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Research Report

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Title of Research Report

On the Crossing Profile of Rectilinear Drawings of Complete Graph

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On the Crossing Profile of Rectilinear Drawings of Complete Graphs

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Abstract

Geometric graph theory is the study of various geometric representations of graphs. We are interested in how the *crossings* between edges of a *drawing* are distributed, extending previous work on the *crossing number* of various graphs. Let $\overline{\mathcal{D}}_n$ denote a *rectilinear drawing* (all edges are drawn as line segments) of the complete graph K_n in $\mathbb{R}^2$. We prove tight lower and upper bounds on the number of edges in $\overline{\mathcal{D}}_n$ that are crossed at most k times by other edges, a quantity denoted by $S_k(\overline{\mathcal{D}}_n)$. We also consider the number of edges in $\overline{\mathcal{D}}_n$ that are crossed exactly k times, denoted by $e_k(\overline{\mathcal{D}}_n)$, and show a non-trivial lower bound for this quantity. Some additional minor results are also presented throughout the paper. Our results greatly expand on prior knowledge about the number of edges in $\overline{\mathcal{D}}_n$ involved in 0 crossings and other work on the number of edges in a (general) drawing of K_n with at most k crossings.

Keywords

Rectilinear drawing, complete graph, crossing.

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Commitments on Academic Honesty and Integrity

We hereby declare that we

1. are fully committed to the principle of honesty, integrity and fair play throughout the competition.
2. actually perform the research work ourselves and thus truly understand the content of the work.
3. observe the common standard of academic integrity adopted by most journals and degree theses.
4. have declared all the assistance and contribution we have received from any personnel, agency, institution, etc. for the research work.
5. undertake to avoid getting in touch with assessment panel members in a way that may lead to direct or indirect conflict of interest.
6. undertake to avoid any interaction with assessment panel members that would undermine the neutrality of the panel member and fairness of the assessment process.
7. observe the safety regulations of the laboratory(ies) where we conduct the experiment(s), if applicable.
8. observe all rules and regulations of the competition.
9. agree that the decision of YHSA is final in all matters related to the competition.

We understand and agree that failure to honour the above commitments may lead to disqualification from the competition and/or removal of reward, if applicable; that any unethical deeds, if found, will be disclosed to the school principal of team member(s) and relevant parties if deemed necessary; and that the decision of YHSA is final and no appeal will be accepted.

(Signatures of full team below)

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Declaration of Academic Integrity

The participating team declares that the paper submitted is comprised of original research and results obtained under the guidance of the instructor. To the team's best knowledge, the paper does not contain research results, published or not, from a person who is not a team member, except for the content listed in the references and the acknowledgment. If there is any misinformation, we are willing to take all the related responsibilities.

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Name of the instructor: Oriol Solé Pi

Signature of the instructor:

Date: 8/24/24

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1 Introduction

A graph consisting of vertices and edges joining its vertices is *planar* if it can be drawn in the plane so that no two edges cross each other. The study of planar graphs dates back to Euler, who first proposed Euler's polyhedron formula in 1750 [1]. The formula relates the number of faces, edges, and vertices of a three-dimensional convex polyhedron and has been extended to planar graphs. More precisely, this result states that if a connected planar graph has v vertices, e edges, and f faces, then the equation $v - e + f = 2$ always holds. Another natural question one might ask about planar graphs is whether or not they can always be drawn in the plane without crossings between edges using only line segments (instead of curves) to represent the edges. It turns out that the answer is always yes, and this result is known as Fáry's theorem [2, 3, 4].

A natural extension on the classical study of planar graphs is to study how often the edges of a non-planar graph drawn in the plane intersect each other. A version of this concept was first explored by Turán during World War II. In Turán's brick factory problem, we are given a factory where wagon tracks directly connect every kiln and storage site. Because wagons are harder to push across the tracks at places where two distinct tracks cross, Turán wanted to minimize the total number of crossings between two edges [5].

Given a *drawing* of a graph G in the plane, a *crossing* is a point where two distinct edges intersect each other. Moreover, the *crossing number* $\mathrm{cr}(G)$ is the smallest possible total number of crossings in a drawing of G across all drawings of G. Turán's brick factory problem is equivalent to determining the crossing number of the complete bipartite graph, a graph where the vertices are split into two sets, and two of them are joined by an edge if and only if they belong to different sets.

As of now, the precise value of the crossing number of the complete bipartite graph $K_{m,n}$ is not known [6]. The same statement holds for the complete graph on n vertices K_n, where every two vertices are connected by an edge, as well as for the corresponding *rectilinear drawings* of the complete bipartite graph and the complete graph, where all edges are represented as line segments. In fact, even the asymptotic behaviour of these quantities is not well understood. More precisely, if $\overline{\mathrm{cr}}(G)$ denotes the smallest possible total number of crossings in a rectilinear drawing of G across all rectilinear drawings of G, then it is known that the limits of the four quantities

$$\frac{\mathrm{cr}(K_n)}{\binom{n}{4}}, \frac{\mathrm{cr}(K_{n,m})}{\binom{n}{4}}, \frac{\overline{\mathrm{cr}}(K_n)}{\binom{n}{4}}, \frac{\overline{\mathrm{cr}}(K_{n,m})}{\binom{n}{4}}$$

exist as n goes to infinity and are non-zero, but the precise values remain unknown [7, 8, 9].

In recent times, the study of crossing numbers has found applications in the field of graph visualization, where the goal is to discover pictographic representations of graphs that are easy to read and quickly convey the important properties and structure of the graph. We refer the reader to the extensive handbook by Tamassia et al. on graph drawing and graph visualization [10].

Geometric graph theory is also connected to very-large-scale integration (VLSI), an area of computer science that aims to design circuits in ways that allow a great number of transistors to fit into a single chip. Some early papers in this direction include [11, 12, 13].

Many more results about crossing numbers and geometric graph theory in general can be found in [14, 15].

1.1 Our Focus and Organization of Paper

Although planar graphs have been studied extensively, far less is known about drawings with crossings between edges. At a high level, the main goal of this work is to improve our understanding of the crossings between the edges in a rectilinear drawing $\overline{\mathcal{D}}_n$ of the complete graph on n vertices K_n. We study the number of edges in $\overline{\mathcal{D}}_n$ that are crossed exactly k times, denoted by $e_k(\overline{\mathcal{D}}_n)$, and the number of edges in $\overline{\mathcal{D}}_n$ that are crossed at most k times, denoted by $S_k(\overline{\mathcal{D}}_n)$.

In Section 2, we provide an introduction to necessary definitions and new notation, review previous results that are related and applicable to our work, and share a few initial observations. In Section 3, we present an intricate geometric construction that achieves $e_k(\overline{\mathcal{D}}_n) = \Omega(n)$ for all positive integers k such that $e_k(\overline{\mathcal{D}}_n) > 0$ is achievable, share an easy upper bound on $e_k(\mathcal{D}(G))$, and provide some speculation on how a $\overline{\mathcal{D}}_n$ achieving a larger value for $e_k(\overline{\mathcal{D}}_n)$ may look from a structural perspective. In Section 4, we share our strongest bounds on $\max e_1(\overline{\mathcal{D}}_n)$ and $\max S_1(\overline{\mathcal{D}}_n)$ and also explore when $\min e_k(\overline{\mathcal{D}}_n) = 0$ is guaranteed to hold. In Section 5, we show tight upper and lower bounds on $S_k(\overline{\mathcal{D}}_n)$, fully resolving the rectilinear versions of problems mentioned and explored in [16, 17].

2 Preliminaries

2.1 Drawings and Crossings

Geometric graph theory is the study of various geometric representations of graphs. In particular, we are interested in certain types of graph embeddings in $\mathbb{R}^2$.

Definition 2.1. Let $G = (V, E)$ be a graph. A *drawing* $\mathcal{D}(G)$ consists of mapping the vertices of G to distinct points in $\mathbb{R}^2$ and the edges of G to simple continuous curves in $\mathbb{R}^2$ that connect their respective endpoints. We also assume that no edge intersects a vertex other than at one of its endpoints, any two edges share at most one point[1] and are never tangent at a point, and no three edges share an interior point.

The stipulation that any two edges intersect at most once is motivated by the following special class of drawings in $\mathbb{R}^2$.

Definition 2.2. Let G be a graph. A *rectilinear drawing* $\overline{\mathcal{D}}(G)$ is a drawing of G such that every edge is a line segment.

In moving onward from planar graphs, we consider the points other than vertices where two edges of a drawing meet.

Definition 2.3. Let $\mathcal{D}(G)$ be a drawing of the graph G. A *crossing* of $\mathcal{D}(G)$ is a shared interior point of two distinct edges in $\mathcal{D}(G)$.

[1] Other authors do not necessarily assume that any two edges of a drawing have at most one point in common, instead using the term *simple drawing* or *good drawing* to encode this condition.

One way to measure how close a graph G is to being planar is by determining the smallest total number of crossings a drawing of G can have.

Definition 2.4. Let G be a graph. The *crossing number* $\operatorname{cr}(G)$ is the smallest possible total number of crossings in $\mathcal{D}(G)$ across all possible drawings of G. Similarly, the *rectilinear crossing number* $\overline{\operatorname{cr}}(G)$ is the smallest possible total number of crossings in $\overline{\mathcal{D}}(G)$ across all possible rectilinear drawings of G.

The following notable result yields a lower bound on the crossing number of any simple graph.

Theorem 2.1 (Crossing lemma, Ajtai et al. and Leighton, [18, 11])**.** *Let $G = (V, E)$ be a simple graph on n vertices. If $\left|E(G)\right| \geq 4n$, then*

$$\operatorname{cr}(G) \geq \frac{\left|E(G)\right|^3}{64n^2}.$$

It follows easily from this inequality that dense simple graphs, i.e., graphs $G = (V, E)$ on n vertices where $\left|E(G)\right| = \Omega(n^2)$, have $\operatorname{cr}(G) = \Omega(n^4)$. Moreover, every simple graph has crossing number at most $O(n^4)$ since simple graphs contain at most $\binom{n}{2}$ edges and any two edges can cross at most once.

2.2 Crossing Profile

We aim to further understand the crossings of a drawing by studying the distribution of crossings across the edges of a graph, namely deducing how many edges can have a fixed number of crossings.

Definition 2.5. Let $\mathcal{D}(G)$ be a drawing of the graph G. For $k \geq 0$, define the *k-crossing set* $E_k(\mathcal{D}(G))$ as the set of edges in $\mathcal{D}(G)$ that are part of k crossings. Additionally, the *k-crossing index* $e_k(\mathcal{D}(G)) = \left|E_k(\mathcal{D}(G))\right|$ is the number of edges in $\mathcal{D}(G)$ that are crossed k times.

Now, we coin a new term that showcases how the edges of $\mathcal{D}(G)$ are distributed across the sets $E_k(\mathcal{D}(G))$.

Definition 2.6. Let $\mathcal{D}(G)$ be a drawing of the graph G. The *crossing profile* of $\mathcal{D}(G)$ is the sequence

$$\operatorname{cp}(\mathcal{D}(G)) = (e_0(\mathcal{D}(G)), e_1(\mathcal{D}(G)), \ldots).$$

Observe that a drawing $\mathcal{D}(G)$ is k-planar if and only if all the non-zero terms of $\operatorname{cp}(\mathcal{D}(G))$ are among the first $k + 1$ entries of the sequence.

2.3 Notational Simplifications and Additional Assumptions

For fixed k, we denote by $\max e_k(\mathcal{D}(G))$ and $\min e_k(\mathcal{D}(G))$ the maximum and minimum values, respectively, attained by $e_k(\mathcal{D}(G))$ across all drawings of G. Analogously, for fixed k, we denote by $\max e_k(\overline{\mathcal{D}}(G))$ and $\min e_k(\overline{\mathcal{D}}(G))$ the maximum and minimum values, respectively, attained by $e_k(\overline{\mathcal{D}}(G))$ across all rectilinear drawings of G. In addition, we always assume that the vertices of $\mathcal{D}(G)$ are in general position, as sufficiently small perturbations can always be made to the vertices to ensure this condition. For the sake of convenience and to avoid overwhelming notation, we let $\overline{\mathcal{D}}_n$ denote a rectilinear drawing of K_n for the remainder of this paper. Finally, we also write $\max e_k(\overline{\mathcal{D}}_n) = \max e_k(\overline{\mathcal{D}}(K_n))$ and $\min e_k(\overline{\mathcal{D}}_n) = \min e_k(\overline{\mathcal{D}}(K_n))$.

In addition, we use the convention $[n] = \{1, 2, \ldots, n\}$ for all $n \in \mathbb{Z}^+$.

2.4 Previous Results on $e_0(\mathcal{D}(K_n))$ and $e_0(\overline{\mathcal{D}}_n)$

Previous explorations of i-crossing indices focused on edges with 0 crossings within both general and rectilinear drawings of complete graphs.

In 1963, Ringel discovered a tight upper bound on $e_0(\mathcal{D}(K_n))$.

Theorem 2.2 (Ringel, [19]). *For $n \geq 4$, we have $\max e_0(\mathcal{D}(K_n)) = 2n - 2$.*

Looking at the other direction, Harborth and Mengersen found the exact values of $\min e_0(\mathcal{D}(K_n))$ for $n \geq 2$.

Theorem 2.3 (Harborth and Mengersen, [20]). *The values of $\min e_0(\mathcal{D}(K_n))$ for $n \geq 2$ are displayed in Table 1.*

n	2	3	4	5	6	7	$\mathbb{Z}_{\geq 8}$
$\min e_0(\mathcal{D}(K_n))$	1	3	4	4	3	2	0

Table 1: Smallest number of edges with 0 crossings in drawings of K_n.

Moreover, they showed that $e_0(\mathcal{D}(K_n))$ can attain every value between $\min e_0(\mathcal{D}(K_n))$ and $\max e_0(\mathcal{D}(K_n))$ inclusive except for $e_0(\mathcal{D}(K_4)) = 5$ and $e_0(\mathcal{D}(K_5)) = 7$ [20].

Now, we shift our attention to the 0-crossing index for rectilinear drawings of K_n. In 1996, Harborth and Thürmann demonstrated that $\max e_0(\overline{\mathcal{D}}_n)$ is the same as $\max e_0(\mathcal{D}(K_n))$ for $n \geq 4$.

Theorem 2.4 (Harborth and Thürmann, [21]). *For $n \geq 4$, we have $\max e_0(\overline{\mathcal{D}}_n) = 2n - 2$.*

We provide an example of their construction attaining $e_0(\overline{\mathcal{D}}_n) = 2n - 2$.

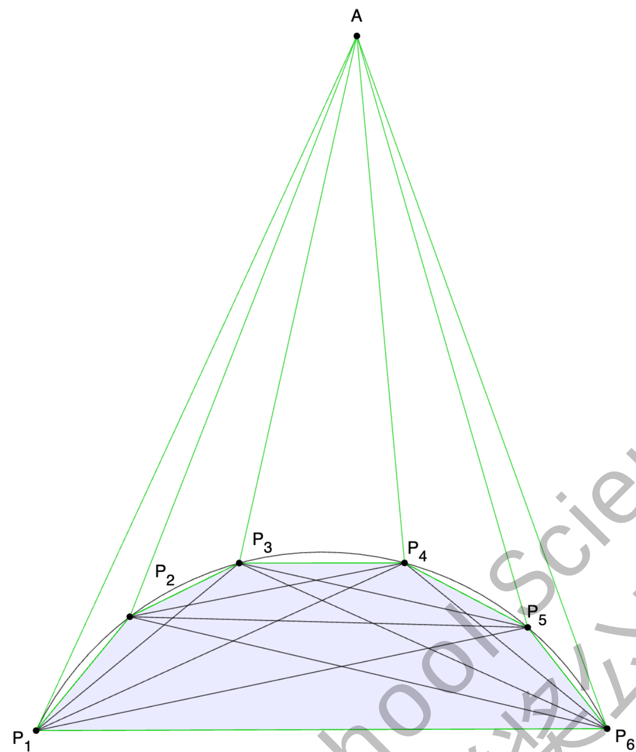

Figure 1: Construction attaining $e_0(\overline{\mathcal{D}}_n) = 2n - 2$ for $n = 7$.

As demonstrated in 1, $\overline{\mathcal{D}}_n$ consists of $\triangle AP_1P_{n-1}$ and a convex polygon $P_1P_2 \ldots P_{n-1}$ contained in the interior of $\triangle AP_1P_{n-1}$. In particular, the $2n-2$ edges of $\overline{\mathcal{D}}_n$ that belong to $E_0(\overline{\mathcal{D}}_n)$ are $\overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_{n-1}P_1}$ and $\overline{AP_1}, \overline{AP_2}, \ldots, \overline{AP_{n-1}}$.

In the same paper, the authors fully characterized $\min e_0(\overline{\mathcal{D}}_n)$ as well for $n \geq 3$.

Theorem 2.5 (Harborth and Thürmann, [21])**.** *We have* $\min e_0(\overline{\mathcal{D}}_n) = 5$ *for* $n \geq 8$ *and the values of* $\min e_0(\overline{\mathcal{D}}_n)$ *for* $n \in [3, 7]$ *are displayed in Table 2.*

n	3	4	5	6	7
$\min e_0(\overline{\mathcal{D}}_n)$	3	4	5	5	6

Table 2: Smallest number of edges with 0 crossings in rectilinear drawings of K_n for $n \in [3, 7]$.

Figure 2 depicts the constructions achieving $e_0(\overline{\mathcal{D}}_n) = 5$ for $n = 10$ and $n = 13$, respectively.

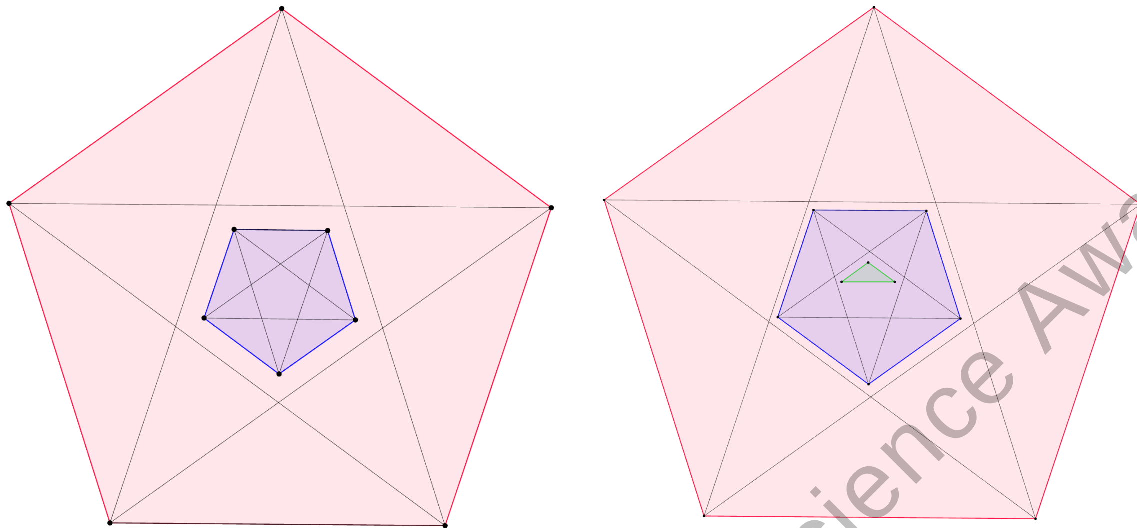

Figure 2: Constructions achieving $e_0(\overline{\mathcal{D}}_n) = 5$ for $n = 10$ (left) and $n = 13$ (right), respectively.

2.5 Partial Sums $S_k(\mathcal{D}(G))$ and Known Results

Instead of analyzing individual entries of the crossing profile, we can also consider the set of edges that are part of at most k crossings, which is equivalent to taking partial sums of the crossing profile sequence.

Definition 2.7. Let $\mathcal{D}(G)$ be a drawing of the graph G. For $k \geq 0$, the *k-crossing sum* $S_k(\mathcal{D}(G))$ is the number of edges in $\mathcal{D}(G)$ that are crossed at most k times. Equivalently, $S_k(\mathcal{D}(G)) = \sum_{i=0}^{k} e_k(\mathcal{D}(G))$.

For fixed k, we let $\max S_k(\mathcal{D}(G))$ and $\min S_k(\mathcal{D}(G))$ denote the maximum and minimum values, respectively, attained by $S_k(\mathcal{D}(G))$ across all drawings of G.

Now, we exhibit previous work on k-crossing sums in drawings. In addition to their work $e_0(\overline{\mathcal{D}}_n)$ seen in Subsection 2.4, Harborth and Mengersen proved strong bounds on $\max S_1(\mathcal{D}(K_n))$ and found the exact values of this quantity for $n \in [9]$.

Theorem 2.6 (Harborth and Mengersen, [16])**.** *For $n \geq 8$, we have*

$$2n + \left\lfloor \frac{n-1}{2} \right\rfloor - 2 \leq \max S_1(\mathcal{D}(K_n)) \leq 2n + \left\lfloor \frac{n-1}{2} \right\rfloor + 7,$$

and the exact values of $\max S_1(\mathcal{D}(K_n))$ for $n \in [9]$ are displayed in Table 3.

n	$1, 2, \ldots, 6$	7	8	9
$\max e_1(\mathcal{D}(K_n))$	$\binom{n}{2}$	18	20	22

Table 3: Largest number of edges with at most 1 crossing in drawings of K_n for $n \in [9]$.

Observe that a graph $G = (V, E)$ is k-planar if and only if $S_k(\mathcal{D}(G)) = |E(G)|$ holds for some drawing $\mathcal{D}(G)$ of G. First, we introduce a known upper bound on the number of edges in a simple k-planar graph.

Theorem 2.7 (Pach and Tóth, [22]). *Let $G = (V, E)$ be a simple k-planar graph on n vertices. For $k \geq 1$, we have*

$$E(G) \leq n\sqrt{16.875k}.$$

We provide an upper bound for $S_k(\mathcal{D}(G))$ using this result about k-planar graphs.

Proposition 2.8. *Let G be a simple graph on n vertices. For all $\mathcal{D}(G)$ and $k \geq 1$, we have*

$$S_k(\mathcal{D}(G)) = O(n\sqrt{k}).$$

Proof. Consider the subgraph H of G that consists of all the edges in $\mathcal{D}(G)$ involved in at most k crossings, i.e., the subgraph of G consisting of edges in $\bigcup_{i=0}^{k} E_i(\mathcal{D}(G))$. H is clearly a k-planar graph, so Theorem 2.7 directly implies H has at most $n\sqrt{16.875k}$ edges, or rather $S_k(\mathcal{D}(G)) \leq n\sqrt{16.875k}$, which suffices. □

2.6 Cutting Lemma

The following result from computational and discrete geometry helps us obtain the lower bound presented in Section 5.

Theorem 2.9 (Cutting lemma, Matoušek, [23]). *Let S be a set of n lines in $\mathbb{R}^2$ and $t \in (1, n)$ be a parameter. Then, $\mathbb{R}^2$ can be subdivided[1] into $r \leq Ct^2$ generalized triangles (regions that are the intersection of three half-planes), where C is an absolute constant, such that the interior of each generalized triangle is intersected by at most $\frac{n}{t}$ lines of S.*

Higher dimensional analogs and various applications of this statement can be found in [24, 25].

3 Constructive Lower Bound and an Easy Upper Bound on $\max e_k(\overline{\mathcal{D}}_n)$ for $k \geq 1$

In this section, we demonstrate a lower bound for $\max e_k(\overline{\mathcal{D}}_n)$ obtained via elaborate geometric constructions, share an established upper bound on $\max e_k(\overline{\mathcal{D}}_n)$, and briefly mention additional instincts concerning the value of $\max e_k(\overline{\mathcal{D}}_n)$.

3.1 Constructions for $\max e_k(\overline{\mathcal{D}}_n) = \Omega(n)$

We first introduce a geometric construction that achieves a linear k-crossing index with respect to n for $k \in \mathbb{Z}^+$ such that $e_k(\overline{\mathcal{D}}_n) > 0$ is achievable.

[1] We assume that the regions are pairwise interior disjoint and cover all of $\mathbb{R}^2$.

Theorem 3.1. *If* $k \in \left[1, \left\lfloor \left(\frac{n-2}{2}\right)^2 \right\rfloor\right]$, *we have* $\max e_k(\overline{\mathcal{D}}_n) = \Omega(n)$.

Remark. Suppose P_1 and P_2 are two vertices of $\overline{\mathcal{D}}_n$ such that there are x vertices on one side of $\overleftrightarrow{P_1P_2}$. Then the number of edges crossing $\overline{P_1P_2}$ is at most $x((n-2)-x) \le \left(\frac{n-2}{2}\right)^2$ by the AM-GM inequality. Thus, this theorem applies to all $k \in \mathbb{Z}^+$ for which $e_k(\overline{\mathcal{D}}_n) > 0$ is attainable.

Proof. Let $k = (m-2)^2 + r$ where $r \in [1, 2m-3]$, whence $m = \left\lceil \sqrt{k} \right\rceil + 1$. First, we produce configurations of $2m-1$ or $2m$ points yielding $\Omega(m)$ edges with k crossings. Afterwards, we string together sufficiently many copies of such configurations such that they do not interfere with each other to obtain $\Omega(n)$ edges in $E_k(\overline{\mathcal{D}}_n)$.

We start with the construction for n even. One can check $k \in \left[1, \left(\frac{n-2}{2}\right)^2\right]$ implies $2m \le n$.

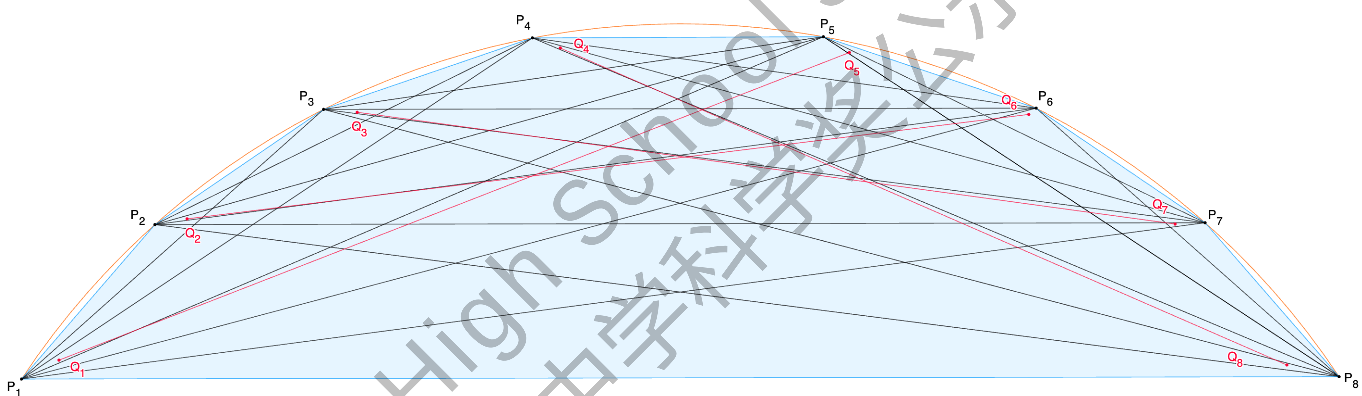

Figure 3: A configuration of 16 points for $m = 8$ and $k = 39$.

Case 3.1. If $m = 2j$, place m vertices $P_1, P_2, \ldots, P_m$ on a minor circular arc in clockwise order such that $\widehat{P_iP_{i+1}}$ are pairwise congruent for $i \in [m-1]$. We also construct m vertices $Q_1, Q_2, \ldots, Q_m$ such that Q_i is sufficiently close to P_i for $i \in [m]$. Henceforth, take all indices modulo m.

Our goal is to perturb $Q_1, Q_2, \ldots, Q_m$ so that the j interior diameters

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_jQ_{2j}}$$

are each crossed k times. Notice that $\overline{Q_iQ_{i+j}}$ is crossed by all $(m-2)^2$ edges of the form $\overline{P_aP_b}$, $\overline{Q_aQ_b}$, $\overline{P_aQ_b}$, and $\overline{Q_aP_b}$ for $a \in [i+1, i+j-1]$ and $b \in [i+j+1, i+2j-1]$. Now, we address the edges that are incident to at least one of P_i and P_{i+j}.

For all $i \in [m]$, we begin by perturbing $Q_1, Q_2, \ldots, Q_m$ so that Q_i lies in between $\overrightarrow{P_iP_{i+j}}$ and $\overrightarrow{P_{i-1}P_i}$. This ensures that the $m-1$ rays

$$\overrightarrow{P_iQ_{i+j-1}}, \overrightarrow{P_iP_{i+j-1}}, \overrightarrow{P_iQ_{i+j-2}}, \overrightarrow{P_iP_{i+j-2}}, \ldots, \overrightarrow{P_iQ_{i+1}}, \overrightarrow{P_iP_{i+1}}, \overrightarrow{P_{i-1}P_i} \tag{1}$$

appear in counterclockwise order with respect to P_i and that the $m-1$ rays

$$\overrightarrow{P_{i+j}Q_{i-1}}, \overrightarrow{P_{i+j}P_{i-1}}, \overrightarrow{P_{i+j}Q_{i-2}}, \overrightarrow{P_{i+j}P_{i-2}}, \ldots, \overrightarrow{P_{i+j}Q_{i+j+1}}, \overrightarrow{P_{i+j}P_{i+j+1}}, \overrightarrow{P_{i+j-1}P_{i+j}} \tag{2}$$

appear in counterclockwise order with respect to P_{i+j}. Moreover, because Q_i and Q_{i+j} lie on different sides of $\overline{P_iP_{i+j}}$, it follows that $\overline{P_iP_{i+j}}$ also crosses $\overline{Q_iQ_{i+j}}$. Now, we proceed with a series of more precise secondary perturbations.

If $r \in [1, m-1]$, then we perturb Q_i so that it lies in between $\overrightarrow{P_iP_{i+j}}$ and $\overrightarrow{P_iQ_{i+j-1}}$. Additionally, we perturb Q_{i+j} so that it lies in between the $(r-1)^{\text{th}}$ and r^{th} rays of 2, where the 0^{th} ray is defined as $\overrightarrow{P_{i+j}P_i}$. This ensures that the set of edges incident to exactly one of P_i and P_{i+j} that cross $\overline{Q_iQ_{i+j}}$ are the edges corresponding to the first $r-1$ rays of 2. Thus, the total number of edges crossing $\overline{Q_iQ_{i+j}}$ is indeed

$$(m-2)^2 + 1 + (r-1) = (m-2)^2 + r = k.$$

Because these secondary perturbations of Q_i and Q_{i+j} do not alter the crossings of any other interior diameters, we can perturb Q_i and Q_{i+j} in this fashion for all $i \in [j]$ to transform

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_jQ_{2j}}$$

into edges with k crossings.

If $r \in [m, 2m-3]$, then we perturb Q_i so that it lies in between the $(r-m+1)^{\text{th}}$ and $(r-m+2)^{\text{th}}$ rays of 1. Additionally, we perturb Q_{i+j} so that it lies in between $\overrightarrow{P_{i+j}P_{i+j+1}}$ and $\overrightarrow{P_{i+j-1}P_{i+j}}$. This ensures that the set of edges incident to exactly one of P_i and P_{i+j} that cross $\overline{Q_iQ_{i+j}}$ are the edges corresponding to the first $r-m+1$ rays of 1 and the edges corresponding to the first $m-2$ rays of 2. Thus, the total number of edges crossing $\overline{Q_iQ_{i+j}}$ is indeed

$$(m-2)^2 + 1 + (r-m+1) + (m-2) = (m-2)^2 + r = k.$$

Once again, because these secondary perturbations of Q_i and Q_{i+j} do not alter the crossings of any other interior diameters, we can perturb Q_i and Q_{i+j} in this fashion for all $i \in [j]$ to transform

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_jQ_{2j}}$$

into edges with k crossings. Thus, we have covered $k \in \left((2j-2)^2, (2j-1)^2\right]$ where $j \in \mathbb{Z}^+$.

Case 3.2. If $m = 2j+1$ and $r \in [2, 2m-4]$, we once again place m vertices $P_1, P_2, \ldots, P_m$ on a minor circular arc in clockwise order such that $\widehat{P_iP_{i+1}}$ are pairwise congruent for $i \in [m-1]$ and construct m vertices $Q_1, Q_2, \ldots, Q_m$ such that Q_i is sufficiently close to P_i for $i \in [m]$. Henceforth, we also take all indices modulo m.

Our goal is for the j interior diameters

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_jQ_{2j}}$$

to each get crossed k times. Observe that $\overline{Q_iQ_{i+j}}$ is crossed by all $(m-1)(m-3)$ edges of the form $\overline{P_aP_b}$, $\overline{Q_aQ_b}$, $\overline{P_aQ_b}$, and $\overline{Q_aP_b}$ for $a \in [i+1, i+j-1]$ and $b \in [i+j+1, i+2j]$. Now, we address edges that are incident to at least one of P_i and P_{i+j}. For all $i \in [m]$, we begin by perturbing $Q_1, Q_2, \ldots, Q_m$ so that Q_i lies in between $\overrightarrow{P_iP_{i+j}}$ and $\overrightarrow{P_{i-1}P_i}$. This ensures that the $m-2$ rays

$$\overrightarrow{P_iQ_{i+j-1}}, \overrightarrow{P_iP_{i+j-1}}, \overrightarrow{P_iQ_{i+j-2}}, \overrightarrow{P_iP_{i+j-2}}, \ldots, \overrightarrow{P_iQ_{i+1}}, \overrightarrow{P_iP_{i+1}}, \overrightarrow{P_{i-1}P_i} \tag{3}$$

appear in counterclockwise order with respect to P_i and that the m rays

$$\overrightarrow{P_{i+j}Q_{i-1}}, \overrightarrow{P_{i+j}P_{i-1}}, \overrightarrow{P_{i+j}Q_{i-2}}, \overrightarrow{P_{i+j}P_{i-2}}, \ldots, \overrightarrow{P_{i+j}Q_{i+j+1}}, \overrightarrow{P_{i+j}P_{i+j+1}}, \overrightarrow{P_{i+j-1}P_{i+j}} \tag{4}$$

appear in counterclockwise order with respect to P_{i+j}. Moreover, because Q_i and Q_{i+j} lie on different sides of $\overline{P_iP_{i+j}}$, it follows that $\overline{P_iP_{i+j}}$ also crosses $\overline{Q_iQ_{i+j}}$. Now, we proceed with a series of more precise secondary perturbations.

If $r \in [2, m-1]$, then we perturb Q_i so that it lies in between $\overrightarrow{P_iP_{i+j}}$ and $\overrightarrow{P_iQ_{i+j-1}}$. Additionally, we perturb Q_{i+j}[1] so that it lies in between the r^{th} and $(r+1)^{\text{th}}$ rays of 4. This ensures that the set of edges incident to exactly one of P_i and P_{i+j} that cross $\overline{Q_iQ_{i+j}}$ are the edges corresponding to the first r rays of 4. Thus, the total number of edges crossing $\overline{Q_iQ_{i+j}}$ is indeed

$$(m-1)(m-3) + 1 + r = (m-2)^2 + r = k.$$

Because these secondary perturbations of Q_i and Q_{i+j} do not alter the crossings of any other interior diameters, we can perturb Q_i and Q_{i+m} in this fashion for all $i \in [j]$ to transform

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_jQ_{2j}}$$

into edges with k crossings.

If $r \in [m, 2m-4]$, then we perturb Q_i so that it lies in between the $(r-m+1)^{\text{th}}$ and $(r-m+2)^{\text{th}}$ rays of 3. Additionally, we perturb Q_{i+j} so that it lies in between $\overrightarrow{P_{i+j}P_{i+j+1}}$ and $\overrightarrow{P_{i+j-1}P_{i+j}}$. This ensures that the set of edges incident to exactly one of P_i and P_{i+j} that cross $\overline{Q_iQ_{i+j}}$ are the edges corresponding to the first $(r-m+1)$ rays of 3 and the edges corresponding to the first $m-1$ rays of 4. Thus, the total number of edges crossing $\overline{Q_iQ_{i+j}}$ is indeed

$$(m-1)(m-3) + 1 + (r-m+1) + (m-1) = (m-2)^2 + r = k.$$

Once again, because these secondary perturbations of Q_i and Q_{i+j} do not alter the crossings of any other interior diameters, we can perturb Q_i and Q_{i+m} in this fashion for all $i \in [j]$ to transform

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_jQ_{2j}}$$

into edges with k crossings.

Case 3.3. If $m = 2j+1$ and $r = 1$, then we start with the previously presented construction consisting of $2m-2$ points for edges with $(m-3)^2 + (m-2)$ crossings[2] and turn it into a configuration consisting of $2m-1$ points that has $\Omega(m)$ edges with $(m-2)^2 + 1$ crossings.

For the sake of completeness, we reproduce a description of this previously presented construction. Place $m-1$ vertices $P_1, P_2, \ldots, P_{m-1}$ on a minor circular arc in clockwise order such that $\widehat{P_iP_{i+1}}$ are pairwise congruent for $i \in [m-2]$ and construct $m-1$ vertices

[1] The condition given in the initial perturbation implies that Q_{i+j} must lie in between $\overrightarrow{P_{i+j}P_{i-1}}$ and $\overrightarrow{P_{i+j-1}P_{i+j}}$, and this stipulation forces $r \geq 2$ to hold for this perturbation of Q_{i+j} to be valid.

[2] Because $m-1$ is even and $(m-3)^2 + (m-2) \in \left((m-3)^2, (m-2)^2\right]$ for $m \geq 3$, it follows that this construction was already covered in 3.1.

$Q_1, Q_2, \ldots, Q_{m-1}$ such that Q_i is sufficiently close to P_i for $i \in [m-1]$. Henceforth, take all indices modulo $m-1$. For all $i \in [j]$, we know Q_i lies in between $\overrightarrow{P_iP_{i+j}}$ and $\overrightarrow{P_iQ_{i+j-1}}$, while Q_{i+j} lies in between $\overrightarrow{P_{i+j}P_{i+j+1}}$ and $\overrightarrow{P_{i+j-1}P_{i+j}}$.

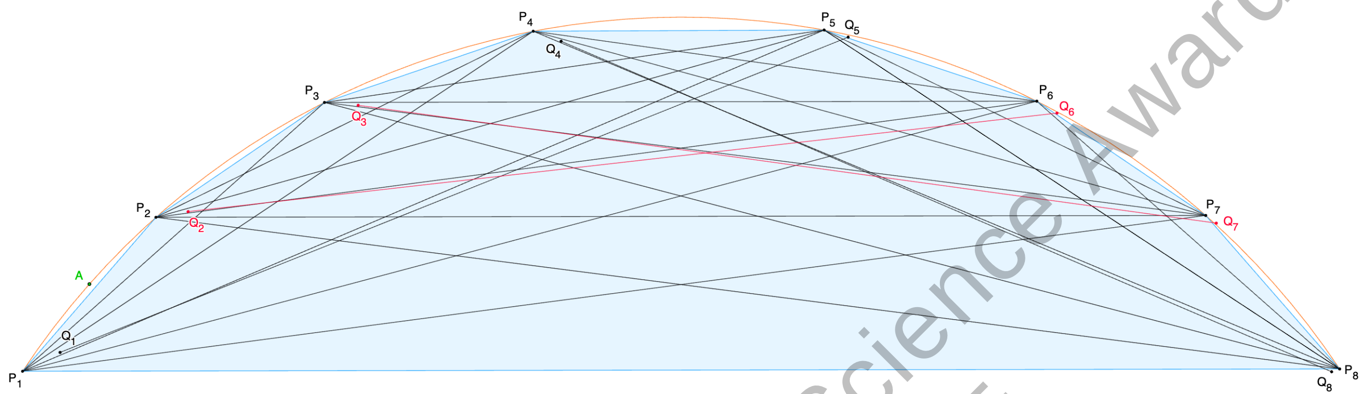

Figure 4: A configuration of 17 points for $m = 9$ and $k = 43$.

Now, we add the arc midpoint A of $\widehat{P_1P_2}$ and consider how many edges incident to A cross the $j-2$ interior diameters

$$\overline{Q_2Q_{j+2}}, \overline{Q_3Q_{j+3}}, \ldots, \overline{Q_{j-1}Q_{2j-1}}.$$

First of all, for each $i \in [2, j-1]$, the $m-2$ points on the opposite side of $\overleftrightarrow{Q_iQ_{i+j}}$ with respect to A are $P_{i+1}, Q_{i+1}, P_{i+2}, Q_{i+2}, \ldots, P_{i+j-1}, Q_{i+j-1}, P_{i+j}$, and the $m-3$ edges

$$\overline{AP_{i+1}}, \overline{AQ_{i+1}}, \overline{AP_{i+2}}, \overline{AQ_{i+2}}, \ldots, \overline{AP_{i+j-1}}, \overline{AQ_{i+j-1}}$$

clearly cross $\overline{Q_iQ_{i+j}}$. Now, we focus on $\overline{AP_{i+j}}$.

For any $i \in [2, j-1]$, it's easy to see A lies in between $\overrightarrow{P_{i+j}P_i}$ and $\overrightarrow{P_{i+j}P_{i+j+1}}$. Now, the characterizations of Q_i and Q_{i+j} given above imply that $\angle Q_iP_{i+j}Q_{i+j}$ fully contains $\angle P_iP_{i+j}P_{i+j+1}$, whence A lies in between $\overrightarrow{P_{i+j}Q_i}$ and $\overrightarrow{P_{i+j}Q_{i+j}}$. This means $\overline{AP_{i+j}}$ does indeed cross $\overline{Q_iQ_{i+j}}$, so the number of edges crossing each of the $j-2$ interior diameters

$$\overline{Q_2Q_{j+2}}, \overline{Q_3Q_{j+3}}, \ldots, \overline{Q_{j-1}Q_{2j-1}}$$

is precisely $(m-3)^2 + (m-2) + (m-2) = (m-2)^2 + 1$, as required.

Case 3.4. Lastly, we consider $r = 2m-3$ or $k = (m-1)^2$. In fact, we just place $2m$ vertices $P_1, P_2, \ldots, P_{2m}$ on a minor circular arc in clockwise order, as in this case

$$\overline{P_1P_{m+1}}, \overline{P_2P_{m+2}}, \ldots, \overline{P_mP_{2m}}$$

are all edges with $k = (m-1)^2$ crossings.

Thus, we have covered all $k \in \left((2j-1)^2, (2j)^2\right]$ where $j \in \mathbb{Z}^+$.

Now, we find working constructions for n odd and $k \in \left[1, \left(\frac{n-3}{2}\right)^2\right]$. Because $2m < n$ holds for k in this interval, we can utilize the appropriate constructions presented above to once again obtain configurations of $2m-1$ or $2m$ points containing $\Omega(m)$ edges with k crossings.

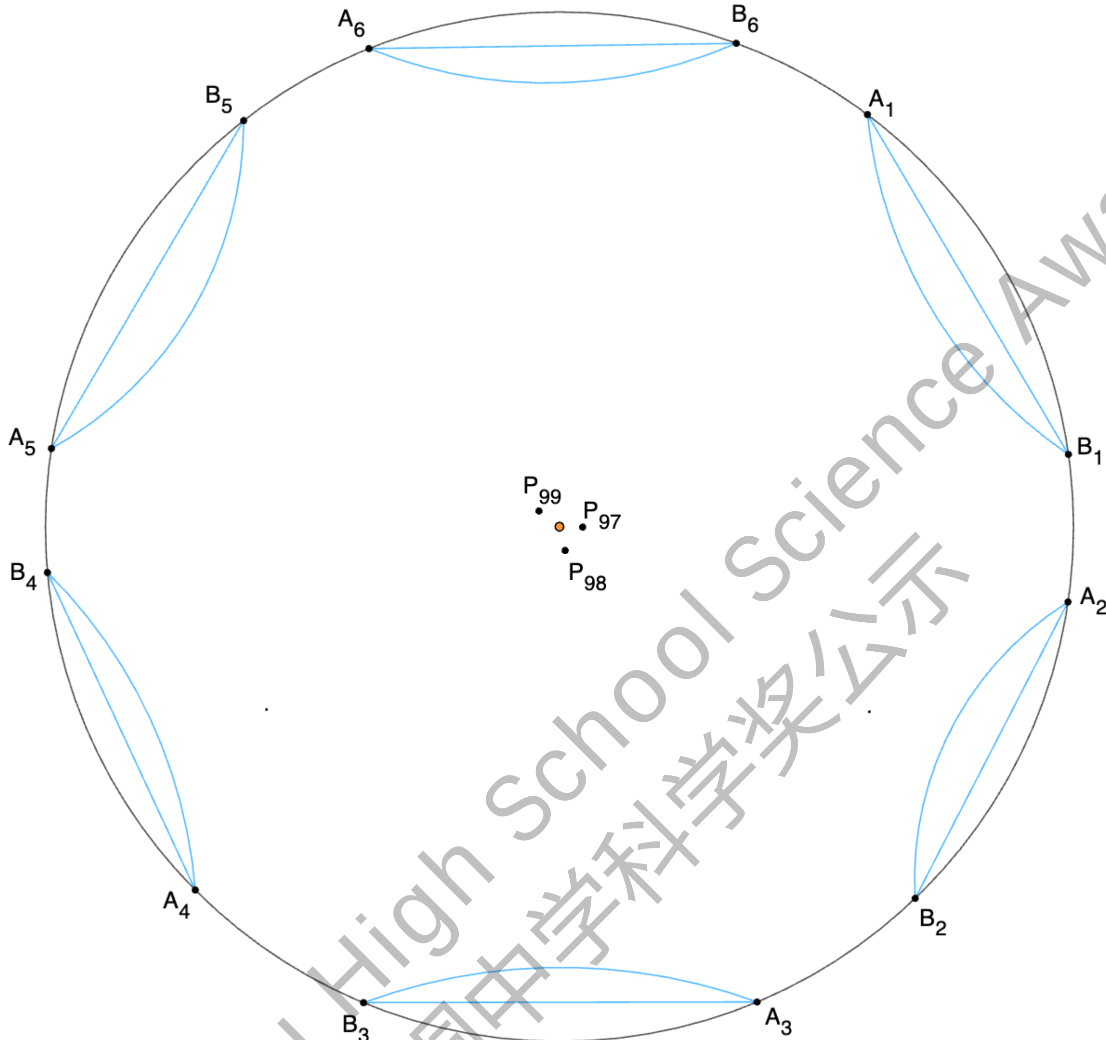

Figure 5: Grouping together 6 copies of a 16 point configuration from 3 for $n = 99$ total vertices.

For all n and k already addressed above, we now describe a method for placing $t = \left\lfloor \frac{n}{2m} \right\rfloor$ configurations of $2m-1$ or $2m$ points together so that no additional edges in such a $\overline{\mathcal{D}}_n$ interfere with any interior diameters of individual configurations. Place points $A_1, B_1, A_2, B_2, \ldots, A_t, B_t$ on a circle C in clockwise order[1]. For each $\overline{A_iB_i}$ where $i \in [t]$, we construct a minor circular arc c_i containing A_i and B_i that is fully contained inside C, as seen in 5, and create a configuration of $2m-1$ or $2m$ points using c_i as the minor circular arc. Finally, place any leftover vertices in a cluster near the center of C.

We can make each c_i sufficiently flat to ensure that no relevant interior diameters of the configuration based at c_i are intersected by additional edges of $\overline{\mathcal{D}}_n$. Thus, all of the relevant

[1] Note that none of these $2t$ points are vertices of $\overline{\mathcal{D}}_n$.

interior diameters of each configuration still have k crossings, meaning for this setup we have

$$e_k(\overline{\mathcal{D}}_n) \geq t \cdot \Omega(m) = \left\lfloor \frac{n}{2m} \right\rfloor \cdot \Omega(m) = \Omega(n),$$

as desired.

Lastly, we invent new constructions for n odd and $k \in \left(\left(\frac{n-3}{2}\right)^2, \left\lfloor \left(\frac{n-2}{2}\right)^2 \right\rfloor \right]$, as $2m \leq n$ does not hold in this case.

Case 3.5. If $\frac{n-3}{2}$ is odd, then set $m = \frac{n-1}{2}$ and write $k = (m-2)^2 + 2(m-1) + d$ where $d \in [0, m-2]$.

We utilize the previously presented construction from 3.1 consisting of $2m$ points for edges with $(m-2)^2 + (m-1) + d$ crossings, noting $(m-2)^2 + (m-1) + d \in \left((m-2)^2, (m-1)^2\right]$, and turn it into a configuration of $2m+1$ points that has $\Omega(m)$ edges with $k = (m-2)^2 + 2(m-1) + d$ crossings. Noting m is even, set $m = 2j$.

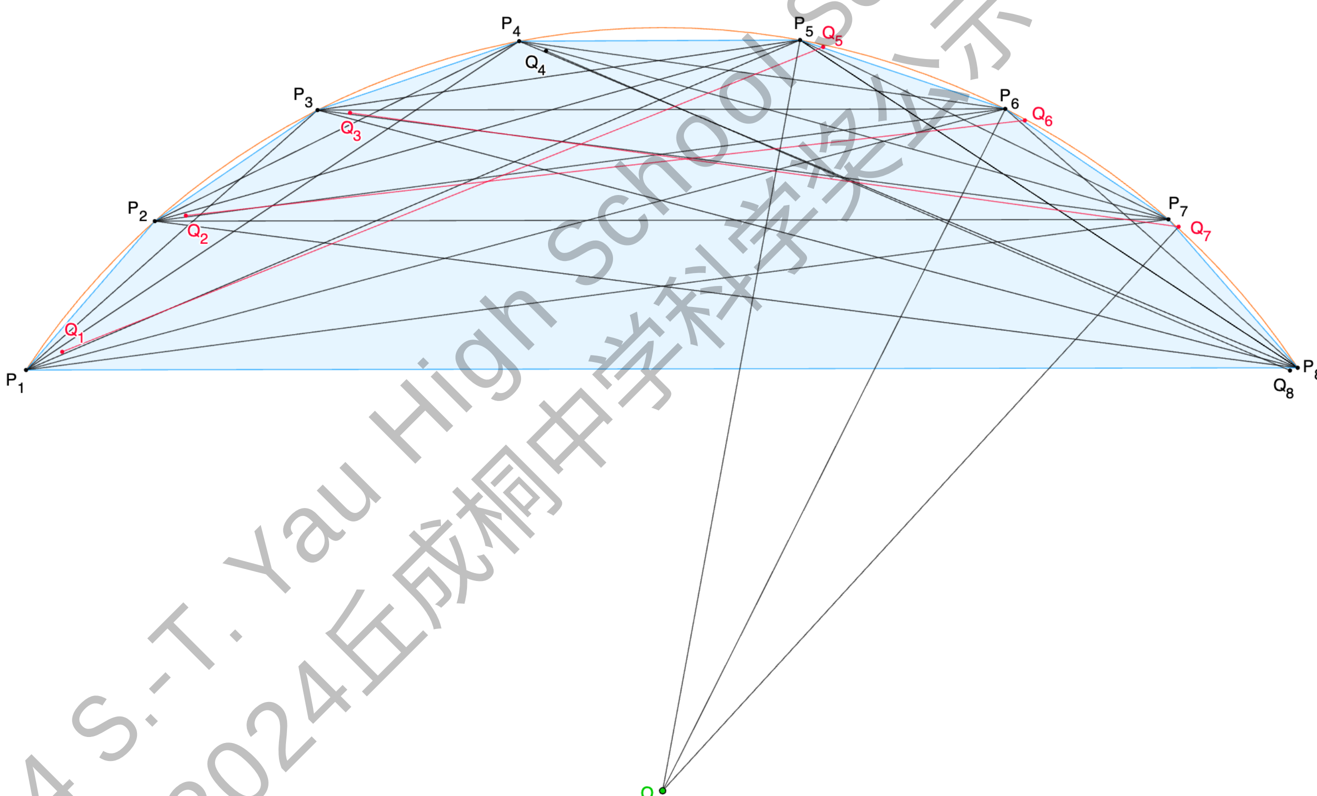

Figure 6: A configuration of 17 points for $m = 8$ and $k = 50$.

We add the center O of $\widehat{P_1 P_2 \dots P_m}$ and consider crossings between edges incident to O and the $j-1$ interior diameters

$$\overline{Q_1 Q_{j+1}}, \overline{Q_2 Q_{j+2}}, \dots, \overline{Q_{j-1} Q_{2j-1}}$$

from the previously presented construction. For each $i \in [j-1]$, we know the $m-1$ points on the opposite side of $\overleftrightarrow{Q_i Q_{i+j}}$ with respect to O are $P_{i+1}, Q_{i+1}, P_{i+2}, Q_{i+2}, \dots, P_{i+j-1}, Q_{i+j-1}, P_{i+j}$.

Furthermore, the $m-2$ edges

$$\overline{OP_{i+1}}, \overline{OQ_{i+1}}, \overline{OP_{i+2}}, \overline{OQ_{i+2}}, \ldots, \overline{OP_{i+j-1}}, \overline{OQ_{i+j-1}}$$

clearly cross $\overline{Q_iQ_{i+j}}$. Now, we focus on $\overline{OP_{i+j}}$.

For any $i \in [j-1]$, it's easy to see O lies in between $\overrightarrow{P_{i+j}P_i}$ and $\overrightarrow{P_{i+j}P_{i+j+1}}$ since $\angle P_iP_{i+j}P_{i+j+1}$ is obtuse. Now, the characterizations of Q_i and Q_{i+j} given in the previously presented construction imply that $\angle Q_iP_{i+j}Q_{i+j}$ fully contains $\angle P_iP_{i+j}P_{i+j+1}$, whence O lies in between $\overrightarrow{P_{i+j}Q_i}$ and $\overrightarrow{P_{i+j}Q_{i+j}}$. This means $\overline{OP_{i+j}}$ does indeed cross $\overline{Q_iQ_{i+j}}$, so the number of edges crossing each of the $j-1$ interior diameters

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_{j-1}Q_{2j-1}}$$

is precisely $\big((m-2)^2+(m-1)+d\big)+(m-1) = (m-2)^2+2(m-1)+d = k$, as required.

Case 3.6. If $\frac{n-3}{2}$ is even and $k \in \left[\left(\frac{n-3}{2}\right)^2+2, \left\lfloor\left(\frac{n-2}{2}\right)^2\right\rfloor\right]$, then set $m = \frac{n-1}{2}$ and write $k = (m-2)^2+(m-1)+m+d$ where $d \in [0, m-3]$.

We utilize a variation the previously presented construction from 3.2 consisting of $2m$ points for edges with $(m-2)^2+(m-1)+d$ crossings, noting $(m-2)^2+(m-1)+d \in \big((m-2)^2, (m-1)^2\big]$, and turn it into a configuration of $2m+1$ points that has $\Omega(m)$ edges with $k = (m-2)^2+(m-1)+m+d$ crossings. Noting m is odd, set $m = 2j+1$.

Place m vertices $P_1, P_2, \ldots, P_m$ on a minor circular arc in clockwise order such that $\widehat{P_iP_{i+1}}$ are pairwise congruent for $i \in [m-1]$ and construct m vertices $Q_1, Q_2, \ldots, Q_m$ such that Q_i is sufficiently close to P_i for $i \in [m]$. Henceforth, take all indices modulo m. For all $i \in [m]$, we perturb so that Q_i lies in between $\overrightarrow{P_iP_{i+j}}$ and $\overrightarrow{P_{i-1}P_i}$. Our goal is to ensure the $j-1$ interior diameters

$$\overline{Q_1Q_{j+2}}, \overline{Q_2Q_{j+3}}, \ldots, \overline{Q_{j-1}Q_{2j}}$$

each have k crossings.

For every $i \in [j+2, 2j]$, we perturb so that Q_i lies in between $\overrightarrow{P_iP_{i+1}}$ and $\overrightarrow{P_{i-1}P_i}$ and Q_{i+j} lies in between the $(d+2)^{\text{th}}$ and $(d+3)^{\text{th}}$ rays of 4. Using reasoning similar to that from 3.2, we can now deduce that the number of edges from the complete graph on the $2m$ existing vertices that cross each of the $j-1$ interior diameters

$$\overline{Q_1Q_{j+2}}, \overline{Q_2Q_{j+3}}, \ldots, \overline{Q_{j-1}Q_{2j}}$$

is precisely

$$(m-1)(m-3)+1+(m-3)+(d+2) = (m-2)^2+(m-1)+d.$$

Now, we add the center O of $\widehat{P_1P_2\ldots P_m}$ and consider crossings between edges incident to O and the $j-1$ interior diameters

$$\overline{Q_1Q_{j+2}}, \overline{Q_2Q_{j+3}}, \ldots, \overline{Q_{j-1}Q_{2j}}.$$

For every $i \in [j+2, 2j]$, we know $P_{i+j+1}, Q_{i+j+1}, P_{i+j+2}, Q_{i+j+2}, \ldots, P_{i-1}, Q_{i-1}, P_i$ are the m points on the opposite side of $\overleftrightarrow{Q_iQ_{i+j}}$ with respect to O. Furthermore, the $m-1$ edges

$$\overline{OP_{i+j+1}}, \overline{OQ_{i+j+1}}, \overline{OP_{i+j+2}}, \overline{OQ_{i+j+2}}, \ldots, \overline{OP_{i-1}}, \overline{OQ_{i-1}}$$

clearly cross $\overline{Q_iQ_{i+j}}$. Now, we focus on $\overline{OP_i}$.

For any $i \in [j+2, 2j]$, it's easy to see O lies in between $\overrightarrow{P_iP_{i+j}}$ and $\overrightarrow{P_iP_{i+1}}$ since $\angle P_{i+j}P_iP_{i+1}$ is obtuse. Now, the characterizations of Q_i and Q_{i+j} presented in this case imply that $\angle Q_{i+j}P_iQ_i$ fully contains $\angle P_{i+j}P_iP_{i+1}$, whence O lies in between $\overrightarrow{P_iQ_{i+j}}$ and $\overrightarrow{P_iQ_i}$. This means $\overline{OP_i}$ does indeed cross $\overline{Q_iQ_{i+j}}$, so the number of edges crossing each of the $j-1$ interior diameters

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_{j-1}Q_{2j-1}}$$

is precisely $\big((m-2)^2 + (m-1) + d\big) + m = k$, as required.

Case 3.7. If $\frac{n-3}{2}$ is even and $k = \left(\frac{n-3}{2}\right)^2 + 1$, then set $m = \frac{n-1}{2}$, which means $k = (m-1)^2 + 1$.

We take the previously presented construction from 3.2 consisting of $2m$ points for edges with $(m-2)^2 + m$ crossings and turn it into a configuration of $2m+1$ points that has $\Omega(m)$ edges with $k = (m-1)^2 + 1$ crossings. Noting m is odd, set $m = 2j + 1$.

We add the center O of $\widehat{P_1P_2 \ldots P_m}$ and consider crossings between edges incident to O and the j interior diameters

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_jQ_{2j}}$$

from the previously presented construction. For each $i \in [j]$, we know the $m-2$ points on the opposite side of $\overleftrightarrow{Q_iQ_{i+j}}$ with respect to O are $P_{i+1}, Q_{i+1}, P_{i+2}, Q_{i+2}, \ldots, P_{i+j-1}, Q_{i+j-1}, P_{i+j}$. Furthermore, the $m-3$ edges

$$\overline{OP_{i+1}}, \overline{OQ_{i+1}}, \overline{OP_{i+2}}, \overline{OQ_{i+2}}, \ldots, \overline{OP_{i+j-1}}, \overline{OQ_{i+j-1}}$$

clearly cross $\overline{Q_iQ_{i+j}}$. Now, we focus on $\overline{OP_{i+j}}$.

For any $i \in [j]$, it's easy to see O lies in between $\overrightarrow{P_{i+j}P_i}$ and $\overrightarrow{P_{i+j}P_{i+j+1}}$ since $\angle P_iP_{i+j}P_{i+j+1}$ is obtuse. Now, the characterizations of Q_i and Q_{i+j} given in the previously presented construction imply that $\angle Q_iP_{i+j}Q_{i+j}$ fully contains $\angle P_iP_{i+j}P_{i+j+1}$, whence O lies in between $\overrightarrow{P_{i+j}Q_i}$ and $\overrightarrow{P_{i+j}Q_{i+j}}$. This means $\overline{OP_{i+j}}$ does indeed cross $\overline{Q_iQ_{i+j}}$, so the number of edges crossing each of the j interior diameters

$$\overline{Q_1Q_{j+1}}, \overline{Q_2Q_{j+2}}, \ldots, \overline{Q_jQ_{2j}}$$

is precisely $\big((m-2)^2 + m\big) + (m-2) = (m-1)^2 + 1 = k$, as desired.

Thus, we have also covered $k \in \left(\left(\frac{n-3}{2}\right)^2, \left\lfloor\left(\frac{n-2}{2}\right)^2\right\rfloor\right]$ for n odd, exhausting all necessary cases, which finishes. □

3.2 Easy Upper Bound on $e_k(\mathcal{D}(G))$ for $k \geq 1$

Now, we present a standard proof of an upper bound on $e_k(\mathcal{D}(G))$ for $k \geq 1$.

Proposition 3.2 (Folklore)**.** *Let G be a graph with n vertices. For all $\mathcal{D}(G)$ and $k \geq 1$, we have $e_k(\mathcal{D}(G)) = O(n\sqrt{k})$.*

Although directly applying Proposition 2.8 to $\mathcal{D}(G)$ gives

$$e_k(\mathcal{D}(G)) \leq S_k(\mathcal{D}(G)) \leq n\sqrt{16.875k},$$

there exists a simpler proof of this result, albeit with a worse constant, based on the crossing lemma, which we share here.

Proof. Let G_k denote the subgraph of G consisting of all the edges in $\mathcal{D}(G)$ that are part of k crossings. We consider the drawing $\mathcal{D}(G_k)$ induced by $\mathcal{D}(G)$. If $e_k(\mathcal{D}(G_k)) < 4n$, then the desired bound follows immediately. Otherwise, we assume $e_k(\mathcal{D}(G_k)) \geq 4n$ and apply the crossing lemma on G_k. Because every edge of $\mathcal{D}(G_k)$ is crossed k times by other edges of $\mathcal{D}(G)$ and $\mathcal{D}(G_k) \subseteq \mathcal{D}(G)$, a simple double counting argument on the crossings of $\mathcal{D}(G_k)$ yields

$$\frac{k \cdot e_k(\mathcal{D}(G))}{2} \geq \mathrm{cr}(G_k) \geq \frac{e_k(\mathcal{D}(G))^3}{64n^2},$$

where the second inequality follows from the crossing lemma. This inequality chain implies $e_k(\mathcal{D}(G)) \leq 4n\sqrt{2k}$, which suffices. □

3.3 Discussion of Further Work on $\max e_k(\overline{\mathcal{D}}_n)$

At this moment, we have no further knowledge on the asymptotic behavior of $\max e_k(\overline{\mathcal{D}}_n)$, although we believe structural results concerning the vertices of $\overline{\mathcal{D}}_n$ under the condition that $e_k(\overline{\mathcal{D}}_n) = \Omega(n\sqrt{k})$ are within reach. In particular, under such a condition, the vertices of $\overline{\mathcal{D}}_n$ should in some sense trace out the border of a smooth convex shape.

4 Progress on $\max e_1(\overline{\mathcal{D}}_n)$, $\max S_1(\overline{\mathcal{D}}_n)$, and $\min e_k(\overline{\mathcal{D}}_n) = 0$

In this section, we present a constructive lower bound on $\max e_1(\overline{\mathcal{D}}_n)$, look at the range of $\max S_1(\overline{\mathcal{D}}_n)$, and show $\min e_k(\overline{\mathcal{D}}_n) = 0$ is true for nearly all k when n is sufficiently large.

4.1 Best Bounds on $\max e_1(\overline{\mathcal{D}}_n)$ and $\max S_1(\overline{\mathcal{D}}_n)$

Claim 4.1. *For $n \geq 8$, we have*

$$\max e_1(\overline{\mathcal{D}}_n) \geq \begin{cases} \frac{7}{5}(n-3), & n \equiv 3 \pmod 5; \\ \frac{7}{5}(n-4), & n \equiv 4 \pmod 5; \\ \frac{7}{5}(n-5), & n \equiv 0 \pmod 5; \\ \frac{7}{5}(n-6)+2, & n \equiv 1 \pmod 5; \\ \frac{7}{5}(n-7)+4, & n \equiv 2 \pmod 5. \end{cases}$$

Proof. We start by constructing for $n \equiv 3 \pmod 5$, setting $n = 5m + 3$.

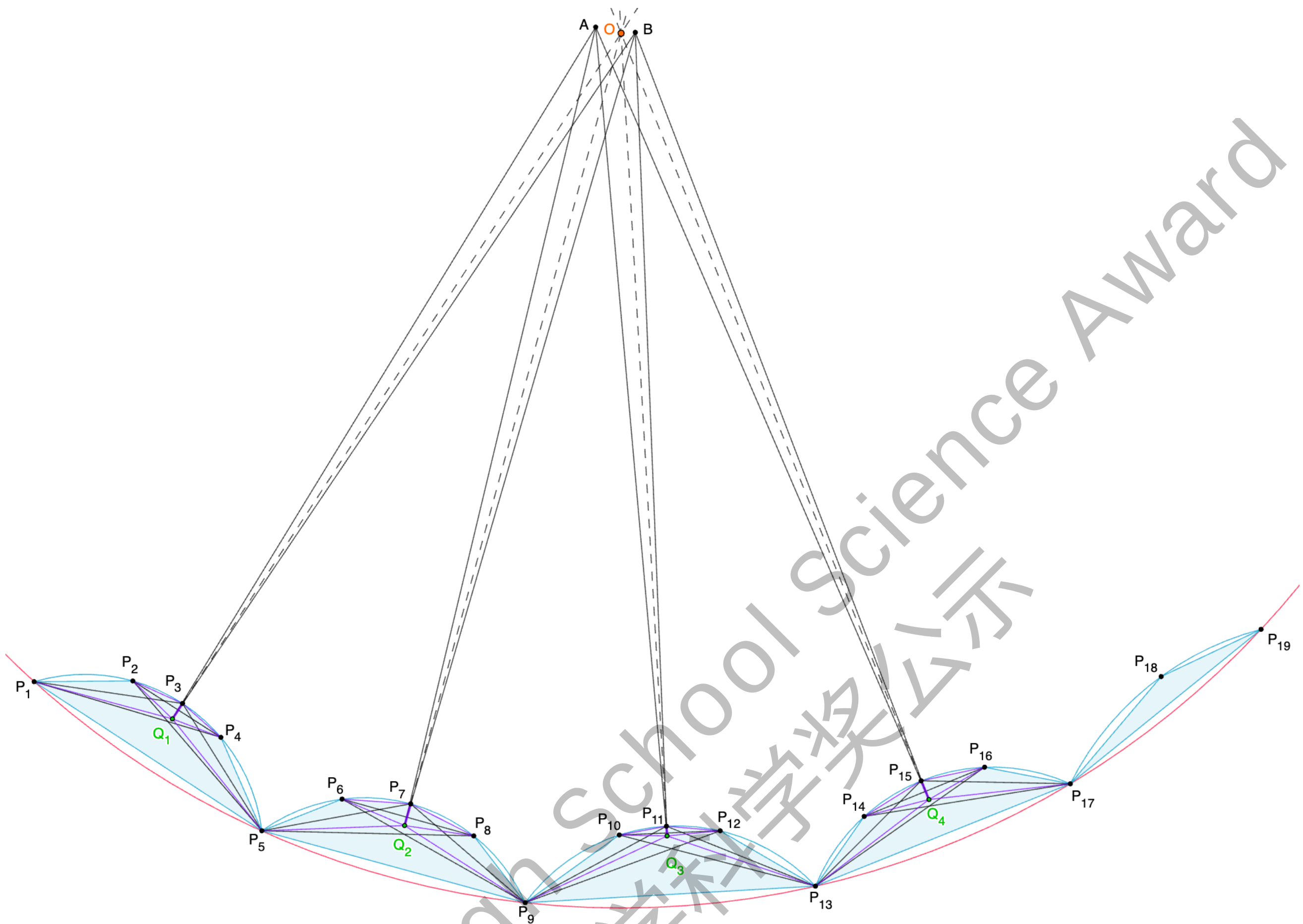

Figure 7: Full construction for $n = 25$ achieving $e_1(\overline{\mathcal{D}}_{25}) = 28$ with the edges of $E_1(\overline{\mathcal{D}}_n)$ marked in purple.

Place points $P_1, P_5, \ldots, P_{4m+1}$ in that order along a large circular arc with center O so that $\overline{P_{4i-3}P_{4i+1}}$ are pairwise congruent for $i \in [m]$ and sufficiently short. Now, we construct congruent inward-facing circular arcs C_i for every $i \in [m]$, each fully contained inside the original circular arc, so that $P_{4i-3}, P_{4i-2}, P_{4i-1}, P_{4i}, P_{4i+1}$ appear in that order on C_i. Moreover, we set P_{4i-1} as the arc midpoint of C_i and place P_{4i-2} and P_{4i} sufficiently close to P_{4i-1}. For every $i \in [m]$, we also add a point Q_i on the perpendicular bisector l_i of $\overline{P_{4i-3}P_{4i+1}}$ and in the interior pentagon formed by the diagonals of pentagon $P_{4i-3}P_{4i-2}P_{4i-1}P_{4i}P_{4i+1}$. Lastly, we consider the region in $\mathbb{R}^2$ that is to the left of all m perpendicular bisectors $l_1, l_2, \ldots, l_m$ and let A be the image of a sufficiently small perturbation of O into this region. Point B is defined analogously for the region that is to the right of all m perpendicular bisectors. The $5m+3$ vertices of $\overline{\mathcal{D}}_n$ consists of all the aforementioned points except for O.

If we make each C_i sufficiently flat, then no edge that spans between two points lying on different inward-facing circular arcs enters the interior of any pentagon. Moreover, no edge between two points not on or inside C_i can enter the interior of pentagon $P_{4i-3}P_{4i-2}P_{4i-1}P_{4i}P_{4i+1}$. This implies that the only edges that can enter $P_{4i-3}P_{4i-2}P_{4i-1}P_{4i}P_{4i+1}$ are incident to Q_i. Thus, the only edge that crosses $\overline{QP_{4i-3}}$ is $\overline{P_{4i+1}P_{4i-2}}$ and similarly for

$\overline{QP_{4i-2}}, \overline{QP_{4i-1}}, \overline{QP_{4i}}, \overline{QP_{4i+1}}$, which means these 5 edges all belong to $E_1(\overline{\mathcal{D}}_n)$.

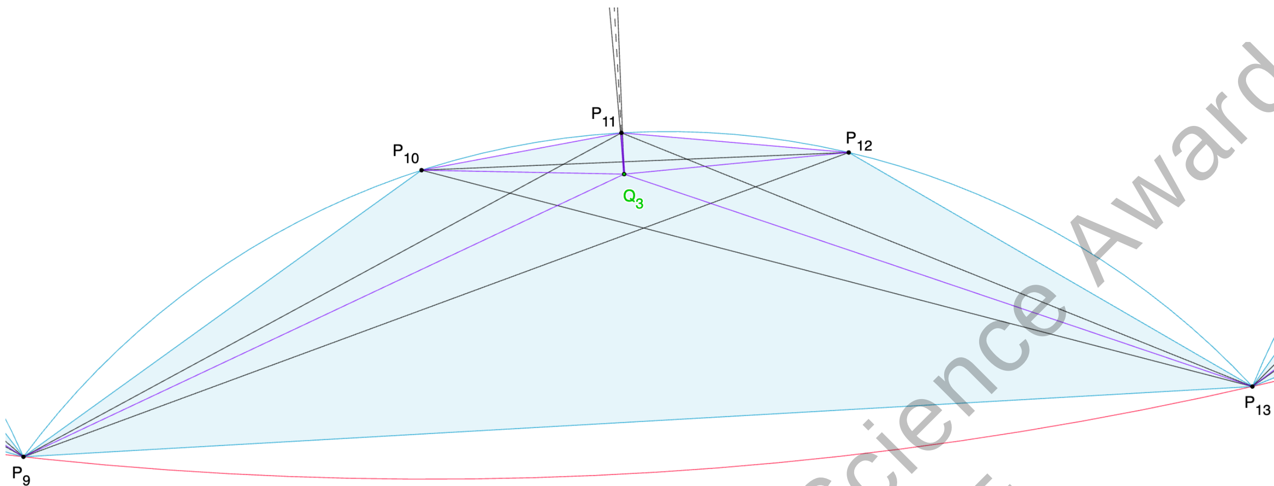

Figure 8: Zoomed-in view of an inward-facing arc with edges of $E_1(\overline{\mathcal{D}}_n)$ marked in purple.

Now, because P_{4i-2} and P_{4i} are sufficiently close to the arc midpoint of C_i in P_{4i-1}, we can assume that all edges spanning from a point on or inside C_i to a point on or inside another inward-facing circular arc either intersects no sides of $P_{4i-3}P_{4i-2}P_{4i-1}P_{4i}P_{4i+1}$, $\overline{P_{4i-3}P_{4i-2}}$, or $\overline{P_{4i}P_{4i+1}}$. However, $\overline{AQ_i}$ clearly intersects $\overline{P_{4i-2}P_{4i-1}}$ and $\overline{BQ_i}$ clearly intersects $\overline{P_{4i-1}P_{4i}}$, which means $\overline{P_{4i-2}P_{4i-1}}$ and $\overline{P_{4i-1}P_{4i}}$ both belong to $E_1(\overline{\mathcal{D}}_n)$ as well. Thus, we have $e_1(\overline{\mathcal{D}}_n) = 7m$ for this case, as desired.

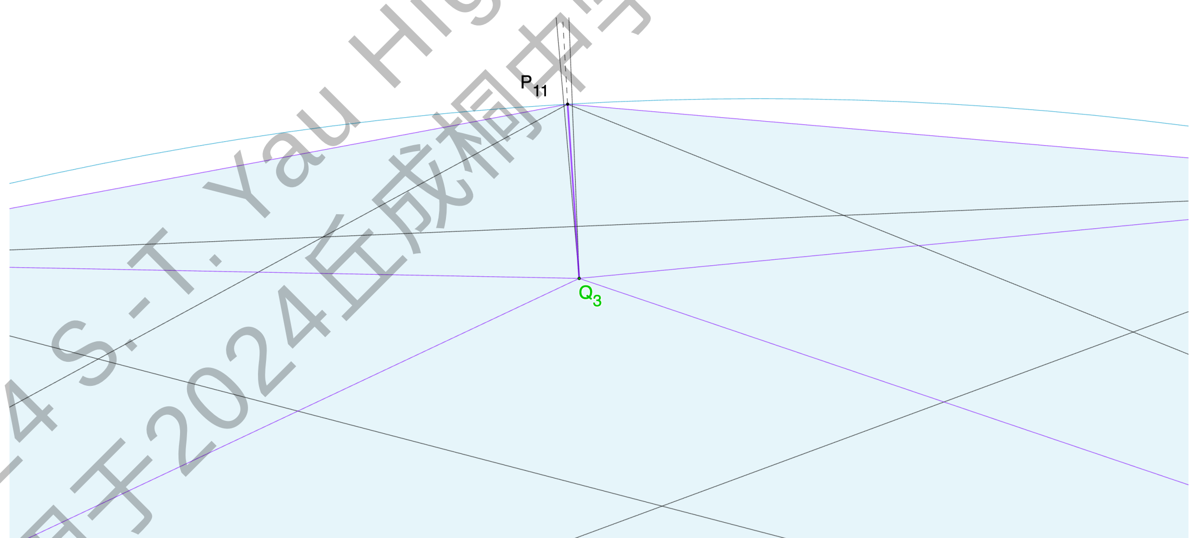

Figure 9: Highly zoomed-in view of an inward-facing arc with edges of $E_1(\overline{\mathcal{D}}_n)$ marked in purple.

When $n \in [5m+4, 5m+7]$, we add another (possibly degenerate) congruent inward-facing circular arc C_{m+1} to the left of C_m that is also anchored on the large circular arc. For

$n = 5m+6$, we obtain $e_1(\overline{\mathcal{D}}_n) = 7m+2$ because convex quadrilateral $P_{4m+1}P_{4m+2}P_{4m+3}P_{4m+4}$ contributes two diagonals with exactly one crossing, respectively. For $n = 5m + 7$, we can place $P_{4m+1}, P_{4m+2}, P_{4m+3}, P_{4m+4}$ on C_{m+1} so that P_{4m+3} is the arc midpoint of C_{m+1}, P_{4m+2} is sufficiently close to P_{4m+3}, and add another point Q_{m+1} that lies on the perpendicular bisector l_{m+1} of $\overline{P_{4m+1}P_{4m+4}}$ and inside the triangle formed by $\overline{P_{4m+1}P_{4m+3}}$, $\overline{P_{4m+2}P_{4m+4}}$, and $\overline{P_{4m+3}P_{4m+4}}$.

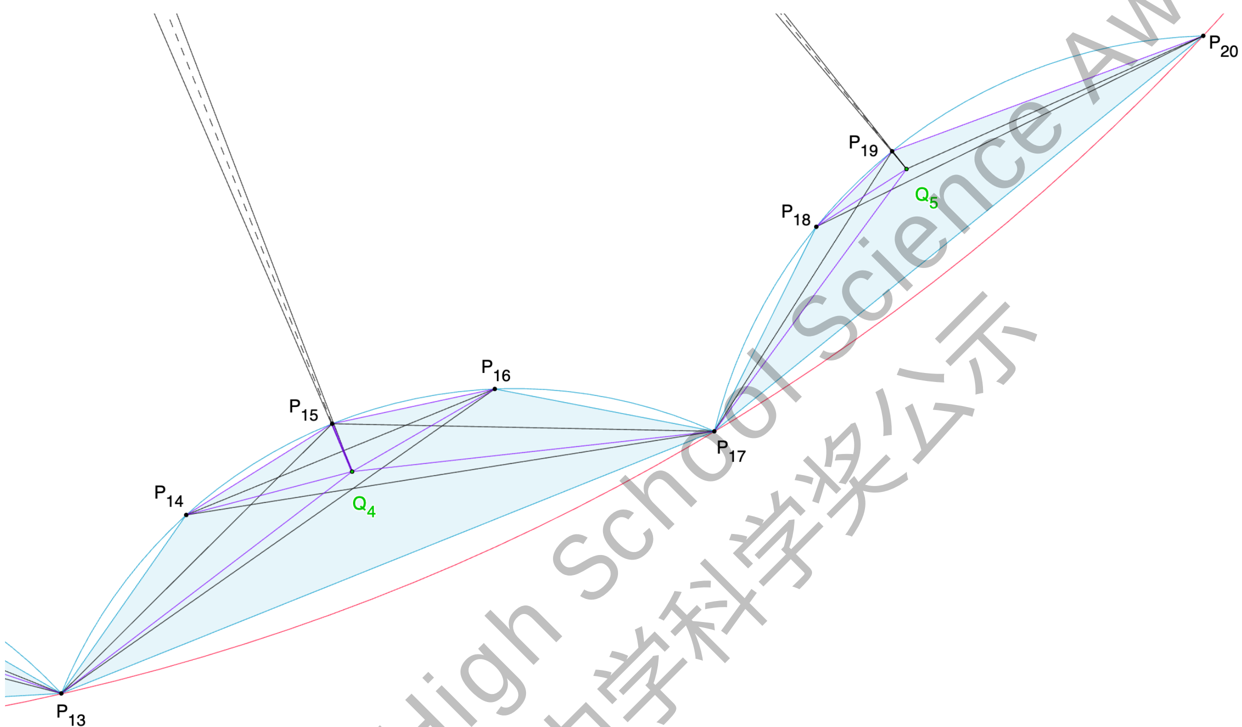

Figure 10: Zoomed-in view of C_5 in the construction for $n = 27$ achieving $e_1(\overline{\mathcal{D}}_{25}) = 28$ with edges of $E_1(\overline{\mathcal{D}}_n)$ marked in purple.

Using reasoning analogous to that from the $n = 5m + 3$ case, one can deduce that $\overline{Q_{m+1}P_{4m+1}}$, $\overline{Q_{m+1}P_{4m+2}}$, $\overline{P_{4m+2}P_{4m+3}}$, and $\overline{P_{4m+3}P_{4m+4}}$ all belong to $E_1(\overline{\mathcal{D}}_n)$, whence $e_1(\overline{\mathcal{D}}_n) = 7m + 4$, as desired. □

Claim 4.2. *For $n \geq 8$, we have*

$$2n + \left\lfloor \frac{n-2}{2} \right\rfloor - 2 \leq \max S_1(\overline{\mathcal{D}}_n) \leq 2n + \left\lfloor \frac{n-1}{2} \right\rfloor + 7.$$

Proof. First of all, observe

$$\max S_1(\overline{\mathcal{D}}_n) \leq \max S_1(\mathcal{D}(K_n)) \leq 2n + \left\lfloor \frac{n-1}{2} \right\rfloor + 7.$$

where we directly apply Theorem 2.6 to obtain the second inequality. Now, we provide constructions that achieve $S_1(\overline{\mathcal{D}}_n) = 2n + \left\lfloor \frac{n-2}{2} \right\rfloor - 2$. If $n = 2m$, then we take the construction for $m+1$ points depicted in 1 and add points $Q_1, Q_2, \ldots, Q_{m-1}$ so that $\overline{AQ_i}$ and $\overline{P_iP_{i+1}}$ cross each other and Q_i is sufficiently close to $\overline{P_iP_{i+1}}$ for $i \in [m-1]$.

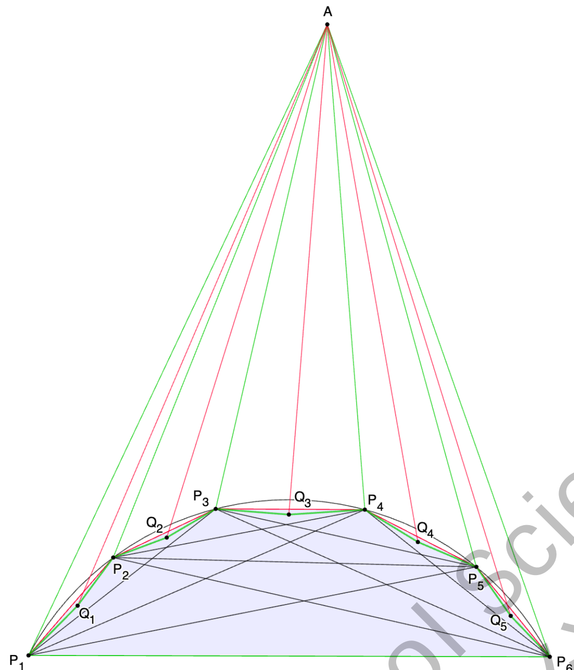

Figure 11: Construction achieving $S_1(\overline{\mathcal{D}}_{12}) = 27$.

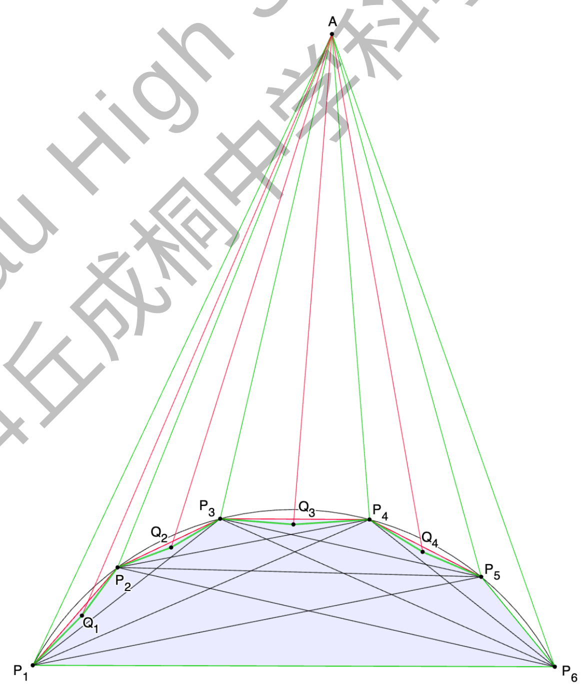

Figure 12: Construction achieving $S_1(\overline{\mathcal{D}}_{11}) = 24$.

If $n = 2m - 1$, then we just alter the aforementioned construction for $n = 2m$ by deleting Q_{m-1}. See the transformation between 11 and 12 for an example of this alteration. $\square$

Note that our bounds on $\max S_1(\overline{\mathcal{D}}_n)$ are nearly identical to those for $\max S_1(\mathcal{D}(K_n))$ stated in Theorem 2.6.

Fully pinpointing the values of $e_1(\overline{\mathcal{D}}_n)$ and $S_1(\overline{\mathcal{D}}_n)$ remains an open problem. In particular, there is no known upper bound on $e_1(\overline{\mathcal{D}}_n)$ other than $e_1(\overline{\mathcal{D}}_n) \leq 2n + \left\lfloor \frac{n-1}{2} \right\rfloor + 7$. However, because this bound is obtained through a long inequality chain, we naturally suspect that it is not tight.

4.2 Achieving $e_k(\overline{\mathcal{D}}_n) = 0$ for $k \geq 1$

Now, we present an approach for achieving $e_k(\overline{\mathcal{D}}_n) = 0$ when n is sufficiently large and $k \notin (n, n^{1+\epsilon})$ that is based on two circular arcs forming a complete bipartite graph.

Lemma 4.3. *For any $\epsilon > 0$, there exists some $N_\epsilon \in \mathbb{Z}^+$ such that for any $n \geq N_\epsilon$, we have $\min e_k(\overline{\mathcal{D}}_n) = 0$ for $k \notin (n, n^{1+\epsilon})$.*

Proof. Clearly, we can assume $\epsilon << 1$.

First, we deal with with $k \in [n]$. Let $P_1, P_2, \ldots, P_n$ denote the n vertices of $\overline{\mathcal{D}}_n$. If $P_1 P_2 \ldots P_n$ is a convex polygon, then $e_k(\overline{\mathcal{D}}_n) = 0$ holds for every $k \geq 1$ other than $k \in T_n$ where

$$T_n = \left\{ a((n-2) - a) \mid a \in \left[1, \left\lfloor \frac{n-2}{2} \right\rfloor\right] \cap \mathbb{Z}^+ \right\}.$$

When $n \geq 9$, $n - 3$ is the only element of T_n less than or equal to n. But one can check via simple bounding that the construction demonstrated in 1 achieves $e_{n-3}(\overline{\mathcal{D}}_n) = 0$ for $n \geq 11$, whence $\min e_k(\overline{\mathcal{D}}_n) = 0$ when $k \in [n]$ and $n \geq 11$.

To address $k \geq n^{1+\epsilon}$, we utilize a result from analytic number theory.

Theorem 4.4 ([26])**.** *For any $\epsilon > 0$, there exists some $r_\epsilon \in \mathbb{Z}^+$ such that all $n \geq r_\epsilon$ satisfy $\tau(n) < n^\epsilon$.*

If $\lfloor n^\epsilon \rfloor = n^{\epsilon_1}$, then we set $N_\epsilon = \max(r_{\epsilon_1}, 11)$. Consider two circular arcs C_1 and C_2 containing m and $n - m$ points, respectively, that are facing each other.

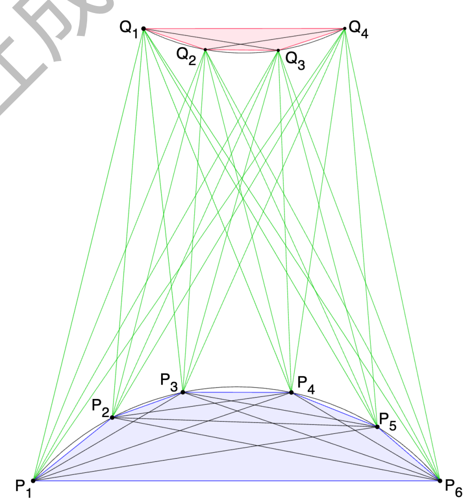

Figure 13: Example of C_1 and C_2 for $n = 10$ and $m = 4$.

Clearly, edges between two vertices of C_1 are only crossed by other edges connecting two vertices of C_1, and similarly for C_2. This means the number of times an edge between two vertices of C_1 is crossed must belong to T_m, and similarly for C_2 and T_{n-m}.

Now, consider the set of arrangements given by $m = 1, 2, \ldots, \lfloor n^\epsilon \rfloor$. Because $k > r_{\epsilon_1}$, the characterization of T_{n-m} and Theorem 4.4 implies there exists at least one arrangement so that no edge between two vertices of C_2 has k crossings. Moreover, the number of times an edge between two vertices of C_1 gets crossed is at most

$$\left(\frac{m-2}{2}\right)^2 \leq \left(\frac{\lfloor n^\epsilon \rfloor - 2}{2}\right)^2 < n < k,$$

so no edge between two vertices of C_1 can have k crossings.

Moreover, edges connecting vertices from C_1 and C_2 only cross other edges between vertices from C_1 and C_2. This means the maximum number of crossings an edge spanning from C_1 to C_2 can have is

$$(m-1)(n-m-1) < mn \leq n^{1+\epsilon} \leq k,$$

whence no edge connecting vertices from C_1 and C_2 can have k crossings.

Thus, we have shown at least one of the $\lfloor n^\epsilon \rfloor$ arrangements contains no edges with k crossings, which suffices. □

We conjecture that our approach involving two circular arcs can be extended to also cover $k \in (n, n^{1+\epsilon})$ for sufficiently large n.

Conjecture 4.5. *For sufficiently large n and all $k \geq 1$, we have* $\min e_k(\overline{\mathcal{D}}_n) = 0$.

5 Tight Upper and Lower Bounds on $S_k(\overline{\mathcal{D}}_n)$

We first prove $S_k(\overline{\mathcal{D}}_n) = O(n\sqrt{k})$ is tight for $k \geq 1$. Then, we demonstrate $S_k(\overline{\mathcal{D}}_n) = \Omega\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{k}{n}\right)\right)$ holds when $C_1 n \leq k = O(n^{\frac{3}{2}})$, where C_1 is an absolute constant characterized in the proof of Claim 5.4, and $S_k(\overline{\mathcal{D}}_n) = \Omega\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{n^2}{k}\right)\right)$ holds for $k = \Omega(n^{\frac{3}{2}})$. Finally, we show that both of these lower bounds are tight in Subsection 5.3.

5.1 Tight Upper Bound on $S_k(\overline{\mathcal{D}}_n)$

Theorem 5.1. *For $k \geq 1$, we have the asymptotically tight upper bound $S_k(\overline{\mathcal{D}}_n) = O(n\sqrt{k})$.*

Proof. Observe that applying Proposition 2.8 to $\overline{\mathcal{D}}_n$ immediately implies $S_k(\overline{\mathcal{D}}_n) = O(n\sqrt{k})$. Now, we formulate a construction for $\overline{\mathcal{D}}_n$ that shows this bound is asymptotically tight.

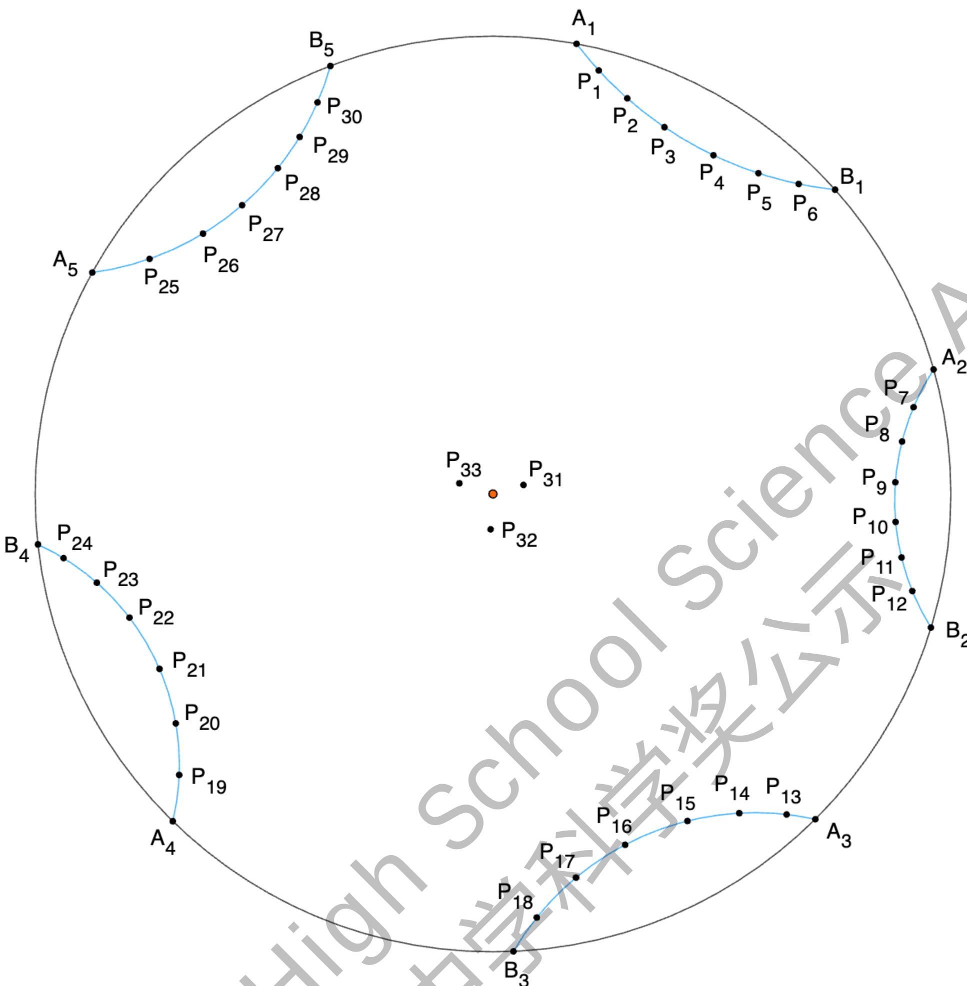

Figure 14: Construction attaining $S_k(\overline{\mathcal{D}}_n) = \Omega(n\sqrt{k})$ for $n = 33$ and $k \in [4, 8]$.

Let $m = 2\lfloor\sqrt{k}\rfloor + 2$ and $j = \lfloor \frac{n}{m} \rfloor$, noting that $k \le \left(\frac{n-2}{2}\right)^2$ ensures $m \le n$. We place points $A_1, B_1, A_2, B_2, \ldots, A_j, B_j$[1] in that order on a circle C so that $\overline{A_iB_i}$ are pairwise congruent for $i \in [j]$. For every $i \in [j]$, erect a minor circular arc c_i with endpoints A_i and B_i that is fully contained inside of C, and let R_i denote the finite region of $\mathbb{R}^2$ bounded by $\overline{A_iB_i}$ and c_i. Finally, we add m distinct vertices to each c_i and place any leftover vertices in a cluster near the center of C.

If each c_i is sufficiently flat, then no edge incident to a vertex that does not lie on c_i intersects the interior of R_i. Thus, the interior of R_i only consists of edges with both endpoints on c_i. Now, it is easy to see that any edge between two vertices of c_i is crossed at most

$$\left(\frac{m-2}{2}\right)^2 = \lfloor\sqrt{k}\rfloor^2 \le k$$

[1]Note that none of these $2j$ points are vertices of $\overline{\mathcal{D}}_n$.

times, whence there are at least

$$j\binom{m}{2} = \Omega(n\sqrt{k})$$

edges in $\overline{\mathcal{D}}_n$ that are crossed at most k times, as desired. □

5.2 Lower Bound on $S_k(\overline{\mathcal{D}}_n)$

Theorem 5.2. *For $C_1 n \leq k = O(n^{\frac{3}{2}})$, where C_1 is an absolute constant characterized in the proof of Claim 5.4, we have $S_k(\overline{\mathcal{D}}_n) = \Omega\left(\frac{k^2}{n^2}\log\left(\frac{k}{n}\right)\right)$. For $k = \Omega(n^{\frac{3}{2}})$, we have $S_k(\overline{\mathcal{D}}_n) = \Omega\left(\frac{k^2}{n^2}\log\left(\frac{n^2}{k}\right)\right)$.*

Before fully proving this theorem, we start by considering a subset of the n vertices of $\overline{\mathcal{D}}_n$ defined by a region of the plane bounded between two lines and show a lower bound on the number of edges between two vertices of the subset that belong to $\bigcup_{i=0}^{k} E_i(\overline{\mathcal{D}}_n)$.

Lemma 5.3. *Let l_1 and l_2 be two vertical parallel lines in $\mathbb{R}^2$ such that l_1 is to the left of l_2 and the open region in between l_1 and l_2 contains the set P of $m \in \left(\frac{2k}{n}, \min\left(\frac{1}{C_2}\left(\frac{k}{n}\right)^2, \frac{n}{2}\right)\right)$ vertices of $\overline{\mathcal{D}}_n$, where C_2 is an absolute constant specified in the proof of Claim 5.4. Suppose $a \leq m$ vertices of $\overline{\mathcal{D}}_n$ lie in the open half-plane R_1 on the left of l_1 and the remaining $n-(a+m)$ vertices of $\overline{\mathcal{D}}_n$ lie in the open half-plane R_2 on the right of l_2. Then there exists $\Omega\left(\left(\frac{k}{n}\right)^2\right)$ edges between two vertices of P that belong to $\bigcup_{i=0}^{k} E_i(\overline{\mathcal{D}}_n)$.*

Proof. We say an edge of $\overline{\mathcal{D}}_n$ is contained in a set of points V if both endpoints of the edge belong to V.

Let L denote the set of edges in $\overline{\mathcal{D}}_n$ that are either incident to a vertex of P or have an endpoint in R_1 and R_2. Observe that no edges between two vertices of R_1 or two vertices of R_2 can cross an edge contained in P. Hence, we only have to consider the edges in L when counting the number of edges contained in P that belong to $\bigcup_{i=0}^{k} E_i(\overline{\mathcal{D}}_n)$. Using this setup, we now prove a claim that is stronger than this lemma.

Claim 5.4. *Let L' be the set of lines in $\mathbb{R}^2$ formed by the extensions of the line segments in L. If Q denotes the set of edges contained in P that are intersected by at most k lines in L', then $|Q| = \Omega\left(\left(\frac{k}{n}\right)^2\right)$.*

Proof. We apply the cutting lemma on the lines of L' with parameter $t = \frac{|L'|}{k}$. First, we check that $t \in (1, |L'|)$ is satisfied. Notice

$$\begin{aligned}\left|L'\right| = m(n-m) + \binom{m}{2} + ab &\geq m(n-m) + \frac{m(m-1)}{2} \\ &= m\left(n - \frac{m+1}{2}\right) \\ &\geq m\left(n - \frac{\frac{n}{2}+1}{2}\right) \\ &\geq \frac{mn}{2},\end{aligned}$$

whence

$$t = \frac{|L'|}{k} \geq \frac{mn}{2k} > 1.$$

By setting $C_1 > 1$, we clearly have

$$t = \frac{|L'|}{k} \leq \frac{|L'|}{C_1 n} < |L'|,$$

which shows $t \in (1, |L'|)$ is satisfied under these conditions.

Let $T_1, T_2, \ldots, T_r$ denote the r generalized triangles formed by the cut. Observe that the interior of T_i is intersected by at most $\frac{|L'|}{t} = k$ lines of L'. Thus, if P_i is the set of points in P that belong closed finite region bounded by T_i, then any edge contained in P_i is intersected by at most k lines of L'. Now, applying Jensen's Inequality yields

$$|Q| \geq \sum_{i=1}^{r} \binom{|P_i|}{2} \geq r\binom{\frac{1}{r}\sum_{i=1}^{r}|P_i|}{2} \geq r\binom{\frac{m}{r}}{2} = \frac{m(\frac{m}{r}-1)}{2} \geq \frac{m^2}{4r}, \tag{5}$$

where the last inequality holds if and only if $\frac{m}{r} \geq 2$.

Now, we check that $\frac{m}{r} \geq 2$ always holds. First, compute

$$\begin{aligned}\left|L'\right| = m(n-m) + \binom{m}{2} + ab &\leq m(n-m) + \frac{m(m-1)}{2} + mb \\ &= m\left(n - \frac{m+1}{2} + b\right) \\ &\leq m(n+b) \\ &\leq 2mn.\end{aligned}$$

The cutting lemma implies that there exists an absolute constant C such that

$$r \leq Ct^2 = C\left(\frac{|L'|}{k}\right)^2 \leq C\left(\frac{2mn}{k}\right)^2 = 4Cm^2\left(\frac{n}{k}\right)^2 < 4Cm^2\left(\frac{1}{C_2 m}\right) = \frac{4Cm}{C_2},$$

so taking $C_2 = 8C$ guarantees $r \leq \frac{m}{2}$.

Now, we verify that the permitted range of m from the statement of Lemma 5.3 contains at least one positive integer by considering the differences between both possible supremums and the infimum of the given open interval. Setting $C_1 \geq 32C$, observe that

$$\frac{1}{C_2}\left(\frac{k}{n}\right)^2 - \frac{2k}{n} \geq \frac{C_1}{8C}\left(\frac{k}{n}\right) - \frac{2k}{n} \geq \frac{4k}{n} - \frac{2k}{n} > 2,$$

where the last inequality follows from $\frac{k}{n} \geq C_1 > 1$. Furthermore, we have

$$\frac{n}{2} - \frac{2k}{n} = \frac{n^2 - 4k}{2n} \geq \frac{n^2 - (n-2)^2}{2n} = \frac{2n-2}{n} \geq 1$$

for $n \geq 2$, which suffices.

Thus, setting $C_1 > \max(1, 32C)$ and $C_2 = 8C$ satisfies all the necessary conditions. To finish, we compute 5

$$|Q| \geq \frac{m^2}{4r} \geq \frac{m^2}{4Ct^2} = \frac{1}{4C}\left(\frac{mk}{|L'|}\right)^2 \geq \frac{1}{4C}\left(\frac{mk}{2mn}\right)^2 = \frac{1}{16C}\left(\frac{k}{n}\right)^2,$$

whence $|Q| = \Omega\left(\left(\frac{k}{n}\right)^2\right)$. □

Because an edge contained in P belongs to Q only if it belongs to $\bigcup_{i=0}^{k} E_i(\overline{\mathcal{D}}_n)$, Lemma 5.3 follows easily. □

Now, we can attack Theorem 5.2 by splitting the vertices of $\overline{\mathcal{D}}_n$ into groups and applying Lemma 5.3 to add up the number of edges contained within each group that belong to $\bigcup_{i=0}^{k} E_i(\overline{\mathcal{D}}_n)$.

Proof of Theorem 5.2. We start by partitioning the vertices of $\overline{\mathcal{D}}_n$ into different subsets with vertical parallel lines via a sweeping line process that starts on the left of all n vertices and progresses rightwards. Then, we obtain a lower bound on $S_k(\overline{\mathcal{D}}_n)$ by applying Lemma 5.3 on each of these subsets and summing across all subsets.

If $\min\left(\frac{1}{C_2}\left(\frac{k}{n}\right)^2, \frac{n}{2}\right) = \frac{n}{2}$, then $k = \Omega(n^{\frac{3}{2}})$. Set $r = \left\lfloor \log_2\left(\frac{n^2}{2k}\right)\right\rfloor$ and use $r+1$ vertical parallel lines to form r consecutive disjoint regions containing $\frac{2k}{n}, \frac{4k}{n}, \ldots, \frac{2^r k}{n}$ vertices respectively. Because each of these r regions fully contains $\Omega\left(\left(\frac{k}{n}\right)^2\right)$ edges in $\bigcup_{i=0}^{k} E_i(\overline{\mathcal{D}}_n)$ by Lemma 5.3, we have

$$S_k(\overline{\mathcal{D}}_n) \geq r \cdot \Omega\left(\left(\frac{k}{n}\right)^2\right) = \Omega\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{n^2}{k}\right)\right),$$

which finishes for $k = \Omega(n^{\frac{3}{2}})$.

If $\min\left(\frac{1}{C_2}\left(\frac{k}{n}\right)^2, \frac{n}{2}\right) = \frac{1}{C_2}\left(\frac{k}{n}\right)^2$, then $k = O(n^{\frac{3}{2}})$. Set $r = \left\lfloor \log_2\left(\frac{k}{C_2 n}\right)\right\rfloor$ and use $r+1$ vertical parallel lines to form r consecutive disjoint regions containing $\frac{2k}{n}, \frac{4k}{n}, \ldots, \frac{2^r k}{n}$ vertices respectively. Because each of these r regions fully contains $\Omega\left(\left(\frac{k}{n}\right)^2\right)$ edges in $\bigcup_{i=0}^{k} E_i(\overline{\mathcal{D}}_n)$ by

Lemma 5.3, we have

$$S_k(\overline{\mathcal{D}}_n) \geq r \cdot \Omega\left(\left(\frac{k}{n}\right)^2\right) = \Omega\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{k}{n}\right)\right),$$

which finishes for $C_1 n \leq k = O(n^{\frac{3}{2}})$, as desired. □

5.3 Tight Construction for Lower Bound on $S_k(\overline{\mathcal{D}}_n)$

We describe a construction for $k \geq C_1 n$ that shows both lower bounds on $S_k(\overline{\mathcal{D}}_n)$ given in Theorem 5.2 are tight.

Theorem 5.5. *For $C_1 n \leq k = O(n^{\frac{3}{2}})$, there exists a $\overline{\mathcal{D}}_n$ attaining $S_k(\overline{\mathcal{D}}_n) = O\left(\frac{k^2}{n^2}\log\left(\frac{k}{n}\right)\right)$. For $k = \Omega(n^{\frac{3}{2}})$, there exists a $\overline{\mathcal{D}}_n$ attaining $S_k(\overline{\mathcal{D}}_n) = O\left(\frac{k^2}{n^2}\log\left(\frac{n^2}{k}\right)\right)$.*

Proof. We will assume that n is a multiple of 3, as it is easy to extend our construction to the general case. Moreover, because the desired bound for this theorem is meaningless when $k = \Omega(n^2)$, we assume $k = o(n^2)$ for the remainder of this proof.

Let $m = \frac{n}{3}$. We place the vertices of $\overline{\mathcal{D}}_n$ in the plane so that they form m concentric equilateral triangles $T_1, T_2, \ldots, T_m$ with center O such that T_{i+1} is the image of T_i under a homothety at O with scale factor $-\epsilon$, where $\epsilon \in (0, 1)$ is sufficiently small.

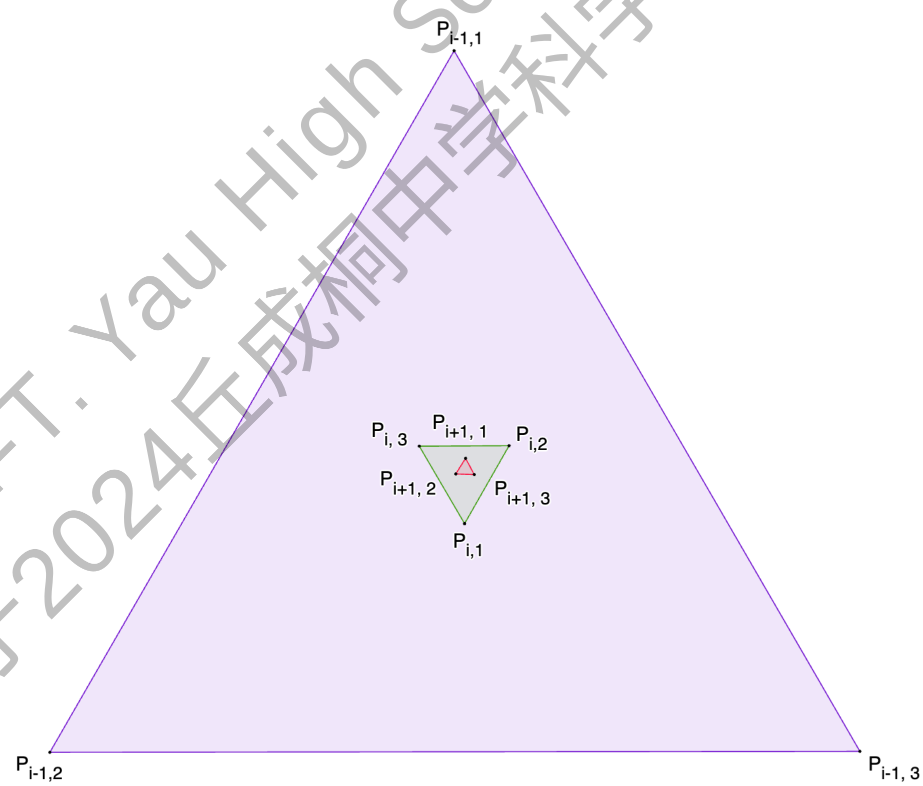

Figure 15: Three consecutive equilateral triangles T_{i-1}, T_i, and T_{i+1}.

Let the vertices of T_i be $P_{i,1}, P_{i,2}, P_{i,3}$ such that for $r \in \{1,2,3\}$, the points $P_{i,r}$ are all collinear[1] where $i \in [m]$. Without loss of generality, assume $\overline{P_{1,2}P_{1,3}}$ is horizontal.

Consider the edge e between P_{i,r_1} and P_{j,r_2} where $i < j$. For the sake of convenience, henceforth we write $a = i-1$, $b = j-i-1$ and $c = m-j$. If $j-i$ is odd, then we have the following:

- For any T_x and T_y where $x \in [i-1]$ and $y \in [i+1, j-1]$, there exists at least one edge spanning from a vertex of T_x to a vertex of T_y that crosses e;
- For any T_x and T_y where $x, y \in [i+1, j-1]$, there exists at least one edge spanning from a vertex of T_x to a vertex of T_y that crosses e;
- For any T_x and T_y where $x \in [i+1, j-1]$ and $y \in [j+1, m]$, there exists at least one edge spanning from a vertex of T_x to a vertex of T_y that crosses e;
- For any T_x and T_y where $x \in [i-1]$ and $y \in [j+1, m]$, there exists at least one edge spanning from a vertex of T_x to a vertex of T_y that crosses e.

In particular, taking ϵ small enough ensures these conditions hold.

When $j-i$ is even, the first three conditions stated above are always true, but the last condition does not necessarily hold when $r_1 = r_2$. Henceforth, we say that e is *bad* if and only if $j-i$ is even and $r_1 = r_2$. Otherwise, we say e is *good*.

Suppose that e is involved in at most k crossings. Because

$$\Omega(b(n-2)) = \Omega(ab) + \Omega(b^2) + \Omega(bc) \leq k$$

must hold regardless of whether e is good or bad, we have $b = O\left(\frac{k}{n}\right)$. But $k = o(n^2)$ gives $\frac{k}{n} = o(n)$, so $b = o(n)$ and thus $a + c = \Omega(n)$. Furthermore, if e is a good edge, then $\Omega(ac) \leq k$ also holds, which in conjunction with the previous asymptotic result implies $\min(a,c) = O\left(\frac{k}{n}\right)$.

Now, we upper bound the number of good e involved in at most k crossings. Without loss of generality, we can assume $\min(a,c) = a$ and multiply by 2 at the end. This assumption yields $a = O\left(\frac{k}{n}\right)$ and $b = O\left(\frac{k}{n}\right)$ holds unconditionally, so the number of good edges is at most

$$2(ab) = 2 \cdot O\left(\frac{k}{n}\right) \cdot O\left(\frac{k}{n}\right) = O\left(\left(\frac{k}{n}\right)^2\right).$$

Next, we upper bound the number of bad edges with at most k crossings. First, we perform an elaborate perturbation on the set of vertices of the form $P_{x,r}$ where $x \in [m]$ has fixed parity and $r \in \{1,2,3\}$ is fixed. Without loss of generality, assume x is odd and $r = 1$.

First, using the relation $b = O\left(\frac{k}{n}\right)$, we take some absolute constant C_3 such that $b \leq \frac{C_3 k}{n}$ holds unconditionally. We construct $q = \left\lceil \frac{8C_3 k}{n} \right\rceil$ disjoint clusters of vertical lines each containing at least $\frac{n^2}{C_3 k}$ vertical lines that are arbitrarily close together. We also ensure all vertical lines are sufficiently close to the vertical line $\overleftrightarrow{OP_{1,1}}$.

[1] Later on in this proof, we will perform an elaborate perturbation on the vertices of $\overline{\mathcal{D}}_n$ which guarantees the n vertices are in general position.

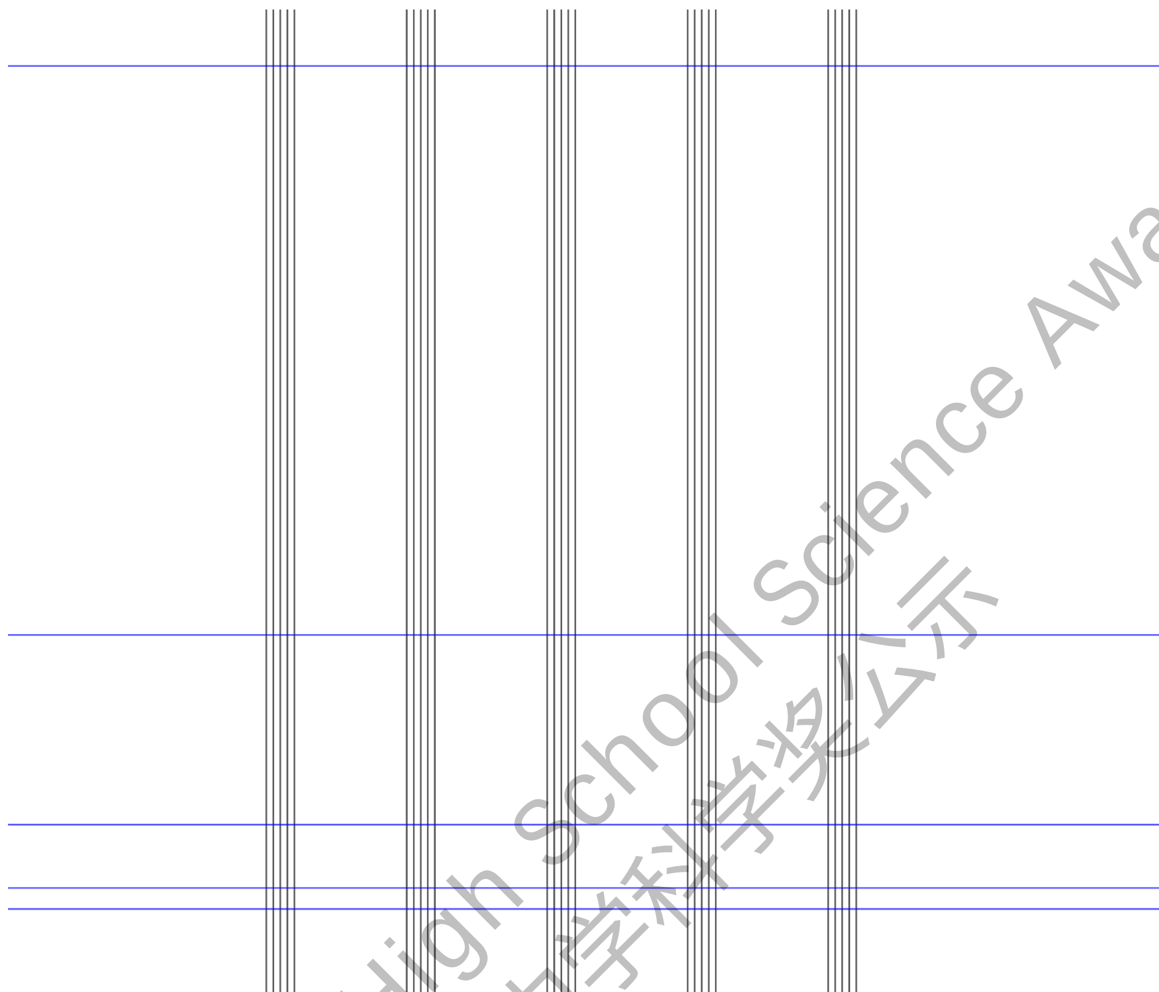

Figure 16: A perturbation setup with 5 disjoint clusters of vertical lines each containing 5 vertical lines. The vertical positioning of 5 consecutive vertices from the set of vertices involved in the perturbation is also shown.

Setting $t = \left\lceil \frac{m}{2} \right\rceil$, we consider the sequence of points $P_{1,1}, P_{3,1}, \ldots, P_{2t-1,1}$. We will not alter the vertical positioning of any of these points through perturbations. Instead, we will perturb them horizontally so that they lie on one of the vertical lines we have constructed. Beginning with $P_{1,1}$ and working our way inwards towards O, if p is the remainder of d modulo q, then we shift the d^{th} point of the sequence so that it lies on an unoccupied vertical line in the p^{th} cluster from the left.

Consider the set U of vertices in this grid-like shape that are below both $P_{2i-1,1}$ and $P_{2j-1,1}$ and also lie in between the vertical lines containing $P_{2i-1,1}$ and $P_{2j-1,1}$, respectively. Because ϵ is sufficiently small, we know the (vertical) distance between consecutive horizontal lines that determine the vertical position of (consecutive) vertices in the grid grows sufficiently fast as we move further out from O. This implies that every single edge between a point in U and a vertice above $P_{2i-1,1}$ crosses $\overline{P_{2i-1,1}P_{2j-1,1}}$. Now, we upper bound the number of edges $\overline{P_{2i-1,1}P_{2j-1,1}}$ that get crossed at most k times within this grid. Without loss of generality, assume $i < j$.

Case 5.1. If $i = \Omega(n)$, then we start by upper bounding the number of vertices in U. Clearly, $|U| \le \frac{k}{i-1}$ or $|U| = O\left(\frac{k}{n}\right)$ must hold.

For $y \in \mathbb{Z}^+$, define the y^{th} *layer* of the grid as the $(yq-(q-1))^{\text{th}}, (yq-(q-2))^{\text{th}}, \ldots, (yq)^{\text{th}}$ points of the aforementioned sequence. Now, recall that

$$j - i - 1 = b \le \frac{C_3 k}{n} \le \frac{q}{8}. \tag{6}$$

This implies that if $P_{2j-1,1}$ belongs to the y^{th} layer, then $P_{2i-1,1}$ belongs to the y^{th} or $(y-1)^{\text{th}}$ layer of the grid such that there are at least $j-i-1$ vertical clusters in between the vertical clusters that $P_{2i-1,1}$ and $P_{2j-1,1}$ are contained in. Thus, if there are x completely full layers below the layer that $P_{2j-1,1}$ belongs to, the inequality $|U| \ge x(j-i-1)$ holds, and combining this with $|U| = O\left(\frac{k}{n}\right)$ yields $j - i = O\left(\frac{k}{nx}\right)$.

This deduction yields two useful conclusions. Firstly, because there are $q = O\left(\frac{k}{n}\right)$ vertices in each layer, the number of edges $\overline{P_{2i-1,1}P_{2j-1,1}}$ with at most k crossings such that $P_{2j-1,1}$ belongs to the y^{th} layer is $O\left(\frac{k^2}{n^2 x}\right)$. Second of all, now we know

$$x = O\left(\frac{k}{n(j-i)}\right) \le O\left(\frac{k}{n}\right), \tag{7}$$

so the total number of layers in the grid is upper bounded by $O\left(\frac{k}{n}\right)$. On the other hand, the total number of layers is also upper bounded by

$$\frac{O(n)}{q} = \frac{O(n)}{\left\lceil \frac{8C_3 k}{n} \right\rceil} = O\left(\frac{n^2}{k}\right).$$

Thus, we have $x \le \min\left(O\left(\frac{k}{n}\right), O\left(\frac{n^2}{k}\right)\right)$, whence the total number of such edges $\overline{P_{2i-1,1}P_{2j-1,1}}$ with at most k crossings is

$$\sum_{y=1}^{\min\left(O\left(\frac{k}{n}\right), O\left(\frac{n^2}{k}\right)\right)} O\left(\frac{k^2}{n^2 x}\right) = \min\left(O\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{n^2}{k}\right)\right), O\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{k}{n}\right)\right)\right), \tag{8}$$

where the equality follows from evaluating a harmonic sum.

Case 5.2. If $i = o(n)$, then a similar argument allows us to conclude there are $O\left(\left(\frac{k}{n}\right)^2\right)$ such edges $\overline{P_{2i-1,1}P_{2j-1,1}}$ with at most k crossings.

It's clear that the highest order term out of

$$O\left(\left(\frac{k}{n}\right)^2\right), \min\left(O\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{n^2}{k}\right)\right), O\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{k}{n}\right)\right)\right)$$

is just $\min\left(O\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{n^2}{k}\right)\right), O\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{k}{n}\right)\right)\right)$. Moreover, because there are only n

edges in $\overline{\mathcal{D}}_n$ between vertices belonging to the same triangle, we know

$$S_k(\overline{\mathcal{D}}_n) = \min\left(O\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{n^2}{k}\right)\right), O\left(\left(\frac{k}{n}\right)^2 \log\left(\frac{k}{n}\right)\right)\right) \tag{9}$$

has been proven, which suffices. □

6 Main Takeaways and Future Directions

In Section 2, we coin the term crossing profile and draw attention to the importance of studying how crossings are distributed across the edges of a graph rather than the global trends of the crossing number. In Section 3, we present a construction that guarantees $\max e_k(\overline{\mathcal{D}}_n) = \Omega(n)$ for all values of k where $\max e_k(\overline{\mathcal{D}}_n) > 0$ is possible. In Section 4, we provide the first lower bound on $\max e_1(\overline{\mathcal{D}}_n)$ and prove $\min e_k(\overline{\mathcal{D}}_n) = 0$ holds for nearly all k when n is sufficiently large. In Section 5, we find tight upper and lower bounds for $S_k(\overline{\mathcal{D}}_n)$ when k is at least linear with respect to n.

In addition to pinpointing the precise value of $\max e_k(\overline{\mathcal{D}}_n)$ and resolving smaller problems such as achieving $e_k(\overline{\mathcal{D}}_n) = 0$ for all k when n is sufficiently large, we are also interested in a variant that was briefly considered during this project. This problem concerns intersections between edges of $\overline{\mathcal{D}}_n$ and lines through two vertices of $\overline{\mathcal{D}}_n$. This variant is closely related to both our original question about crossings between edges of $\overline{\mathcal{D}}_n$ and the well-studied k-sets problem [27, 28]. During our brief exploration of this variant, we discovered tight bounds on the number of edges that are crossed by at most k lines defined by two vertices of $\overline{\mathcal{D}}_n$, which is the analog of $S_k(\overline{\mathcal{D}}_n)$ and Section 5 for this variant.

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