$27 - (2 + 2) \times 6 = 3$

s_1	s_2	s_3	$\mathcal{O}$
2	2	7	$(2 + 2) \times 7 - 27 = 1$
2	2	8	$(2 + 2) \times 8 - 27 = 5$
2	4	4	$2 \times 4 \times 4 - 27 = 5$
2	4	10	$27 - 2 \times 4 - 10 = 9$
2	6	7	$27 - 6 \times 7 \div 2 = 6$
2	6	8	$27 - 2 \times 6 - 8 = 7$
2	6	10	$27 - 2 \times 10 - 6 = 1$
2	7	7	$27 - 2 \times 7 - 7 = 6$
2	7	8	$27 - 2 \times 7 - 8 = 5$
2	8	8	$27 - 2 \times 8 - 8 = 3$
2	8	10	$27 - 2 \times 8 - 10 = 1$
4	4	4	$27 - 4 \times 4 - 4 = 7$
4	4	10	$27 - 4 \times 4 - 10 = 1$
4	10	10	$27 - 4 - 10 - 10 = 3$
5	8	8	$27 - 5 - 8 - 8 = 6$
5	8	10	$27 - 5 - 8 - 10 = 4$
6	7	7	$27 - 6 - 7 - 7 = 7$
6	7	8	$27 - 6 - 7 - 8 = 6$
6	8	8	$27 - 6 - 8 - 8 = 5$
6	8	10	$27 - 6 - 8 - 10 = 3$
7	7	7	$27 - 7 - 7 - 7 = 6$
7	7	8	$27 - 7 - 7 - 8 = 5$
8	8	8	$27 - 8 - 8 - 8 = 3$
8	8	10	$27 - 8 - 8 - 10 = 1$

Table 4: *Table of choices of* $\mathcal{O}$ *against* $S = \{s_1, s_2, s_3\}$

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Declaration of Previous Submission

This project was submitted previously for the following science fairs/competitions:

Name of Competition	Date	Awards Won
Singapore Science and Engineering Fair (SSEF)	March 2024	Distinction Award (Project)
Singapore Mathematics Project Festival (SMPF)	May 2024	Foo Kean Pew Memorial Prize with Excellent Presentation Prize