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INVESTIGATION ON THE TRENDS OF PARTICULAR COSINE PRODUCTS

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ABSTRACT. This paper aims to investigate the trend exhibited in the particulate cosine products of the general form $\prod_{k=1}^{0.5(n-1)} \cos\left(\frac{m\pi k}{n}\right)$. It will be shown that this particular product of cosines is related to the exponential of 2 and the greatest common divisor between the non-zero integers n, m where n is odd. Hence, the product can be connected to the greatest common divisor function, the prime counting function and other topics in number theory.

Keywords. Number Theory, Cosines, Trigonometry

CONTENTS

1. INTRODUCTION

The following cosine product identities can be found in several papers concerned with the factorization of Fibonacci numbers [**1**], [**2**]:

$$\prod_{k=1}^{(n-1)/2} 2\cos\left(\frac{\pi k}{n}\right) = 1, \text{ n odd} \tag{1.1}$$

where the relationship between similar trigonometric expressions and the Fibonacci, Lucas numbers have been extensively studied. However, the aesthetically pleasing expression (1.1), referred to as the “Grandma’s Identity” by Steve Humble in his 88.62 note on the Cambridge Mathematical Gazette [**3**], have been far less studied, being regarded as simply an aside in other investigations. This paper aims to expand upon previous discoveries regarding the identity, deriving the following general expression of the product for all nonzero integers n, m:

$$\prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{m\pi k}{n}\right) = \begin{cases} (-1)^{\lceil \frac{t-1}{4} \rceil} 2^{\frac{t-n}{2}}, & \text{if } m \text{ odd} \\ (-1)^{\lceil \frac{t-1}{4} \rceil + \lceil \frac{n-1}{4} \rceil} 2^{\frac{t-n}{2}}, & \text{if } m \text{ even} \end{cases} \tag{1.2}$$

where n, m are nonzero integers, n is odd and $t = \gcd(n, m)$. By rearranging, we can obtain the following trigonometric expression for the greatest common divisor between n, m:

$$\gcd(m, n) = n + \frac{2}{\log 2} \sum_{k=1}^{\frac{n-1}{2}} \log\left|\cos\left(\frac{m\pi k}{n}\right)\right| \tag{1.3}$$

which leads to the following equation, where the roots of the equation $p = p_1, p_2, \ldots$ are the odd prime numbers.

$$\left(\frac{p-1}{2}\right)^2 \log 2 + \sum_{1 \le i,j \le (p-1)/2} \log\left|\cos\left(\frac{ij\pi}{p}\right)\right| = 0 \tag{1.4}$$

2. Proof for $m = 1$

We will start the proof of the general case (1.2) with a proof for the case where $m = 1$, which is equivalent to the expression in (1.1). The simplest way of proving this expression, as suggested by [**3**] is a proof involving the roots of unity. Given the solutions of the complex equation $z^n = 1$ are $z_1, z_2 \cdots z_n$, it is known that the equation can be factored into:

$$z^n - 1 = (z - z_1)(z - z_2) \cdots (z - z_n) = 0$$

The solutions of the equation $z^n = 1$ are referred to as the roots of unity and have the form $z_k = \exp\left(\frac{2\pi i k}{n}\right)$ where $k = 1, 2, \cdots n$ and $i = \sqrt{-1}$. Then by the Euler's Formula we know that:

$$z_k = \cos\left(\frac{2\pi i k}{n}\right) + i \sin\left(\frac{2\pi i k}{n}\right) \tag{2.1}$$

We see that $z = 1$ is a solution to the equation when $k = n$, therefore we can substitute $z_n = 1$ to derive the following expression:

$$\frac{z^n - 1}{z - 1} = (z - z_1)(z - z_2) \cdots (z - z_{n-1}) = \prod_{k=1}^{n-1} (z - z_k)$$

Then by substituting (2.1) for z_k and using the double angle identities for sine and cosine we obtain the following:

$$\frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1} \left(z + 1 - 2\cos^2\left(\frac{\pi k}{n}\right) - 2i \sin\left(\frac{\pi k}{n}\right) \cos\left(\frac{\pi k}{n}\right)\right)$$

As $\forall z \in \mathbb{R}$ the above equation is satisfied, we can set $z = -1$:

$$\frac{(-1)^n - 1}{-2} = \prod_{k=1}^{n-1} \left[-2\cos^2\left(\frac{\pi k}{n}\right) - 2i \sin\left(\frac{\pi k}{n}\right) \cos\left(\frac{\pi k}{n}\right)\right]$$

By rearranging and using Euler's Formula again, we obtain:

$$\frac{(-1)^n - 1}{-2} = (-2)^{n-1} \prod_{k=1}^{n-1} \cos\left(\frac{\pi k}{n}\right) \prod_{k=1}^{n-1} \exp\left(\frac{i\pi k}{n}\right) \tag{2.2}$$

Using property of exponents, we can simplify the latter product, as shown:

$$\prod_{k=1}^{n-1} \exp\left(\frac{i\pi k}{n}\right) = \exp\left(\sum_{k=1}^{n-1} \frac{i\pi k}{n}\right) = \exp\left(\frac{i\pi}{n} \sum_{k=1}^{n-1} k\right) = e^{\frac{i\pi(n-1)}{2}} = (-1)^{\frac{n-1}{2}}$$

Substituting the above into (2.2), we obtain the following expression:

$$\frac{(-1)^n - 1}{-2} = (-2)^{n-1}(-1)^{\frac{n-1}{2}} \prod_{k=1}^{n-1} \cos\left(\frac{\pi k}{n}\right)$$

Rearranging, we derive the following:

$$\prod_{k=1}^{n-1} \cos\left(\frac{\pi k}{n}\right) = 2^{1-n}(-1)^{\frac{1-n}{2}} \tag{2.3}$$

From trigonometric identities, we obtain the following property of cosines:

$$\cos(x \pm \pi y) = (-1)^y \cos(x) \qquad \forall\ y \in \mathbb{Z} \tag{2.4}$$

Using this property, we see that:

$$\prod_{k=1}^{n-1} \cos\left(\frac{\pi k}{n}\right) = \prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{\pi k}{n}\right) \prod_{k=\frac{n+1}{2}}^{n-1} \cos\left(\frac{\pi k}{n}\right) = \prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{\pi k}{n}\right) \prod_{k=\frac{n+1}{2}}^{n-1} - \cos\left(\frac{\pi(n-k)}{n}\right)$$

Note that for $l = n - k$:

$$\prod_{k=\frac{n+1}{2}}^{n-1} \cos\left(\frac{\pi(n-k)}{n}\right) = \prod_{l=1}^{\frac{n-1}{2}} \cos\left(\frac{\pi l}{n}\right) = \prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{\pi k}{n}\right)$$

By substituting and rearranging we arrive at:

$$\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{n}\right) = (-1)^{\frac{n-1}{2}} \left[\prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{k\pi}{n}\right)\right]^2$$

Finally, by substituting (2.3) for the cosine product, we arrive at our desired equation (1.1):

$$\prod_{k=1}^{\frac{n-1}{2}} \cos\frac{\pi k}{n} = 2^{\frac{1-n}{2}}$$

3. Proof for general case $m \in \mathbb{Z}$

3.1. **Proof for Magnitude.** It can be noted from the generalized expression (1.2) that the absolute value of the cosine product equals the following:

$$\left|\prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{m\pi k}{n}\right)\right| = \left|\prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{m'\pi k}{n'}\right)\right| = 2^{\frac{t-n}{2}} \tag{3.1}$$

where $t = \gcd(m, n)$, $m = tm'$, $n = tn'$. We will attempt to prove this $\forall m, n \in \mathbb{Z}^+$, which can be expanded to $m, n \in \mathbb{Z}$ where $m, n \neq 0$ by the even property of the cosine function.

Lemma 3.1. *Let $n' \in \mathbb{Z}^+$ and $m' \in \mathbb{Z}^+$ satisfy $\gcd(m', n') = 1$. Let the remainder function $R : \mathbb{Z}^+ \to \{0, \ldots, n' - 1\}$ be:*

$$R(x) := m'k - n'\left\lfloor\frac{m'k}{n'}\right\rfloor$$

Then, the Lemma states that:

$$\{\, R(m'k) \mid k = 1, \ldots, n' - 1 \,\} = \{\, 1, \ldots, n' - 1 \,\} = S$$

Proof. Let a, b be distinct integers in the set S. Without loss of generality, we can set $b > a$. As both $a, b \in S$, we know that $b - a < n'$ or that $b = a + ln'$ where $0 < l < 1$. Then we can construct the following equalities:

$$\begin{aligned} m'b &= pn' + R(m'b) \quad p \in \mathbb{Z} \\ m'a &= qn' + R(m'a) \quad q \in \mathbb{Z} \\ \iff m'(b-a) &= (p-q)n' + R(m'b) - R(m'a) \end{aligned}$$

Assume that $R(m'b) = R(m'a)$ for some a, b. Then, we see that:

$$m'(b-a) = m'ln' = (p-q)n' \iff m'l = p - q \in \mathbb{Z}$$

This implies:

$$\frac{m'}{n'} = \frac{m'l}{n'l} = \frac{p-q}{b-a} = \frac{M}{N}$$

Where $M, N \in \mathbb{Z}$ and $M = m'l < m'$ and $N = n'l < n'$, as $l \in (0, 1)$. This suggests that the fraction $\frac{m'}{n'}$ can be further simplified to $\frac{M}{N}$, a contradiction to the original condition $\gcd(m', n') = 1$. Therefore, assumption $R(m'b) = R(m'a)$ is false, or $R(m'b) \neq R(m'a)\ \forall\, a, b \in S$. As $R(m'k) \neq 0$ for $k \in S$, $R(m'k)$ would give distinct values from the set S, a set of all possible nonzero remainders when divided by n', for $k \in S$, i.e.:

$$\{\, R(m'k) \mid k = 1, \ldots, n' - 1 \,\} = \{\, 1, \ldots, n' - 1 \,\}$$

$\square$

From Equation (2.4), we derive that:

$$\left|\prod_{k=1}^{n'-1}\cos\left(\frac{m'\pi k}{n'}\right)\right| = \left|\prod_{k=1}^{n'-1}\cos\left(\frac{(m'k-\gamma n')\pi}{n'}\right)\right| \quad \forall\gamma\in\mathbb{Z}$$

If we let γ be the quotient when $m'k$ is divided by n', $m'k-\gamma n' = R(m'k)$. Then by using Lemma (3.1), we derive that:

$$\left|\prod_{k=1}^{n'-1}\cos\left(\frac{m'\pi k}{n'}\right)\right| = \left|\prod_{k=1}^{n'-1}\cos\left(\frac{R(m'k)\pi}{n'}\right)\right| = \left|\prod_{k=1}^{n'-1}\cos\left(\frac{k\pi}{n'}\right)\right|$$

From replacing n with n' in (2.3), we see that the above is equal to the following:

$$\left|\prod_{k=1}^{n'-1}\cos\left(\frac{m'\pi k}{n'}\right)\right| = 2^{1-n'} \tag{3.2}$$

We already know from (2.4) that $\left|\cos\left(\frac{m'\pi k}{n'}\right)\right| = \left|\cos\left(\frac{m'\pi(n'-k)}{n'}\right)\right|$. Then:

$$\left|\prod_{k=\frac{1}{2}(n'+1)}^{n'-1}\cos\left(\frac{m'\pi k}{n'}\right)\right| = \left|\prod_{k=\frac{1}{2}(n'+1)}^{n'-1}\cos\left(\frac{m'\pi(n'-k)}{n'}\right)\right| = \left|\prod_{k'=1}^{\frac{1}{2}(n'-1)}\cos\left(\frac{m'\pi k'}{n'}\right)\right|$$

for $n'\geq 3$, where $k'\mapsto n'-k$. Using (3.2), this implies that:

$$\left|\prod_{k=\frac{1}{2}(n'+1)}^{n'-1}\cos\left(\frac{m\pi k}{n}\right)\right| = \left|\prod_{k=1}^{\frac{n'-1}{2}}\cos\left(\frac{m'\pi k}{n'}\right)\right|^2 = 2^{1-n'}$$

$$\iff \left|\prod_{k=1}^{\frac{n'-1}{2}}\cos\left(\frac{m'\pi k}{n'}\right)\right| = \left|\prod_{k=\frac{1}{2}(n'+1)}^{n'-1}\cos\left(\frac{m'\pi(n'-k)}{n'}\right)\right| = 2^{\frac{1-n'}{2}} \tag{3.3}$$

For the case in which $t=1$, $n'=n, m'=m$ and the above equation simply becomes the desired Equation (3.1):

$$\left|\prod_{k=1}^{\frac{n'-1}{2}}\cos\left(\frac{m'\pi k}{n'}\right)\right| = \left|\prod_{k=1}^{\frac{n-1}{2}}\cos\left(\frac{m\pi k}{n}\right)\right| = 2^{\frac{1-n}{2}} = 2^{\frac{t-n}{2}}$$

In the case where $t>1$, we can consider the following equality:

$$\left|\prod_{k=\alpha n'+1}^{\frac{1}{2}((2\alpha+1)n'-1)}\cos\left(\frac{m'\pi k}{n'}\right)\right| = \left|\prod_{k=\alpha n'+1}^{\frac{1}{2}((2\alpha+1)n'-1)}\cos\left(\frac{m'\pi(k-\alpha n')}{n'}\right)\right| = \left|\prod_{k=1}^{\frac{n'-1}{2}}\cos\left(\frac{m'\pi k}{n'}\right)\right| = 2^{\frac{1-n'}{2}}$$

where $k-\alpha n'\mapsto k$ and $\alpha\in\mathbb{Z}$. Similarly, we know that:

$$\left|\prod_{k=\frac{1}{2}((2\alpha+1)n'+1)}^{(\alpha+1)n'}\cos\left(\frac{m'\pi k}{n'}\right)\right| = 2^{\frac{1-n'}{2}}$$

Then we can construct the following equality for $\alpha \geq 1$:

$$\left|\prod_{k=1}^{\frac{1}{2}((2\alpha+1)n'-1)} \cos\left(\frac{m'\pi k}{n'}\right)\right| =$$

$$\left|\left[\prod_{k=\alpha n'+1}^{\frac{1}{2}((2\alpha+1)n'-1)} \cos\left(\frac{m'\pi k}{n'}\right)\right] \prod_{\beta=0}^{\alpha-1}\left[\prod_{k=\beta n'+1}^{\frac{1}{2}((2\beta+1)n'-1)} \cos\left(\frac{m'\pi k}{n'}\right) \prod_{k=\frac{1}{2}((2\beta+1)n'+1)}^{(\beta+1)n'} \cos\left(\frac{m'\pi k}{n'}\right)\right]\right|$$

$$= \left|2^{\frac{1-n'}{2}} \prod_{\beta=0}^{\alpha-1} 2^{\frac{1-n'}{2}} 2^{\frac{1-n'}{2}}\right| = \left|2^{\frac{1-n'}{2}} \cdot 2^{(1-n')\alpha}\right| = 2^{\left(\frac{1-n'}{2}\right)(2\alpha+1)} \tag{3.4}$$

As n is odd, we know that $t = \gcd(m,n)$ must be odd and therefore, we can set $t = 2\alpha+1$ for $t \geq 3$. Then the above becomes:

$$\left|\prod_{k=1}^{\frac{1}{2}(tn'-1)} \cos\left(\frac{m'\pi k}{n'}\right)\right| = \left|\prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{m\pi k}{n}\right)\right| = 2^{\left(\frac{1-n'}{2}\right)t} = 2^{\frac{t-n}{2}} \tag{3.5}$$

as desired.

3.2. **Proof for Sign.** The proof for the sign of the product (1.2) will follow a similar progression with the proof for the magnitude but with the sign changes of the cosines considered. The following expressions will be used in the proof:

$$A := \prod_{k=\frac{n'+1}{2}}^{n'} \cos\left(\frac{m'\pi k}{n'}\right), \qquad a := \operatorname{sgn}(A) \tag{3.6}$$

$$B := \prod_{k=1}^{\frac{n'-1}{2}} \cos\left(\frac{m'\pi k}{n'}\right), \qquad b := \operatorname{sgn}(B) \tag{3.7}$$

$$C := AB = \prod_{k=1}^{n'} \cos\left(\frac{m'\pi k}{n'}\right), \qquad c := \operatorname{sgn}(C) \tag{3.8}$$

Where sgn is the sign function returning 1,0,-1. It can be seen that:

$$A = \prod_{k=\frac{n'+1}{2}}^{n'} \cos\left(\frac{m'\pi k}{n'}\right) = \cos(m'\pi) \prod_{k=\frac{n'+1}{2}}^{n'-1} \cos\left(\frac{m'\pi k}{n'}\right)$$

$$= (-1)^{m'} \prod_{k=\frac{n'+1}{2}}^{n'-1} (-1)^{m'} \cos\left(\frac{m'\pi(n'-k)}{n'}\right) = (-1)^{\frac{m'(n'+1)}{2}} B$$

$$A = (-1)^{\frac{m'(n'+1)}{2}} B \tag{3.9}$$

If we substitute (3.9) into (3.8), we see that:

$$C = AB = (-1)^{\frac{m'(n'+1)}{2}} B^2$$

$$\iff c = (-1)^{\frac{m'(n'+1)}{2}} \tag{3.10}$$

By using (2.4), we see that:

$$\prod_{\frac{1}{2}((2\alpha+1)n'+1)}^{(\alpha+1)n'} \cos\left(\frac{m'\pi k}{n'}\right) = \prod_{\frac{1}{2}((2\alpha+1)n'+1)}^{(\alpha+1)n'} (-1)^{m'\alpha} \cos\left(\frac{m'\pi(k-\alpha n')}{n'}\right) = (-1)^{\frac{m'\alpha(n'+1)}{2}} A$$

$$\prod_{\frac{1}{2}((2\alpha+1)n'+1)}^{(\alpha+1)n'} \cos\left(\frac{m'\pi k}{n'}\right) = (-1)^{\frac{m'\alpha(n'+1)}{2}} A \tag{3.11}$$

Similarly,

$$\prod_{k=\alpha n'+1}^{\frac{1}{2}((2\alpha+1)n'-1)} \cos\left(\frac{m'\pi k}{n'}\right) = (-1)^{\frac{m'\alpha(n'-1)}{2}} B \tag{3.12}$$

3.2.1. *Proof for $t = 1$.* We will first begin by proving for the case where $t = 1$. It can be noted that for $t = 1$:

$$\prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{m\pi k}{n}\right) = \prod_{k=1}^{\frac{n'-1}{2}} \cos\left(\frac{m'\pi k}{n'}\right) = B \tag{3.13}$$

Therefore, the sign of the final cosine product will be equivalent to $b = \text{sgn}(B)$. From (2.4) we know that:

$$B = \prod_{k=1}^{\frac{n'-1}{2}} \cos\left(\frac{m'\pi k}{n'}\right) = \prod_{k=1}^{\frac{n'-1}{2}} (-1)^q \cos\left(\frac{(m'k - qn')\pi}{n'}\right)$$

We know from appendix A that for $q = \lceil \frac{m'k}{n'} + \frac{1}{2} \rceil - 1$, $\cos\left(\frac{(m'k-qn')\pi}{n'}\right) > 0$. This means that:

$$b = \prod_{k=1}^{\frac{n'-1}{2}} (-1)^q = \prod_{k=1}^{\frac{n'-1}{2}} (-1)^{\lceil \frac{m'k}{n'} + \frac{1}{2} \rceil - 1}$$

We know from appendix B that $\lceil \frac{m'k}{n'} + \frac{1}{2} \rceil = \lceil \frac{1}{2} \lfloor \frac{2m'k}{n'} \rfloor \rceil + 1$. Substituting this result to the above equation, we obtain the following:

$$b = \prod_{k=1}^{\frac{n'-1}{2}} (-1)^{\lceil \frac{1}{2} \lfloor \frac{2m'k}{n'} \rfloor \rceil} \tag{3.14}$$

Let q_k and r_k be the quotient and remainder respectively, when $m'k$ is divided by n'. This implies:

$$m'k = q_k n' + r_k, \qquad q_k = \lfloor \frac{m'k}{n'} \rfloor \tag{3.15}$$

It follows that:

$$\lceil \frac{1}{2} \lfloor \frac{2m'k}{n'} \rfloor \rceil = q_k + \lceil \frac{1}{2} \lfloor \frac{2r_k}{n'} \rfloor \rceil \tag{3.16}$$

As r_k is the remainder when divided by n', it follows that $0 < r_k < n'$. Then from simple casework, it can be shown that:

$$1\{r_k > \frac{n'-1}{2}\} = \lceil \frac{1}{2} \lfloor \frac{2r_k}{n'} \rfloor \rceil = \begin{cases} 0, & r_k \le \frac{n'-1}{2} \\ 1, & r_k > \frac{n'-1}{2} \end{cases} \tag{3.17}$$

where $1\{r_k > \frac{n'-1}{2}\}$ is the indicator (Iverson bracket). We can then introduce S:

$$S := \sum_{k=1}^{\frac{n'-1}{2}} \lceil \frac{1}{2} \lfloor \frac{2m'k}{n'} \rfloor \rceil, \quad b = (-1)^S \tag{3.18}$$

where the parity of S will decide the value of b. By substituting (3.17) into (3.16) and then substituting the resulting expression into (3.18), we obtain:

$$S = \sum_{k=1}^{\frac{n'-1}{2}} \lceil \frac{1}{2} \lfloor \frac{2m'k}{n'} \rfloor \rceil = \sum_{k=1}^{\frac{n'-1}{2}} q_k + 1\{r_k > \frac{n'-1}{2}\} \tag{3.19}$$

Furthermore, by summing over k in (3.15), we obtain:

$$\sum_{k=1}^{\frac{n'-1}{2}} m'k = \sum_{k=1}^{\frac{n'-1}{2}} q_k n' + r_k \iff m'\left(\frac{n'^2-1}{8}\right) \equiv \sum_{k=1}^{\frac{n'-1}{2}} q_k + r_k \pmod 2$$

$$\sum_{k=1}^{\frac{n'-1}{2}} q_k \equiv m'\left(\frac{n'^2-1}{8}\right) + \sum_{k=1}^{\frac{n'-1}{2}} r_k \pmod 2$$

By substituting the above into (3.19) we derive:

$$S \equiv m'\left(\frac{n'^2-1}{8}\right) + \sum_{k=1}^{\frac{n'-1}{2}} r_k + 1\{r_k > \frac{n'-1}{2}\} \pmod 2$$

which can be simplified further through casework:

$$S \equiv \begin{cases} \sum_{k=1}^{\frac{n'-1}{2}} r_k + 1\{r_k > \frac{n'-1}{2}\} \pmod 2, & m \text{ even} \\ \frac{n'^2-1}{8} + \sum_{k=1}^{\frac{n'-1}{2}} r_k + 1\{r_k > \frac{n'-1}{2}\} \pmod 2, & m \text{ odd} \end{cases} \tag{3.20}$$

Lemma 3.2. *Let $R = \{r_k \,|\, k \le \frac{n'-1}{2}\}$ be a set of remainders r_k when divided by n'. For some remainder $r \in R$ its complementary remainder $n' - r \notin R$.*

Proof. Without loss of generality, let $r_{k_1} = r$ and $r_{k_2} = n' - r$. It follows that:

$$m'k_1 \equiv r \pmod{n'}$$
$$m'k_2 \equiv n' - r \equiv -r \pmod{n'}$$
$$m'(k_1 + k_2) \equiv 0 \pmod{n'}$$
$$\iff k_1 + k_2 \equiv 0 \pmod{n'}$$

as m', n' relatively prime. However, as $1 \le k_1, k_2 < n' - 1$, in order for $k_1 + k_2$ to be a multiple of n', it must sum up to n', which implies $k_2 = n' - k_1$. If $k_1 \le \frac{n'-1}{2}$, $k_2 > \frac{n'-1}{2}$ and vice versa, meaning only one of them can be included in R. □

In order to compute $r_k + 1\{r_k > \frac{n'-1}{2}\}$, we start with by partitioning the nonzero remainders of n' into pairs $P_r = \{r, n'-r\}$ where $r = 1, 2, \ldots \frac{n'-1}{2}$. It can be seen that $r \le \frac{n'-1}{2}$, $n'-r > \frac{n'-1}{2}$ and $r, n'-r$ have opposite parity, due to n' being odd. Lemma 3.2 essentially states that one element from each pair P_r will be chosen when summing through $k = 1, 2, \ldots \frac{n'-1}{2}$. Let R be the remainder from the pair that is chosen. It can be seen that:

(1) If $R = n'-r$, $1\{R > \frac{n'-1}{2}\} = 1$:
$r_k + 1\{r_k > \frac{n'-1}{2}\} = n' - r + 1 \equiv 1 - r + 1 \equiv r \pmod 2$

(2) Similarly, if $R = r$, $1\{R < \frac{n'-1}{2}\} = 0$:
$r_k + 1\{r_k > \frac{n'-1}{2}\} = r + 0 \equiv r \pmod 2$

Therefore, regardless of which R is picked, $r_k + 1\{R < \frac{n'-1}{2}\} \equiv r \pmod 2$, where $r = 1, 2, \ldots \frac{n'-1}{2}$. Therefore:

$$\sum_{k=1}^{\frac{n'-1}{2}} r_k + 1\{r_k > \frac{n'-1}{2}\} \equiv \sum_{r=1}^{\frac{n'-1}{2}} r \equiv \lceil \frac{n'-1}{4} \rceil \pmod 2 \tag{3.21}$$

From further casework highlighted in appendix C and appendix D, it can be seen that $\lceil \frac{n'-1}{4} \rceil + \frac{n'^2-1}{8} \equiv 0 \pmod 2$ and $\lceil \frac{n'-1}{4} \rceil \equiv \lceil \frac{n-1}{4} \rceil + \lceil \frac{t-1}{4} \rceil \pmod 2$. This leads to the following final equation for S:

$$S \equiv \begin{cases} \lceil \frac{n-1}{4} \rceil + \lceil \frac{t-1}{4} \rceil \pmod 2, & m \text{ even} \\ 0 \pmod 2, & m \text{ odd} \end{cases}$$

which gives the following equation for b:

$$b = \begin{cases} (-1)^{\lceil \frac{n-1}{4} \rceil + \lceil \frac{t-1}{4} \rceil}, & m \text{ even} \\ 1, & m \text{ odd} \end{cases} \tag{3.22}$$

Recalling (3.13), we see that the sign of the main cosine product is equivalent to the following expression for $t = 1$:

$$\operatorname{sgn}\left(\prod_{k=1}^{\frac{n'-1}{2}} \left(\frac{m\pi k}{n} \right) \right) = \begin{cases} (-1)^{\lceil \frac{n-1}{4} \rceil}, & m \text{ even} \\ 1, & m \text{ odd} \end{cases} \tag{3.23}$$

3.2.2. *Proof for $t \ge 3$.* By rewriting (3.4) without the absolute value signs, we see that:

$$\prod_{k=1}^{\frac{1}{2}((2\alpha+1)n'-1)} \cos\left(\frac{m'\pi k}{n'}\right) =$$

$$\left[\prod_{k=\alpha n'+1}^{\frac{1}{2}((2\alpha+1)n'-1)} \cos\left(\frac{m'\pi k}{n'}\right) \right] \prod_{\beta=0}^{\alpha-1} \left[\prod_{k=\beta n'+1}^{\frac{1}{2}((2\beta+1)n'-1)} \cos\left(\frac{m'\pi k}{n'}\right) \prod_{k=\frac{1}{2}((2\beta+1)n'+1)}^{(\beta+1)n'} \cos\left(\frac{m'\pi k}{n'}\right) \right]$$

By substituting (3.11), (3.12) and using (3.8), we see that:

$$\prod_{k=1}^{\frac{1}{2}((2\alpha+1)n'-1)} \cos\left(\frac{m'\pi k}{n'}\right) = (-1)^{\frac{m'\alpha(n'-1)}{2}} B \prod_{\beta=0}^{\alpha-1} (-1)^{m'n'\beta} C$$

As $t \geq 3$, we can set $t = 2\alpha + 1$ for $\alpha \geq 1$. Then the above simply becomes:

$$\prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{m\pi k}{n}\right) = (-1)^{m'\left(\frac{t-1}{4}\right)(n'-1)} BC^{\frac{t-1}{2}} \prod_{\beta=0}^{\frac{t-3}{2}} (-1)^{m'n'\beta} \tag{3.24}$$

For the product on the right side, it can be seen that when $m \equiv m' \equiv 0 \pmod 2$, $m'n'\beta \equiv 0 \pmod 2$ for all index β and the product reduces to 1. When m' is odd, $m'n' = 1 \pmod 2$. Therefore:

$$\prod_{\beta=0}^{\frac{t-3}{2}} (-1)^{m'n'\beta} = \prod_{\beta=0}^{\frac{t-3}{2}} (-1)^{\beta} = (-1)^{\lceil \frac{t-3}{4} \rceil}$$

This results in the following final equation:

$$\prod_{\beta=0}^{\frac{t-3}{2}} (-1)^{m'n'\beta} = \begin{cases} 1, & m \text{ even} \\ (-1)^{\lceil \frac{t-3}{4} \rceil}, & m \text{ odd} \end{cases} \tag{3.25}$$

Taking the sign of both sides in (3.24) and substituting equations (3.7), (3.8), (3.10), (3.22) and (3.25) appropriately, we derive:

$$\operatorname{sgn}\left(\prod_{k=1}^{\frac{n-1}{2}} \left(\frac{m\pi k}{n}\right)\right) = (-1)^{m'\left(\frac{t-1}{4}\right)(n'-1)} b (-1)^{\frac{m'(n'+1)(t-1)}{4}} \prod_{\beta=0}^{\frac{t-3}{2}} (-1)^{m'n'\beta}$$

$$= \begin{cases} (-1)^{\lceil \frac{n-1}{4} \rceil + \lceil \frac{t-1}{4} \rceil}, & m \text{ even} \\ (-1)^{\frac{t-1}{4}(n'-1+m'(n'+1)) + \lceil \frac{t-3}{4} \rceil} = (-1)^{\frac{t-1}{2} + \lceil \frac{t-3}{4} \rceil}, & m \text{ odd} \end{cases}$$

Finally, using the derivation from appendix E, we see that $\frac{t-1}{2} + \lceil \frac{t-3}{4} \rceil \equiv \lceil \frac{t-1}{4} \rceil \pmod 2$, resulting in the final expression for $t \geq 3$:

$$\operatorname{sgn}\left(\prod_{k=1}^{\frac{n-1}{2}} \left(\frac{m\pi k}{n}\right)\right) = \begin{cases} (-1)^{\lceil \frac{n-1}{4} \rceil + \lceil \frac{t-1}{4} \rceil}, & m \text{ even} \\ (-1)^{\lceil \frac{t-1}{4} \rceil}, & m \text{ odd} \end{cases} \tag{3.26}$$

By substituting $t = 1$, we see that the equation matches (3.23) for the $t = 1$ case. Therefore, for all possible values of t (3.26) holds. Combining with the absolute value of the product derived in (3.5), we arrive at the final expression:

$$\prod_{k=1}^{\frac{n-1}{2}} \cos\left(\frac{m\pi k}{n}\right) = \begin{cases} (-1)^{\lceil \frac{t-1}{4} \rceil} 2^{\frac{t-n}{2}}, & \text{if } m \text{ odd} \\ (-1)^{\lceil \frac{t-1}{4} \rceil + \lceil \frac{n-1}{4} \rceil} 2^{\frac{t-n}{2}}, & \text{if } m \text{ even} \end{cases}$$

as desired.

4. Extensions

From taking the log on both sides of (1.2) it can be seen that:

$$\log\left|\prod_{k=1}^{\frac{n-1}{2}}\cos\left(\frac{m\pi k}{n}\right)\right| = \frac{t-n}{2}\log 2$$

this can be rearranged to:

$$t = \frac{2\log\left(\left|\prod_{k=1}^{\frac{n-1}{2}}\cos\left(\frac{m\pi k}{n}\right)\right|\right)}{\log 2} + n$$

this can be simplified to the following:

$$t = gcd(m,n) = n + 2\sum_{k=1}^{\frac{n-1}{2}}\log_2\left|\cos\left(\frac{m\pi k}{n}\right)\right| \tag{4.1}$$

As only odd prime number $p = n$, will satisfy $t = 1$ for $m = 1, 2 \ldots \frac{p-1}{2}$, we can see that:

$$\text{p odd prime} \iff \sum_{m=1}^{\frac{p-1}{2}}\left[p + 2\sum_{k=1}^{\frac{p-1}{2}}\log_2\left|\cos\left(\frac{m\pi k}{p}\right)\right| - 1\right] = 0$$

Rearranging, one arrives at:

$$\left(\frac{p-1}{2}\right)^2\log 2 + \sum_{1\le i,j\le (p-1)/2}\log\left|\cos\left(\frac{ij\pi}{p}\right)\right| = 0$$

Therefore, the set of integer solutions p to this equation will be equivalent to the set of all odd prime numbers.

5. Appendix

Appendix A.

Claim: $\forall\ m', n', k \in \mathbb{Z}, \quad \exists\, q = \lceil \frac{m'k}{n'} + \frac{1}{2} \rceil - 1 \in \mathbb{Z}$ such that $|m'k - qn'| < \frac{n'}{2}$ or $\cos\left(\frac{(m'k - qn')\pi}{n'}\right) > 0$.

Proof.

$$|m'k - qn'| < \frac{n'}{2} \iff -\frac{1}{2} < \frac{m'k}{n'} - q < \frac{1}{2} \iff \frac{m'k}{n'} - \frac{1}{2} < q < \frac{m'k}{n'} + \frac{1}{2}$$

As the range for q is less than 1, there exists exactly one integer q that satisfies the inequality. The value for this q will be:

$$\frac{m'k}{n'} - \frac{1}{2} = \frac{m'k}{n'} + \frac{1}{2} - 1 \leq q = \lceil \frac{m'k}{n'} + \frac{1}{2} \rceil - 1 \leq \frac{m'k}{n'} + \frac{1}{2}$$

□

Appendix B.

Claim: $\lceil \frac{m'k}{n'} + \frac{1}{2} \rceil = \lceil \frac{1}{2} \lfloor \frac{2m'k}{n'} \rfloor \rceil + 1$

Proof. Let $q = \lfloor \frac{2m'k}{n'} \rfloor$ be the quotient when $m'k$ is divided by $\frac{n'}{2}$ and $r = m'k - q\frac{n'}{2}$ the remainder. This implies that $0 < r < \frac{n'}{2} \iff 0 < \frac{r}{n'} < \frac{1}{2}$. It can be seen that:

$$\lceil \frac{m'k}{n'} + \frac{1}{2} \rceil = \lceil \frac{r}{n'} + \frac{q+1}{2} \rceil = \begin{cases} \frac{q+1}{2} + 1 = \lceil \frac{q}{2} \rceil + 1, & \text{if } q \text{ odd} \\ \frac{q}{2} + 1 = \lceil \frac{q}{2} \rceil + 1, & \text{if } q \text{ even} \end{cases}$$

$$\lceil \frac{m'k}{n'} + \frac{1}{2} \rceil = \frac{q}{2} + 1 = \lceil \frac{1}{2} \lfloor \frac{2m'k}{n'} \rfloor \rceil + 1$$

□

Appendix C.

Claim: $\lceil \frac{n'-1}{4} \rceil + \frac{n'^2-1}{8} \equiv 0 \pmod 2$

Proof. As n' is odd, $\frac{n'-1}{2} \in \mathbb{Z}$ and the following casework can be considered:

(1) If $\frac{n'-1}{2} \equiv 0 \pmod 2$, then $\frac{n'+1}{2} \equiv 1 \pmod 2$ and $\lceil \frac{n'-1}{4} \rceil = \frac{n'-1}{4}$: $\lceil \frac{n'-1}{4} \rceil + \frac{n'^2-1}{8} = \frac{n'-1}{4} + \frac{n'-1}{4}\frac{n'+1}{2} \equiv \frac{n'-1}{2} \equiv 0 \pmod 2$, then

(2) If $\frac{n'-1}{2} \equiv 1 \pmod 2$, then $\frac{n'+1}{2} \equiv 0 \pmod 2$ and $\lceil \frac{n'-1}{4} \rceil = \frac{n'+1}{4}$: $\lceil \frac{n'-1}{4} \rceil + \frac{n'^2-1}{8} = \frac{n'+1}{4} + \frac{n'-1}{2}\frac{n'+1}{4} = \frac{n'+1}{4}\frac{n'+1}{2} \equiv 0 \pmod 2$

Therefore, regardless of the parity of $\frac{n'-1}{2}$:

$$\lceil \frac{n'-1}{4} \rceil + \frac{n'^2-1}{8} \equiv 0 \pmod 2$$

□

Appendix D.

Claim: $\lceil\frac{n'-1}{4}\rceil \equiv \lceil\frac{n-1}{4}\rceil + \lceil\frac{t-1}{4}\rceil \pmod 2$

Proof. For $t=1$, the claim is trivial as $n'=n$ and $\lceil\frac{t-1}{4}\rceil = 0$. For $t \geq 3$, we note that $\lceil\frac{n-1}{4}\rceil = \lceil\frac{n'-1}{4} + n'\frac{t-1}{4}\rceil$. Then consider the following casework:

(1) If $\frac{t-1}{2} \equiv 0 \pmod 2$, $\frac{t-1}{4} \in \mathbb{Z}$:
$\lceil\frac{n-1}{4}\rceil = \lceil\frac{n'-1}{4}\rceil + n'\frac{t-1}{4} \equiv \lceil\frac{n'-1}{4}\rceil + \lceil\frac{t-1}{4}\rceil \pmod 2$
$\iff \lceil\frac{n'-1}{4}\rceil = \lceil\frac{n'-1}{4}\rceil + n'\frac{t-1}{4} \equiv \lceil\frac{n-1}{4}\rceil + \lceil\frac{t-1}{4}\rceil \pmod 2$

(2) If $\frac{t-1}{2} \equiv 1 \pmod 2$, $\frac{t-1}{4} + \frac{1}{2} = \frac{t+1}{4} \in \mathbb{Z}$:

$$\lceil\frac{n-1}{4}\rceil = \lceil\frac{n'-1}{4} + n'\frac{t+1}{4} - \frac{n'}{2}\rceil \equiv \lceil -\frac{n'+1}{4}\rceil + \lceil\frac{t+1}{4}\rceil = \begin{cases} -\lceil\frac{n'+1}{4}\rceil + \lceil\frac{t+1}{4}\rceil \pmod 2, & \frac{n'+1}{2} \text{ even} \\ -\lceil\frac{n'-1}{4}\rceil + \lceil\frac{t-1}{4}\rceil \pmod 2, & \frac{n'+1}{2} \text{ odd} \end{cases}$$

In the case where $\frac{n'+1}{2}$ is even, $\lceil\frac{n'+1}{4}\rceil = \lceil\frac{n'+1}{4} - \frac{1}{2}\rceil = \lceil\frac{n'-1}{4}\rceil$, therefore in general: $\lceil\frac{n'-1}{4}\rceil \equiv -\lceil\frac{n-1}{4}\rceil + \lceil\frac{t-1}{4}\rceil \equiv \lceil\frac{n-1}{4}\rceil + \lceil\frac{t-1}{4}\rceil \pmod 2$

Therefore regardless of the parity of $\frac{t-1}{2}$:

$$\lceil\frac{n'-1}{4}\rceil \equiv \lceil\frac{n-1}{4}\rceil + \lceil\frac{t-1}{4}\rceil \pmod 2$$

□

Appendix E.

Claim: $\frac{t-1}{2} + \lceil\frac{t-3}{4}\rceil \equiv \lceil\frac{t-1}{4}\rceil \pmod 2$

Proof. As t odd, $\frac{t-1}{2} \in \mathbb{Z}$. We can consider the following casework:

(1) When $\frac{t-1}{2} \equiv 0 \pmod 2$, $\frac{t-1}{4} \in \mathbb{Z}$:
$\frac{t-1}{2} + \lceil\frac{t-3}{4}\rceil = \frac{t-1}{2} + \lceil\frac{t-1}{4} - \frac{1}{2}\rceil = 3\frac{t-1}{4} + \lceil -\frac{1}{2}\rceil \equiv \frac{t-1}{4} = \lceil\frac{t-1}{4}\rceil \pmod 2$

(2) When $\frac{t-1}{2} \equiv 1 \pmod 2$, $\frac{t-1}{4} + \frac{1}{2} \in \mathbb{Z}$:
$\frac{t-1}{2} + \lceil\frac{t-3}{4}\rceil = \frac{t-1}{2} + \frac{t-3}{4} \equiv 1 + \frac{t-3}{4} = \frac{t+1}{4} \pmod 2$
as $\frac{t+1}{4} \in \mathbb{Z}$, we know that:
$\frac{t-1}{2} + \lceil\frac{t-3}{4}\rceil \equiv \frac{t+1}{4} = \lceil\frac{t+1}{4} - \frac{1}{2}\rceil = \lceil\frac{t-1}{4}\rceil \pmod 2$

Therefore regardless of the parity of $\frac{t-1}{2}$:

$$\frac{t-1}{2} + \lceil\frac{t-3}{4}\rceil \equiv \lceil\frac{t-1}{4}\rceil \pmod 2$$

□

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Acknowledgement

1. The motivation for the research topic came from a problem that the author encountered through a friend that involved the simplified identity highlited in Equation (1.1), which can be found in several papers concerning the factorization of Fibonacci and Lucas numbers, as mentioned briefly in the introduction section. The author conducted the mathematical body of the research independently without external support, other than the references listed in the references section. The advisor was responsible solely for providing assistance or guidance in writing and editing the thesis (paper), playing no role in the derivation of the mathematical contents presented in the thesis. The advisor was not compensated or paid for their guidance or any role in the research. No assitance from those other than the advisor was given or used. Digital tools like Desmos and ChatGPT were used for algebra and checking results within the investigation which were originally derived by the author. Certain unmentioned references were used in formatting the thesis in LaTeX. Some more information about the advisor is given in the advisor form.

2. The motivation for the research topic has already been given in (1). Given the nature of the research – no data acquisition, data analysis or computation, experimental design and implementation was necessary. The author, after deciding upon the research topic, used tools including Desmos and Python to find the trend in the product, including the trend in its sign and the magnitude for different values of n and m. After the general trends were discovered, the author attempted to prove them mathematically by proving for the simple m=1 case first (which he later discovered has already been proven in several papers included in the references, most notably reference [3]), moving onto the generalized case with changing signs. The order of the sections in the thesis roughly follow the chronological progression of the author in the investigation. As mentioned previously, the entirety of the research component, including all the mathematical derivations, were done exclusively by the single author, without assitance from teammates, other students, the advisor, relatives or any other individual / entity. The advisor provided guidance in registering for the competition and giving suggestions in writing the thesis. Several difficulties were encountered during the incremental research progress, mostly having to deal with the author failing to prove certain sections, most notably determining the sign of (3.22). The author also initially tried using the roots of unity method for different values of m, but later realized the result is misguided and found that a different approach is necessary. There were also several instances where the author realized that certain parts of the proof were incomplete or wrong altogether, needing adjustment. As it is often the case with mathematical research, the challenges were simply overcomed by trying different approaches, refining previous approaches over time.

Other Notes: the research has not been published anywhere previously, but the author do intend at the current moment to use an extended part of the research for the IB Math IA.

修改说明:

Page 2 - “As the above expression is satisfied for all z in R”-> “As for all z in R the above equation is satisfied”

Page 3 - “Finally, by substituting (2.3), ” -> “Finally, by substituting (2.3) for the cosine product,”

Page 10 - equation 3.26 changed “n’ -> n” in the product from k=1 to 0.5(n'-1) , correcting type error in original paper.

Page 11 - “Only for odd prime number p= n,” -> “As only odd prime number p= n, will satisfy”