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论文题目：On an inverse problem of Bermond's conjecture

On an inverse problem of Bermond's conjecture

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Abstract

A Hamilton cycle is a spanning cycle that visits each vertex exactly once and the Hamilton cycle problem is the problem of determining whether a Hamilton cycle exists in a given graph. The Hamilton cycle problem is one of the crucial problems in graph theory and the Hamilton decomposability of line graphs has been studied extensively. A long-standing conjecture of Bermond stated that the line graph of a Hamilton decomposable graph is Hamilton decomposable. In this paper, we study an inverse problem of Bermond's conjecture and characterize the root graphs whose line graphs can be completely decomposed into three Hamilton cycles.

Keywords: Hamilton cycle; Bermond's conjecture; Line graph; Hamilton cycle decomposition

Contents

1 Introduction

Throughout this paper, a graph $G = (V(G), E(G))$ is a finite undirected graph with no multiple edges or loops. We say that a cycle in a graph G is a *Hamilton cycle* when it contains all vertices of G without repetition. A graph is defined as *Hamiltonian* if it contains a Hamilton cycle. The *line graph* of a graph G, written as $L(G)$, is constructed by assigning a vertex to an edge of G, with two vertices connected if and only if their associated edges in G share a common endpoint. The graph G is the *root graph* of $L(G)$. If the edges of a graph G can be decomposed into a set of Hamilton cycles in G so that every edge belongs to exactly one cycle, this collection is called a *Hamilton decomposition* of G. A graph is *Hamilton decomposable* if it possesses a Hamilton decomposition. The Hamilton cycle problem concerns deciding if a Hamiltonian cycle is present in a specified graph. It is well-known that these problems are very hard to solve, which are NP-complete. For the standard terminology and notation, we follow [2].

In 1988, Bermond [1] proposed the following famous conjecture in graph theory,

Conjecture 1.1. *A Hamilton decomposable graph has a line graph that is Hamilton decomposable.*

Here, we present the progress of previous works that support the conjecture of Bermond. Kotzig's 1964 result [9] states that a cubic graph is Hamiltonian exactly when its line graph is Hamilton decomposable. In 1983, Jaeger [8] proved that whenever a simple graph G with an even number of edges has a Hamiltonian decomposition of its edge set, the line graph of G is 1-factorizable. Moreover, if a graph G can be completely decomposed into an even (respectively, odd) number of Hamilton cycles, then $L(G)$ can be decomposed into Hamilton cycles (respectively, into Hamilton cycles along with a 2-factor). In addition, for a $2k$-regular graph G that contains a Hamilton cycle, $L(G)$ admits a decomposition into Hamilton cycles together with a 2-factor (A. Muthusamy and P. Paulraja, 1995) [10]. This settles Bermond's conjecture for all graphs which have a Hamilton cycle decomposition with even number cycles. It was also proved that when a bipartite $(2k+1)$-graph G is decomposable, then $L(G)$ is also Hamilton decomposable (Pike, 1995) [11]. It was further proved that some bipartite Hamilton decomposable graphs which are regular graphs and with degree $\delta \equiv 2\ (mod\ 4)$ have Hamilton decomposable line graphs. It settles the conjecture of Bermond for every bipartite Hamilton decomposable graph G with vertex connectivity $\kappa(G) = 2$ (Pike, 2005) [12]. Recently, Bryant et al. [3] claimed that they have fully proved Bermond's conjecture and the paper can be found in arXiv. Other related results can be found in [4, 5, 7].

In 2022, Sivaraman and Zaslavsky [13] asked an inverse problem of Bermond's conjecture:

Problem 1. *If $L(G)$ is decomposable into k Hamilton cycles. What is G?*

In the same paper, they studied the case when $k = 2$.

Theorem 1.1. *[13] The graph G of Problem 1, when $k = 2$, is either $K_{1,5}$, or the first subdivision of a 4-regular graph G' that decomposes into two Hamilton cycles, or a Hamilton cubic graph.*

In this paper, we study the Problem 1 when $k = 3$ and characterize the root graphs G whose line graphs can be decomposable into three Hamilton cycles.

2 Preliminary and main results

Extensive work has been carried out by various researchers on Hamiltonian cycles of line graphs. The first contribution to this topic was given by Harary and Nash-Williams in 1965 [6].

Theorem 2.1. *[6] For a graph G, $L(G)$ is Hamiltonian if and only if G has a dominating closed trail.*

Here, we define the three *quasi-compatible dominating trails* for (5,3)-biregular graphs and 4-regular graph respectively.

A *quasi-compatible dominating trial* for (5, 3)-biregular graphs T_i ($i = 1, 2, 3$) is defined as follows: T_i visits all 5-valent vertices of the root graph G at least once. If a vertex u is not in T_i, it is in both the other two dominating trails. Additionally, each edge is visited either not or twice by the three dominating trails.

The three *quasi-compatible dominating trails* for 4-regular graphs T_i ($i = 1, 2, 3$) satisfy the following. First, each 2-path can be visited at most three times. Second, each vertex v is visited by at least two of the quasi-compatible dominating trails. Third, each edge can only be visited once or three times by the dominating trails. Fourth, two vertices that are visited only by two of the dominating trails and visited by the same pair can not be adjacent to the same vertex that is visited twice in a dominating trail at a specific occurrence. Fifth, two vertices that are visited only by two of the dominating trails and visited by the two distinct pairs can not be adjacent to the same vertex that is visited twice in a dominating trail at the two distinct occurrences.

The following is our main result. We characterize the root graphs whose line graphs can be decomposable into three Hamilton cycles.

Theorem 2.2. *The graph G of Problem 1, when $k = 3$, is either $K_{1,7}$, or the first subdivision of a 6-regular graph G' that decomposes into three Hamilton cycles, a $(3, 5)$-biregular graph that contains three quasi-compatible dominating trails, or a 4-regular graph with three quasi-compatible dominating trails.*

SG is the first subdivision of a graph G, obtained by subdividing every edge into a path of length 2.

The first case gives line graph K_7, which is obviously a solution to Problem 1 when $k = 3$.

In the second case, Sivaramana and Zaslavskyb [8] gave a construction of the decomposition of $L(G)$ for $G = SG'$ of a graph $2h - regular\ G'$, $h \geq 2$, which decomposes into h Hamilton cycles $H'_1, \ldots, H'_h$.

For the third and the fourth case, this paper will explain how the line graph decomposes in terms of the three quasi-compatible dominating graphs.

3 Construction

We describe the construction of the Hamilton decomposition of our 6-regular $L(G)$ in each case other than K_7.

Example 1. Suppose G is a (3, 5)-biregular graph with three quasi-compatible dominating trails.

There are only four different ways to partition the five edges at a 5-valent vertex v. As each edge incident to v can only be visited either not or twice and each vertex is visited by each quasi-compatible dominating trail at least once, at least three 2 paths and at most five 2-path is visited at the v. The first case is three 2-paths in the dominating trails at v in G, visiting three

Figure 1: Case 1 and Case 2 in Example 1

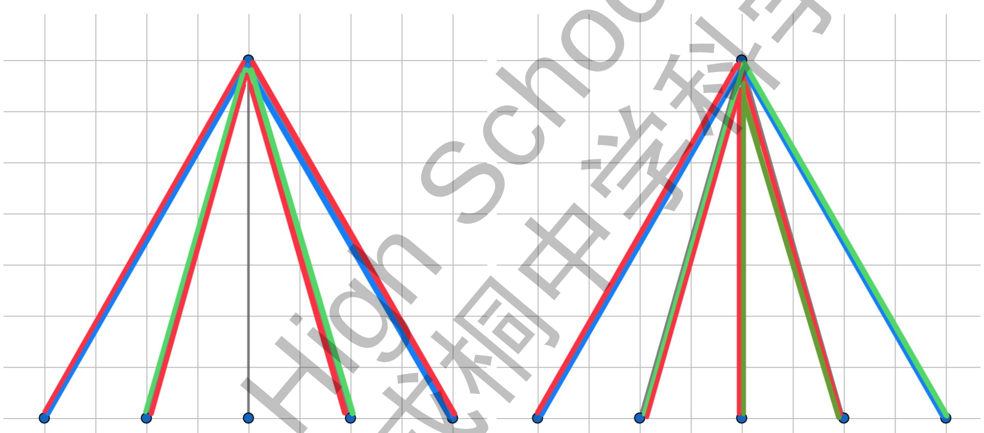

Figure 2: Case 3 and Case 4 in Example 1

edges incident to v twice each. It is straightforward to verify that no 2-path can be visited twice in this case (see Fig. 1 Case 1, different colors represent different dominating trails). The second case is four distinct 2-path in the dominating trails at v in G, visiting four edges incident to v twice each, while all 2-path is visited once (see Fig. 1 Case 2). The third case is two distinct 2-paths with no overlapping edges in the dominating trails at v in G, visited twice each (see Fig. 2 Case 3). The fourth case is five 2-paths in the dominating trails at v in G. By inspection, we know that the 2-paths have to be distinct (see Fig. 2 Case 4).

Each partition of the five edges at the 5-valent vertex v in G can correspond to one way of partitioning the 5-clique $L(v)$ in the line graph.

Here, we demonstrate one method of partitioning the 5-cliques in $L(G)$ into H_1, H_2 and H_3 according to how the three quasi-compatible dominating trails T_1, T_2 and T_3 partitioned the 2-paths at 5-valent vertices. Let there be $5n$ 3-valent vertices and $3n$ 5-valent vertices. Consider a 5-valent vertex v, let the 3-valent vertices adjacent to v in G be u_1, u_2, u_3, u_4 and u_5, thus $L(v)$ is

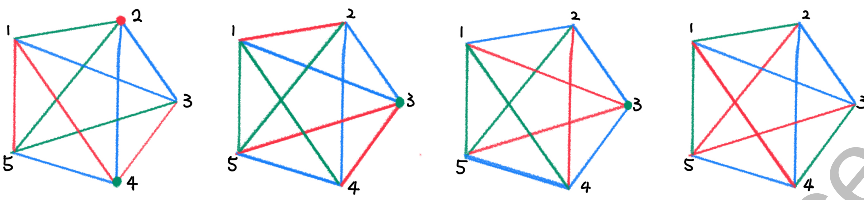

Figure 3: Coloring the edges and the vertices of the 5-cliques (Case 1 to 4, from left to right)

the 5-clique with vertices vu_1, vu_2, vu_3, vu_4 and vu_3 (referred to as e_1, e_2, e_3, e_4, e_5 respectively in the rest of the Example 1). Let T_i, T_j and T_k be the three distinct dominating trails ($i \neq j \neq k$).

We start by coloring the edges and the vertices of the 5-cliques with the colors c_p^q ($p = 1, 2, 3$ and $q = 1, 2$). We demonstrate a mapping from e_1, e_2, e_3, e_4, e_5 to the vertices numbered 1,2,3,4 and 5 in Fig. 3.

If the partitioning of the 2-paths at v is isomorphic to Case 1 in Fig. 1: let the colored 2-paths be in the three dominating trails respectively, while $u_2 \in T_i$ and $u_4 \in T_j$, since u_2 and u_4 are both contained by two distinct dominating trails. As shown in the partitioning of $K_5(v)$ of Case 1 in Fig. 3, the red, green and blue path are colored in c_i, c_j and c_k respectively, where e_1 is the edge in both T_j and T_k, e_3 is the edge in both T_i and T_j and e_5 is the edge in both T_i and T_k. Additionally, e_1 is colored c_j^1 and c_k^1; e_3 is colored c_i^1 and c_j^1; e_5 is colored c_i^1 and c_k^1; e_2 is colored c_i^2 and e_4 is colored c_i^2 in $L(G)$.

If the partitioning of the 2-paths at v is isomorphic to Case 2 or 3 in Fig. 1 and Fig. 2: let the colored 2-paths be in the three dominating trails respectively, and vu_3 be the edge not visited by the three dominating trails. Let T_i be the dominating trail that visited v twice. As each u is included by at least two distinct dominating trails, let $u_3 \in V(T_j)$. Thus, as shown in the partitioning of $K_5(v)$ in $L(G)$ of Case 2 or Case 3 (respectively) in Fig. 3, the red, green and blue path are colored in c_i, c_j and c_k respectively, where in e_1 and e_5 are the two edges in both T_i and T_k and e_2 and e_4 are the two edges in both T_i and T_j. Differently, in Case 2 e_1 and e_2 are adjacent in T_i while in Case 3 e_1 and e_5 are adjacent in T_i. Additionally, e_1 and e_5 are colored c_i^1 and c_k^1, e_2 and e_4 are colored c_i^1 and c_j^1, and e_3 is colored c_j^2 in $L(G)$.

If the partitioning of the 2-paths at v is isomorphic to Case 4 in Fig. 2: let the colored 2-paths be in the three dominating trails respectively. Let T_i be the dominating trail that only visited v once. As shown in the partitioning of $K_5(v)$ in $L(G)$ of Case 4 (respectively) in Fig. 3, the blue, red and green path are colored in c_i, c_j and c_k respectively, where e_1 is the edge in both T_i and T_j, e_5 is the edge in both T_i and T_k, and e_2, e_3 and e_4 are the edges in both T_j and T_k. Additionally, e_2 and e_5 are adjacent in T_k while e_1 and e_4 are adjacent in T_j. We color e_1 with both c_i^1 and c_j^1, e_5 with both c_i^1 and c_k^1, and $e_2, 2_3$ and e_4 with both c_j^1 and c_k^1 in $L(G)$.

Then, we partition the edges in the 3-cliques in $L(G)$ according to our coloring. First, in each 3-clique, we partition each edge that connect the vertices colored in c_i^1 ($i = 1, 2, 3$) into H_i. Then, if the other vertex is colored in c_i^2, we substitute the edge with the two other edge in the 3-clique.

In the line graph $L(G)$ partitioned into H_1, H_2, H_3, in each 5-clique, the vertices are either visited by paths through the 5-clique or colored and visited by a 2-path in the 3-clique. Thus all vertices are visited exactly once by each Hamilton cycle, by the paths in 5-cliques and 3-cliques. The paths are connected as the terminal vertices are connected because only the terminal vertices of a path partitioned in H_i ($i = 1, 2, 3$) is colored c_i^1 and made the terminal vertices of the path partitioned into H_i in the 3-cliques. Thus, we have proved that H_1, H_2 and H_3 constructed are three distinct Hamilton cycles of $L(G)$.

Example 2. Suppose G is a 4-regular graph with three quasi-compatible dominating trials.

We know that the three quasi-compatible dominating trails satisfy the following conditions: first, each 2-path can be visited at most three times; second, each vertex v is visited by at least two of the quasi-compatible dominating trails; third, each edge can only be visited once or three times by the dominating trails; fourth, two vertices that are visited only by two of the dominating trails and visited by the same pair can not be adjacent to the same vertex that is visited twice in a dominating trail at a specific occurrence; fifth, two vertices that are visited only by two of the dominating trails and visited by the two distinct pairs can not be adjacent to the same vertex that is visited twice in a dominating trail at the two distinct occurrences.

There are only are only five different ways to partition the four edges each vertex v. As each edge incident to v can only be visited either once or trice and each vertex is visited by at least two distinct quasi-compatible trails. Thus each v is visited by at least two 2-paths and at most six 2-paths. Fig. 4 and Fig. 5 below are the five cases, with each color representing 2-paths in one dominating trails and different shade of the same color signify two different visits of v by the same dominating trails. The first case is two 2-paths, visiting each edge incident to v once (see Fig. 4 Case 1). The second case is three 2-path, visiting one edge incident to v trice and the three other edges incident to v once (see Fig. 4 Case 2). There are two cases when there are four 2-paths of the three quasi-compatible dominating trails at v. The third case has only two distinct 2-paths, with one visited three times by the three dominating trails (see Fig. 5 Case 3). The fourth case has only three distinct 2-paths, with one visited twice (see Fig. 5 Case 4). It can be proven that v cannot be visited for five times by the three dominating trails. Since there are five 2-paths from the three dominating trails and that each edge can only be visited once or trice, thus, there must be one edge visited once and three edges visited trice. Since each dominating trail can only visit each edge once, three edges are in all three of the dominating trails. Thus, the vertex must be visited twice by each dominating trail, resulting in six 2-paths instead of five. In the fifth case, v is visited six times by the three quasi-compatible dominating trails. From the 4-clique $K_4(v)$ in the line graph $L(G)$, there is only 6 edges and thus only one way to partition it into 6 paths. Since each dominating trail can only visit each edge once, the only case when v is visited six times is when all six distinct 2-paths at v is visited once (see Fig. 5 Case 5).

Each partitioning of the four edges at the vertices in the root graph G corresponds to ways of partitioning the 4-cliques in the line graph.

We demonstrate one method of partitioning the 4-cliques in $L(G)$ into H_1, H_2 and H_3 according to how the three quasi-compatible dominating trails T_1, T_2 and T_3 partitioned the 2-paths at the vertices. Let there be n ($n \geq 3$) vertices. Consider a vertex v, let the vertices adjacent to v in the root graph G be v_1, v_2, v_3 and v_4. Thus $K_4(v)$ is the 4-clique with the vertices vv_1, vv_2, vv_3 and vv_4 (referred to as $e_{i_1}, e_{i_2}, e_{i_3}$ and e_{i_4} ($\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$) respectively in the rest of

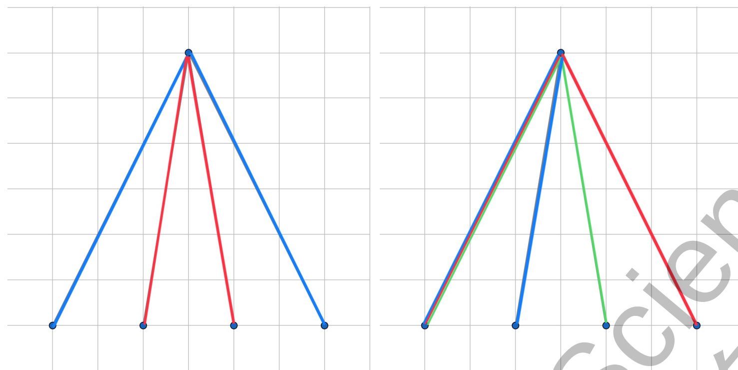

Figure 4: Case 1 and Case 2 in Example 2

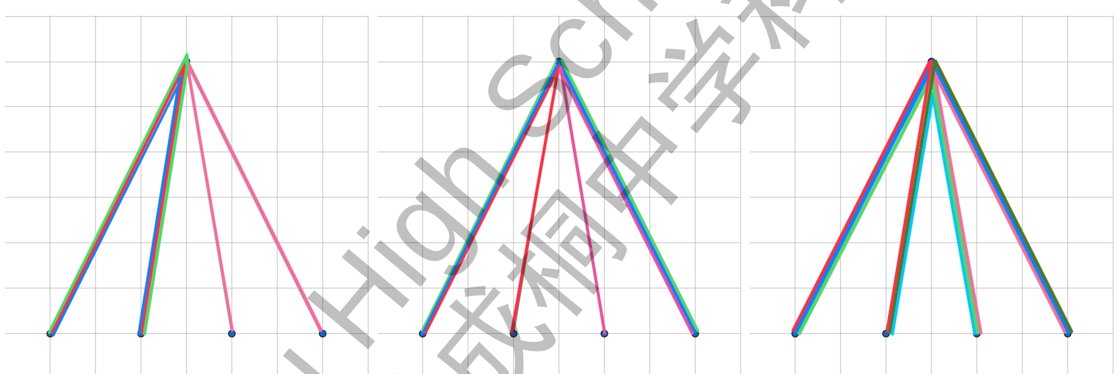

Figure 5: Case 3, Case 4 and Case 5 in Example 2

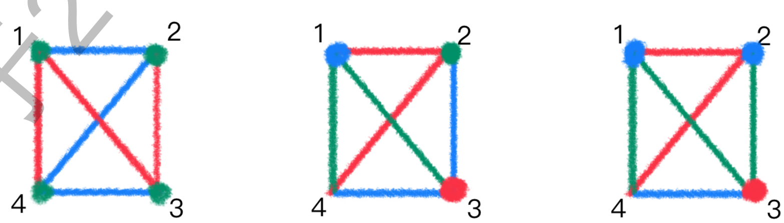

Figure 6: Coloring the edges and the vertices of the 4-cliques (Case 1 to 3, from left to right)

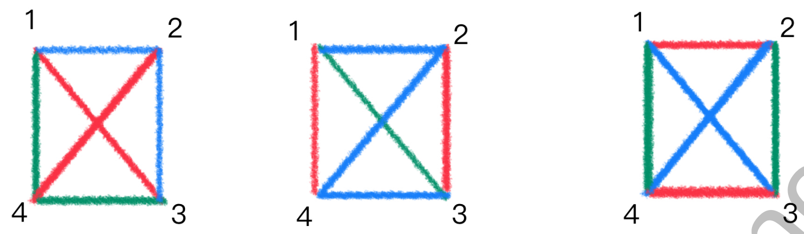

Figure 7: Coloring the edges and the vertices of the 4-cliques (Case 4 to 6, from left to right)

the Example 2). Let T_i, T_j and T_k be the three distinct dominating trails ($i \neq j \neq k$).

First, we color some of the vertices in the line graph $L(G)$ in color c_i ($i = 1, 2, 3$). An edge e in the root graph is incident to two vertices. If the edge is only visited once by the three dominating trails and only one of the vertex that its incident to is visited by all three of the dominating trails. Let the other vertex incident to e be visited by T_i and T_j, we color the vertex e in the line graph with c_k. Next, we partition the 4-cliques according to the three quasi-compatible dominating trails.

If a vertex v is visited by one dominating trail twice in G, let $e_{i_1} v e_{i_2}$ and $e_{i_3} v e_{i_4}$ be the two distinct 2-paths visited by T_i. Then, in the line graph $L(G)$, the edges $e_{i_1} e_{i_2}$ and $e_{i_3} e_{i_4}$ in $K_4(v)$ are partitioned into H_i.

If the partitioning of the 2-paths at v is isomorphic to Fig. 5 Case 5: the six 1-paths in $K_4(v)$ in $L(G)$ is already partitioned into H_1, H_2 and H_3 in the previous step.

We partition all the 4-cliques $K_4(v)$ of vertex v that is isomorphic to Fig. 4 Case 1. If a vertex v in the root graph G is only visited by two of the dominating trails (Case 1): the paths are only partitioned into two of the dominating cycles. Let e_1 and e_3 be the edges in T_i and e_2 and e_4 be the edges in T_j. As shown in the partitioning of $K_4(v)$ in $L(G)$ of Case 1 in Fig. 6, the blue and red paths are partitioned into H_i and H_j respectively. In the line graph, we color e_1, e_2, e_3, e_4 in c_k.

If the partitioning of the 2-paths at v is isomorphic to Fig. 4 Case 2: let e_4 be the edge incident to v that is visited by all three of the dominating trails in the root graph, and let e_1, e_2, e_3 be in T_i, T_j, T_k in the root graph respectively. If e_1, e_2, e_3 are colored by three distinct colors: let $e_{i_1}, e_{i_2}, e_{i_3}$ ($\{i_1, i_2, i_3\} = \{1, 2, 3\}$) be the vertices in $L(G)$ that are colored c_i, c_j, c_k respectively. e_{i_1} can not be e_1 because we only vertices not visited by one dominating trail can be painted with the corresponding color. Similarly, $e_2 \neq e_{i_2}$ and $e_3 \neq e_{i_3}$. We partition the path that pass through the vertices e_1, e_{i_1}, e_4 in $K_4(v)$ in $L(G)$ into H_i and color the other vertex in c_i, e_2, e_{i_2}, e_4 into H_j and color the other vertex in c_j, and e_3, e_{i_3}, e_4 into H_k and color the other vertex in c_k (see Case 2 in in Fig. 6). If e_1, e_2, e_3 are colored by two distinct colors: let e_{i_1} be colored c_i and e_{i_2}, e_{i_3} be colored c_j. e_{i_1} can not be e_1 because we only vertices not visited by one dominating trail can be painted with the corresponding color. Similarly, $e_{i_2}, e_{i_3} \neq e_2$. We partition the path that pass through the vertices e_1, e_{i_1}, e_4 in $K_4(v)$ in $L(G)$ into H_i and color the other vertex in c_i, $e_2, e_{i_2}, e_{i_3}, e_4$ into H_j, and e_3, e_4 into H_k and color the other two vertices in c_k (see Case 3 in in Fig. 6).

If the partitioning of the 2-paths at v is isomorphic to Fig. 5 Case 3: let T_i be the dominating trail that visited v twice, two 1-path is already partitioned into H_i in the previous step. Let e_1 and e_3 be the two edges in G visited by all three of the dominating trails. Let e_2 be colored in c_j. We partition the path that pass through the vertices e_1, e_2, e_3 in $K_4(v)$ in $L(G)$ into H_j and e_1, e_4, e_3 into H_k (see Case 4 in Fig. 7). e_2, e_4 can not be colored in the same color because, by definition of the quasi-compatible dominating trails, two vertices that are visited only by two of the dominating trails and visited by the same pair can not be adjacent to the same vertex that is visited twice in a dominating trail at a specific occurrence. Additionally, we color e_2, e_4 in c_k, c_j respectively.

If the partitioning of the 2-paths at v is isomorphic to Fig. 5 Case 4: let T_i be the dominating trail that visited v twice, two 1-path is already partitioned into H_i in the previous step. Let e_1 and e_3 be the two edges in G visited by all three of the dominating trails. Let e_2, e_4 be colored in c_j. We partition the path that pass through the vertices e_1, e_2, e_4, e_3 in $K_4(v)$ in $L(G)$ into H_j and e_1, e_3 into H_k. e_2, e_4 can not be colored in two distinct colors because by definition of the quasi-compatible dominating trails, two vertices that are visited only by two of the dominating trails and visited by the two distinct pairs can not be adjacent to the same vertex that is visited twice in a dominating trail at the two distinct occurrences. Additionally, we color e_2, e_4 in c_k.

After partitioning all 4-cliques in $L(G)$ for vertices isomorphic to Fig. 5 Case 1 and Fig. 6 Case 5 in the root graph G. We partition the 4-cliques in $L(G)$ in which all vertices e, corresponding to an edge that is not included by all three dominating trails, are colored. If all the 4-cliques in $L(G)$ are not colored: for vertex in the root graph as Fig. 5 Case 2, we arbitrarily color the vertices $e_{i_1}, e_{i_2}, e_{i_3}$ ($\{i_1, i_2, i_3\} = \{1, 2, 3\}$) in c_i, c_j, c_k respectively (see Case 2 in Fig. 6); for vertex in the root graph as Fig. 6 Case 3, we color the vertices e_{i_1}, e_{i_2} ($\{i_1, i_2\} = \{2, 4\}$) in c_j, c_k respectively (see Case 4 in Fig. 7); for vertex in the root graph as Fig. 6 Case 4, we color the vertices e_{i_1}, e_{i_2} ($\{i_1, i_2\} = \{2, 4\}$) in c_j (see Case 5 in Fig. 7).

Now, we prove that the paths in partitioned into each H_i ($i = 1, 2, 3$) are connected and are Hamilton cycles. In the line graph $L(G)$ partitioned into H_1, H_2, H_3, each vertex either correspond to an edge is visited by one or three distinct dominating trails in the root graph G. All the vertex that correspond to an edge that is visited by all three distinct dominating trails in G are the terminal vertices for three paths partitioned into each of H_1, H_2, H_3 respectively in both 4-cliques that it is in. The vertices that correspond to an edge that is visited only by T_i is the end terminal of a path partitioned into H_i for both 4-cliques that it is in. Let the two 4-cliques that it is in be $K_4(v)$ and $K_4(u)$, partitioned in this order. It must be visited in the path partitioned into H_j ($j \in \{1, 2, 3\}$ and $j \neq i$). Thus, it will be colored in the color corresponding to c_k ($k \in \{1, 2, 3\}$ and $k \neq i, j$). Thus, it will be visited by the path partitioned into H_k in $K_4(u)$ by the partitioning method above. Thus H_1, H_2, H_3 passes through each vertex in $L(G)$ exactly once and are three edge-disjoint 2-factors of $L(G)$. As each 4-cliques in $L(G)$ is visited by each H_i in the order of the corresponding vertex in G in T_i, each H_i is a complete Hamilton cycle.

4 Proof

We prove that the examples in Theorem 2 are the only ones. We assume given a line graph $L = L(G)$, but not the root graph G, and a decomposition of L into Hamilton cycles H_1, H_2 and H_3. A root graph can be quickly calculated from L, thus the vertex cliques in L are known. Because the line graph can be decomposed completely into three Hamilton cycles, the line graph is 6-regular. Thus each edge in G is adjacent with 6 distinct edges. In particular, we know when $G = K_{1,7}$ because then L is a 7-clique. If it is not, then we know when G is (6,2)-biregular by the existence of 6-cliques in L, G is (5, 3)-biregular by the existence of 5-cliques and 3-cliques, and G is 4-regular in the remaining case. Thus, we can consider each type of root graph separately. $(L = K_7 \ldots)$

Case 1.(Sivaraman and Zaslavsky, 2022) [13] *Suppose G is* (6, 2)*-biregular.* We can reconstruct G'', a graph isomorphic to G', directly from the 6-cliques in L. The vertices k_i of G'' are the 6-cliques in L, and there is an edge $k_i k_j$ of G'' for each edge between a vertex of L in $L(k_i)$ and one in $L(k_j)$. We know $G'' \cong G'$ because each vertex of G'' is the vertex clique of $L = L(G)$ that corresponds to a quadrivalent vertex in SG', that is, a vertex of G'.

The graph G'' enables us to deduce the two Hamilton cycles H_i'' of G'' from H_i in L. Since a vertex in G'', considered as a 6-clique in L, is quadrivalent, each Hamilton cycle $H_i$of L must enter and leave that 6-clique exactly once using separate edges from those used by the other one. Thus, H_i acts as a divalent subgraph H_i'' of G''. This H_i'' is connected because H_i is, hence it is a Hamilton cycle of G''. That proves the characterization of G in the (6, 2)-biregular case of L. Note that, if we begin with G' and its Hamilton cycles H_i', construct $L = L(SG')$ and its Hamilton cycles H_i' as in Example 1, and then construct H_i'' in G'', then the natural isomorphism of G'' to G carries each H_i'' to H_i.

Case 2 *Suppose G is (5, 3)-biregular* We reconstruct the dominating trails in G from L. Let G be a (5, 3)-biregualr graph with $5n$ 3-valent vertices and $3n$ 5-valent vertices. The vertices $v_j (0 < j \leq 3n, j \in \mathbb{N})$ $(u_k (0 < k \leq 5n, k \in \mathbb{N})$, respectively) of G are the 5-cliques (3-cliques) in $L(G)$. There is edge $v_j u_k$ for each edge between a vertex of $L(G)$ in $L(v_j)$ and one in $L(u_k)$.

Let the trails T_i (i = 1, 2, 3) be reconstructed from the Hamilton cycles H_i (i = 1, 2, 3) in L. There is an edge $v_j u_k \in T_i$ in G for each edge between a vertex of H_i of L in $L(v_j)$ and one in $L(u_k)$. For each H_i $(i = 1, 2, 3)$, for each 3-clique $L(u)$ in $L(G)$, the two 5-cliques connected by the longest path of H_i in $L(u)$ is connected in the root graph G by a 2-path through u.

Each 5-clique has 10 edges; since only 4 edges can be visited without repeating vertices, we know that each 5-clique had to be partitioned into all three of the Hamilton cycles of L. Thus, each Hamilton cycle passed through all 5-cliques in L. Therefore, each trail reconstructed from the Hamilton cycles contain all 5-valent vertices and is thus a dominating trail. Each 3-clique can be partitioned into three 1-path, one in each Hamilton cycle, or one 1-path and one 2-path in two distinct Hamilton cycles. Thus each 3-valent vertex in the root graph is contained by at least two dominating trails. Finally, since all of the vertices have a degree of four in the 5-cliques in the line graph $L(G)$ it can only be the terminal vertex of an even number of paths in the 5-clique. Additionally, there are only three Hamilton cycles, and thus at most three terminal vertex of three distinct paths, each vertex in the 5-clique can only be the terminal vertex of zero or two paths. Therefore, each edge in the root graph G can only be visited not or twice by the three quasi-compatible dominating trails.

Case 3 *Suppose G is 4-regular* We reconstruct the dominating trials in G from L. Let G be a 4-regular graph with n ($n \geq 5$) vertices. The vertices are the 4-cliques in $L(G)$. There is an edge v_iv_j in the root graph G for each edge between a vertex of $L(G)$ in $K_4(v_i)$ and one in $K_4(v_j)$.

Let the dominating trails T_i (i = 1, 2, 3) be reconstructed from the Hamilton cycles H_i (i = 1, 2, 3) in $L(G)$. There is an edge $v_jv_k \in T_i$ in G for each edge between a vertex of H_i of L in $K_4(v_j)$ and one in $K_4(v_k)$. In each 4-clique in $L(G)$, the edges can be either partitioned into two or three Hamilton cycles. Thus, each vertex v in the root graph is visited by at least two of the quasi-compatible dominating trails. As the root graph is 4-regular, the structure of the $L(G)$ is connected 4-cliques. Inside each 4-clique in the line graph, each vertex has a valency of 3 thus a 2-paths can not be visited more than three times. By experimentation, it is possible for a 2-path to be visited three times. Additionally, there are only three Hamilton cycles, and thus each vertex can only be at most the terminal vertex for three distinct paths in a 4-clique. Because the vertices in 4-cliques in the line graph $L(G)$ are 3-valent, each vertex can only be the terminal vertex of and odd number of paths, thus one or three paths. Therefore, each edge in the root graph G can only be visited once or trice by the three quasi-compatible dominating trails.

We prove that the two conditions left are true by contradiction. Let the three Hamilton cycles of $L(G)$ be H_i, H_j, H_k. First, if two vertices that are visited only by two of the dominating trails and visited by the same pair are adjacent to the same vertex v that is visited twice in a dominating trail at a specific occurrence. Thus, the 4-clique $K_4(v)$ in $L(G)$ is partitioned into two 1-paths in an Hamilton cycle and two paths in each on the two other Hamilton cycles left. Two of its vertices must be the terminal vertex of three distinct paths partitioned into each of the Hamilton cycles. We number the vertices of the 4-clique $K_4(v)$ in $L(G)$ be e_1, e_2, e_3 and e_4. Let the paths e_1e_2 and e_3e_4 be partitioned into H_i in the line graph and let e_1, e_2 be the two vertices that are the terminal vertex of three paths. Thus, e_3, e_4 are each included in a 4-clique beside $K_4(v)$ that is visited by only two Hamilton cycles. We know from the condition that the two Hamilton cycles are the same pair, H_i and H_j, because e_3, e_4 are both terminal vertex of a path in H_i in $K_4(v)$. Thus, e_3, e_4 are internal vertices of a path in H_j in the other 4-cliques they are in. Therefore, they must be the internal vertices of a path in H_k in $K_4(v)$ and must include the paths $e_1e_3e_2$ and $e_1e_4e_2$, which revisits e_1 and e_2 and contradicts with the nature of Hamilton cycle H_k. Two vertices that are visited only by two of the dominating trails and visited by the same pair **can not** be adjacent to the same vertex v that is visited twice in a dominating trail at a specific occurrence. Second, if two vertices that are visited only by two of the dominating trails and visited by the two distinct pairs are adjacent to the same vertex that is visited twice in a dominating trail at the two distinct occurrences. Thus, the 4-clique $K_4(v)$ in $L(G)$ is partitioned into two 1-paths in an Hamilton cycle and two paths in each on the two other Hamilton cycles left. Two of its vertices must be the terminal vertex of three distinct paths partitioned into each of the Hamilton cycles. We number the vertices of the 4-clique $K_4(v)$ in $L(G)$ be e_1, e_2, e_3 and e_4. Let the paths e_1e_2 and e_3e_4 be partitioned into H_i in the line graph and let e_1, e_3 be the two vertices that are the terminal vertex of three paths. Thus, e_2, e_4 are each included in a 4-clique beside $K_4(v)$ that is visited by only two Hamilton cycles. We know from the condition that the two Hamilton cycles are the distinct pair, (H_i, H_j) and (H_i, H_k), because e_3, e_4 are both terminal vertex of a path in H_i in $K_4(v)$. Thus, e_2, e_4 are internal vertices of a path in H_j, H_k respectively in the other 4-cliques they are in. Therefore, e_2, e_4 must be the internal vertices of a path in H_k, H_j respectively in $K_4(v)$. Thus the paths $e_1e_4e_2$ and e_3, e_2e_4 must be in H_j, H_k respectively. As H_j and H_k are two edge-disjoint Hamilton cycles of $L(G)$, e_2, e_4 can not be in both Hamilton cycles. Therefore, two vertices that are visited only by two of the dominating trails and visited by the two distinct pairs **can not** be adjacent to the same vertex that is visited twice in a dominating trail at the two distinct occurrences.

Thus, we have prove that the three dominating trails visited each 2-paths at most three times, visited each vertex v at least two of the quasi-compatible dominating trails, and each edge can only be visited once or trice by the dominating trails. Two vertices that are visited only by two of the dominating trails and visited by the same pair can not be adjacent to the same vertex that is visited twice in a dominating trail at a specific occurrence. Two vertices that are visited only by two of the dominating trails and visited by the two distinct pairs can not be adjacent to the same vertex that is visited twice in a dominating trail at the two distinct occurrences.

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