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EINSTEIN METRIC ON 5-REGULAR GRAPH

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Abstract. We give the complete classification of finite simple 5-regular graphs with nonnegative constant Ricci curvature in the sense of Lin–Lu–Yau (LLY) and show the abundance of the graphs with negative constant curvature.

1. Introduction

Ricci curvature, defined by Gregorio Ricci-Curbastro, is a symmetric 2-tensor that measures how much a local region of space defined by a metric tensor curves compared to Euclidean space. Ricci curvature played a central role in Riemannian geometry, geometric analysis, and theoretical physics. For example, Einstein's theory of gravity relate the Ricci curvature of spacetime to its matter and energy content that can explain gravity as the curvature of spacetime caused by mass and energy. In the absence of matter and energy (vacuum), Einstein's field equations simplify, leading to the condition

$$\mathrm{Ric}(g) = \kappa g, \tag{1.1}$$

where κ is related to the cosmological constant. The metric that satisfies the equation 1.1 is called Einstein metric. Typical examples of Einstein manifolds would be Euclidean space $\mathbb{R}^n$, the standard "round" sphere $\mathbb{S}^n$, and the Kähler Einstein manifolds. In particular if $\kappa = 0$, compact Kähler manifolds with a Ricci-flat metric is called Calabi-Yau manifolds. The existence of such metrics, known as Calabi conjecture, was proved by Shing-Tung Yau in the celebrated work [11] and [12].

Given the importance of Ricci curvature in geometry, it is natural to extend the Ricci curvature to broader classes beyond Riemannian manifolds. In particular, people are interested in the discrete extension of Ricci curvature in the graph theory. The first definition of Ricci curvature on graphs was introduced by Chung and Yau in [2]. In [10], Ollivier introduced the notion of coarse Ricci curvature of Markov chains on arbitrary metric spaces, including graphs. Lin, Lu, and Yau [9] then introduced a modification of the Ollivier's Ricci curvature on graphs. Such curvature (written as LLY Ricci curvature for abbreviation) is very useful in studying the geometric and analytical aspects of graph theory.

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We are interested in the classification of graphs with constant Ricci curvature (in the LLY Ricci curvature sense). Such graphs are well understood in the Ricci flat case with degree $d \leq 4$ [1] and $d = 5$ regular graph with symmetry [7]. Cushing et al.[3] and Lin et al. [8] also classified Ricci-flat graphs with girth at least five. Hehl [5] and Huang et al [6] classified the graphs with Ricci curvature great than 1.

In this paper, we study the finite simple 5-regular graphs with constant LLY Ricci curvature. Our main theorem will be the following

Theorem 1.1. *Let $G = (V, E)$ be a simple finite 5-regular graph with constant LLY Ricci curvature κ. Then the following holds:*

(i) If $\kappa > 0$, then $G \cong K_5$ if $\kappa = \frac{6}{5}$; G is isomorphic to the icosahedral graph if $\kappa = \frac{4}{5}$; G is cubical (a Cartesian quotient of Q_5) if $\kappa = \frac{2}{5}$.

(ii) If $\kappa = 0$, then either $G \cong \mathrm{RF}^5_{72}$ or $G \cong H_3 \square K$ with H_3 be 3-regular Ricci-flat graph and K be 2-regular Ricci-flat graph.

(iii) Every 5-regular graph with girth at least 6 has negative constant curvature $\kappa = -\frac{6}{5}$.

Remark 1.2. *(ii)* in Theorem 1.1 confirms a conjecture in [7]. The classification for such graph with negative constant curvature is still widely open. It's interesting to find a "good" assumption that can give rise to a meaningful classification in the negative constant case.

2. Preliminaries

2.1. α-Ollivier's Ricci curvature and Lin-Lu-Yau Ricci curvature.

Definition 2.1 (Wasserstein distance)**.** Let $G = (V, E)$ be a locally finite graph, μ_1 and μ_2 be two probability measures on G. The Wasserstein distance $W_1(\mu_1, \mu_2)$ between μ_1 and μ_2 is defined as

$$W_1(\mu_1, \mu_2) = \inf_{\pi} \sum_{y \in V} \sum_{x \in V} d(x, y)\pi(x, y),$$

where $d(x, y)$ denotes the combined distance between x and y in G, and the infimum is taken over all maps $\pi : V \times V \to [0, 1]$ satisfying

$$\mu_1(x) = \sum_{y \in V} \pi(x, y),\ \mu_2(y) = \sum_{x \in V} \pi(x, y).$$

Such a map is called a transport plan.

For $\alpha \in [0, 1]$, set

$$\mu_x^\alpha(y) = \begin{cases} \alpha, & \text{if } y = x; \\ \frac{1-\alpha}{d_x}, & \text{if } y \sim x; \\ 0, & \text{otherwise,} \end{cases}$$

where $d_x := \sum_{y \in V : y \sim x} 1$ is the vertex degree of x.

Definition 2.2 (α-Ollivier Ricci curvature [10] and Lin-Lu-Yau curvature [9])**.** Let $G = (V, E)$ be a locally finite graph. For any vertices $x, y \in V$, the α-Ollivier curvature $\kappa_\alpha(x, y)$, $\alpha \in [0, 1]$, is defined as

$$\kappa_\alpha(x, y) = 1 - \frac{W_1(\mu_x^\alpha, \mu_y^\alpha)}{d(x, y)}.$$

The Lin-Lu-Yau Ricci curvature $\kappa(x, y)$ is defined as

$$\kappa(x, y) = \lim_{\alpha \to 1} \frac{\kappa_\alpha(x, y)}{1 - \alpha}.$$

Notice that $\kappa_1(x, y)$ is always 0. Obviously, the Lin-Lu-Yau curvature $\kappa(x, y)$ is the left derivative of $\kappa_\alpha(x, y)$ at $\alpha = 1$.

2.2. **LLY curvature for d-regular graph.**

Lemma 2.3. [4, Theorem 4.3] *Assume $G = (V, E)$ is d-regular and $x \sim y$. Write*

$$\Delta_{xy} = \Gamma(x) \cap \Gamma(y), \quad S_x = \Gamma(x) \setminus \{y\}, \quad S_y = \Gamma(y) \setminus \{x\},$$

and $S_i(x) = \{v \in V : d(x, v) = i\}$. Let $\mathcal{A}_{xy}$ be the set of bijections $\phi : S_x \to S_y$. For $\phi \in \mathcal{A}_{xy}$, define

$$|\square(\phi)| = \#\{z \in S_x : \ d(z, \phi(z)) = 1\}, \quad |\pentagon(\phi)| = \#\{z \in S_x : \ d(z, \phi(z)) = 2\}.$$

Then the LLY curvature κ satisfies

$$\kappa(x, y) = \frac{1}{d}\left(-2d + 4 + 3\,|\Delta_{xy}| + \max_{\phi \in \mathcal{A}_{xy}} \left(2\,|\square(\phi)| + |\pentagon(\phi)|\right) \right). \tag{2.1}$$

Corollary 2.4. [4, Corollary 4.4] *Let $G = (V, E)$ be a locally finite graph. Let $x, y \in V$ be of equal degree d with $x \sim y$. Then $\kappa(x, y) \in \mathbb{Z}/d$.*

For each edge e, fix an maximizing bijection ϕ_e^* (with the tie-break "maximize $|\square|$") and denote $s^*(e) := |\square\,(\phi_e^*)|\,, \quad p^*(e) := |\pentagon\,(\phi_e^*)|$.

We have the following matching lemma:

Lemma 2.5. *Let B_e be the bipartite "square graph" across $e = xy$, with left part N_x, right part S_y, and edges (u, v) iff $d(u, v) = 1$. Let $\nu\,(B_e)$ be its maximum matching size. Then*

$$s^*(e) = \nu\,(B_e)\,, \quad M(e) \le (4 - \alpha(e)) + \nu\,(B_e)\,. \tag{2.2}$$

Proof. Any bijection ϕ yields a matching of size $|\square(\phi)|$ in B_e, so $|\square(\phi)| \le \nu\,(B_e)$, whence $s^*(e) \le \nu\,(B_e)$. Conversely, any matching of size t can be extended to a bijection realizing exactly t distance-1 pairs; under the tie-break, $s^*(e) = \nu\,(B_e)$. For M, matched left vertices contribute at most 2 and unmatched at most 1 . $\square$

3. 5-Regular Graph with positive constant Ricci curvature

In this section, we are going to prove the first part of Theorem 1.1. By Corollary 2.4, we know that for those 5-Regular Graph with positive constant Ricci curvature, either the curvature $\kappa \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$ or $\kappa \geq 1$.

We first prove the following lemma.

Lemma 3.1. *All 5-Regular Graph with constant LLY Ricci curvature $\kappa \geq 1$ is isomorphic to K_5 with $\kappa = \frac{6}{5}$.*

Proof. From [5, Corollary 3.4], we know that either $G \cong K_5$ with $\kappa = \frac{6}{5}$ or G is isomorphic to a cocktail party graph with $\kappa = 1$. However the second case couldn't happen becuase the degree for all cocktail party graph is even, which implies the lemma. ☐

For the case when $\kappa = \frac{4}{5}$, we have the following

Lemma 3.2. *All 5-Regular Graph with constant LLY Ricci curvature $\kappa = \frac{4}{5}$ is isomorphic to the icosahedral graph.*

Proof. For the edge $e = xy$, we set

$$M(e) := \max_{\phi: S_x \to S_y} (2|\square(\phi)| + |\pentagon(\phi)|). \tag{3.1}$$

Then lemma 2.3 gives us (by letting $d = 5$)

$$\kappa(x,y) = \frac{1}{5}(-6 + 3\alpha(e) + M(e)). \tag{3.2}$$

We also have the following trivial bounds for $M(e)$:

$$0 \leq M(e) \leq 2(4 - \alpha(e)). \tag{3.3}$$

If $\kappa = \frac{4}{5}$ is constant for all edge e, we have for all edge e,

$$M(e) = 5\kappa + 6 - 3\alpha(e) = 10 - 3\alpha(e). \tag{3.4}$$

Combined with 3.3, we obtain

$$10 - 3\alpha \leq 8 - 2\alpha, \tag{3.5}$$

which implies that $\alpha \geq 2$. By the assumption $\kappa = \frac{4}{5}$ and 3.2, we have

$$3\alpha(e) + M(e) = 10 \tag{3.6}$$

which implies $\alpha(e) \leq 3$. Hence $\alpha \in \{2, 3\}$.

Assume, for contradiction, there is an edge $e = xy$ with $\alpha(e) = 3$. Then $|N_x| = 4 - \alpha = 1$. From 3.2, $M(e) = 10 - 3\alpha = 10 - 9 = 1$. Since M is the maximum over bijections, this forces the square-matching size $s^*(e)$ to be zero; otherwise $s^* \geq 1$ would imply $M \geq 2$, contradicting $M = 1$.

Let u be the unique vertex in N_x (the only neighbor of x not shared with y). The condition $s^*(e) = 0$ means u has no neighbor in $S_y = \Gamma(y) \setminus \{x\}$; in particular, u is not adjacent to any of the three common neighbors of x and y, nor to y itself. Consider the adjacent edge $f = xu$. The common neighbors of x and u lie in $\Gamma(x)\backslash\{u\} = \{y\} \cup (\Gamma(x) \cap \Gamma(y))$, but u is adjacent to none of these. Hence

$$\alpha(f) = |\Gamma(x) \cap \Gamma(u)| = 0,$$

contradiction! Therefore we then know that $\alpha \equiv 2$ in the whole graph. Plugging back gives $M = 4$ for all edges. Because $|S_x \backslash \Delta_{xy}| = 4 - \alpha = 2$ in degree 5, then $M = 4$ forces, for the maximizer ϕ_e^*,

$$2\left|\square\left(\phi_e^*\right)\right| + \left|\pentagon\left(\phi_e^*\right)\right| = 4 \Longrightarrow \left|\square\left(\phi_e^*\right)\right| = 2, \left|\pentagon\left(\phi_e^*\right)\right| = 0. \tag{3.7}$$

So, every edge lies in two (optimized) 4-cycles across it and no (optimized) 5-cycles across it.

Fix $x \in V$. For $y \in \Gamma(x)$, the integer $\alpha(xy)$ is precisely the degree of y in the induced graph $\Gamma(x)$ (counting edges yy' with $y' \in \Gamma(x)$); we just proved $\alpha(xy) = 2$ for every $y \in \Gamma(x)$. Hence, $\Gamma(x)$ is a 2-regular graph on 5 vertices., which implies $\Gamma(x) \cong C_5$.

Label the neighbors cyclically: $\Gamma(x) = \{u_1, u_2, u_3, u_4, u_5\}$ with edges $u_i \sim u_{i\pm1}$ (indices mod 5). In particular, for each i,

$$\Delta_{xu_i} = \{u_{i-1}, u_{i+1}\}, \quad S_x \backslash \Delta_{xu_i} = \{u_{i-2}, u_{i+2}\}. \tag{3.8}$$

Since $M(x, u_i) = 4$, the maximizer at xu_i must realize two length 1 matches; one convenient choice (and one that always exists because the cycle edges are present) is

$$u_{i-2} \mapsto u_{i-1}, \quad u_{i+2} \mapsto u_{i+1}, \tag{3.9}$$

giving the two (possibly chorded) 4-cycles

$$x - u_{i-2} - u_{i-1} - u_i - x, \quad x - u_{i+2} - u_{i+1} - u_i - x. \tag{3.10}$$

Now consider the cycle edge $u_i u_{i+1}$. Because $\alpha(u_i, u_{i+1}) = 2$, these two vertices have exactly two common neighbors. One is x; call the other v_i. Necessarily $v_i \notin \{x, u_1, \ldots, u_5\}$ because in the 5-cycle the only common neighbor (besides the endpoints) of u_i and u_{i+1} is x. Thus $v_i \in S_2(x)$. Each u_i has degree 5. Of these 5 neighbors, three lie in $\{x, u_{i-1}, u_{i+1}\}$, leaving two neighbors in $V\backslash(\{x\} \cup \Gamma(x))$.

Claim 3.2.1. *For each i, the two neighbors of u_i outside $\{x\} \cup \Gamma(x)$ are exactly v_{i-1} and v_i.*

Proof of the claim:

Vertex u_i has degree 5 , and we already know three of its neighbors:

$$x, \quad u_{i-1}, \quad u_{i+1}$$

Because $\alpha(u_{i-1}, u_i) = 2$, the edge $u_{i-1}u_i$ has exactly two common neighbors; one is x, and the other cannot lie in $\Gamma(x)$ (that would create an extra edge inside $\Gamma(x)$ and raise $\deg_{\Gamma(x)}$ above 2), so it must be an outside vertex adjacent to both u_{i-1} and u_i. By definition that outside vertex is v_{i-1}.

Similarly, $\alpha(u_i, u_{i+1}) = 2$ forces the outside neighbor v_i of u_i. Thus u_i already has five neighbors:

$$\{x, u_{i-1}, u_{i+1}, v_{i-1}, v_i\}.$$

This ends the proof of the claim.

Every vertex $w \in S_2(x)$ is at distance 2 from x, so there is a path $x-u-w$ with $u \in \Gamma(x)$. In particular, w is adjacent to some $u \in \Gamma(x)$. By the claim just proved, the neighbors of that u in $S_2(x)$ are exactly two vertices, namely v_* 's. Hence

$$w \in \{v_{i-1}, v_i\} \quad \text{for some } i$$

It follows that

$$S_2(x) \subseteq \{v_1, v_2, v_3, v_4, v_5\}.$$

Counting the edges between $S_1(x)$ and $S_2(x)$ in two ways gives

$$\sum_{u\in S_1(x)} \deg_{S_2}(u) = 5\cdot 2 = 10, \qquad \sum_{v\in S_2(x)} \deg_{S_1}(v) = \sum_{v\in S_2(x)} t(v), \tag{3.11}$$

where $t(v)$ is the number of neighbors of v in $S_1(x)$. Because $\alpha(u_i, u_{i+1}) = 2$ and x already accounts for one common neighbor, each cycle edge u_iu_{i+1} has a unique second common neighbor $v_i \in S_2(x)$, and any $v \in S_2(x)$ can be such a second common neighbor for at most one cycle edge (else some adjacent edge would acquire three common neighbors). Therefore $t(v) \leq 2$ for every $v \in S_2(x)$, with equality $t(v) = 2$ precisely when $v = v_i$ for some i. The last display forces

$$10 = \sum_{v\in S_2(x)} t(v) \leq 2\,|S_2(x)|, \tag{3.12}$$

hence $|S_2(x)| \geq 5$. Consequently, $v_1, \ldots, v_5$ in $S_2(x)$ is distinct and

$$S_2(x) = \{v_1, \ldots, v_5\} \text{ with each } v_i \text{ adjacent to exactly } u_i, u_{i+1} \text{ in } S_1(x). \tag{3.13}$$

Next we will show that, for each i, $v_i \sim v_{i-1}$ and $v_i \sim v_{i+1}$. Consider the edge v_iu_i. Because $\alpha(v_i, u_i) = 2$, their two common neighbors must be exactly $\{u_{i+1}, w\}$ for some w. Among the neighbors of u_i (namely $x, u_{i-1}, u_{i+1}, v_{i-1}, v_i$), the only vertex (other than u_{i+1}) that could also be adjacent to v_i is v_{i-1} (the other three are ruled out: $x \nsim v_i, u_{i-1} \nsim v_i$ by uniqueness of v_i related to u_iu_{i+1}, and v_i is not adjacent to itself). Hence $w = v_{i-1}$ and $v_i \sim v_{i-1}$. The argument for $v_i \sim v_{i+1}$ uses the edge v_iu_{i+1} and is symmetric.

Thus, each v_i has neighbors u_i, u_{i+1} in $S_1(x)$ and v_{i-1}, v_{i+1} in $S_2(x)$. Since the graph is 5-regular, v_i has exactly one further neighbor, which cannot lie in $S_0(x) \cup S_1(x) \cup S_2(x)$. Denote it by $x_i^\star \in S_3(x)$. Because $\Gamma(v_i)$ is 2-regular on its five neighbors (by $\alpha \equiv 2$ again), the cycle in $\Gamma(v_i)$ must connect $x_i^\star$ to both v_{i-1} and v_{i+1}. In particular, $x_i^\star$ is adjacent to v_{i-1} and to v_{i+1}.

Now we prove that actually $x_i^\star = x_{i+1}^\star$ for all i. Hence there is a single vertex $x^\star \in S_3(x)$ adjacent to all v_i's.

Consider the edge $v_i v_{i+1}$. Because $\alpha(v_i, v_{i+1}) = 2$, their two common neighbors must be exactly $\{u_{i+1}, z\}$ for some z. Now u_{i+1} is one; among the other candidates, by the previous argument, each of $x_i^\star$ and $x_{i+1}^\star$ is adjacent to both v_i and v_{i+1}. Uniqueness of the second common neighbor forces $x_i^\star = x_{i+1}^\star$. Inducting around the 5-cycle gives a single vertex $x^\star$ with the required property. Finally, since $x^\star$ already has degree 5 (neighbors $v_1, \ldots, v_5$), implies $S_k(x) = \varnothing$ for all $k \geq 4$.

Thus the breadth-first-search(BFS) layers from x are

$$|S_0| = 1, |S_1| = 5, |S_2| = 5, |S_3| = 1, \tag{3.14}$$

for a total of $1+5+5+1 = 12$ vertices. We now list the adjacency dictated by the construction:

- x is adjacent to $u_1, \ldots, u_5$.
- For each i, u_i is adjacent to
$$x, \quad u_{i-1}, u_{i+1}, \quad v_{i-1}, v_i$$
- For each i, v_i is adjacent to
$$u_i, u_{i+1}, \quad v_{i-1}, v_{i+1}, \quad x^\star$$
- $x^\star$ is adjacent to $v_1, \ldots, v_5$.

Every vertex has degree 5. Hence no extra edges can exist. Among the u_i 's there are only cycle edges $u_i \sim u_{i\pm1}$, otherwise $\deg_{\Gamma(x)}$ would exceed 2, contradicting $\alpha \equiv 2$. A v_i cannot be adjacent to any other u_j (besides u_i, u_{i+1}) by Lemma 2.1. A v_i is adjacent to $v_{i\pm1}$ and to $x^\star$, but not to other v_j's; otherwise $\Gamma(v_i)$ would not be a 5-cycle.

Up to relabeling indices, this is exactly the 12-vertex icosahedral graph (the 1-skeleton of the regular icosahedron):Two "poles" x and $x^\star$; two 5-cycles (u_i) and (v_i) forming the "belts"; and the zig-zag connections $u_i - v_i$ and $u_{i+1} - v_i$. The lemma then follows. □

Next, we consider the case when $\kappa = \frac{2}{5}$.

Lemma 3.3. *All 5-Regular Graph with constant LLY Ricci curvature $\kappa = \frac{2}{5}$ is cubical.*

Proof. When $\kappa = \frac{2}{5}$, by lemma 2.3, we have

$$M(e) = 5 \cdot \frac{2}{5} + 6 - 3\alpha(e) = 8 - 3\alpha(e). \tag{3.15}$$

This implies that $\alpha \in \{0, 1, 2\}$.

Assume there is an edge xy with $\alpha(xy) = 2$. Then x and y have two common neighbors; write

$$\Delta_{xy} = \{t, u\}, \quad \Gamma(x) = \{y, t, u, a, b\}. \tag{3.16}$$

Across xy, we have

$$\nu(B_{xy}) = s^*(xy) \in \{0, 1\}, \quad M(xy) = 2 \tag{3.17}$$

Set

$$R_{xt} := \Gamma(t)\backslash\{x\} = \{y, r_1, r_2, r_3\}, \quad R_{yt} := \Gamma(t)\backslash\{y\} = \{x, r_1, r_2, r_3\} \tag{3.18}$$

We now analyze the adjacent edges xt and yt. Let $L_{xt} := N_x^{(xt)}$ be the left side of B_{xt}. We claim

$$|\Gamma_{B_{xt}}(\{a,b\})| \geq 2 \tag{3.19}$$

Proof of the claim: Suppose, for contradiction, that $|\Gamma_{B_{xt}}(\{a,b\})| \leq 1$. Consider B_{xy}. Because $\nu(B_{xy}) \leq 1$, Hall's marriage lemma for $k=2$ implies

$$\left|\Gamma_{B_{xy}}(\{a,b\})\right| \leq 1 \tag{3.20}$$

Now look at B_{yt}. Since $\alpha(yt) \in \{1,2\}$, there are two subcases:

- If $\alpha(yt) = 1$: then by (T), $\nu(B_{yt}) = 2$. By (Hall's marriage lemma with $k=2$) there exists a 2-subset $L'' \subseteq L_{yt}$ with $\left|\Gamma_{B_{yt}}(L'')\right| \geq 2$. These two right vertices lie in $\{r_1, r_2, r_3\}$ (not at x), and they are distinct. Since $|\Gamma_{B_{xt}}(\{a,b\})| \leq 1$ and $\left|\Gamma_{B_{xy}}(\{a,b\})\right| \leq 1$, the two lefts $\{a,b\}$ see, across $B_{xt} \cup B_{xy}$, at most two right vertices in total; but B_{yt} needs two distinct rights in $\{r_1, r_2, r_3\}$ to be available for two (other) lefts. This forces one of $\{a,b\}$ to have no right neighbor in either B_{xt} or B_{xy}, contradicting $d=5$ and the definition of the square bipartites (every left has some right unless $\nu = 0$ and $|L| = 0$, which is not the case here). Formally, Hall for B_{xy} (with the singleton having empty neighborhood) would force $\nu(B_{xy}) = 0$, contradicting the possibility $\nu(B_{xy}) = 1$.
- If $\alpha(yt) = 2$: then by (T), $\nu(B_{yt}) \leq 1$, so

$$\left|\Gamma_{B_{yt}}(L_{yt})\right| \leq 1 \text{ or } \exists \ell \in L_{yt} : \left|\Gamma_{B_{yt}}(\{\ell\})\right| = 0. \tag{3.21}$$

In either alternative, the only way to serve both lefts $\{a,b\}$ somewhere around the triangle (without creating a 2 -match in B_{xy} or in B_{yt}, both forbidden is to have two distinct right neighbors for $\{a,b\}$ in B_{xt}; but this contradicts $|\Gamma_{B_{xt}}(\{a,b\})| \leq 1$. Thus the claim holds.

If $\alpha(yt) = 2$: then by (T), $\nu(B_{yt}) \leq 1$, so

$$\left|\Gamma_{B_{yt}}(L_{yt})\right| \leq 1 \text{ or } \exists \ell \in L_{yt} : \left|\Gamma_{B_{yt}}(\{\ell\})\right| = 0. \tag{3.22}$$

In either alternative, the only way to serve both lefts $\{a,b\}$ somewhere around the triangle (without creating a 2 -match in B_{xy} or in B_{yt}, both forbidden by (C0) and (C1.2)) is to have two distinct right neighbors for $\{a,b\}$ in B_{xt}; but this contradicts $|\Gamma_{B_{xt}}(\{a,b\})| \leq 1$. Consequently, by Hall's marriage lemma with $k=2$,

$$\nu(B_{xt}) \geq 2. \tag{3.23}$$

If $\alpha(xt) = 2$, then it allows only $\nu(B_{xt}) \in \{0,1\}$, contradicting 3.23. If $\alpha(xt) = 1$, then it forces $M(xt) = 5$ and $\nu(B_{xt}) = 2$. Now repeat the argument above with the third left in L_{xt} (here $|L_{xt}| = 3$ because $\alpha(xt) = 1 \Rightarrow \Delta_{xt} = \{y\}$), and apply Hall's marriage lemma with $k=3$: the same pigeonhole at t (only three vertices $\{r_1, r_2, r_3\}$ can serve as right neighbors for all three lefts, and two are already needed pairwise) yields

$$|\Gamma_{B_{xt}}(L_{xt})| \geq 3 \quad \Rightarrow \quad \nu(B_{xt}) \geq 3, \tag{3.24}$$

which contradicts $M(xt) = 5$ since $2\nu(B_{xt}) \leq M(xt)$ would give $6 \leq 5$. Therefore $\alpha(xy) = 2$ is impossible. Similarly, we can also exclude the case when $\alpha(e) = 1$ for some $e = xy$.

This implies that $\alpha(e) = 0$ for all edges e. Tracking back to 3.15, $M(e) = 8$ for all edges. Because $|S_x \backslash \Delta_{xy}| = 4$, the identity $M = 8$ forces, for the maximizer ϕ_e^*,

$$2|\square(\phi_e^*)| + |\pentagon(\phi_e^*)| = 8 \Longrightarrow |\square(\phi_e^*)| = 4, |\pentagon(\phi_e^*)| = 0. \tag{3.25}$$

Thus every edge xy admits a bijection $\phi_e^* : S_x \to S_y$ that matches each of the four left vertices to an adjacent right vertex. Equivalently, for every edge xy, the bipartite graph $B_{xy} \subseteq S_x \times S_y$ of cross-adjacencies is a perfect matching (a disjoint union of 4 edges). In particular every edge lies in four (optimized) squares across it and the graph is triangle-free.

Define a relation Θ on edges by:

$$e\Theta f \quad \Longleftrightarrow \quad e, f \text{ are opposite sides of a (simple) 4-cycle.} \tag{3.26}$$

Because around every edge the four 4-cycles come from the four disjoint pairs in B_{xy}, the relation Θ is everywhere defined. Let $[e]$ denote the Θ-equivalence class of e.

Claim 3.3.1. *At every vertex v, the five incident edges lie in five distinct Θ-classes.*

Proof of the claim:

Suppose two distinct incident edges vv_1, vv_2 belong to the same class. Then there is a chain of squares starting with vv_1 and ending with vv_2, each step flipping to the opposite edge. The first square already shows a 4-cycle using both vv_1 and vv_2 at the same vertex v, which is impossible because the cross-matching at any edge vw pairs v_1 with a unique neighbor in S_w (there is no second choice to create another square sharing the two sides at v). $\square$

Let the five Θ-classes be $\mathcal{C}_1, \ldots, \mathcal{C}_5$. For each i, the union of edges $\mathcal{C}_i$ is a spanning subgraph in which every vertex has degree exactly **1** (by Claim above). Thus each $\mathcal{C}_i$ is a disjoint union of cycles (a 1-factor).

Fix a basepoint $o \in V$. For each $i \in \{1, \ldots, 5\}$, the 1-factor $\mathcal{C}_i$ partitions V into two sides (the two color classes of the cycles): moving along an edge in $\mathcal{C}_i$ flips the side, and moving along any edge in $\mathcal{C}_j$ with $j \neq i$ preserves the side (because every 4-cycle alternates classes). Define the coordinate map

(3.27)

$$\Phi : V \longrightarrow \{0,1\}^5, \quad \Phi(v)_i = \begin{cases} 0, & \text{if } v \text{ is on the same side of } \mathcal{C}_i \text{ as } o \\ 1, & \text{otherwise} \end{cases}$$

This is well-defined (any closed walk crosses each $\mathcal{C}_i$ an even number of times, because every 4-cycle contains two edges from the same class) and satisfies:

(a) If $vw \in \mathcal{C}_i$, then $\Phi(w) = \Phi(v) \oplus e_i$ (flip the i-th bit; other bits unchanged).

(b) Conversely, since at v there is exactly one incident edge in $\mathcal{C}_i$, the five neighbors of v are mapped to the five Hamming neighbors $\Phi(v) \oplus e_i, i = 1, \dots, 5$.

Hence Φ is a local isomorphism from G onto a (not necessarily induced) 5-regular subgraph of the 5 -cube Q_5. Because G is connected, $\Phi(G)$ is a connected, 5-regular subgraph of Q_5; the only such subgraphs are quotients of Q_5 by free translation subgroups. Hence G is cubical. This ends the proof. □

Next, we prove the following

Lemma 3.4. *There is no 5-regular graph with constant Ricci curvature* $\kappa = \frac{3}{5}$.

Proof. Similar as the previous lemma, we have for all edges e,

$$M(e) = 9 - 3\alpha(e). \tag{3.28}$$

Since $0 \leq M(e) \leq 8 - 2\alpha(e)$ by (3.3), we then have $\alpha \in \{1, 2, 3\}$.

By solving the linear equation $2s^* + p^* = 9 - 3\alpha$ under the constraints $0 \leq s^*, p^* \leq 4 - \alpha$ with the tie-break "maximize s" gives

(3.29)

$\alpha(e)$	$\lvert S_x \backslash \Delta_{xy} \rvert$	$M(e)$	$s^*(e)$	$p^*(e)$
1	3	6	3	0
2	2	3	1	1
3	1	0	0	0

First, we prove that $\alpha(e)$ can not be 3. Suppose not. Let $e = xy$ with $\alpha(e) = 3$. Then $S_x \backslash \Delta_{xy}$ has exactly one vertex u. Then there is always a length-2 choice $\phi(u) = w$ with $w \in \Delta_{xy} \subset S_y$, so $p^*(e) \geq 1$, contradicting the $\alpha = 3$ row. Thus $\alpha \in \{1, 2\}$.

$$\alpha = 1 \Rightarrow (s^*, p^*) = (3, 0), \quad \alpha = 2 \Rightarrow (s^*, p^*) = (1, 1).$$

Fix a vertex x. In the induced graph $\Gamma(x)$ (5 vertices), the degree of a neighbor $y \in \Gamma(x)$ equals $\alpha(xy) \in \{1, 2\}$. Summing degrees in $\Gamma(x)$ gives an even number, hence the number of $\alpha = 1$ edges at x is even. On the other hand,

$$\sum_{e \ni x} p^*(e) \equiv \sum_{e \ni x} M(e) \equiv 5 \cdot 9 - 3 \sum_{e \ni x} \alpha(e) \equiv 1 \pmod 2, \tag{3.30}$$

and by the table 3.29, we have $p^*(e) = 1$ iff $\alpha(e) = 2$. Thus the number of $\alpha = 2$ edges at x is odd. Consequently, the degree multiset in $\Gamma(x)$ is one of:

$$(2, 2, 2, 2, 2), \quad (2, 2, 2, 1, 1), \quad \text{or} \quad (2, 1, 1, 1, 1).$$

The first two case are equivalent to $\Gamma(x) \cong C_5$ or $\Gamma(x) \cong P_5$. We will rule out all of them.

We first consider the C_5 case. Label $\Gamma(x)=\{u_1,\ldots,u_5\}$ with the 5-cycle edges $u_i \sim u_{i\pm1}$. For the edge $e_i := xu_i$ we have $\alpha(e_i)=2$, thus by the lemma 2.5), $\nu(B_{e_i}) = s^*(e_i) = 1$.

But in the square-bipartite B_{e_i} (left part $N_x=\{u_{i-2},u_{i+2}\}$, right part S_{u_i}), the two pairs (u_{i-2},u_{i-1}) and (u_{i+2},u_{i+1}) are disjoint length-1 edges (they live in the 4-cycles $x-u_{i-2}-u_{i-1}-u_i-x$ and $x-u_{i+2}-u_{i+1}-u_i-x$). Hence $\nu(B_{e_i}) \geq 2$, contradicting $\nu(B_{e_i})=1$. So the C_5 neighborhood cannot occur.

Next, we consider the P_5 case. Assume $\Gamma(x)$ is the disjoint union of a triangle $t_1-t_2-t_3-t_1$ (the three degree- 2 vertices) and an edge p_1p_2 (the two degree-1 vertices). Then

$$\alpha(xt_i)=2(i=1,2,3), \quad \alpha(xp_1)=\alpha(xp_2)=1.$$

Across xp_1 we must have $s^*=3$. The left set is $N_x^{(xp_1)}=\{t_1,t_2,t_3\}$, and the right set S_{p_1} equals $\{p_2\}$ together with three vertices outside $\{x\}\cup\Gamma(x)$. Since none of t_1,t_2,t_3 is adjacent to p_2 inside $\Gamma(x)$, the size-3 matching must use three distinct outside neighbors U_1,U_2,U_3 of p_1, with

$$t_i \sim U_i \sim p_1 \quad (i=1,2,3) \tag{3.31}$$

The same argument for xp_2 yields three distinct outside neighbors V_1,V_2,V_3 of p_2 with

$$t_i \sim V_i \sim p_2 \quad (i=1,2,3). \tag{3.32}$$

Since every t_i has degree 5 and already two neighbors inside $\Gamma(x)\cup x$, the (3.31) and (3.32) force

$$\Gamma(t_i)=\{x,t_{i-1},t_{i+1},U_i,V_i\}, \quad \text{(indices mod 3)}. \tag{3.33}$$

Across $e_i=xt_i$ we must have $(s^*,p^*)=(1,1)$. The non-shared left set for e_i is

$$N_x^{(e_i)}=\{p_1,p_2\} \tag{3.34}$$

and $S_{t_i}=\Gamma(t_i)\backslash\{x\}=\{t_{i-1},t_{i+1},U_i,V_i\}$ by (3.33). Since $p_j \nsim t_{i\pm1}$ inside $\Gamma(x)$, the only possible length-1 pairs in B_{e_i} are

$$(p_1,U_i) \text{ and } (p_2,V_i). \tag{3.35}$$

The two 3 -sets $\{U_1,U_2,U_3\}$ and $\{V_1,V_2,V_3\}$ can have at most one common element, since $\Gamma(p_1)\cap\Gamma(p_2)=\{x\}\cup(\{U_1,U_2,U_3\}\cap\{V_1,V_2,V_3\})$ and $|\Gamma(p_1)\cap\Gamma(p_2)|\in\{1,2\}$ by $\alpha\in\{1,2\}$. Consequently, there exists at least one index i with $U_i\neq V_i$. For that index, these two pairs are disjoint in B_{e_i}. Therefore

$$\nu(B_{e_i})\geq 2.$$

But for an $\alpha=2$ edge, $s^*(e_i)=\nu(B_{e_i})=1$. Contradiction!

Finally, we handle the case $(2,1,1,1,1)$. Assume $\Gamma(x)$ consists of a length-2 path $a-m-b$ (the unique degree-2 vertex is m) and a disjoint edge $c-d$. Then

$$\alpha(xm)=2, \quad \alpha(xa)=\alpha(xb)=\alpha(xc)=\alpha(xd)=1. \tag{3.36}$$

We first record the forced adjacencies required (existence of the size-3 matchings at the $\alpha=1$ edges).

- For xa: the left set is $N_x^{(xa)}=\{b,c,d\}$ and S_a contains m and three outside neighbors of a. Because $b\sim m$ in $\Gamma(x)$, one length-1 pair is (b,m). The other two must use two distinct outside neighbors of a, call them A_c,A_d, with

$$c\sim A_c\sim a, \quad d\sim A_d\sim a. \tag{3.37}$$

- For xb: symmetrically there exist distinct $B_c,B_d\in S_b\backslash\{m\}$ with

$$c\sim B_c\sim b, \quad d\sim B_d\sim b. \tag{3.38}$$

- For xc: the left set is $N_x^{(xc)}=\{a,b,d\}$, but neither a nor b is adjacent to c in $\Gamma(x)$, and d cannot be matched to the right vertex d (distance 0). Thus all three length-1 pairs must use three distinct outside neighbors of c, call them C_a,C_b,C_d, with

$$a\sim C_a\sim c, \quad b\sim C_b\sim c, \quad d\sim C_d\sim c. \tag{3.39}$$

- For xd_1: likewise, there exist three distinct outside neighbors D_a,D_b,D_c of d with

$$a\sim D_a\sim d, \quad b\sim D_b\sim d, \quad c\sim D_c\sim d. \tag{3.40}$$

Now consider the unique $\alpha=2$ edge $e:=xm$. Its non-shared left set is $N_x^{(e)}=\{c,d\}$, and

$$S_m=\Gamma(m)\backslash\{x\}=\{a,b,M_1,M_2\} \tag{3.41}$$

where M_1,M_2 are the two neighbors of m outside $\{x\}\cup\Gamma(x)$. Inside $\Gamma(x)$ we know $c\nsim a,b$ and $d\nsim a,b$, so the only potential length- 1 pairs in B_e are (c,M_i) and (d,M_j) with $i,j\in\{1,2\}$. To realize the required $s^*(e)=1$ we need at least one of these adjacencies to exist; to violate $s^*(e)=1$ we would need $\nu(B_e)\geq 2$, i.e. both c and d adjacent to $\{M_1,M_2\}$ with disjoint choices.

We now show that, given (3.37)-(3.40), $\nu(B_e)\geq 2$ is forced.

From (3.39), c is adjacent to three distinct outside vertices C_a,C_b,C_d. At least one of these three must be nonadjacent to a and b simultaneously (they already are by construction) and free with respect to the pairs used in (3.37)-(3.38). Since every vertex is degree-5, m has exactly two outside neighbors M_1,M_2. If neither M_1 nor M_2 is among $\{C_a,C_b,C_d\}$, then across xa,xb,xc the three distinct outside neighbors demanded for c already account for three neighbors of c in $V\backslash(\{x\}\cup\Gamma(x))$; but then xd still needs a fourth outside neighbor of c (namely D_c from (3.40)), contradicting degree of $c=5$. Hence

$$\{M_1, M_2\}\cap\{C_a, C_b, C_d\}\neq\varnothing \tag{3.42}$$

Therefore c is adjacent to at least one of $\{M_1, M_2\}$.

The same argument with d and (3.40) shows d is adjacent to at least one of $\{M_1, M_2\}$.

If the two vertices c, d happen to be adjacent to the same M_i, then (since $c\sim d$ in $\Gamma(x)$) the edge cd would have

$$\alpha(c,d)\geq|\{x, M_i\}|=2. \tag{3.43}$$

If $\alpha(c,d)=2$, then $(s^*,p^*)(cd)=(1,1)$. But the left non-shared set for cd contains two vertices (e.g. some of $\{A_c, B_c\}$ on the c-side and $\{A_d, B_d\}$ on the d side), and the constructions (3.37)-(3.40) produce two disjoint length-1 pairs across cd (one via a neighbor linked through a, the other via a neighbor linked through b), forcing $\nu(B_{cd})\geq 2$, contrary to $\nu=1$ at $\alpha=2$ edges. Thus c and d cannot both hit the same M_i.

Consequently, c is adjacent to one of M_1, M_2 and d is adjacent to the other. Therefore the bipartite B_e across $e=xm$ contains the two disjoint length-1 pairs

$$(c, M_i) \quad \text{and} \quad (d, M_j) \quad (i\neq j), \tag{3.44}$$

so $\nu(B_e)\geq 2$. But at an $\alpha=2$ edge, $s^*(e)=\nu(B_e)=1$. Contradiction. Hence the $(2,1,1,1,1)$ case cannot occur. The lemma then follows. □

Finally, we consider the case when $\kappa=\frac{1}{5}$.

Lemma 3.5. *There is no 5-regular graph with constant Ricci curvature $\kappa=\frac{1}{5}$.*

Proof. By lemma 2.3, we have

$$M(e)=7-3\alpha(e). \tag{3.45}$$

Since $0\leq M(e)\leq 2(4-\alpha(e))$, we obtain that $\alpha\in\{0,1,2\}$.

(3.46)

$\alpha=0\Rightarrow(s^*,p^*)=(3,1)$, $\alpha=1\Rightarrow(s^*,p^*)=(2,0)$, $\alpha=2\Rightarrow(s^*,p^*)=(0,1)$.

We first prove that $\alpha=2$ case can not happen.

Suppose not, then there exists a edge e with $\alpha(e)=2$. Then N_x has two vertices u_1, u_2 and $\Delta_{xy}=\{w_1, w_2\}\subset S_y$. Define a bijection ϕ by $\phi(u_i)=w_i$ for $i=1,2$ and mapping the two shared left vertices to the two non-shared right vertices arbitrarily. Then $|\pentagon(\phi)|\geq 2$ (both u_i-x-w_i are length-2), so $2|\square(\phi)|+|\pentagon(\phi)|\geq 2$, contradicting $M(e)=1$. Hence

$$\alpha(e)\in\{0,1\} \quad \text{for all } e. \tag{3.47}$$

In particular, $\Gamma(x)$ (the induced graph on the 5 neighbors of x) is a matching plus isolated vertices (vertex-degrees 0 or 1 only).

Next, we show that $\alpha = 0$ case can not happen. Suppose not, then there exists a edge $e = xy$ with $\alpha(e) = 0$. By (3.45), $M(e) = 7$. Since $\alpha(e) = 0$, we have $N_x = S_x = \Gamma(x)\backslash\{y\}$ (size 4). Consider any right vertex $v \in S_y = \Gamma(y)\backslash\{x\}$. Because $v \sim y$ and $y \sim x$, the vertex y lies in $\Gamma(x) \cap \Gamma(v)$, hence

$$\alpha(xv) = |\Gamma(x) \cap \Gamma(v)| \geq 1. \tag{3.48}$$

By the argument above, $\alpha(xv)$ cannot be 2 ; thus $\alpha(xv) = 1$.

Take any $u \in N_x = \Gamma(x)\backslash\{y\}$ and any $v \in S_y$. If $u \sim v$, then both y and u would be common neighbors of x and v, which would give $\alpha(xv) \geq 2$, contradicting $\alpha(xv) = 1$. Hence no pair (u, v) with $u \in N_x, v \in S_y$ is adjacent. Equivalently, the "square bipartite" B_{xy} (left N_x, right S_y, edges $u \sim v$) has no edges. Thus for every bijection ϕ, $|\Box(\phi)| = 0$.

By definition, $|\pentagon(\phi)|$ counts how many $u \in N_x$ satisfy $d(u, \phi(u)) = 2$. Each u is matched to one right vertex under ϕ, so each u can contribute at most one to $|\pentagon(\phi)|$. Therefore

$$|\pentagon(\phi)| \leq |N_x| = 4 \tag{3.49}$$

and

$$2|\Box(\phi)| + |\pentagon(\phi)| \leq 0 + 4 = 4. \tag{3.50}$$

and hence $M(e) \leq 4$. This contradicts $M(e) = 7$. Hence $\alpha = 0$ case can not happen.

Now fix x. Summing M around x,

$$\sum_{e \ni x} M(e) = \sum_{e \ni x} (7 - 3\alpha(e)) = 35 - 3 \sum_{e \ni x} \alpha(e), \tag{3.51}$$

and since $\sum_{e \ni x} \alpha(e)$ equals the degree-sum in $\Gamma(x)$, it is even. Therefore

$$\sum_{e \ni x} M(e) \equiv 1 \pmod 2 \quad \Longrightarrow \quad \sum_{e \ni x} p^*(e) \equiv 1 \pmod 2, \tag{3.52}$$

because $\sum 2s^*$ is even. We also know that $p^*(e) = 1$ iff $\alpha(e) = 0$. Hence the number $t(x)$ of $\alpha = 0$ edges at x is odd: $t(x) \in \{1, 3, 5\}$. Accordingly, $\Gamma(x)$ is one of:

- type **I**: two disjoint edges + one isolated vertex (degree multiset $(1, 1, 1, 1, 0)$, here $t(x) = 1$);
- type **J**: one edge + three isolated vertices (degree multiset $(1, 1, 0, 0, 0)$, here $t(x) = 3$);
- type **K**: five isolated vertices (degree multiset $(0, 0, 0, 0, 0)$, here $t(x) = 5$).

Each type contains $\alpha = 0$, which can not happen by previous argument. The lemma then follows. $\square$

The (i) of Theorem 1.1 then follows from Lemma 3.1–3.5.

4. 5-Regular Graph with nonpositive Ricci curvature

In this section, we are going to study the case when LLY Ricci curvature $\kappa \leq 0$.

We first handle the second part of Theorem 1.1, i.e. the case when LLY Ricci curvature $\kappa = 0$.

By lemma 2.3, we know that for any edge $e = xy$,

$$M(e) = 6 - 3\alpha(e). \tag{4.1}$$

We first prove the following

Lemma 4.1. *If $\kappa \equiv 0$, then $\alpha(e) \neq 2$ for every edge $e = xy$.*

Proof. If $\alpha(e) = 2$, then $|N_x| = 2$ and $\Delta_{xy} = \{w_1, w_2\} \subset S_y$. Define a bijection ϕ by $\phi(u_i) = w_i$ for the two $u_i \in N_x$; these two pairs have distance 2 via $u_i - x - w_i$, so $2|\square(\phi)| + |\pentagon(\phi)| \geq 2$. But (4.1) says $M(e) = 6 - 3 \cdot 2 = 0$, contradiction! $\square$

Hence $\alpha(e) \in \{0, 1\}$ for all edges. Then (4.1) implies

$$\alpha = 0 \Rightarrow M = 6, \quad \alpha = 1 \Rightarrow M = 3. \tag{4.2}$$

The second lemma characterize the edges e with $\alpha(e) = 0$.

Lemma 4.2. *If $\alpha(e) = 0$, then $2 \leq \nu(B_e) \leq 3$.*

Proof. By lemma 2.5, $6 = M(e) \leq (4-0) + \nu(B_e) = 4 + \nu(B_e)$, so $\nu(B_e) \geq 2$. If $\nu(B_e) = 4$, there is a perfect matching; extending it gives a bijection with $|\square(\phi)| = 4$ and $M = 8$, contradicting to the fact that $M = 6$. Thus $\nu(B_e) \leq 3$. $\square$

Definition 4.3. We call an $\alpha = 0$ edge type **A** if $\nu(B_e) = 3$ (so a maximizer can realize $s^* = 3, p^* = 0$), and type **B** if $\nu(B_e) = 2$ (so a maximizer must realize $s^* = 2, p^* = 2$).

For $\alpha = 1$, (4.1) only requires $2s^* + p^* = 3$ with $s^* \leq \nu(B_e)$ and $s^* \leq 1$; both $s^* = 1, p^* = 1$ and $s^* = 0, p^* = 3$ are possible (the tie-break picks $s^* = 1$ if $\nu(B_e) \geq 1$).

We then have the following result

Proposition 4.4. *If G contains at least one type **B** edges, then $G \cong H_3 \square C_n$, where H_3 be a 3-regular graph and C_n be a cycle of length $n \geq 6$.*

Proof. Let $e_0 = xy$ be a fixed type **B** edge. Choose and fix a size-2 matching

$$M_{xy} = \{(xu_1, yv_1), (xu_2, yv_2)\} \subseteq E(B_{xy}) \tag{4.3}$$

(so $x - u_i - v_i - y - x$ are two 4-cycles, disjoint on S_x and on S_y).

Definition 4.5. Give color $\mathcal{D}$ to the two edges of G opposite to xy in those two 4-cycles (i.e., u_1v_1 and u_2v_2); color xy itself and the two remaining edges at x and at y by color $\mathcal{C}$. Now, whenever we traverse an edge ab of G (in a BFS spanning tree), we transport colors across a chosen size-2 matching $M_{ab} \subseteq B_{ab}$ as follows:

- the two edges opposite to ab in the 4-cycles specified by M_{ab} receive color $\mathcal{D}$;
- the other three edges incident to each endpoint receive color $\mathcal{C}$ (so every vertex has a $3+2$ split).

This defines a 2-coloring of all edges once we visit them (because every $\alpha = 0$ edge has $\nu \geq 2$; for $\alpha = 1$ we do not use any 4-cycle to transport a color and simply keep whatever color it receives when discovered-consistency will be checked below).

Claim 4.5.1. *In every 4-cycle of G, opposite edges receive the same color.*

Proof of the claim:

If the 4-cycle contains a tree edge ab, the color was imposed when ab was processed, by definition. If a 4-cycle contains no tree edge, we reach it after both of its opposite tree-edges have been processed; opposite edges then already share color, and we never recolor.

Claim 4.5.2. *The partition of E into $\mathcal{C}$ and $\mathcal{D}$ is well-defined and coincides with the two connected components of the "opposition graph" whose vertices are E and whose edges link opposite edges in a 4-cycle.*

Proof of the claim:

Well-definedness follows from previous claim: any two ways to propagate colors differ by detouring around 4-cycles, and opposite edges retain the same color. Since we start from the type-B edge xy and color its two opposite edges differently from xy itself, we obtain two nonempty classes; connectedness in the opposition graph forces that there are exactly these two classes.

At this point we know, every vertex v has its 5 incident edges split as three of color $\mathcal{C}$ and two of color $\mathcal{D}$. For each color $X \in \{\mathcal{C}, \mathcal{D}\}$, define

$$N_X : V \to V, \quad N_X(v) = \text{the unique neighbor of } v \text{ joined by an edge of color } X.$$

Uniqueness holds because at v the opposition across the two 4-cycles through a tree edge pairs two edges of color $\mathcal{D}$ at the other vertex but only one of color $\mathcal{D}$ at v continues "forward"; the propagation is set up so that every vertex has exactly one edge of color $\mathcal{D}$ in the "forward" direction (and therefore one of color $\mathcal{C}$ singled out similarly). Formally, the construction yields a 1-factor in each color class.

Since Colors are undirected labels: the unique $\mathcal{C}$ edge leaving v is the same undirected edge used to come back from $N_{\mathcal{C}}(v)$, we know that

$$N_{\mathcal{C}}^2 = \text{id and } N_{\mathcal{D}}^2 = \text{id}. \tag{4.4}$$

We also have for each vertex v,

$$N_{\mathcal{C}}\left(N_{\mathcal{D}}(v)\right) = N_{\mathcal{D}}\left(N_{\mathcal{C}}(v)\right). \tag{4.5}$$

This is because, from v, take the two edges $v \to N_{\mathcal{C}}(v)$ and $v \to N_{\mathcal{D}}(v)$. By construction (we always transported colors using a size-2 matching), there is a 4-cycle

$$v \xrightarrow{\mathcal{C}} N_{\mathcal{C}}(v) \xrightarrow{\mathcal{D}} \star \xrightarrow{\mathcal{C}} N_{\mathcal{D}}(v) \xrightarrow{\mathcal{D}} v$$

with opposite edges of the same color (by previous claim). This shows that the endpoint of " $\mathcal{C}$ then $\mathcal{D}$ " equals the endpoint of " $\mathcal{D}$ then $\mathcal{C}$ ". Thus $N_{\mathcal{C}}, N_{\mathcal{D}}$ are commuting involutions.

Let $G_{\mathcal{C}} = (V, E_{\mathcal{C}})$ be the subgraph with the $\mathcal{C}$ edges, and $G_{\mathcal{D}} = (V, E_{\mathcal{D}})$ the subgraph with the $\mathcal{D}$ edges. By argument above, $G_{\mathcal{C}}$ is 3-regular and $G_{\mathcal{D}}$ is 2-regular (hence a disjoint union of cycles). Denote by

$$\mathcal{D}\text{-cycles } = \{\mathrm{S}\} \quad \text{and} \quad \mathcal{C}\text{-components } = \{\mathrm{X}\} \tag{4.6}$$

the vertex sets of the connected components of $G_{\mathcal{D}}$ and $G_{\mathcal{C}}$ respectively. For a vertex $v \in V(G)$, write

$$\mathrm{S}(v) \in \{\mathrm{S}\}, \quad \mathrm{X}(v) \in \{\mathrm{X}\} \tag{4.7}$$

for the $\mathcal{D}$-cycle containing v and the $\mathcal{C}$-component containing v. the vertex sets of the connected components of $G_{\mathcal{D}}$ and $G_{\mathcal{C}}$ respectively. For a vertex $v \in V(G)$, write

$$\mathrm{S}(v) \in \{\mathrm{S}\}, \quad \mathrm{X}(v) \in \{\mathrm{X}\} \tag{4.8}$$

for the $\mathcal{D}$-cycle containing v and the $\mathcal{C}$-component containing v.

Fix a $\mathcal{D}$-cycle S and choose an orientation of S. Let

$$\text{succs } : S \to S \tag{4.9}$$

be the successor map (advance by one $\mathcal{D}$-edge along S). Every $\mathcal{C}$ edge respects squares with $\mathcal{D}$-edges: if $v \in$ S and $v \xrightarrow{\mathcal{C}} w$, then succs$(v)$ has a unique $\mathcal{C}$-neighbor $\text{succs}_{\mathcal{S}'}(w)$ in the $\mathcal{D}$-cycle $\mathbf{S}' = \mathbf{S}(w)$, and the four vertices

$$v \xrightarrow{\mathcal{D}} \text{succs}(v) \xrightarrow{\mathcal{C}} \text{succ}_{\mathcal{S}'}(w) \xrightarrow{\mathcal{D}} w \xrightarrow{\mathcal{C}} v \tag{4.10}$$

form a color-alternating 4-cycle.

Consequently, for each $\mathcal{C}$-edge $e : v \to w$ with $\mathrm{S}(v) = \mathrm{S}$ and $\mathrm{S}(w) = \mathrm{S}'$ there is a uniquely defined cycle isomorphism $f_e : \mathrm{S} \xrightarrow{\cong} \mathrm{S}'$ satisfying

$$f_e\left(\text{succs}_{\mathrm{S}}(u)\right) = \text{succ}_{\mathrm{S}'}\left(f_e(u)\right) \tag{4.11}$$

for all $u \in$ S, and $f_e(v) = w$. That is: f_e is the unique bijection that sends v to w. In particular, S and S$'$ have the same length, so all $\mathcal{D}$-cycles in G have the same length $n \geq 3$.

Choose, for each $\mathcal{D}$-cycle S , a cyclic labeling

$$\text{pos}_{\mathrm{S}} : \mathrm{S} \longrightarrow \mathbb{Z}/n\mathbb{Z} \tag{4.12}$$

so that for every $\mathcal{C}$-edge $e : v \to w$ (with $\mathrm{S}(v) = \mathrm{S}, \mathrm{S}(w) = \mathrm{S}'$) we have

$$\text{pos}_{\mathrm{S}'}\left(f_e(u)\right) = \text{pos}_{\mathrm{S}}(u) \quad \text{for all } u \in \mathrm{S}. \tag{4.13}$$

Here, we obtain such labeling by picking one fixed cycle S_0, fixing any numbering there, and propagating it uniquely to all other cycles using the isomorphisms f_e along a spanning tree of the $\mathcal{C}$-adjacency among $\mathcal{D}$-cycles.

Commutation on 4-cycles guarantees that the label transported along any closed loop returns to itself.

Let H be the graph whose vertices are the $\mathcal{D}$-cycles $\{\mathrm{S}\}$. Put an (undirected, simple) edge $\mathrm{S} \sim \mathrm{S}'$ in H iff there exists a $\mathcal{C}$-edge in G joining some vertex of S to some vertex of S′. Because every $v \in V(G)$ has exactly three $\mathcal{C}$ neighbors (P1) and every $\mathcal{C}$-edge preserves the C-coordinate (2.2), the set of $\mathcal{D}$ cycles adjacent to $\mathrm{S}(v)$ in H depends only on $\mathrm{S}(v)$, not on the specific vertex $v \in \mathrm{S}(v)$. Hence $\deg_H(\mathrm{S}) = 3$ for every S.

Now for $v \in V(G)$, define

$$\Phi(v) := \Big(\mathrm{S}(v), \mathrm{pos}_{\mathrm{S}(v)}(v)\Big) \in V(H) \times \mathbb{Z}/n\mathbb{Z}. \tag{4.14}$$

This is obviously well-defined (we already fixed $\mathrm{S}(v)$ and $\mathrm{pos}_{\mathrm{S}}$). We now verify that Φ is a graph homomorphism whose local behavior is exactly that of a Cartesian product:

- If $v \sim w$ is a $\mathcal{D}$-edge, then $\mathrm{S}(w) = \mathrm{S}(v)$ and $\mathrm{pos}(w) = \mathrm{pos}(v) \pm 1$. Hence
$$\Phi(w) = (\mathrm{S}(v), \mathrm{pos}(v) \pm 1) \quad \text{is a } C_n\text{-neighbor of } \Phi(v).$$
- If $v \sim w$ is a $\mathcal{C}$-edge, then $\mathrm{S}(w)$ is one of the three neighbors of $\mathrm{S}(v)$ in H and, by (2.2), $\mathrm{pos}(w) = \mathrm{pos}(v)$. Hence
$$\Phi(w) = (\mathrm{S}(w), \mathrm{pos}(v)) \quad \text{is an } H\text{-neighbor of } \Phi(v).$$

So every edge of G maps to an edge of $H \square C_n$ changing exactly one coordinate.

Fix $v \in V(G)$. Its five neighbors split as:

- three $\mathcal{C}$-neighbors w_i (one into each of the three H-neighbors of $\mathrm{S}(v)$);
- two $\mathcal{D}$-neighbors (the predecessor and successor in $\mathrm{S}(v)$, hence the two neighbors of $\mathrm{pos}(v)$ in C_n).

By the observations as previous paragraph, Φ maps these five neighbors bijectively to the five neighbors of $\Phi(v) = (\mathrm{S}(v), \mathrm{pos}(v))$ in $H \square C_n$:

$$\mathrm{Nbr}_{H\square C_n}(\Phi(v)) = \big\{\big(\mathrm{S}', \mathrm{pos}(v)\big) : \mathrm{S}' \sim_H \ \mathrm{S}(v)\big\} \cup \{(\mathrm{S}(v), \mathrm{pos}(v) \pm 1)\}.$$

Thus Φ is a local isomorphism at every vertex. In other words, Φ is a (graph) covering map onto its image.

Finally, because G is finite and connected, and because every $\mathcal{D}$-cycle has been labeled pos bijectively onto $\mathbb{Z}/n\mathbb{Z}$, the map Φ is surjective onto $V(H) \times \mathbb{Z}/n\mathbb{Z}$. (Each pair (S, c) is the image of exactly one vertex of S with $\mathrm{pos} = c$.) Hence the image of Φ is all of $H \square C_n$.

Because Φ is a covering and is bijective on vertices (each (S, c) has exactly one preimage), Φ is, in fact, a graph isomorphism

$$G \cong H \square C_n, \quad \deg H = 3, \deg C_n = 2.$$

The proposition then follows. □

Finally, we have the following

Proposition 4.6. *Let G be 5-regular, Ricci-flat, triangle-free, and suppose every edge is type* $\mathbf{A}$ *(i.e. $\alpha(e) = 0$ and $\nu(B_e) = 3$ for all e). Then G is unique up to isomorphism; in particular $|V(G)| = 72$. Such graph is RF^5_{72}, which first discovered and classified in* [7].

Proof. Since $\alpha \equiv 0$, no two neighbors of x are adjacent, so $|S_1| = 5$, S_1 is independent. For each xu with $u \in S_1$, since $(s^*, p^*) = (3, 0)$, this implies three disjoint length-1 pairs across xu. Because S_1 is independent, these three squares must use right vertices outside $\{x\} \cup S_1$.

Every $u \in S_1$ has degree 5 , with one neighbor x and four neighbors in S_2. Hence the number of edges between S_1 and S_2 is $e(S_1, S_2) = 5 \cdot 4 = 20$.

Claim 4.6.1. *Every $v \in S_2$ has exactly two neighbors in S_1.*

Proof of the claim:

Let $v \in S_2$, and pick $u \in S_1$ with $u \sim v$. Across the edge xu, one of its three squares is $x - u - v - w - x$ for some $w \in S_1 \backslash \{u\}$ (because all three squares must use right vertices outside $\{x\} \cup S_1$), so $v \sim w$ as well. Thus $d_{S_1}(v) \geq 2$. On the other hand, if $d_{S_1}(v) \geq 3$, then for the three distinct $u_1, u_2, u_3 \in S_1 \cap \Gamma(v)$, each edge xu_i would have a square using v; but the three squares across xu_i must use three distinct right vertices (because $s^* = 3$), forcing v to be used by at most one among the three. Contradiction! Hence $d_{S_1}(v) = 2$. Therefore,

$$|S_2| = \frac{e(S_1, S_2)}{2} = 10.$$

Moreover, the bipartite graph between S_1 and S_2 is 2-regular on the S_2 side and 4-regular on the S_1 side.

Claim 4.6.2. *S_2 is independent.*

Proof of the claim:

Suppose $v, v' \in S_2$ are adjacent. Let their two S_1-neighbors be $\{u, w\}$ and $\{u', w'\}$ (each be a 2-set by claim 4.6.1). Consider the edge xu. One of its three squares is $x - u - v - w - x$. Similarly, the edge xu' has a square $x - u' - v' - w' - x$. The edge vv' now lies in a 5-cycle $v - u - x - u' - v' - v$, and across the middle edge xu one can realize a fourth square (using v') in addition to the other three already from $s^* = 3$, contradicting $\nu(B_{xu}) = 3$. Hence no such vv' exists.

Each $v \in S_2$ has degree 5 , with two neighbors in S_1 and three neighbors beyond S_2; therefore all those three lie in S_3. Counting edges between S_2 and S_3 gives $e(S_2, S_3) = |S_2| \cdot 3 = 30$. We next constrain the number of S_2-neighbors of a given $z \in S_3$.

Claim 4.6.3. *Every $z \in S_3$ has exactly three neighbors in S_2.*

Proof of the claim:

Let $z \in S_3$. It has at least one neighbor $v \in S_2$. Consider the three edges xu for $u \in S_1 \cap \Gamma(v)$ (there are two by claim 4.6.1). Across each such edge

xu, the vertex v already accounts for one of the three squares; the other two squares must be disjoint and use right vertices that are S_2-neighbors of u distinct from v. Chasing this around the two S_1-neighbors of v shows z must be adjacent to two more S_2-vertices (otherwise we cannot realize the third square across the corresponding xu without creating an extra square across some other xw.) Thus $d_{S_2}(z) \geq 3$. On the other hand, if $d_{S_2}(z) \geq 4$, pick four S_2-neighbors $v_1, \ldots, v_4$. Each v_i has exactly two S_1-neighbors; by pigeonhole there exist $i \neq j$ sharing an S_1 neighbor u. Then across xu we would have four distinct right vertices adjacent to u (namely v_i, v_j and the two others required by the assumption $s^* = 3$, contradicting $\nu(B_{xu}) = 3$. Hence $d_{S_2}(z) = 3$.

Therefore $|S_3| = 30/3 = 10$, and S_3 is independent.

Because S_3 is independent and $d_{S_2}(z) = 3$ for each $z \in S_3$, every $z \in S_3$ has exactly two neighbors in S_4. Thus

$$e(S_3, S_4) = \sum_{z \in S_3} d_{S_4}(z) = 10 \cdot 2 = 20$$

We then know the following:

- For all $w \in S_4, d_{S_3}(w) \geq 2$. (If $d_{S_3}(w) = 1$, then in B_{zw} with the unique $z \in S_3$ the three left S2-neighbors of z have no right neighbors (cannot connect to S_5), so $\nu(B_{zw}) \leq 1$, contradicting $\nu = 3$.)
- For all $w \in S_4, d_{S_3}(w) \leq 3$. (If $d_{S_3}(w) \geq 4$, then for any $z \in S_3 \cap \Gamma(w)$ the left set N_z has 3 vertices in S_2 and 1 vertex in $S_4 \backslash \{w\}$, while the right set has at least 3 vertices in $S_3 \backslash \{z\}$ and one in S_5; this furnishes a 4-matching in B_{zw}, contradicting $\nu = 3$.)

Hence

$$d_{S_3}(w) \in \{2, 3\} \quad \text{for all } w \in S_4, \quad \sum_{w \in S_4} d_{S_3}(w) = 20 \tag{4.15}$$

If there exists $w \in S_4$, such that $r(w) = 3$, then w has only two neighbors left outside S_3, i.e. $d_{S_4}(w) + d_{S_5}(w) = 2$. Now inspect simultaneously the three bipartites $B_{z_i w}$ (one for each $z_i \in \Gamma(w) \cap S_3$) and the one or two bipartites B_{wu} for $u \in \Gamma(w) \cap (S_4 \cup S_5)$. Each $B_{z_i w}$ needs its "third" match to come from $w_i' \mapsto r_i$ with $r_i \in \Gamma(w) \cap (S_4 \cup S_5)$; and each B_{wu} needs two disjoint matches from z's into $\Gamma(u) \cap S_4$ (because S_3 left cannot touch S_6). A direct count shows that with only two vertices in $\Gamma(w) \cap (S_4 \cup S_5)$, it is impossible to supply all these disjoint matches without creating a 4-matching in one of the four bipartites (either a $B_{z_i w}$ or a B_{wu}). In other words, at least one of those edges would fail the type-$\mathbf{A}$ condition $\nu = 3$. Hence, $d_{S^3}(w) = 2$ and $|S_4| = 20$. From here, the rest of the proof is same as lemma 3.2, we omit it here. $\square$

For the case $\kappa < 0$, we have the following

Proposition 4.7. *Every 5-regular graph with girth at least 6 has negative constant curvature* $\kappa = -\frac{6}{5}$.

Proof. Since the graph has the girth at least 6, we know that $\alpha = 0$, $|\square|(\phi) = 0$ and $|\pentagon|(\phi) = 0$ for any bijection ϕ. Hence, by lemma 2.3

$$\kappa(x,y) = \frac{1}{d}\left(-6 + 3\,|\Delta_{xy}| + \max_{\phi\in\mathcal{A}_{xy}}\left(2\,|\square(\phi)| + |\pentagon(\phi)|\right)\right) = -\frac{6}{5}. \tag{4.16}$$

□

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The successful completion of this research is primarily attributable to the dedicated guidance and substantial support provided by our mentor, Ms. Jiaxin Li. Ms. Li served as Jiajing's mathematics instructor during junior high school and was the advisor for our advanced mathematics club. The initial inspiration for this research originated during junior high school when Ms. Li introduced us to the world of Riemannian geometry within the context of the advanced mathematics club she guided. Our mentor subsequently encouraged us to collaboratively explore more advanced topics, including discrete Ricci curvature. We began with the simplest case, the regular graph, which served as our initial focus. This endeavor was undertaken solely for academic purposes, and mentorship was provided without any financial compensation. The insights obtained at that time planted the foundational seeds for the research conducted today.

The research background of this project lies in the extension of Ricci curvature from Riemannian geometry to discrete structures. In classical geometry, Ricci curvature measures how the geometry of a space deviates from flat Euclidean space, and it became fundamental in Einstein's general relativity. In recent decades, mathematicians have generalized Ricci curvature to graphs, with the Lin–Lu–Yau (LLY) definition emerging as both rigorous and practical. Graphs with constant LLY Ricci curvature can be regarded as discrete analogues of Einstein manifolds. The classification of such graphs has been achieved for small degrees ($d \leq 4$), but the degree 5 case is widely open, allowing both positive and negative curvature. This project therefore aimed to complete the classification of finite simple 5-regular graphs with constant curvature, identifying positive curvature families, establishing uniqueness for Ricci-flat cases, and proving that graphs with large girth necessarily exhibit negative curvature.

The division of responsibilities was as follows: Yuchen Yin focused on technical derivations in the positive curvature cases, analyzing values such as $\kappa = 6/5$, $\kappa = 4/5$, and $\kappa = 2/5$, and connecting them to K_5, the icosahedral graph, and cubical quotients of the hypercube. Jiajing Zhang worked on the Ricci-flat and negative curvature cases, proving that the 72-vertex graph RF^5_{72} is unique and that graphs with girth at least six necessarily have constant negative curvature. She also handled edge cases such as $\kappa = \frac{1}{5}$ and $\kappa = \frac{3}{5}$, showing they cannot occur. Both authors collaborated on writing, editing, and refining the overall structure, with Yuchen focusing on LaTeX formatting and Jiajing on diagrams and clarity.

During this research, we face several difficulties. At one stage, our attempt to classify graphs with curvature $\kappa = \frac{1}{5}$ led to contradictions, which initially puzzled us. Our mentor advised us to retrace the assumptions and test edge cases systematically, eventually revealing the impossibility of such

graphs. Another challenge was in understanding the isomorphism structures in the cubical case; by examining the Θ-equivalence classes of edges and their relation to hypercubes, we were able to resolve this.

5. Revision summary

In the proof of lemma 3.2, we add a paragraph indicating how to exclude the case when $\alpha(e) = 3$ for some edges e. (See the red color paragraph labeled)

In the proof of lemma 3.3, we make the correction for excluding the case when $\alpha(e) = 1$ or $\alpha(e) = 2$ for some edges e. (See the red color paragraph labeled)

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