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SCHEME-THEORETIC AND SET-THEORETIC COMPLETE INTERSECTION OF POINTS

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Abstract. This paper explores whether a general set of closed points in the projective space is an intersection of n hypersurfaces. We study both the scheme-theoretic and the set-theoretic intersection. The scheme-theoretic part is understood by Bezout's theorem and computations in Chow ring, while extra details are needed in exploiting the set-theoretic intersection part of this topic.

Keywords: algebraic geometry, complete intersections, projective space, interpolation, Bezout's theorem

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1. Introduction

Among all projective varieties, (scheme-theoretic) complete intersections are particularly important because of both simplicity and generality. They are more structured and well behaved (for example, they are Gorenstein, hence Cohen-Macaulay), yet still general enough to include many interesting examples. In addition, the computational nature for complete intersections is also better. For example, the normal bundle and canonical bundle can be directly written down using the degrees of the intersected hypersurfaces: Suppose X is a complete intersection of degree d_1, d_2, ...,d_m, then the conormal bundle is $\mathcal{N}^\vee = \mathcal{I}/\mathcal{I}^2 = \mathcal{O}_X(-d_1) \oplus \mathcal{O}_X(-d_2) ... \oplus \mathcal{O}_X(-d_n)$ and hence by adjunction formula, the canonical bundle $\mathcal{K}_X = \mathcal{O}_X(d_1 + d_2 ... + d_n - n - 1)$.

Set-theoretic complete intersections are less researched by mathematicians but they still motivate a family of interesting open questions, among which the most famous one is whether each irreducible curve in $\mathbb{P}^3$ is a set-theoretic intersection of two surfaces. This question remains open even for smooth curves in $\mathbb{P}^3$ with genus 0 and degree 4.

Determining all complete intersection of given dimension d is a difficult moduli problem. While in lower dimensions, mathematicians have managed to introduce some tools to understand why some specific variety is a scheme-theoretic complete intersection or not, the problem of determining set-theoretic complete intersections in most cases, as stated in the previous paragraph, is still wildly open.

In this paper we use elementary arguments to give a complete description of the integer m such that given m general points on $\mathbb{P}^n$, whether we can always write them as a complete intersection. Also, we prove that in $\mathbb{P}^2$, any set of points can be a set-theoretic complete intersection. Now we state our theorems here:

Theorem. *A general set of m discrete points on $\mathbb{P}^n$ is a complete intersection if and only if one of the following condition holds:*
(1) $n \geq 4$, $m = 1$ *or* 2 (2) $n = 3$, $m \in \{1, 2, 8\}$ (3) $n = 2$, $m \in \{1, 2, 4\}$

Theorem. *Any set of points on $\mathbb{P}^2$ is a set-theoretic complete intersection.*

These theorem will also be stated in chapter 4 and the sketch of the proof will be introduced there.

A simple example is a set of 3 general distinct points on $\mathbb{P}^2$. If it is a complete intersection, then it must be an intersection of two curves of degree 1 and 3, which means these three points must be colinear, so if the three points are general enough (not colinear), then this set is not a complete intersection. However, for any three points, one can always find a conic passing through all three points, and choose two lines connecting two of them, so that this set is the intersection of a conic and a degenerate quadratic surface consisting of the union of those two lines.

(a) Complete intersection iff colinear (b) Always set-theoretic complete intersection

We will also make comments to some consequential results and new questions emerged in the last chapter. Some part of this paper is also closely related to the interpolation theory, which is a very interesting and active topic in algebraic geometry. For example in [LV23], Eric Larson and Isabel Vogt determine the number of general points through which a Brill–Noether curve of fixed degree and genus in any projective space can be passed.

The main tool we use in this paper is Bezout's theorem and its general version on $\mathbb{P}^n$.

2. Preliminaries

2.1. **Algebraic sets and ring of regular functions.** We start by introducing basic concepts in elementary algebraic geometry. For simplicity, we always suppose that our groundfield is $\mathbb{C}$ but keep in mind all results could work in any algebraically closed field (at least for char 0).

Denote $\mathbb{A}^n$ by the vector space $\mathbb{C}^n$ equipped with polynomial functions. In order to study zeros of polynomials, we make the following definition:

Definition 2.1. A subset Z of $\mathbb{A}^n$ is called an *algebraic set* if there exist finitely many polynomials $f_1, f_2, ..., f_m$ such that Z is the set of common zeros of them on $\mathbb{A}^n$.

If we allow ring theory to play a role here, this definition is equivalent to Z *is the set of common zeros of the polynomials in the ideal* $I = (f_1, f_2, ..., f_m)$, in which case we write $Z = \mathcal{Z}(I)$. Now the following theorem makes sure that the finiteness in the definition is always satisfied for any ideal I.

Theorem 2.2. (Hilbert Basis Theorem) Any ideal in a polynomial ring is finitely generated. This property is called *noetherian*.

For a subset S of $\mathbb{A}^n$, we can define $\mathcal{I}(S)$ to be all polynomials vanishing on S. This operation, together with the operation $\mathcal{Z}$ we have just defined, have following properties.

Proposition 2.3. *Suppose I_1, I_2 are ideals of $\mathbb{C}[x_1, x_2, ...x_n]$, S_1, S_2 are subsets on $\mathbb{A}^n$, then we have:*

(1) If $I_1 \subseteq I_2$, then $\mathcal{Z}(I_1) \supset \mathcal{Z}(I_2)$. Similarly, If $S_1 \subseteq S_2$, then $\mathcal{I}(S_1) \supset \mathcal{I}(S_2)$.
(2) $\mathcal{Z}(I_1 I_2) = \mathcal{Z}(I_1) \cup \mathcal{Z}(I_2)$, $\mathcal{Z}(I_1 + I_2) = \mathcal{Z}(I_1) \cap \mathcal{Z}(I_2)$
(3) $\mathcal{I}(S_1 \cup S_2) = \mathcal{I}(S_1) \cap \mathcal{I}(S_2)$

For an algebraic set Z in $\mathbb{A}^n$, we define its *coordinate ring* by $A(Z) = \mathbb{C}[x_1, x_2, ..., x_n]/\mathcal{I}(Z)$. These are equivalent classes of functions on Z in the sense of $f \sim g$ if f and g restrict to the same function to Z. One important property relating algebra and geometry tells that the "algebraic" maps between algebraic sets are completely determined by these maps on coordinate rings.

Proposition 2.4. *Suppose $S, T \in \mathbb{A}^n$ are algebraic sets with coordinate rings $A(S), A(T)$, then there is an one to one correspondence between polynomial maps from T to S and maps of $\mathbb{C}$-algebra from $A(S)$ to $A(T)$, the correspondence is given by composition $f \to f.(\phi)(p) = \phi \circ f(p)$.*

2.2. **Zariski topology and algebraic varieties.** Recall that a *topology* on a set is given by a family of subsets $\{A_i\}_{i \in I}$, which we call *closed sets*, such that an arbitrary intersection of closed sets is closed, a finite union of closed sets is closed. It's easy to verify that the algebraic sets gives a topology on $\mathbb{A}^n$ where the closed sets are the algebraic sets. We call this topology *Zariski Topology.*

For a locally closed subspace of A^n, we can restrict our topology and also call it Zariski topology. One can check this coincide with the topological custom that closed sets are zeros of "regular" functions.

A set is called *irreducible* if it cannot be written as the union of two proper closed subsets. One checks that a Zariski closed set is irreducible if and only if its coordinate ring is an integral domain, and in that case we call it an *algebraic variety.*

There are deep relations between Zariski closed sets on $\mathbb{A}^n$ and *radical* ideals in $\mathbb{C}[x_1, ...x_n]$. Recall that an ideal I is called radical if and only if $\mathrm{Rad}(I) = I$, where

$$\mathrm{Rad}(I) = \{f \in \mathbb{C}[x_1, x_2, ..., x_n] | \text{there exists } n \text{ such that } f^n \in I.\}$$

We state this important correspondence here:

Theorem 2.5. (Hilbert's Nullstellensatz) For any ideal I in $A(\mathbb{A}^n) = \mathbb{C}[x_1, x_2, ..., x_n]$, we have $\mathcal{I}(\mathcal{Z}(I)) = \mathrm{Rad}(I)$. Hence the two operations: $S \to \mathcal{I}(S)$ and $I \to \mathcal{Z}(I)$, induce an one-to-one correspondence between Zariski closed sets on $\mathbb{A}^n$ and radical ideals in $A(\mathbb{A}^n) = \mathbb{C}[x_1, x_2, ..., x_n]$. This result also extends to all algebraic sets, say that if S is an algebraic set with coordinate ring $A(S)$, then the Zariski closed sets in S one-to-one corresponds to radical ideals in $A(S)$. Furthermore, by remark above, the algebraic varieties correspond to prime ideals.

We only make comments on the last correspondence: Suppose X is reducible, then $X = Z_1 \cup Z_2$, then we can choose $f_1 \in \mathcal{I}(Z_1) - \mathcal{I}(Z_2)$, $f_2 \in \mathcal{I}(Z_2) - \mathcal{I}(Z_1)$, while $f_1 f_2$ is in $\mathcal{I}(Z)$, we see that $\mathcal{I}(Z)$ cannot be prime. Conversely, if Z is irreducible, suppose $\mathcal{I}(Z)$ is not prime, then there is f, g not in $\mathcal{I}(Z)$ such that

$$(\mathcal{I}(Z), f)(\mathcal{I}(Z), g) = \mathcal{I}(Z)$$

hence $Z = \mathcal{Z}(\mathcal{I}(Z)) = \mathcal{Z}(\mathcal{I}(Z), f) \cup \mathcal{Z}(\mathcal{I}(Z), g)$ is a union of two proper closed subsets, a contradiction.

One further comment is that, for any algebraic set Z in $\mathbb{A}^n$, we can associate it with the quotient ring $\mathbb{C}[x_1, x_2, ..., x_n]/\mathcal{I}(Z)$ and call this ring its *coordinate ring* $A(Z)$. It's easy to verify that in Theorem 2.5, if we replace $\mathbb{A}^n$ by Z, $\mathbb{C}[x_1, x_2, ..., x_n]$ by $A(Z)$, the statement remains true.

2.3. **Graded structure, Projective space.** The complex projective space $\mathbb{P}^n$ (or $\mathbb{CP}^n$) is defined as the set of all lines through origin in $\mathbb{A}^{n+1}$. We can parametrize it by homogeneous coordinates $[x_0, x_1, ..., x_n]$, where x_i's are not all zero and $[\lambda x_0, \lambda x_1, ..., \lambda x_n] = [x_0, x_1, ..., x_n]$ for any $\lambda \in \mathbb{C}^*$.

Elements in polynomial ring $\mathbb{C}[x_0, x_2, ..., x_n]$ are not well-defined functions on $\mathbb{P}^n$, but we can still talk about "zeros" of homogeneous polynomials since their zero sets in $\mathbb{A}^{n+1}$ are stable multiplying by scalars. Hence we can define the Zariski closed sets as the zero sets of homogeneous ideals, which are namely the ideals generated by homogeneous elements. In this case we still have Hilbert Nullstellensatz correspondence, but one should notice that both the unit ideal and the ideal $(x_0, x_1, ..., x_n)$ correspond to empty set since the origin in $\mathbb{A}^{n+1}$ does not appear in any equivalent class in $\mathbb{P}^n$. In addition, it's easy to check that Proposition 2.3 also holds.

One important idea is to regard $\mathbb{P}^n$ as the compactification or completion of $\mathbb{A}^n$. This is by the following observation: Denote U_{x_i} the open subset defined by $x_i \neq 0$, then one see that

$$[x_1, x_2, ..., x_{n+1}] \to (\frac{x_1}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_{n+1}}{x_i})$$

defines an one to one correspondence between U_{x_i} and $\mathbb{A}^n$. While the rest part $\mathcal{Z}(x_i)$ is obviously isomorphic to $\mathbb{P}^{n-1}$, we see that $\mathbb{P}^n$ is constructed by adding an "infinite line" to $\mathbb{A}^n$. One also notices that $U_{x_i},\ i = 0, 1, 2, ..., n$ covers $\mathbb{P}^n$, so $\mathbb{P}^n$ is "glued up" by $\mathbb{A}^n$.

In the following contexts, we also call the irreducible locally closed subsets (namely, closed in an open subset) in $\mathbb{P}^n$ (quasi-projective) algebraic varieties, and we will call a variety *affine* if it can be embedded as a closed subset in $\mathbb{A}^n$ by fractional-polynomial maps.

Modern algebraic geometry establishes conventions for schemes. However, in our case this classical package of language consists of algebraic sets, varieties, coordinate rings and ideals works well, so we choose not to introduce schemes here.

2.4. **Localization, Local rings.** Consider $X = \mathbb{A}^1$, we wish to consider the open subset $U = \mathbb{A}^1 - \{0\}$. It is open in X, but we can define a bijection of U with the hyperbola $xy = 1$ in $\mathbb{A}^2$:

$$x \in U \to (x, \frac{1}{x}) \in \mathcal{Z}(xy - 1)$$

So we can give U a variety structure with coordinate ring $\mathbb{C}[x,y]/(xy-1)$. Notice that this ring is the same as making x invertible in $A(\mathbb{A}^1)$.

In another situation, we want to investigate the behavior of a single point $x = 1$ in $\mathbb{A}^1$. Then we want ignore all other points, which means by discussion above, $(x-a)$ are all invertible for $a \neq 1$. Then for any f having nonzero value at $x = 1$, by fundamental theorem of algebra, f splits into linear factors $(x-x_1)(x-x_2)...(x-x_n)$ where $x_i \neq 1$ for all i, which means it should also be invertible. And we can actually find such a ring: in the function field $\mathbb{C}(x)$, let R be the ring consisting of fractions $\frac{f}{g}$ with $g(1) \neq 0$.

Based on these two examples, we can make the following definition:

Definition 2.6. Suppose R is a coordinate ring of some variety (or simply some algebraic set), U is a subset of R closed under multiplication, define $U^{-1}R$ to be the set of equivalent classes of fractions $\dfrac{r}{u}$ under the equivalence relation:

$$\frac{r}{u} \sim \frac{s}{v} \quad \text{iff} \quad w(rv - su) = 0 \ \text{ for some } w \in U$$

It's easy to verify that the usual addition and multiplication of fractions are well-defined for these equivalent classes, which makes $U^{-1}R$ a ring. We call this ring *the localization of R by U*.

For any affine algebraic variety X with coordinate ring R and $f \in R$, let $U = \{f^i | i \in \mathbb{Z}\}$, we see that this $\mathbb{R}_f$ is the coordinate ring of the variety in $X \times \mathbb{A}^1$ defined by equation $yf(x) = 1$, which can be identified with the open subset $U_f = \{x \in X | f(x) \neq 0\}$ in X by bijection as above, hence gives U_f an affine structure. One also verifies by noetherian property that U_f form a basis for Zariski topology on X, so we have a nice family of "distinguished neighborhoods".

Continue with the set up above, For any point p on X, if we wish to investigate the local nature near p, we should take the intersection of all such neighborhood containing p, that is to make all $f \in R$, $f(p) \neq 0$ invertible. Now by Hilbert Nullstellensatz, p corresponds to a unique maximal ideal $\mathfrak{m}$ in R, f vanishes at p if and only if $f \in \mathfrak{m}$, hence localizing all $f \in R$, $f(p) \neq 0$ is the same as localizing all elements outside $\mathfrak{m}$. Hence we can define the *local ring* $R_\mathfrak{m}$ near p as the localization $(R - \mathfrak{m})^{-1}R$. Here is an important property for $R_\mathfrak{m}$:

Proposition 2.7. *Prime ideals in $R_\mathfrak{m}$ are one-to-one correspond to prime ideals in R contained in $\mathfrak{m}$. In particular, $R_\mathfrak{m}$ has a unique maximal ideal $\mathfrak{m}R_\mathfrak{m}$*

One can also find analogy in this to projective space: for any projective variety X with projective coordinate ring R (depends on embedding) and $f \in R$ a homogeneous element, let $U_f = \{x \in X | f(x) \neq 0\}$, then one can show that U_f is affine with coordinate ring $R_{(f)}$, the zero degree part of the localization of R by f. Similarly, for any point $p \in X$, we define the local ring to be $R_{(\mathfrak{m})}$, the zero degree part of the localization $R_\mathfrak{m}$. One checks easily that when restricting to affine atlas, this is compatible with our previous affine constructions.

With those terminology we are able to talk about further topics, such as the multiplicity and intersection multiplicity, which is crucial in solving our problem.

3. Intersection Theory

In this chapter we introduce some most basic intersection theory, including the intersection multiplicity and Bezout's theorem.

3.1. **Dimension, Hypersurfaces and linear system.** We start by talking about dimension:

Definition 3.1. The *dimension* of a topological space is the largest integer n such that there is an strict including chain of irreducible closed subsets: $Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq ...Z_n$.

For example, In the category of algebraic sets, the zero dimensional objects are points, 1 dimensional objects are curves. For an algebraic variety X, algebraic geometers are particularly interested in subvariety of dimension $\dim X - 1$, which they call *divisors*. We only care about the most simple divisors, namely the divisors for $\mathbb{A}^n$ and $\mathbb{P}^n$:

Definition 3.2. A *hypersurface* in $\mathbb{A}^n$ (respectively, $\mathbb{P}^n$) is an $(n-1)$-dimensional algebraic variety.

The following characterization of hypersurface is a consequence of Krull's principal ideal theorem:

Proposition 3.3. *X is a hypersurface in $\mathbb{A}^n$ (respectively, $\mathbb{P}^n$) if and only if X is defined by a single irreducible polynomial (respectively, irreducible homogeneous polynomial).*

This gives a reason for us to extend the definition of hypersurface to all *algebraic sets of pure dimension* $n-1$, i.e., algebraic sets that are union of irreducible hypersurfaces. Then the hypersurfaces in $\mathbb{A}^n$ (respectively, $\mathbb{P}^n$) are exactly the zeros of polynomials (respectively homogeneous polynomials), and by Hilbert Nullstellensatz, two polynomials f and g define the same hypersurface if and only if $f = \lambda g$ for some nonzero λ.

For $\mathbb{P}^n$, as we have shown that the "moduli" of hypersurfaces is actually the projective space $\bigcup_{d=0}^{\infty} \mathbb{P}(S_d)$, where S_d is the $\mathbb{C}$-vector space of homogeneous polynomials of degree d in $S = \mathbb{C}[x_0, x_1, x_2, ..., x_n]$, it is reasonable to define *degree of a hypersurface* to be the degree of the polynomial determining it. Notice that if $H = H_1 \cup H_2$, then $\deg H = \deg H_1 + \deg H_2$.

The following terminology is from modern algebraic geometry but it's useful in our situation:

Definition 3.4. A *linear system* $\mathfrak{d}$ of hypersurfaces of degree d on $\mathbb{P}^n$ is the family of hypersurfaces determined by the polynomials in a vector subspace V of S_d. If $V = S_d$, we call it a *complete linear system*.

By definition and comments above, a linear system is naturally parametrized by $\mathbb{P}(V)$, the projective space consisting of all lines passing through origin in V. We close this section by introducing base point and base component,

Definition 3.5. A variety X on $\mathbb{P}^n$ is called to be in a *base component* of a linear system $\mathfrak{d}$ if all hypersurfaces in $\mathfrak{d}$ contain X. The maximal ones of such X are called *base components*. If X is a point that lies in all hypersurfaces in $\mathfrak{d}$, it is called a *base point*.

3.2. **Multiplicity and intersection multiplicity.** Let's start from $\mathbb{A}^2$. Without loss of generality, suppose C is a curve defined by binary polynomial f which passes through origin $(0,0)$. Then the homogeneous decomposition of f is $f = f_d + f_{d-1} + ... + f_k$ where $d = \deg f$ and $k \geq 1$, $f_k \neq 0$. Then all i^{th} order of partial derivatives of f vanishes at origin for $i = 0, 1, 2, ..., k-1$, and there is at least one k^{th} order of derivative of f that does not vanish at origin, so it is reasonable for us to say $(0,0)$ is a zero of f of degree k, or the curve passes through $(0,0)$ with multiplicity k.

Generalize this idea, we get

Definition 3.6. Suppose f defines a hypersurface H in $\mathbb{A}^n$ and $P \in H$. Translate P to origin $O = (0, 0, ..., 0)$ and suppose f have homogeneous decomposition $f = f_d + f_{d-1} +$

$... + f_k$ where $k \geq 1$, $f_k \neq 0$. Then the *multiplicity* $m_P(H)$ *of* H *at* P is defined as the integer k.

For a hypersurface H in $\mathbb{P}^n$ and a point $P \in H$, the multiplicity is defined by firstly restricted to one affine atlas U_{x_i} (defined in the previous chapter) that contains P and obtain multiplicity of $H \cap U_{x_i}$ at P. One checks easily that this definition is independent of the choice for x_i, it is actually the unique integer k such that $f \in \mathfrak{m}^k$ while $f \notin \mathfrak{m}^{k+1}$ in the local ring $\mathcal{O}_P$ with maximal ideal $\mathfrak{m}$.

Going back to our example in $\mathbb{A}^2$, suppose that two different curves $C : f = 0$ and $D : g = 0$ intersect at P, then as long as P is isolated (means P is not contained in any common component of C and D), then it follows from results in commutative algebra (artinian local rings) that the quotient ring $\mathcal{O}_p/(f, g)$ is a finite dimensional $\mathbb{C}$ vector space. Now if $f = y$, then C is the x-axis, which is isomorphic to $\mathbb{A}^1$, and $l = \dim_{\mathbb{C}} \mathcal{O}_P/(f, g)$ is the multiplicity of $g|_{\mathbb{A}^1}$ at P. So it's reasonable to say "C and D" intesect at P with multiplicity l. This leads to the definition:

Definition 3.7. Suppose $H_1, H_2, ..., H_n$ are n hypersurfaces defined by $f_1, f_2, ..., f_n$ intersecting at P which is isolated, then the $\mathbb{C}$ vector space $\mathcal{O}_P/(f_1, f_2, ..., f_n)$ is finite-dimensional. The *intersection multiplicity* of $H_1, H_2, ..., H_n$ is then defined by

$$m_P(H_1, H_2, ..., H_n) = \dim_{\mathbb{C}} \mathcal{O}_P/(f_1, f_2, ..., f_n)$$

Similarly, the intersection multiplicity in $\mathbb{P}^n$ is defined by firstly restricting to one affine atlas and then obtaining the local intersection multiplicity as our intersection multiplicity, it is also independent of the atlas you choose.

One important relation between those two multiplicities is given in the following theorem:

Theorem 3.8. *Let $H_1, H_2, ..., H_n$ be n hypersurfaces in $\mathbb{P}^n$ whose intersection set is 0-dimensional, then $m_P(H_1, H_2, ..., H_n) \geq m_P(H_1) \cdot m_P(H_2) ... \cdot m_P(H_n)$*

The proof is by irritating the inequality $m_P(X, Y) \geq m_P(X) \cdot m_P(Y)$ in [Ful84], chapter 12.

3.3. **Bezout's theorem and Chow ring.**

One can generalize the notion of degree to all subvarieties in $\mathbb{P}^n$ as a numerical data computed by Hilbert polynomial, which we don't bother introducing the whole theory here but make comment that *the degree of a set of points S is the number of points in S counting with multiplicity.* A fundamental result for intersection theory is cited here from [Har77] chapter 1:

Theorem 3.9. *Let Y be a variety of dimension ≥ 1 in $\mathbb{P}^n$, let H be a hypersurface not containing Y. Let $Z_1, Z_2, ..., Z_s$ be the irreducible components of $Y \cap H$. Then*

$$\sum_{j=1}^{s} m_{Z_i}(Y, H) \cdot \deg Z_i = \deg Y \cdot \deg H$$

Here $m_{Z_i}(Y, H)$ is also a generalization of intersection multiplicity at a point to the notion along a subvariety in the intersection. Now for the case that $H_1, H_2, ..., H_n$ be n hypersurfaces whose intersection set is 0-dimensional, Irritating this formula we get:

Corollary 3.10. *Suppose $H_1, H_2, ..., H_n$ are n hypersurfaces in $\mathbb{P}^n$ whose intersection set is 0-dimensional, then*

$$\deg(H_1 \cap H_2 \cap ... \cap H_n) = \deg H_1 \cdot \deg H_2 ... \cdot \deg H_n$$

Take $n = 2$ we get the most famous and classical result in intersection theory:

Corollary 3.11. (Bezout's Theorem) *Let Y, Z be curves in $\mathbb{P}^2$ of degree d and e, if $Y \cap Z = \{P_1, P_2, ..., P_s\}$ is zero dimensional, then:*

$$\sum_{i=1}^{s} m_{P_i}(Y, Z) = de$$

One can also understand this series of results using *Chow ring*: Define the *Chow group* $Ch(\mathbb{P}^n)$ of $\mathbb{P}^n$ as the free abelian group generated by all subvarieties in $\mathbb{P}^n$ modulo the rational equivalence: Two subvariety X and Y is rationally equivalent if they are different sections of a subvariety $Z \in \mathbb{P}^n \times \mathbb{P}^1$ at two points s and t on $\mathbb{P}^1$. Then $Ch(\mathbb{P}^n)$ has a natural grading by dimension. Intersection theorists have given a complete description of this object,

Theorem 3.12. *$Ch(\mathbb{P}^n)$ has a ring structure such that $[A][B] = [A \cap B]$ for any two subvarieties A and B. We have an isomorphism*

$$Ch(\mathbb{P}^n) \cong \mathbb{Z}[\sigma]/(\sigma^{n+1})$$

of rings, and under this isomorphism, σ represents a hyperplane (hypersurface defined by linear function) in $\mathbb{P}^n$.

A proof of which can be found in [EH16]. Using this tool, one can translate (3.7∼ 3.9) by regarding all varieties as the linear variety rationally equivalent to it and then calculating the intersection using linear algebra.

4. Main result and sketch of the proof

As stated in the Introduction, we are naturally interested in whether a given algebraic set X in $\mathbb{P}^n$ can be represented as a intersection of a codimension number of hyperplanes. By Hilbert Nullstellensatz, this means $\mathcal{I}(X)$ is the radical of an ideal generated by codim$X$ elements. It is a relative easy condition that $\mathcal{I}(X)$ itself can be generated by codim$X$ elements, so we distribute the idea into two cases and make the following definition:

Definition 4.1. Suppose X is an algebraic set on $\mathbb{P}^n$ with dimension r. Denote the ideal of vanishing function $\mathcal{I}(X)$, then:

(1) If $\mathcal{I}(X)$ can be generated by $n-r$ elements, we say X is a *scheme-theoretic complete intersection* or simply *complete intersection*.

(2) If there exist hypersurfaces $H_1, H_2, ..., H_{n-r}$ such that $X = H_1 \cap H_2 ... \cap H_{n-r}$ as subsets on $\mathbb{P}^n$, we say X is a *set-theoretic complete intersection*.

Determining whether a set of dimension ≥ 1 is a complete intersection or set-theoretic complete intersection is a wildly open question, so in this paper we focus on the zero-dimensional algebraic sets, i.e. the sets of discrete points. The core question this paper cares about is: Given an integer m and a general set S of discrete points on $\mathbb{P}^n$ with $|S| = m$, it is possible to S as a complete intersection or set-theoretic complete intersection? For scheme-theoretic part, we have figured out the possible m for all n and for set-theoretic part, we work out the situation for $n = 2$. We restate our main theorems here:

Theorem 4.2. *A general set of m discrete points on $\mathbb{P}^n$ is a complete intersection if and only if one of the following condition holds:*

(1) $n \geq 4$, $m = 1$ *or* 2 (2) $n = 3$, $m \in \{1, 2, 8\}$ (3) $n = 2$, $m \in \{1, 2, 4\}$

Theorem 4.3. *Any set of points on $\mathbb{P}^2$ is a set-theoretic complete intersection.*

The idea for proving Theorem 4.2 comes from the observation: any 2 points determine a line, 5 points determine a conic,... a degree d curve will be determined by $\frac{(d+1)(d+2)}{2} - 1$

points on it. More generally, a hypersurface of degree d in $\mathbb{P}^n$ should be determined by $\binom{n+d}{n}-1$ general points on it, this will be proved in chapter 5 as part of the "interpolation theory". Hence in a complete intersection, if the degrees of the hypersurface is given, then this gives a upper bound for the number of points in the intersection. Simultaneously, one have an equality indicated by Bezout's theorem, or more explicitly Corollary 3.10 for $n \geq 3$, so comparing these two restrictions one can get some numerical relations, which leads to the result of Theorem 4.2.

The proof of Theorem 4.3 follows the similar idea but to manually "assign" each point a multiplicity, so that there are hypersurfaces passing through them by multiplicity that is big enough such that by Bezout's theorem, no more intersection is allowed excepts those given points. This method could possibly generated to dimension n, however, as we will see in chapter 7, it is a hard work to make sure the intersection has dimension zero. There in chapter 7 we use some technical details to show that we can construct linear system of curves determined by passing those given points by some multiplicities without fixed components, so there must be two curves having 0-dimensinoal intersection.

5. Interpolation using hypersurfaces

The proof of both Theorem 4.2 and Theorem 4.3 requires interpolating points using hypersurfaces, so we isolate a whole chapter talking about this.

5.1. **dimension reduction while interpolating points.** Suppose $S = \mathbb{C}[x_0, x_1, ..., x_n]$. As discussed after Proposition 3.3, hypersurfaces of degree d are one-to-one correspond to 1-dimensional linear subspace of degree d part S_d of S, which has dimension $\binom{n+d}{n}$. We call a homogeneous polynomial of degree d a *d-form* on $\mathbb{P}^n$. Now suppose we are given m different points $p_i = [x_0^{(j)}, x_1^{(j)}, ..., x_n^{(j)}], j = 1, 2, ..., m$ in $\mathbb{P}^n$. Given a d-form

$$f = \sum_{i_0+i_1+...+i_n=d} a_{i_0 i_1 ... i_n} x_0^{i_0} x_1^{i_1} ... x_n^{i_n}$$

the hypersurface H determined by f passes through all p_i if and only if the vector consisting of all coefficients of f is contained in the null space of the matrix

$$M(p_1, p_2, ..., p_m) = \Big((x_0^{(j)})^{i_0}((x_1^{(j)})^{i_1}...((x_n^{(j)})^{i_n}\Big)_{m\times\binom{n+d}{n}}$$

where the rows are indexed by j and columns are indexed by $(i_1, i_2, ..., i_n)$ such that $i_0 + i_1 + ... + i_n = d$ in the same order as the coefficient vector for f.

An important result about $M(p_1, p_2, ..., p_m)$ which leads to the dimension reduction is:

Proposition 5.1. *For general m points $p_1, p_2, ...p_m$, $M(p_1, p_2, ..., p_m)$ has full rank.*

Proof. It suffices to check $m = \binom{n+d}{n}$ since in that case when $m < \binom{n+d}{n}$, it is the first m rows of $M(p_1, p_2, ..., p_{\binom{n+d}{n}})$ for any general $p_{m+1}, p_{m+2} ..., p_{\binom{n+d}{n}}$ and full rank of the later matrix implies linear independence; when $m > \binom{n+d}{n}$, the square matrix of first $\binom{n+d}{n}$ rows is $M(p_1, p_2, ..., p_{\binom{n+d}{n}})$, which generally has rank equal to $\binom{n+d}{n}$.

If $m = \binom{n+d}{n}$, it suffice to find a set of m such points such that the determinant of $M(p_1, p_2, ..., p_m)$ does not vanish. For this, let $x_0^{(j)} = 1$ for all j and $x_k^{(j)} = (x_1^{(j)})^{(d+1)^k}$. Then

$$(x_0^{(j)})^{i_0}(x_1^{(j)})^{i_1}...(x_n^{(j)})^{i_n} = (x_1^{(j)})^{i_0+i_1(d+1)+i_2(d+1)^2+...+i_n(d+1)^n}$$

Since $i_k \leq d$ and $d \geq 1$, those integers $i_0 + i_1(d+1) + i_2(d+1)^2 + ... + i_n(d+1)^n$ for different $(i_0, i_1, ..., i_n)$ are naturally different and are greater than 1 by the uniqueness of $(d+1)$-adic representation of integers. Hence if we denote $x_1^{(j)} = a_j$ we have an

$\binom{n+d}{n} \times \binom{n+d}{n}$ or $m \times m$ matrix with rows $(a_j^{t_1}, a_j^{t_2}, a_j^{t_3}, ..., a_j^{t_m})$ and t_i's are distinct (here $t_1 = 0$). The determinant of this matrix is a linear combination of $a_1^{t_{\sigma(1)}} a_2^{t_{\sigma(2)}} ... a_m^{t_{\sigma(m)}}$ with coefficient $(-1)^\sigma$ (here σ runs through all permutations σ of $\{1, 2, ..., m\}$). Since those exponents are different, those terms are linear independent in the polynomial ring $\mathbb{C}[a_1, a_2, ..., a_m]$. Hence the determinant is not a zero polynomial in $a_1, a_2, ..., a_m$, there must be a set of value for $a_1, a_2, ..., a_m$ such that the determinant is not zero, completing the proof. □

As a consequence of this proposition, we can verify the comment for Theorem 4.2 in the previous chapter:

Corollary 5.2. *If $m \leq \binom{n+d}{n}$, for general points $p_1, p_2, ..., p_m$, the dimension of the vector space of d-forms vanishing on those points is $\binom{n+d}{n} - m$, hence the dimension of the linear subvariety in $\mathbb{P}(S_d)$ consisting of all hypersurface passing through these points is $\binom{n+d}{n} - m - 1$.*

Proof. Obvious by previous proposition and basic linear algebra. □

5.2. **dimension reduction while adding multiplicity condition.** In the previous subsection, we discussed the "dimension loss" when you add interpolating condition: Passing through one point "reduces the dimension by 1" for the moduli space of hypersurfaces. Since we have discussed the multiplicity of a point, we can naturally talk about the dimension loss when requiring a hypersurface passing through one point of given degree, and we have the following result:

Proposition 5.3. *For hypersurfaces in $\mathbb{P}^n$, The condition "passing through a point p with multiplicity a reduce the dimension at most by $\binom{n+a-1}{a-1}$.*

Proof. Since multiplicity is local, we may restrict to $\mathbb{A}^n$ and suppose $p = (0, 0, ..., 0)$. The restrictions on d-forms are just linear equations

$$\frac{\partial(f|_{\mathbb{A}^n})}{\partial^I Y}(0, 0, ..., 0) = 0, \text{ for all } |I| \leq a - 1,$$

here $Y = (y_1, ..., y_n)$ are the local coordinates in $\mathbb{A}^n$ and $I = (i_1, i_2 ..., i_n)$, $|I| = i_1 + i_2 ... + i_n$, $\partial^I Y = \partial^{i_1} y_1 \partial^{i_2} y_2 ... \partial^{i_n} y_n$. Hence we have in total

$$\sum_{k=0}^{a-1} \binom{n-1+k}{k} = \sum_{k=0}^{a-1} \binom{n+k}{k} - \binom{n+k-1}{k-1} = \binom{n+a-1}{a-1}$$

linear functions, which reduce the dimension at most by $\binom{n+a-1}{a-1}$. □

6. 0-dimensional scheme-theoretic compelete intersection on $\mathbb{P}^n$

In this chapter we prove Theorem 4.2.

Suppose we are given a general set of m points on $\mathbb{P}^n$ that is a complete intersection of hypersurfaces $H_1, H_2, ..., H_n$, then with degree $d_1, d_2, ..., d_n$. Then by Corollary 5.2, in order to really have hypersurface of degree d_i pass through these points, one should have $\binom{n+d_i}{n} - m - 1 \geq 0$. If we suppose $d_1 \leq d_2 \leq ... \leq d_n$, then this is equivalent to $m \leq \binom{n+d_1}{n} - 1$ or simply $m < \binom{n+d_1}{n}$. While Corollary 3.10 tells us $m = d_1 d_2 ... d_n$, based on d_1 is minimal among d_i's we have an inequality

$$\binom{n+d_1}{n} > m = d_1 d_2 ... d_n \geq d_1^n$$

which is

$$\frac{(d_1+n)(d_1+n-1)...(d_1+1)}{n\cdot(n-1)...\cdot 1} > d_1^n$$

hence equivalent to

$$(\frac{1}{d_1}+\frac{1}{n})(\frac{1}{d_1}+\frac{1}{n-1})...(\frac{1}{d_1}+1) > 1 \tag{$\star$}$$

If $n \geq 4$ and $d_1 \geq 2$, then the $LHS \leq (\frac{1}{2}+1)(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}+\frac{1}{3})(\frac{1}{2}+\frac{1}{4}) = \frac{15}{16} < 1$, a contradiction. Hence for $n \geq 4$, we should have $d_1 = 1$, but this means $m \leq \binom{n+d_1}{n} - 1 = n$ and H_1 is a hyperplane containing it. With the isomorphism $H_1 \cong \mathbb{P}^{n-1}$ and the fact that d-forms on H_1 are exactly restrictions of d-forms on $\mathbb{P}^n$, we see that this reduce to the case of $n-1$. If $n-1 \geq 4$, we can repeat our argument to get $d_2 = 1$ and $m \leq n-1$. So finally we can reduce to the case that $n = 3$ and $m \leq 4$.

Now we suppose $n = 3$, then $(\star)$ implies $(\frac{1}{d_1}+\frac{1}{3})(\frac{1}{d_1}+\frac{1}{2})(\frac{1}{d_1}+1) > 1$. This holds only for $d_1 = 1$ or 2 since $(\frac{1}{3}+\frac{1}{3})(\frac{1}{3}+\frac{1}{2})(\frac{1}{3}+1) = \frac{40}{54} < 1$.

If $d_1 = 1$, we see $m \leq \binom{n+d_1}{n} - 1 = 3$ and the set is contained in a plane, which is $\mathbb{P}^2$. While in $\mathbb{P}^2$, if $m = 3$, any three point is a complete intersection if and only if it is an intersection of a cubic curve and a line, which means the three points must be colinear, so we have lost our generality for this point set. For $m = 2$, and two point can be represented by a complete intersection of a conic passing through them and the line connecting them, so $m = 2$ works. For $m = 1$, obviously any points can be represented by intersection of two different lines going through it. Hence $m = 1$ or 2 in the case $d_1 = 1$.

If $d_1 = 2$, then $m \leq \binom{n+d_1}{n} - 1 = 9$, while $m \geq d_1^n = 8$, we see $m = 8$ or 9. However, by $m = d_1 d_2 ... d_n$, $d_1 = 2$ is a factor of m, so the only possible choice is $m = 8 = 2\cdot 2\cdot 2$. Now the problem is to show general 8 points on $\mathbb{P}^3$ is always a complete intersection of three quadratic surfaces. To see this, we have to use some tools from modern algebraic geometry.

Suppose given general points $p_1, p_2..., p_8$ on $\mathbb{P}^3$, consider the linear system of quadratic surfaces $\mathfrak{d}$. By proposition 5.1, we see that this linear system has dimension $\binom{3+2}{3} - 8 - 1 = 1$. For any general point q on $\mathbb{P}^3$, we see that the condition

$$\text{rank } M(p_1, p_2, ..., p_8, q) = \text{rank } M(p_1, p_2, ..., p_8)$$

imposes $\binom{10}{9} = 10$ restrictions on coordinates for q, and one can easily come up with an example for $p_1, p_2, ..., p_8$ when the points satisfies this 10 condition has dimension 0 (for example, find three quadratic surfaces that intersect at dimension zero, then the linear system of quadratic surfaces containing those points does not have fixed components). Hence generally, the set of points satisfying $\text{rank}M(p_1, p_2, ..., p_8, q) = \text{rank}M(p_1, p_2, ..., p_8)$ have dimension 0, which means the linear system $\mathfrak{d}$ does not have fixed component, so there are three quadratic surfaces whose intersection is zero dimensional and contains $p_1, p_2, ..., p_8$. Again by Corollary 3.10, we see that $p_1, p_2, ..., p_8$ are the only points in their intersection, hence general 8 points on $\mathbb{P}^3$ is always a complete intersection of three quadratic surfaces.

Thus we can conclude all situation for $n \geq 3$ now: for $n = 3$, we have $m \in 1, 2, 8$, for $n \geq 4$, we have $m \leq 4$ and all points is contained in a some linear subspace isomorphic to $\mathbb{P}^3$, hence $m = 1$ or 2.

It remains to check $n = 2$. In this case, inequality $(\star)$ implies $(\frac{1}{d_1}+1)(\frac{1}{d_1}+\frac{1}{2}) > 1$, then $d_1 \leq 3$, which means $m \leq \binom{3+2}{2} - 1 = 9$. Now we can check each possible m by hand:

For $m=1$, any point is a intersection of 2 different lines through it, so $m=1$ works;

For $m=2$, and two point can be represented by a complete intersection of a conic passing through them and the line connecting them, so $m=2$ works;

For $m=3$, $m=d_1d_2$ implies $d_1=1$, contradict to $m \leq \binom{n+d_1}{n}-1=2$ (or equivalently, general three points are not colinear).

For $m=4$, any general 4 points is a complete intersection of two conics passing through them, so $m=4$ works.

For $m=5$, $m=d_1d_2$ implies $d_1=1$, contradicts to $m \leq \binom{n+d_1}{n}-1=2$.

For $m=6$, $m=d_1d_2$ implies $d_1 \leq 2$, contradicts to $m \leq \binom{n+d_1}{n}-1 \leq 5$.

For $m=7$, $m=d_1d_2$ implies $d_1=1$, contradicts to $m \leq \binom{n+d_1}{n}-1=2$.

For $m=8$, $m=d_1d_2$ implies $d_1 \leq 2$, contradicts to $m \leq \binom{n+d_1}{n}-1 \leq 5$.

For $m=9$, $m=d_1d_2$ implies $d_1 \leq 3$. If $d_1 \leq 2$, this contradicts to $m \leq \binom{n+d_1}{n}-1 \leq 5$. If $d_1=3$, then $d_2=3$. However, for general 9 points $p_1,p_2.,.,p_9$, the matrix $M(p_1,p_2,...,p_9)$ has full rank by Proposition 5.1, so the null space of it has dimension 1, i.e., there is only one cubic curve passing through all those points, a contradiction. This completes the proof.

7. 0-dimensional set-theoretic complete intersection on $\mathbb{P}^2$

In the previous chapter, we see that the reason why a set of points cannot be represented by a complete intersection is that the restriction $m=d_1d_2...d_n$ imposes an upper bound for the degree passing through those points, but this upper bound could prevent a general such hyperplane passing through so many points.

However, set-theoretic intersection allows us to have higher degree hypersurfaces: If we suppose $H_1,H_2,...,H_n$ intersect at $p_1,p_2,...,p_m$ with intersection multiplicity $s_1,s_2,...,s_m$, then Corollary 3.10 becomes

$$d_1d_2...d_n = s_1+s_2+...+s_m$$

This gives possibility for d_i to be bigger than before.

In order to control the intersection multiplicity, we can use Theorem 3.8 to impose higher multiplicity for the hypersurfaces passing through thoses points, namely, suppose H_i passes through p_j with multiplicity t_{ij} then by Theorem 3.8, $s_j \geq t_{1j}t_{2j}...t_{nj}$, hence $d_1d_2...d_n \geq \sum_{j=1}^m t_{1j}t_{2j}...t_{nj}$.

Let consider this situation in another direction: If we can find n hypersurfaces $H_1,H_2,...,H_n$ of degree $d_1,d_2,...,d_n$ which passes through $p_1,p_2,...,p_m$ with multiplicity t_{ij} for H_i passes p_j, and we have $d_1d_2...d_n=\sum_{j=1}^m t_{1j}t_{2j}...t_{nj}$ as a prerequisite, then if $H_1 \cap H_2... \cap H_n$ is 0-dimensional, then Theorem 3.8 will make sure that there is no point other than $p_1,p_2,...,p_m$ lying in the intersection $H_1 \cap H_2... \cap H_n$. Since by Proposition 5.3, passing through p_j with multiplicity t_{ij} would cause at most $\binom{n+t_{ij}-1}{t_{ij}-1}$ dimension lost for the space of degree d_i hypersurfaces, it would help us a lot to give a set of integers $d_i, t_{ij}, i=1,2,...n;\ j=1,2,...,m$ such that

$$(1)\ d_1d_2...d_n = \sum_{j=1}^m t_{1j}t_{2j}...t_{nj}; \qquad (2)\ \binom{n+d_i}{d_i} - \sum_{j=1}^m \binom{n+t_{ij}-1}{t_{ij}-1} \geq 1$$

However this is generally not enough: Firstly, if two tuple of integers $(d_i,t_{i1},t_{i2},...,t_{im})$ and $(d_j,t_{j1},t_{j2},...,t_{jm})$ equal each other, then we should have at least two different hypersurfaces in the linear system consisting of degree d_i hypersurfaces passing through p_j with multiplicity t_{ij}, so condition (2) should be (2*) $\binom{n+d_i}{d_i}-\sum_{j=1}^m \binom{n+t_{ij}-1}{t_{ij}-1} \geq 2$. However, since we are manually choosing $d_i, t_{ij}, i=1,2,...n;\ j=1,2,...,m$, it shouldn't be hard to avoid this situation or to replace (2) by (2*) for some i. The really difficult part is to find a hypersurface in each linear system such that their intersection is 0-dimensional, which is possible by some local argument but far beyond this paper's level if $n \geq 3$.

For $n = 2$, The problem would be solved if we prove the following statement:

Proposition 7.1. *Given general points $p_1, p_2, ..., p_m$ on $\mathbb{P}^n$, there exists a set of positive integers $d, a_1, a_2, ..., a_m$ such that $d^2 = a_1^2 + a_2^2 + ... + a_m^2$ and the linear system $\mathfrak{d}$ consisting of degree d curves passing through p_i with multiplicity a_i is nonempty and does not have base component of dimension 1.*

If this statement is true, then it suffices to pick two general curves in $\mathfrak{d}$ and then they should intersect at a zero dimensional set, which means a set of points. Then Corollary 3.10 and $d^2 = a_1^2 + a_2^2 + ... + a_m^2$ tells us that $p_1, p_2, ..., p_m$ are the only intersection of these two curves, so this set of points is represented as a set-theoretic complete intersection of this two curves, which will complete the proof for Theorem 4.3. Thus it suffices for us to prove this proposition.

In order to prove Proposition 7.1, we need the following lemma:

Lemma 7.2. *Suppose $\mathfrak{d}$ is a linear system of degree d curves on $\mathbb{P}^2$ and p is a base point of $\mathfrak{d}$ such that all curves passes through p by multiplicity a. Then if the order a tangent directions of the curves in $\mathfrak{d}$ at p can be arbitrary, then p is not contained in any 1-dimensional base component of $\mathfrak{d}$.*

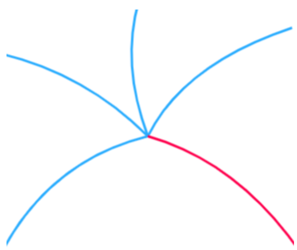
We should firstly make clear what are the "tangent directions" of order a:

Definition 7.3. Suppose a curve $C : f = 0$ on $\mathbb{A}^2$ passes through p with multiplicity a. Then after removing p to $(0, 0)$, the *order a tangent directions of C at p* is defined as the linear factors of f_a, where $f = f_d + f_{d-1} ... + f_a$ is the homogeneous decomposition of f.

For this notion in $\mathbb{P}^2$ one firstly localize in some affine atlas and get the tangent directions there, the result should be independent of the atlas you choose since it's equivalent to factorize in the local ring.

Proof of Lemma 7.2. Let's consider locally, without loss of generality we can suppose $P = (0, 0)$ in $\mathbb{A}^2$. If there is a fixed component passing through p, namely some $h = 0$ for some polynomial h, then any curves in $\mathfrak{d}$ should be of the form $f = gh = 0$ for some polynomial g, so the lowest degree part of f can be divided by the lowest degree part of h, which cannot be arbitrary. □

This lemma essentially shows that if there is a base component E passing through p, then tangent directions of any curve in $\mathfrak{d}$ at p has a tangent direction coming from E, as showed in the following graph,



There must be one tangent direction coming from base component

Now we can prove Proposition 7.1:

Proof of Theorem 7.1. Suppose we are given positive integers $d, a_1, a_2, ..., a_m$ and the linear system $\mathfrak{d}$ consists of degree d curves passing through p_i with multiplicity a_i. Then for any point p_i and a given complete set T of order a tangent directions (which means you have exactly a directions in your set), the condition "the order a tangent directions of the curves in $\mathfrak{d}$ at p is T " imposes exactly a linear restriction on the linear system, hence in order to have all possible T as the set of tangent directions of curves in $\mathfrak{d}$ at any p_i, it suffices to have $\dim \mathfrak{d} - a_i \geq 0$. By proposition 5.3, $\dim \mathfrak{d} \geq \binom{d+2}{d} - \sum_{i=1}^{m} \binom{a_i+1}{a_i-1} - 1 = \binom{d+2}{d} - \sum_{i=1}^{m} \binom{a_i+1}{2} - 1$, so if we suppose $a_1 \geq a_2 \geq ... \geq a_m$, then it suffice for us to find the following data: A set of positive integers $(d, a_1, a_2, ..., a_m)$ with $a_1 \geq a_2 \geq ... \geq a_m$ such that

(1) $d^2 = \sum_{i=1}^{m} a_i^2$ (2) $\binom{d+2}{2} - \sum_{i=1}^{m} \binom{a_i+1}{2} - a_1 - 1 \geq 0$

While (1) holds, (2) is equivalent to $3d - 3a_1 - \sum_{i=2}^{m} a_i \geq 0$, so these two conditions turns into homogeneous condition:

(1) $\sum_{i=1}^{m} (\frac{a_i}{d})^2 = 1$ (2) $3\frac{a_1}{d} + \sum_{i=2}^{m} \frac{a_i}{d} \leq 3.$

Hence it suffices for us to find positive rational numbers $x_1, x_2, ..., x_m$ with $x_1 \geq x_2 \geq ... \geq x_m$ such that $\sum_{i=1}^{m} x_i^2 = 1$ and $3x_1 + \sum_{i=2}^{m} x_i \leq 3$ and then we can let d to be the least common multiple of denominators of x_i's and let $a_i = d_i x_i$. The last step is to find such a set of positive rational numbers, it is equivalent to find a rational point $(x_1, x_2, ..., x_m)$ on the unit sphere $S^{m-1} \subset \mathbb{R}^m$ such that $x_i > 0$, $3x_1 + \sum_{i=2}^{m} x_i \leq 3$.

Now consider the set

$$R = \{(x_1, x_2, ..., x_m) | x_i > 0, 3x_1 + \sum_{i=2}^{m} x_i \leq 3\}$$

it has nonempty interior: the point $(\sqrt{\frac{1-(m-2)\epsilon^2}{2}}, \sqrt{\frac{1-(m-2)\epsilon^2}{2}}, \epsilon, \epsilon ..., \epsilon)$ is contained in R° for sufficiently small $\epsilon > 0$ since

$$\lim_{\epsilon \to 0^+} 3\sqrt{\frac{1-(m-2)\epsilon^2}{2}} + \sqrt{\frac{1-(m-2)\epsilon^2}{2}} + (m-2)\epsilon = 2\sqrt{2} < 3$$

By a very classic result in [oAAD], rational points on the sphere are dense, so there must be a rational point contained in R°, completing the proof.

□

8. Further explorations

As we showed in Theorem 4.2, not all set of points is a complete intersection of hypersurfaces. Then a natural question emerged:

Question 8.1. *Can we construct a moduli space for all 0-dimensional complete intersection of given index?*

If we consider the sequence of points, i.e., we do not ignore the order, then this could be a subset of $\mathbb{P}^n \times \mathbb{P}^n .... \times \mathbb{P}^n$, If we only consider "set", then it could be a subvariety (or simply subset) in the punctured Hilbert scheme $\mathcal{H}^m_{\mathbb{P}^n}$.

Another similar question is

Question 8.2. *Given a set of points that is a complete intersection, construct a moduli space of all tuples of hypersurfaces $(H_1, H_2, ..., H_n)$ whose intersections are compelete and consists of those points.*

Based on our method proving Theorem 4.3, we can make the following conjecture:

Conjecture 8.3. *Any set of points on $\mathbb{P}^n$ is a set-theoretic complete intersection.*

And there should be a similar way controlling the lowest degree of the forms at given terms to avoid base components. This work could be possible but tedious, so we leave it for future exploration.

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I am deeply grateful to my advisor, Dr. Yue Zhao, for her guidance throughout this project. She introduced me to foundational tools in commutative algebra and algebraic geometry, and helped me situate the problem in its broader context. In particular, a motivating open question in intersection theory asks whether every irreducible curve in $\mathbb{P}^3$ is a set-theoretic intersection of two surfaces. My project was inspired by a simpler, related direction: whether a general finite set of closed points in $\mathbb{P}^2$ is a set-theoretic intersection of two curves, and whether a general finite set of points in $\mathbb{P}^n$ can be realized as a scheme-theoretic complete intersection of n hypersurfaces.

After studying the necessary background, I established the scheme-theoretic complete intersection result in $\mathbb{P}^2$. With minor suggestions from my advisor—primarily about organizing the argument and verifying dimension counts—I extended the result to $\mathbb{P}^n$ for $n > 2$, using Bézout's theorem and computations in the Chow ring to control degrees and intersection multiplicities.

For the set-theoretic part, I derived two conditions under which a set of m points in $\mathbb{P}^2$ is a complete intersection of two curves of the same degree: (i) a numerical equality matching the global intersection number with the sum of local intersection multiplicities, and (ii) a dimension inequality ensuring the existence of a positive-dimensional linear system with the prescribed multiplicities. A substantive difficulty arose in ruling out the existence of a fixed one-dimensional base component of the linear system $\mathfrak{d}$ passing through the given points. With my advisor's feedback, I addressed this by imposing general order-a jet conditions at the points (i.e., allowing arbitrary tangent directions for the order-a parts). This ensures that, for two general members of $\mathfrak{d}$, the local intersection multiplicity at each base point achieves the expected product of orders and that no unintended common component persists. This resolves the base-locus issue and validates the set-theoretic complete intersection construction.

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