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Research Report

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Title of Research Report

The Geometric Properties of 2-Knot Immersions and

Their Effects on Invariants

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Abstract

We investigate the geometric and topological properties of 2-knots embedded in a four-dimensional real space. Starting from the fundamental group of the complement of an embedded 2-sphere in $\mathbb{R}^4$, we provide an intuitive geometric interpretation and analyze how local immersion structures of surfaces influence this group. This viewpoint is extended to complements of general 2-knots, where we establish its invariance under Reidemeister moves (R-moves), illustrated by the spun trefoil knot. Emphasis is placed on the geometric meaning of algebraic invariants: we examine the interplay between 2-knot diagrams, the Alexander polynomial, and the Alexander matrix, interpreting the latter via the associated chain complex. We also study the geometric implications of relator substitutions, clarify the relationship between the Burau representation and the Alexander matrix, and outline potential applications of these structures in both mathematics and applied contexts.

Keywords:2-knot theory, fundamental group, Fox calculus, Alexander invariant.

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We hereby declare that we

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3. observe the common standard of academic integrity adopted by most journals and degree theses.
4. have declared all the assistance and contribution we have received from any personnel, agency, institution, etc. for the research work.
5. undertake to avoid getting in touch with assessment panel members in a way that may lead to direct or indirect conflict of interest.
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Contents

1 Introduction

Knot theory is a fundamental area of topology that studies embeddings of manifolds in higher-dimensional spaces. Classical one-dimensional knot theory investigates embeddings of S^1 in S^3, while *2-knot theory* is the natural higher-dimensional generalization concerning embeddings of S^2 in S^4. This field connects multiple branches of mathematics, including algebraic topology, combinatorial group theory, and low-dimensional topology, and holds potential applications in topological quantum computation, molecular structure analysis, image compression, and more. However, existing research on 2-knots has largely focused on algebraic or combinatorial approaches (e.g., Carter–Saito's algebraic methods), with relatively less emphasis on geometric intuition.

In the study of 2-knots, the fundamental group of the knot complement is a crucial invariant. Yet, its geometric meaning—especially its relation to local immersion structures—remains insufficiently explored and is primarily interpreted algebraically, such as in Carter–Saito's work [3]. Current methods for computing invariants like the Alexander polynomial mostly rely on algebraic derivations, making it challenging to intuitively understand their behavior under isotopies and Roseman-type Reidemeister moves (R-moves). Therefore, this research aims to establish an analytical framework that bridges algebraic invariants with geometric constructions, addressing this gap.

This paper adopts a constructive and geometrically intuitive approach to 2-knot theory, presenting the following main contributions:

(1) Provide a geometric interpretation of the fundamental group of the 2-knot complement based on oriented crossing data.

(2) Prove the invariance of this fundamental group under R-moves.

(3) Derive a skein-form Alexander polynomial for 2-knots.

(4) Analyze the effect of relation substitutions on the Alexander matrix and establish its connection with the Burau representation of braid groups.

(5) Explore the extension to multi-component 2-knots (links) and their applications in quantum computation and cryptography.

The structure of this paper is as follows: Section 2 reviews preliminaries including 2-knots, fundamental groups, Fox calculus, and Alexander invariants; Section 3 surveys related previous works and their connection to this study; Section 4 presents the main theorems and proofs, including geometric interpretations of the fundamental group, its invariance, and properties of the Alexander matrix; Section 5 discusses applications in topological quantum computation and cryptography; Section 6 concludes with a summary and future research directions.

2 Preliminaries

Definition 2.0.1 (2-Knot)**.** Consider the embedding of the 2-sphere S^2 in $\mathbb{R}^4$. From this we deduce that a 2-knot is connected, orientable, and has one componet, genus zero.

Here we use the definition of 2-knot diagram in [13]

Definition 2.0.2 (Generic Projection)**.** Define the map $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$\pi(x, y, z, w) = (x, y, z).$$

If all points in $\pi(K)$ are regular points, double points, triple points or branch points, then $\pi(K)$ is called *generic*. We denote by $Dp(K)$, $Tp(K)$, and $Bp(K)$ the sets of double points, triple points, and branch points respectively.

Definition 2.0.3 (2-Knot Diagram)**.** Suppose $\pi_K : K \to \mathbb{R}^3$ and let $p \in Dp(K)$. The preimage $\pi_K^{-1}(p)$ contains two points p_- and p_+ satisfying $w(p_+) > w(p_-)$. We call p_- the *lower point* of p and p_+ the *upper point* of p.

Define $\Lambda_-(K)$ as the closure of the set of all lower points. Let N be a neighborhood of $\Lambda_-(K)$ as illustrated. The diagram of K is then given by

$$D(K) = \pi(K) - \pi(\operatorname{int} N).$$

In the diagram $D(K)$, the lower disk near each double point is divided into two halves, where the double points on the lower and upper disks are depicted by solid and dashed lines respectively, as shown in Figure 1.

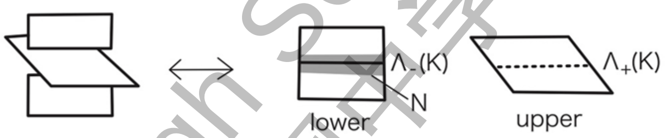

Figure 1: Illustration adapted from [13].

If $t \in Tp(K)$, then $\pi^{-1}|_K(t) = \{t_1, t_2, t_3\}$ with $w(t_1) < w(t_2) < w(t_3)$.
The points t_1, t_2, t_3 are called the *lower*, *middle*, and *upper* points of t, respectively. The disks containing these three points are correspondingly called the *lower*, *middle*, and *upper disks* of t. In the diagram $D(K)$, the middle and lower disks are divided into four and two regions respectively, as illustrated in Figure 2.

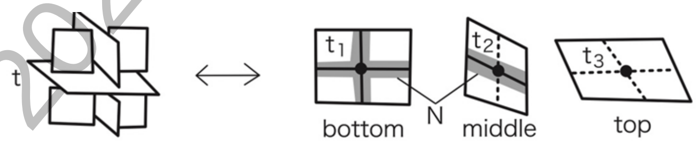

Figure 2: Illustration adapted from [13].

If $b \in Bp(K)$, then

$$\pi^{-1}|_K(b) = \{b\},$$

and b is an endpoint of a double point arc, as illustrated in Figure 3.
The elements of a 2-knot diagram are called *sheets*, and the *sheet number* refers to the number of sheets in the 2-knot diagram. We define the *sheet number* of a 2-knot as the minimal sheet number among all its possible 2-knot diagrams.

Figure 3: Illustration adapted from [13].

Definition 2.0.4 (Roseman moves)**.** From Roseman's study [12], we know that the ambient isotopies between closed surfaces can be categorized into seven types of moves, as illustrated in Figures 4 and 5. These seven moves are called (Roseman's) Reidemeister moves, and are denoted by $\mathcal{R}$. It is known that if there exists a composition of R-moves between two closed surfaces, then they are ambient isotopic.

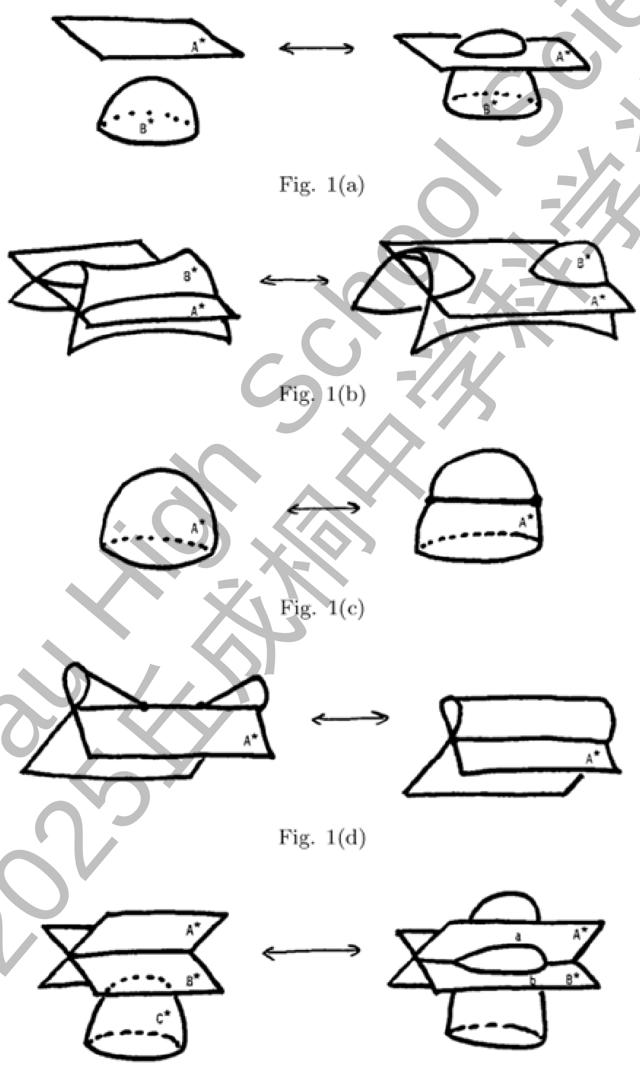

Figure 4: Illustration adapted from [12].

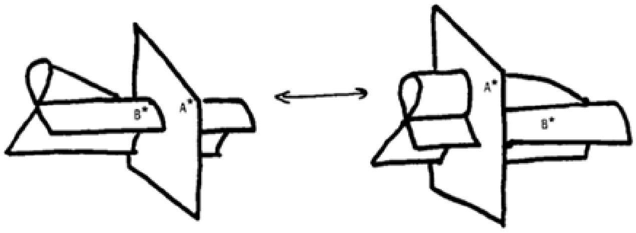

Fig. 1(f)

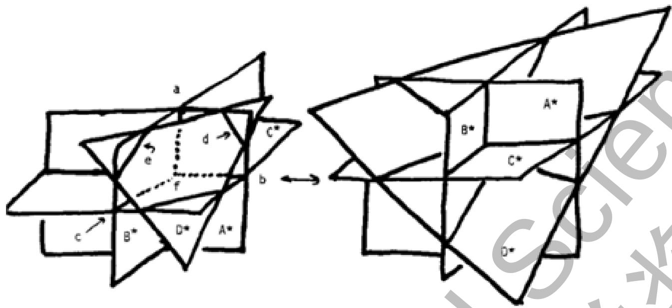

Fig. 1(g)

Figure 5: Illustration adapted from [12].

Definition 2.0.5 (fundamental group)**.** Let X be a topological space with a fixed base point p. A *loop* is a continuous map

$$\gamma : [0,1] \to X$$

such that $\gamma(0) = \gamma(1) = p$. Consider the homotopy class $[\gamma]$ of γ in X. The concatenation of two loops f and g is defined as:

$$(f * g)(t) = \begin{cases} f(2t) & x \in [0,1/2] \\ g(2t-1) & x \in [1/2,1] \end{cases}$$

It is clear that the homotopy class $[f * g]$ depends only on the homotopy classes $[f]$ and $[g]$. Therefore, we can define a binary operation. $* : \pi_1(X,p) \times \pi_1(X,p) \to \pi_1(X,p)$ by concatenation.

Furthermore, we can construct the identity element and inverses, and prove associativity. Hence, the set of homotopy classes of loops based at p forms a group under the operation $*$, called the *fundamental group* of X with base point p, denoted by

$$\pi_1(X,p).$$

Definition 2.0.6 (Free group)**.** A group G is called a *free group* if there exists a subset $S \subseteq G$ such that every element of G can be uniquely expressed as a product of elements of S and their inverses, where two elements in G are distinct unless they are equal by the group axioms (i.e., reductions involving an element and its inverse). Such a group G is denoted by F_S, and the elements of S are called the *generators* (or factors) of F_S. The cardinality of S is called the *rank* of F_S.

Given two groups A and B, their *free product* $A * B$ is the free group generated by the elements of both A and B.

Here we present the most basic statement; proofs and detailed discussions can be found in [7].

Theorem 2.0.7 (Seifert–Van Kampen's Theorem). *Let X be the union of two open sets $U, V \subseteq X$ satisfying:*

1. *U and V are path-connected,*
2. *$U \cap V$ is also path-connected,*
3. *X, U, V, and $U \cap V$ share a common base point.*

Then the fundamental group of X is the amalgamation of the fundamental groups of U and V over $U \cap V$, i.e.,

$$\pi_1(X) \cong \pi_1(U) * \pi_1(V)/\langle N \rangle,$$

where N is the identification of $\pi_1(U \cap V)$ in $\pi_1(U) * \pi_1(V)$.

Definition 2.0.8 (Fox Differential). This involves translating the knot group relations into derivatives, which are used to study group representations and invariants in topology. The specific operations are as follows:

$$\frac{\partial 1}{\partial x} = 0, \quad \frac{\partial x}{\partial x} = 1, \quad \frac{\partial y}{\partial x} = 0 \quad \text{if } y \neq x,$$

$$\frac{\partial (uv)}{\partial x} = \frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x},$$

$$\frac{\partial x^{-1}}{\partial x} = -x^{-1}.$$

Definition 2.0.9 (Alexander matrix). Given a 2-knot K in the space. Set:

$$\pi_1(\mathbb{R}^4 - K) = \langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_{n-1} \rangle$$

Let

$$M = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_{n-1}}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_1}{\partial x_n} & \cdots & \frac{\partial r_{n-1}}{\partial x_n} \end{bmatrix}$$

The Alexander matrix A is the result of M when we abelianize the group $\pi_1(\mathbb{R}^4 - K)$.

Definition 2.0.10 (Alexander polynomial).

$$\Delta_K(t) = \gcd(\det A_i(t)).$$

(Here we define $\det(A_i(t)) \in \mathbb{R}[t, t^{-1}]$)
where $A_i(t)$ denotes the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row from A.

Definition 2.0.11 (Artin representation of braid group)**.** We can represent braid group in free group form with isomorphism $\phi : B_n \to Aut(F_n)$

$$\phi(\sigma_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j \mapsto x_j, \quad \text{for } j \neq i, i+1. \end{cases}$$

Definition 2.0.12 (Burau representation)**.** Here we only consider the reduced case. A linear representation of the braid group is given by an isomorphism$\rho : B_n \to GL_n\big(\mathbb{Z}[t, t^{-1}]\big)$ defined on the generators σ_i by

$$\rho(\sigma_i) = \begin{pmatrix} I_{i-1} & & 0 \\ & \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix} & \\ 0 & & I_{n-i-1} \end{pmatrix},$$

where I_k denotes the $k \times k$ identity matrix.

3 Literature review

In the early 20th century, topology gradually emerged as an important branch of mathematics. Knot theory, as one of its core research areas, was fundamentally shaped by Alexander's introduction of the Alexander polynomial in 1928, laying the groundwork for knot invariants [1].

From the 1980s through the early 21st century, the study of 2-knots (2-dimensional knots) began to flourish, with research directions broadly divided into two categories: one focusing on visualization and geometric representations of 2-knots, exemplified by Shin Satoh and Dennis Roseman, and the other on algebraic descriptions, represented by Seiichi Kamada. Moreover, scholars such as J. Scott Carter and Masahico Saito developed a systematic theoretical framework bridging these approaches.

In 1998, Roseman introduced the original formulation of Reidemeister-type moves for surfaces in four dimensions [12]. Subsequently, Carter and Saito systematically organized Roseman moves and explored the relations between 2-knot diagrams and the fundamental groups of their complements in [3]. In 1994, Kamada and Fernández-Acuña respectively established foundational theories characterizing 2-knot groups in [9] and [4], providing theoretical bases for later works by Damiani, Kádár, and others.

The relationship between 2-knot diagrams and fundamental groups of their complements proposed in this paper is consistent with the results of Carter and Saito [3], but emphasizes a more visual and constructive approach, aiming to offer a more intuitive and operational interpretation. Furthermore, our proof concerning the connection between the Burau representation and the Artin representation adopts a linear algebra perspective, contrasting with the group isomorphism approach used in [2], thereby demonstrating a different proof strategy and viewpoint.

4 Main Results

4.1 Geometric Interpretation of the Fundamental Group

Lemma 4.1.1. $\pi_1(S^1) \cong \mathbb{Z}, \quad \pi_1(S^n) \cong 0 \quad \textit{for } n \geq 2.$

For a detailed proof of this lemma, see [7], pp. 38–40 and p. 44.

Before discussing the geometric meaning of the fundamental group of $\mathbb{R}^4 - K$, it is helpful to understand what the fundamental group is under the most general condition (the sphere).

Lemma 4.1.2. $\pi_1\left(\mathbb{R}^4 - S^2\right) \cong \mathbb{Z}$

Proof. Since $\mathbb{R}^4 - S^2$ is homotopy equivalent to $S^4 - \{p\} \cup S^2$ with $p \notin S^2$, and $S^4 - \{p\} \cup S^2$ is homotopy equivalent to the wedge sum $S^3 \vee S^1$ (where $\vee$ denotes the wedge sum, i.e., S^3 and S^1 intersecting at a point), we have

$$\pi_1\left(\mathbb{R}^4 - S^2\right) \cong \pi_1\left(S^3 \vee S^1\right).$$

By *Seifert–Van Kampen's Theorem*, it follows that

$$\pi_1\left(S^3 \vee S^1\right) \cong \pi_1(S^3) * \pi_1(S^1).$$

Since

$$\pi_1(S^n) \cong 0 \quad \forall n \in \mathbb{N}, n > 1,$$

we conclude

$$\pi_1\left(\mathbb{R}^4 - S^2\right) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

□

Lemma 4.1.3. *Under the preservation of the immersion structure, any graph can be regarded as an enclosing trace.*

Proof. Given an immersion $f : S^2 \looparrowright M$, of the 2-sphere, the self-intersection set $\Sigma \subset S^2$ partitions the sphere. After cutting along Σ, the sphere $S^2 \setminus \Sigma$ decomposes into several connected components, each homeomorphic to a sphere with holes (i.e., a sphere with finitely many open disks removed). This structure is illustrated in Figure 6.
Moreover, the existence of ambient isotopies allows each sphere to be moved to different positions, which completes the proof. □

Consequently, we obtain a new representation. Although this representation may not effectively help us visualize 2-knots, it aids in understanding their algebraic structure.

Lemma 4.1.4. *For 2-knots with only double points, if we consider connectivity as the criterion, then the 2-knot can be represented as a tree.*

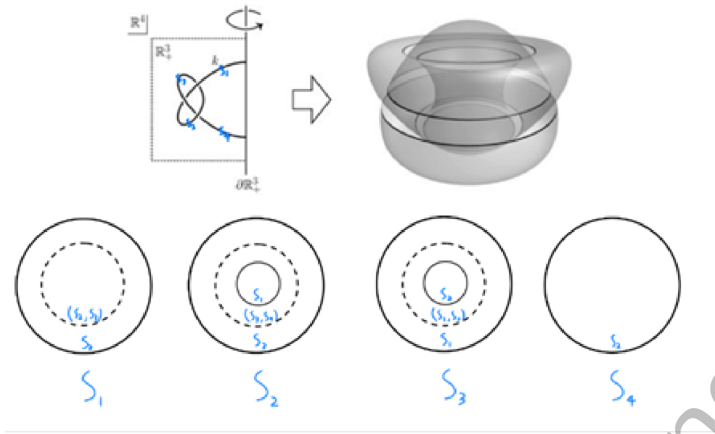

Figure 6: Illustration adapted from [8].

Proof. If there exists a cycle, then the genus is nonzero. Moreover, we observe that the leaves of the tree are homeomorphic to $S^2 \setminus D$, where D is an open disk, and other points are homeomorphic to $S^2 \setminus (D_1 \cup D_2)$, where D_1, D_2 are open disks with $D_1 \cap D_2 = \emptyset$. □

Proposition 4.1.5. $\pi_1\left(\mathbb{R}^4 - \left(S^1 \times y\text{-}axis\right)\right) \cong \mathbb{Z}$.

Proof.

$$\pi_1\left(\mathbb{R}^4 - \left(S^1 \times y\text{-axis}\right)\right) \cong \pi_1\left((\mathbb{R}^3 - S^1) \times \mathbb{R}\right) \cong \pi_1(\mathbb{R}^3 - S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z} \times 0 \cong \mathbb{Z}.$$

□

Based on the conclusion of **Lemma 4.1.2**: $\pi_1\left(\mathbb{R}^4 - S^2\right) \cong \pi_1(S^1)$. We may construct this using an oriented circle:

Theorem 4.1.6 (Geometric Meaning of $\pi_1(\mathbb{R}^4 - S^2)$)**.** *The oriented linking number k between loops and surfaces determines the elements in the fundamental group $\pi_1(\mathbb{R}^4 - S^2)$ of the complement space.*

For convenience in the subsequent proof, we provide here a coordinate description.

Theorem 4.1.7. *In $\mathbb{R}^4$ with coordinates (x, y, z, w), consider an orientable loop a passing through an orientable surface s. That is, there exist four points $p_{1a}, p_{1s}, p_{2a}, p_{2s}$ satisfying*

$$(x_{p_{1a}}, y_{p_{1a}}, z_{p_{1a}}) = (x_{p_{1s}}, y_{p_{1s}}, z_{p_{1s}}), \quad (x_{p_{2a}}, y_{p_{2a}}, z_{p_{2a}}) = (x_{p_{2s}}, y_{p_{2s}}, z_{p_{2s}}).$$

If p_{js} and p_{ja} have the same orientation, then $w(p_{js}) < w(p_{ja})$; if they have opposite orientation, then $w(p_{js}) > w(p_{ja})$. We denote this relation by α.

This situation is illustrated in Figure 7:

The statement for α^{-1} is the opposite: if the orientations are the same, then $w(p_{js}) > w(p_{ja})$, and if the orientations are opposite, then $w(p_{js}) < w(p_{ja})$.

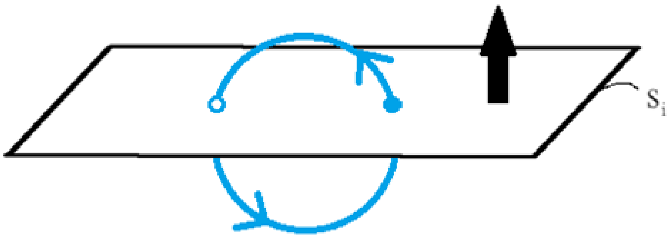

Figure 7

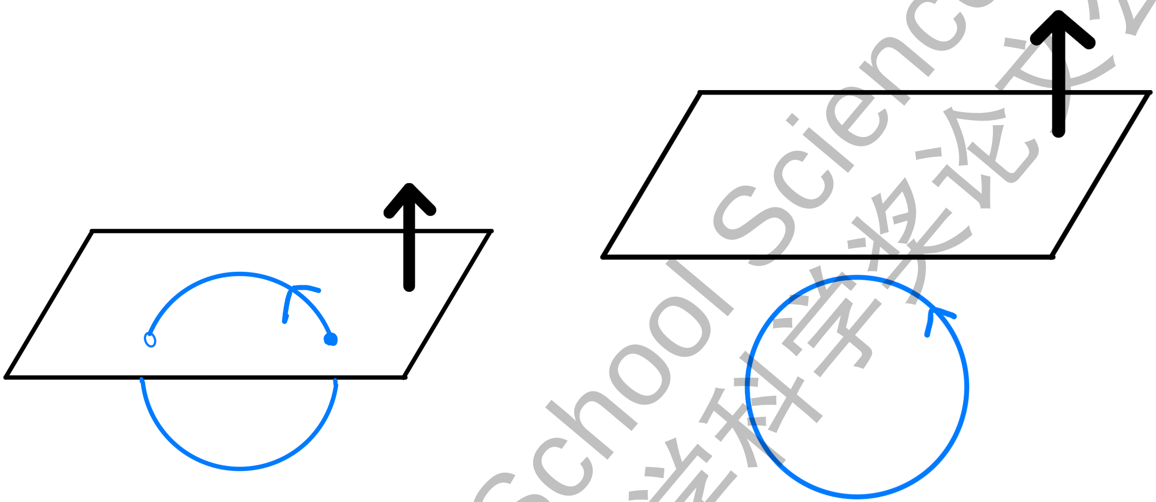

Figure 8: Left:A, Right:B

Proof. Suppose, for the sake of contradiction, that A and B share the same enclosing trace. Consider the segment joining p_{1a} and p_{2a}, and let τ denote its projection onto s. Since a is a continuous oriented loop, and either

$$w\left(p_{1s}\right) < w\left(p_{1a}\right) \wedge w\left(p_{2s}\right) < w\left(p_{2a}\right)$$

or

$$w\left(p_{1s}\right) > w\left(p_{1a}\right) \wedge w\left(p_{2s}\right) > w\left(p_{2a}\right),$$

there must exist a point p on a_i such that

$$w\left(p_i\right) = w\left(\tau\left(p_i\right)\right).$$

It follows that, within the corresponding homotopy, there exist embeddings carrying A into both $S^2 \vee S^1$ and $S^2 \cup S^1$. This would imply that $S^2 \vee S^1$ and $S^2 \cup S^1$ are homeomorphic, a contradiction. Therefore, our assumption is false, and the claim is established. □

With the geometric interpretation of $\pi_1(\mathbb{R}^4 - S^2)$ established, we can further explore the geometric meaning of surface immersions.

Theorem 4.1.8 (Local Geometric Meaning of Surface Immersions.)**.** *If, in a 2-knot K, there* *exist local structures s_i, s_j, s_k as shown in Figure 9, then in $\pi_1(\mathbb{R}^4 - K)$ there is the relation*

$$\alpha_k\, \alpha_i\, \alpha_k^{-1}\, \alpha_j^{-1} = e.$$

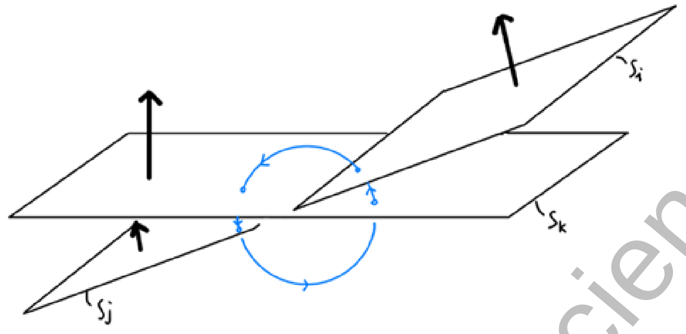

Figure 9

Proof. We now let K be the local structure of the Figure 9 and discuss its relation to the space $\mathbb{R}^4$. Without loss of generality, we place the 2-knot without double points in the subspace $w = 0$, and send the neighborhood of each double point to $w = k$ with $k < 0$. Furthermore, we can homotope this 2-knot so that it has a thickness $\epsilon_1 > 0$ in the (x, y, z)-directions, and a thickness $\epsilon_2 > 0$ (interval) in the w-direction.

Consider the spaces

$$A = \{(x, y, z, w) \mid x, y, z, w \in \mathbb{R},\ w > k - \epsilon_2\}, \quad B = \{(x, y, z, w) \mid x, y, z, w \in \mathbb{R},\ w < k\}.$$

From *Lemma 4.1.2*, *Lemma 4.1.4*, and *Seifert–Van Kampen's Theorem*, we know

$$\pi_1(A - K) = \langle \alpha_i, \alpha_j, \alpha_k \rangle.$$

We also observe that

$$\pi_1(B) \cong 0, \quad \pi_1(A \cap B) \text{ is a rank 1 free group.}$$

The elements of $\pi_1(A \cap B)$ are of the form

$$\alpha_i \alpha_k \alpha_i^{-1} \alpha_j^{-1},$$

which correspond to the identity e in $\pi_1(B)$. Therefore,

$$\pi_1(\mathbb{R}^4 - K) \cong \pi_1(A \cup B) \cong \pi_1(A) * \pi_1(B)/N \cong \langle \alpha_i, \alpha_j, \alpha_k \rangle / \langle \alpha_k \alpha_i \alpha_k^{-1} \alpha_j^{-1} \rangle.$$

$\square$

We now proceed to examine the types of relators that arise from triple points and branch points.

Theorem 4.1.9 (The effect of branch points on the fundamental group of the complement space of a 2-knot)**.** *In a 2-knot K, the structure of branch points does not correspond directly to any relators in* $\pi_1\left(\mathbb{R}^4 - K\right)$.

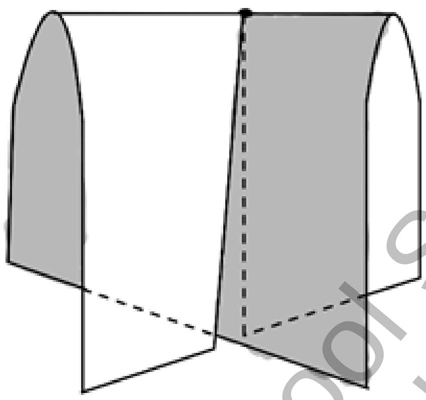

Figure 10

Proof. We decompose the structure of the branch points along the double point arcs into three parts as shown in Figure 11 (yellow, green, and blue) for analysis. Move the blue part K_b along the w-direction so that $w(K_b) < \epsilon_1$, move the yellow part K_y so that $w(K_y) > \epsilon_2$, where $\epsilon_1 > \epsilon_2$, and the green part K_g is the intersection of the yellow and blue parts, i.e., $w(K_g) \in (\epsilon_2, \epsilon_1)$.

Under this setup, analyzing the complement spaces of these three structures and applying *Seifert–Van Kampen's Theorem* yields

$$\pi_1\left(\mathbb{R}^4 - (K_b \cup K_y)\right) \cong \pi_1\left((\mathbb{R}^3 \times (w < \epsilon_1)) - K_b\right) * \pi_1\left((\mathbb{R}^3 \times (w > \epsilon_2)) - K_y\right) / \langle \alpha_{ib}\alpha_{iy}^{-1} \rangle.$$

Since

$$\pi_1\left((\mathbb{R}^3 \times (w < \epsilon_1)) - K_b\right) \cong \mathbb{Z}, \quad \pi_1\left((\mathbb{R}^3 \times (w > \epsilon_2)) - K_y\right) \cong \mathbb{Z},$$

we have

$$\pi_1\left(\mathbb{R}^4 - (K_b \cup K_y)\right) \cong \mathbb{Z} * \mathbb{Z} / \langle \alpha_{ib}\alpha_{iy}^{-1} \rangle \cong \mathbb{Z}.$$

□

In simple terms, since the fundamental groups of the blue and yellow regions are identified as equivalent via the identification in the green region, their elements correspond.

From an intuitive perspective, we can regard the double point arcs with branch points as self-intersecting double point arcs.

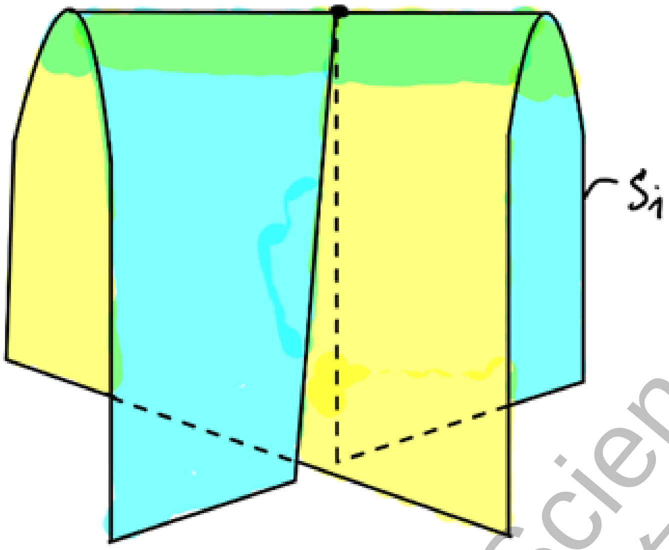

Figure 11

Theorem 4.1.10 (The effect of triple points on the fundamental group of the complement space of a 2-knot)**.** *In a 2-knot K, the structure of triple points does not directly correspond to relators in* $\pi_1\left(\mathbb{R}^4 - K\right)$.

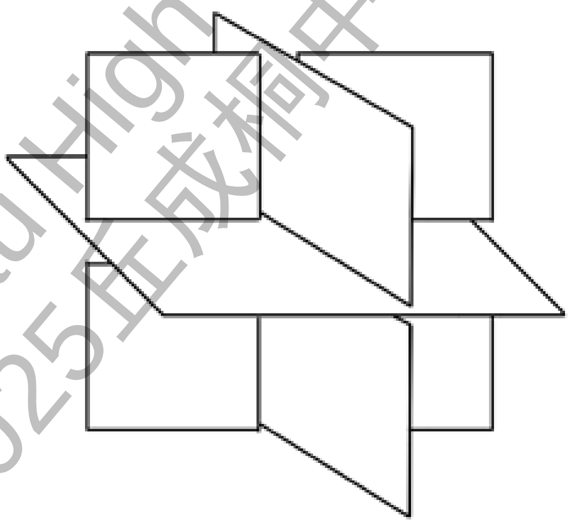

Figure 12

Proof. Here, we label each disk in this construction as shown in Figure 13:
In the structure of a triple point, we classify the double point arcs. Since the double point arcs intersecting the middle and top disks can be moved sufficiently high along the w-direction, we can simplify the situation using *Seifert–Van Kampen's Theorem* to reduce it to the case in

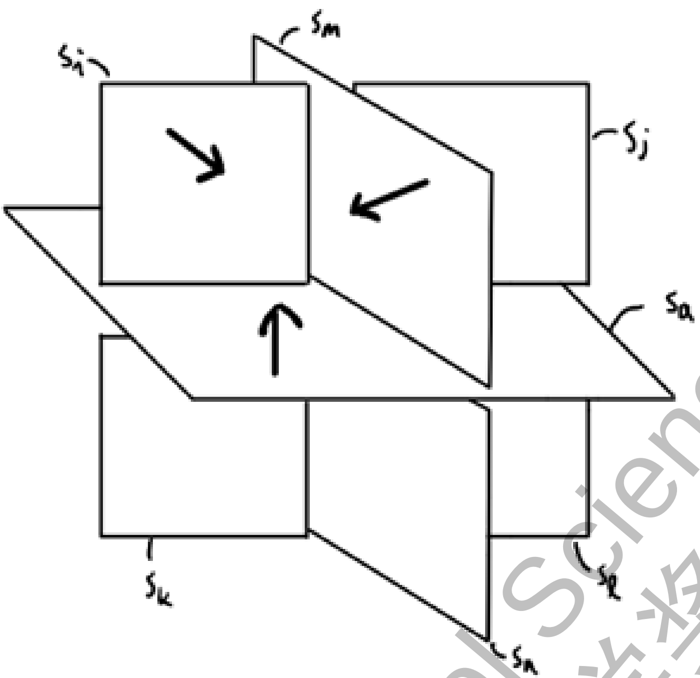

Figure 13

Theorem 4.1.8 and obtain the same conclusion. Therefore, we focus on the double point arcs intersecting the bottom disk, as well as those intersecting the middle and top disks.

The neighborhood of the bottom disk is homeomorphic to $\mathbb{R}^4 - A \times (w\text{-axis})$, where A (see Figure 14) is two intersecting loops (viewing the double point arcs with branch points as self-intersecting double point arcs). From *Lemma 4.1.5* and *Theorem 4.1.8*, we deduce

$$\pi_1\left(\mathbb{R}^4 - A \times (w\text{-axis})\right) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$$

(as illustrated in Figure 15). Corresponding back to the entire space, we obtain the following

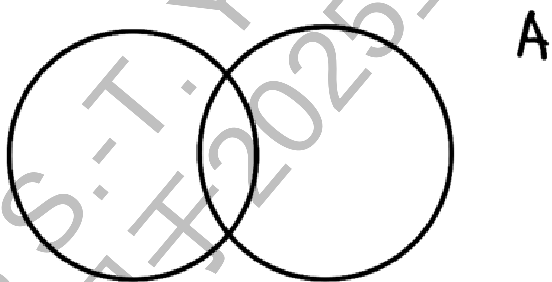

Figure 14

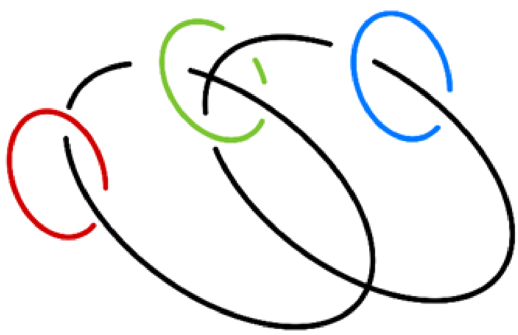

Figure 15

ten relators:

$$\alpha_a\alpha_i\alpha_a^{-1}\alpha_k^{-1}, \quad \alpha_a\alpha_j\alpha_a^{-1}\alpha_l^{-1}, \quad \alpha_m\alpha_i\alpha_m^{-1}\alpha_j^{-1}, \quad \alpha_n\alpha_k\alpha_n^{-1}\alpha_l^{-1},$$

$$\alpha_n\alpha_a\alpha_i\alpha_a^{-1}\alpha_n^{-1}\alpha_l^{-1}, \quad \alpha_n\alpha_k\alpha_n^{-1}\alpha_a\alpha_j^{-1}\alpha_a^{-1}, \quad \alpha_m\alpha_i\alpha_a^{-1}\alpha_n^{-1}\alpha_l^{-1}\alpha_a^{-1}, \quad \alpha_m\alpha_a^{-1}\alpha_k^{-1}\alpha_n^{-1}\alpha_a\alpha_j,$$

$$\alpha_a\alpha_m\alpha_i\alpha_m^{-1}\alpha_a^{-1}\alpha_l^{-1}, \quad \alpha_a\alpha_m^{-1}\alpha_j\alpha_m\alpha_a^{-1}\alpha_k^{-1}$$

and combined with the relator from the top and middle disks:

$$\alpha_a\alpha_m\alpha_a^{-1}\alpha_n^{-1}.$$

We know that six of these relators can be derived from the other four by combination (e.g., substituting $\alpha_j = \alpha_m\alpha_i\alpha_m^{-1}$ into $\alpha_a\alpha_j\alpha_a^{-1}\alpha_l^{-1}$ yields $\alpha_a\alpha_m\alpha_i\alpha_m^{-1}\alpha_a^{-1}\alpha_l^{-1}$).

Therefore, when computing the fundamental group of the complement space with triple point structure, it suffices to consider only the relators arising from the double point arcs. □

Intuitively, paths in the triple point structure can all be expressed as combinations of four basic paths. From the above *Seifert–Van Kampen's Theorem, Theorems 4.1.8, 4.1.9, and 4.1.10* we deduce the following property:

Proposition 4.1.11. *For a 2-knot K, the fundamental group of the complement $\mathbb{R}^4 - K$ can be presented in the form*

$$\pi_1(\mathbb{R}^4 - K) \cong \langle \alpha_1, \ldots, \alpha_n \rangle / \langle \alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1} \rangle$$

for some triple (i, j, k).

Proof. Decompose the complement of K into finitely many connected, open regions

$$A_1, A_2, \ldots, A_m,$$

so that each region contains the neighborhood of a collection of (pairwise) nonconnected double-point arcs. Apply *Seifert-Van Kampen's Theorem* successively to glue these regions together. When two distinct regions A_i and A_j meet along a disk s_k, the identification in the intersection produces the relator

$$\alpha_{ki}\alpha_{kj}^{-1},$$

which, by the original identification, represents the identity element in the glued group.

Hence all relators coming from identifications between different regions are of the form $\alpha_{ki}\alpha_{kj}^{-1}$ and impose no new restrictions beyond those identifications. It follows that it suffices to consider only the relations internal to each region $\pi_1(A_i)$. By *Theorems 4.1.8, 4.1.9 and 4.1.10* (which analyze the local contributions of double points, branch points and triple points), each local fundamental group admits a presentation generated by meridional elements $\alpha_1, \ldots, \alpha_n$ with relations of the form

$$\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1}.$$

Gluing these local presentations together along the trivial identifications yields a global presentation of the form

$$\pi_1(\mathbb{R}^4 - K) \cong \langle \alpha_1, \ldots, \alpha_n \rangle / \langle \alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1} \rangle$$

for some triple (i, j, k), as claimed. □

This proposition allows us to compute the fundamental group of the complement directly from the 2-knot diagram.

Property 4.1.12 ($\pi_1\left(\mathbb{R}^4 - K\right)$ is an invariant of the 2-knot K under R-moves.)**.** *Proof.* Ω_1 :

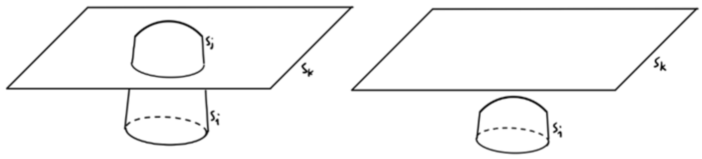

Figure 16

We know in the fundamental group on the left side:

$$\alpha_k \alpha_i^{-1} \alpha_k^{-1} \alpha_j = e,$$

α_i can be expressed as

$$\alpha_i = \alpha_k^{-1} \alpha_j \alpha_k.$$

Therefore,

$$\begin{aligned} \pi_1(\mathbb{R}^4 - A) &\cong \langle \alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n \rangle / \langle \alpha_k \alpha_i \alpha_k^{-1} \alpha_j^{-1}, C \rangle \\ &\cong \langle \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n \rangle / \langle C \rangle \cong \pi_1(\mathbb{R}^4 - B), \end{aligned}$$

where C denotes other relations.
Ω_2 :

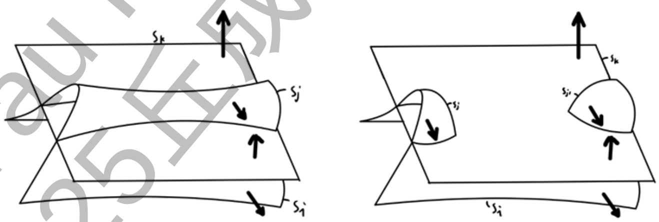

Figure 17

According to the above logic, in $\pi_1(\mathbb{R}^4 - A)$ we have

$$\alpha_j = \alpha_k^{-1} \alpha_i \alpha_k,$$

and similarly in $\pi_1(\mathbb{R}^4 - B)$,

$$\alpha_j = \alpha_k^{-1} \alpha_i \alpha_k = \alpha_{j'},$$

thus we conclude that

$$\pi_1(\mathbb{R}^4 - A) \cong \pi_1(\mathbb{R}^4 - B).$$

Ω_5 :

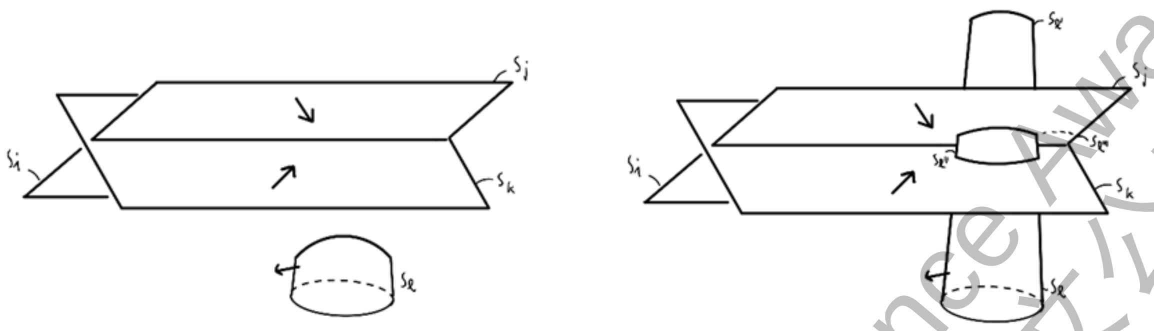

Figure 18

$$\langle \alpha_i, \alpha_k, \alpha_j, \alpha_l, \alpha_{l'}, \alpha_{l''}, \alpha_{l'''} \rangle$$
$$/\left\langle \alpha_k\alpha_i\alpha_k^{-1}\alpha_i^{-1},\ \alpha_l\alpha_k\alpha_{l''}^{-1}\alpha_k^{-1},\ \alpha_{l'}\alpha_j\alpha_{l''}^{-1}\alpha_j^{-1},\ \alpha_{l'''}\alpha_k\alpha_{l'}^{-1}\alpha_k^{-1},\ \alpha_{l'''}\alpha_i\alpha_l^{-1}\alpha_i^{-1} \right\rangle$$

By substituting $\alpha_{l'}$, $\alpha_{l''}$, and $\alpha_{l'''}$, we obtain:

$$\langle \alpha_i, \alpha_k, \alpha_j, \alpha_l \rangle / \langle \alpha_k\alpha_i\alpha_k^{-1}\alpha_i^{-1} \rangle.$$

For Ω_3, Ω_4, and Ω_6, by the properties of branch points in *Theorem 4.1.9*, we know the fundamental groups before and after the moves are the same.

Since the case of Ω_7 is more complicated, we analyze it via the structure: it corresponds to multiple triple points, so by the conclusion of *Theorem 4.1.10*, the fundamental group remains unchanged.

□

Below, we provide an example:

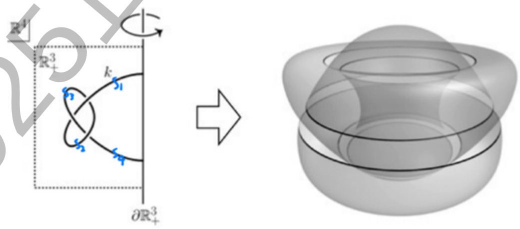

Figure 19: spun trefoil knot. (Illustration adapted from [8].)

Example 4.1.13 (spun trefoil knot).

$$\begin{aligned} \pi_1\left(\mathbb{R}^4 - K\right) &\cong \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle / \langle \alpha_1\alpha_3\alpha_2^{-1}\alpha_3^{-1},\ \alpha_1\alpha_3^{-1}\alpha_1^{-1}\alpha_2,\ \alpha_2\alpha_4^{-1}\alpha_2^{-1}\alpha_3 \rangle \\ &\cong \langle \alpha_2, \alpha_3 \rangle / \langle \alpha_3\alpha_2\alpha_3^{-1}\alpha_2\alpha_3^{-1}\alpha_2 \rangle \\ a = \alpha_3^{-1}\alpha_2, \quad b = \alpha_3 \implies &\cong \langle a, b \mid a^3 = b^2 \rangle \end{aligned}$$

4.2 Alexander Polynomial of 2-Knots

Proposition 4.2.1. *For any 2-knot K (an embedded S^2) in $\mathbb{R}^4$,*

$$H_1(\mathbb{R}^4 - K) \cong \mathbb{Z}.$$

Proof. From *Proposition 4.1.11* we have a presentation

$$\pi_1(\mathbb{R}^4 - K) \;\cong\; \langle \alpha_1, \ldots, \alpha_n \rangle / \langle\, \alpha_k \alpha_i \alpha_k^{-1} \alpha_j^{-1}, \ldots \rangle.$$

Passing to the abelianization (recall $H_1(X) \cong \pi_1^{\mathrm{ab}}(X)$), each relator of the form $\alpha_k \alpha_i \alpha_k^{-1} \alpha_j^{-1} = e$ becomes, in additive/abelian notation,

$$a_k + a_i - a_k - a_j = 0 \quad \Longrightarrow \quad a_i = a_j.$$

Hence every relation of this type identifies the corresponding abelian generators. By *Lemma 4.1.4* the connectivity of the underlying diagram (viewed as a tree) ensures that all generators $\alpha_1, \ldots, \alpha_n$ are identified in the abelianization. Therefore

$$\pi_1^{\mathrm{ab}}(\mathbb{R}^4 - K) \;\cong\; \langle a_1, \ldots, a_n \rangle_{\mathrm{ab}} / \langle a_1 = a_2 = \cdots = a_n \rangle \;\cong\; \langle a \rangle \cong \mathbb{Z}.$$

Consequently,

$$H_1(\mathbb{R}^4 - K) \cong \pi_1^{\mathrm{ab}}(\mathbb{R}^4 - K) \cong \mathbb{Z}.$$

(One can give an explicit isomorphism by sending each meridional class α_i to the generator $1 \in \mathbb{Z}$.) □

We note that directly abelianizing the fundamental group leads to excessive simplification. Therefore, to study the chain complex of the 2-knot complement, we introduce the concept of Fox derivatives, which extend the group to a ring. This approach avoids the problem of over-simplification caused by relators during abelianization. Next, we will discuss the Alexander matrix and Alexander polynomial in more detail. Furthermore, from the proof of *Proposition 4.2.1*, we know that by applying Fox derivatives to any relator and then abelianizing, one obtains a single-variable Laurent polynomial with integer coefficients, i.e., an element of $\mathbb{Z}[t, t^{-1}]$.
We now define a map P to capture the effect of the Fox derivative after abelianization.

Definition 4.2.2. Define a map

$$P : \mathbb{R}(\alpha_1, \ldots, \alpha_n) \to \mathbb{Z}[t, t^{-1}]$$

such that

$$P(\alpha_i) = t, \quad P(1) = 1,$$

and for all $u, v \in \mathbb{R}(\alpha_1, \ldots, \alpha_n)$,

$$P(uv) = P(u) \cdot P(v),$$

$$P(u + v) = P(u) + P(v).$$

First, we compute the relation obtained from the previously derived relator $\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1}$ after applying the Fox derivative followed by abelianization:

Proposition 4.2.3. *For the 2-knot relator $\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1}$, applying the Fox derivative followed by abelianization yields the following equalities:*

$$P\left(\frac{\partial(\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1})}{\partial\alpha_k}\right) = 1-t,\ P\left(\frac{\partial(\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1})}{\partial\alpha_i}\right) = t,\ P\left(\frac{\partial(\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1})}{\partial\alpha_j}\right) = -1.$$

Proof. We compute:

$$\frac{\partial(\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1})}{\partial\alpha_k} = 1-\alpha_k\alpha_i\alpha_k^{-1},$$
$$\frac{\partial(\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1})}{\partial\alpha_i} = \alpha_k,$$
$$\frac{\partial(\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1})}{\partial\alpha_j} = -\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1}.$$

Thus, applying the map P, we have:

$$P\left(\frac{\partial(\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1})}{\partial\alpha_k}\right) = 1-t,$$
$$P\left(\frac{\partial(\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1})}{\partial\alpha_i}\right) = t,$$
$$P\left(\frac{\partial(\alpha_k\alpha_i\alpha_k^{-1}\alpha_j^{-1})}{\partial\alpha_j}\right) = -1.$$

□

Next, we proceed to prove the uniqueness of the Alexander polynomial for 2-knots.

Theorem 4.2.4. *For a 2-knot, the Alexander polynomial satisfies*

$$|\Delta_K(t)| = |\gcd(\det(A_i(t)))| = |\det(A_i(t))|,$$

(Here we define $\det(A_i(t)) \in \mathbb{R}[t, t^{-1}]$*)*
where the absolute value notation is defined as follows:

$$|P(t)| = \begin{cases} P(t), & \textit{if the leading coefficient of } P(t) > 0, \\ -P(t), & \textit{if the leading coefficient of } P(t) < 0. \end{cases}$$

Proof. Note that the sum of each row is zero, i.e.,

$$\sum_{i=1}^{n} v_i = 0.$$

Without loss of generality, consider the relationship between $A_1(t)$ and $A_i(t)$:

$$|\det(A_1(t))| = \left|\det\begin{pmatrix} v_2 \\ \vdots \\ v_n \end{pmatrix}\right| = \left|\det\begin{pmatrix} v_2 \\ \vdots \\ v_j + \sum_{\substack{i=2 \\ i\neq j}}^{n} v_i \\ \vdots \\ v_n \end{pmatrix}\right| = \left|\det\begin{pmatrix} v_2 \\ \vdots \\ -v_1 \\ \vdots \\ v_n \end{pmatrix}\right| = |\det(A_j(t))|.$$

Therefore,

$$|\det(A_1(t))| = |\det(A_2(t))| = \cdots = |\det(A_n(t))|,$$

which implies

$$|\Delta_K(t)| = |\gcd(\det(A_i(t)))| = |\det(A_i(t))|.$$

Hence, we conclude that for each 2-knot quandle, there exists a unique Alexander polynomial.

$\square$

Next, we proceed to discuss the Alexander polynomial of 2-knots in the skein form:

Theorem 4.2.5. *The skein relation for the Alexander polynomial of 2-knots is given by:*

$$\Delta(S_+) - \Delta(S_-) = (t-1)\Delta(S_0),$$

where:

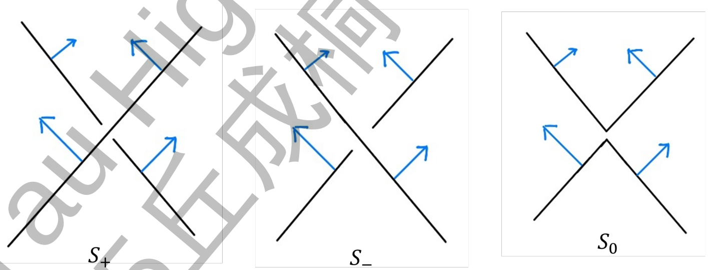

Figure 20

Proof. Without loss of generality, assume the three planes are s_1, s_2, s_3. From this, we obtain the following matrices:

$$M(S_+) = \begin{pmatrix} 1 & v_1 & 1 \\ -t & v_2 & -1 \\ t & u_1 & 0 \\ -1 & u_2 & 0 \\ 0 & \dots & 0 \\ \vdots & N & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \quad M(S_-) = \begin{pmatrix} 1 & v_1 & 0 \\ -t & v_2 & 0 \\ t & u_1 & 1 \\ -1 & u_2 & -1 \\ 0 & \dots & 0 \\ \vdots & N & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \quad M(S_0) = \begin{pmatrix} v_1 & 1 & 0 \\ v_2 & 0 & 1 \\ u_1 & 0 & -1 \\ u_2 & 1 & 0 \\ & 0 & 0 \\ N & \vdots & \vdots \\ & 0 & 0 \end{pmatrix}$$

where v_1, v_2, u_1, u_2 are vectors and N is a matrix.

Consider the matrices A_i obtained by deleting the same row from each $M(\cdot)$, and denote the corresponding reduced matrix N_i replacing N.

We observe that

$$\det\bigl(A_i(S_+)\bigr) - \det\bigl(A_i(S_-)\bigr) = \det \begin{pmatrix} 1 & v_1 & 1 \\ -t & v_2 & -1 \\ t & u_1 & 1 \\ -1 & u_2 & -1 \\ 0 & \dots & 0 \\ \vdots & N_i & \vdots \\ 0 & \dots & 0 \end{pmatrix} = (t-1) \cdot \det \begin{pmatrix} u_1 + v_2 \\ u_2 + v_1 \\ N_i \end{pmatrix}$$

and

$$\det\bigl(A_i(S_0)\bigr) = \det \begin{pmatrix} v_1 & 1 & 0 \\ v_2 & 0 & 1 \\ u_1 & 0 & -1 \\ u_2 & 1 & 0 \\ & 0 & 0 \\ N & \vdots & \vdots \\ & 0 & 0 \end{pmatrix} = \det \begin{pmatrix} u_1 + v_2 \\ u_2 + v_1 \\ N_i \end{pmatrix}$$

Thus, for each A_i, the relation

$$\det\bigl(A_i(S_+)\bigr) - \det\bigl(A_i(S_-)\bigr) = (t-1) \cdot \det\bigl(A_i(S_0)\bigr)$$

always holds.

□

4.3 Transformations of the Alexander Matrix

Theorem 4.3.1. *Suppose that an element α_i of the fundamental group of the knot or 2-knot complement is replaced once by a relation element s_j, expressed as an element u. Then, for other elements in the Alexander matrix, the following transformation holds:*

$$x_{kq} \to x_{kq} + x_{iq} \cdot P\left(\frac{\partial u}{\partial \alpha_k}\right) \quad (k \neq i \wedge q \neq j).$$

Proof. Define

$$\deg(\alpha_i) = 1, \quad \deg(e) = 0, \quad \deg(uv) = \deg(u) + \deg(v).$$

Notice that each relation element s_k satisfies $\deg(s_k) = 0$. If from s_j we derive $\alpha_i = u$ (e.g. $\alpha_1\alpha_2\alpha_1^{-1}\alpha_3^{-1} \implies \alpha_3 = \alpha_1\alpha_2\alpha_1^{-1}$ or $\alpha_1 = \alpha_3\alpha_1\alpha_2^{-1}$), then

$$\deg(u) - \deg(\alpha_i) = \deg(s_k) = 0 \implies \deg(u) = 1.$$

Let s'_q be the relation obtained after substituting α_i by u in s_q. Consider the Fox derivative

$$P\left(\frac{\partial s'_q}{\partial \alpha_k}\right)$$

By the Leibniz rule for Fox derivatives,

$$\frac{\partial(wv)}{\partial g_i} = \frac{\partial w}{\partial g_i} + w\frac{\partial v}{\partial g_i},$$

we separate the part of α_k outside u, denoted by α'_k. Since $\deg(u) = \deg(\alpha_i)$, we have

$$P\left(\frac{\partial s'_q}{\partial \alpha'_k}\right) = P\left(\frac{\partial s_q}{\partial \alpha_k}\right).$$

Next, consider the part of α_k inside u, denoted by β_k. Since $\deg(u) = \deg(\alpha_i)$ and applying the Leibniz rule again, we find that $P\left(\frac{\partial s'_q}{\partial \beta_k}\right)$ can be expressed as a multiple of $P\left(\frac{\partial u}{\partial \beta_k}\right)$, and the multiple is exactly $P\left(\frac{\partial s_q}{\partial \alpha_i}\right)$.

Hence,

$$P\left(\frac{\partial s'_q}{\partial \beta_k}\right) = P\left(\frac{\partial s_q}{\partial \alpha_i}\right) P\left(\frac{\partial u}{\partial \beta_k}\right).$$

Therefore,

$$P\left(\frac{\partial s'_q}{\partial \alpha_k}\right) = P\left(\frac{\partial s'_q}{\partial \alpha'_k}\right) + P\left(\frac{\partial s'_q}{\partial \beta_k}\right) = P\left(\frac{\partial s_q}{\partial \alpha_k}\right) + P\left(\frac{\partial s_q}{\partial \alpha_i}\right) P\left(\frac{\partial u}{\partial \alpha_k}\right).$$

Q.E.D. □

Here, we can explain this from geometric and graph-theoretic perspectives. Consider α_k as a path, where the multiplication of elements corresponds to the concatenation of paths, and each relation corresponds to a loop. The term $P\left(\frac{\partial v}{\partial \alpha_k}\right)$ calculates how many paths are traversed before each occurrence of α_k or α_k^{-1} in the path v.

When α_i is replaced by u, this corresponds to substituting α_i by u in the original relation. Therefore, when recalculating $P\left(\frac{\partial s'_q}{\partial \alpha_k}\right)$, besides considering the original paths, one must also account for occurrences of α_k and α_k^{-1} in u, as well as how many paths are traversed before occurrences of α_i and α_i^{-1} in the original path.

Hence, the above conclusion follows.

Next, we discuss the relationship between the Artin representation of the braid group and the Alexander matrix.

Remark. Artin representation

$$\sigma_i : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j \mapsto x_j, \quad \text{for } j \neq i, i+1. \end{cases}$$

Theorem 4.3.2 (Isomorphism of Artin representation on the Alexander matrix)**.**

$$\sigma_i \mapsto \sigma_i'(A)$$

$$= \begin{pmatrix} I_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 0 & t^{-1} \\ 1 & 1-t^{-1} \end{pmatrix} & \vdots \\ 0 & \cdots & I_{n-i-1} \end{pmatrix} \left(A + \begin{pmatrix} 0_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \vdots \\ 0 & \cdots & 0_{n-i-1} \end{pmatrix} \right) - \begin{pmatrix} 0_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \vdots \\ 0 & \cdots & 0_{n-i-1} \end{pmatrix}$$

Proof. Consider the row vectors of the Alexander matrix as $v_1, v_2, \ldots, v_n$.

Due to the Artin representation

$$\sigma_i : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, \end{cases}$$

we observe the homomorphism $\varphi : B_n \to GL_n\left(\mathbb{Z}[t, t^{-1}]\right)$ defined by

$$\varphi(\sigma_i) : \begin{cases} v_i \mapsto t^{-1} v_{i+1} + \left(0, \ldots, 0, -1, t^{-1}, 0, \ldots, 0\right), \\ v_{i+1} \mapsto (1-t^{-1}) v_{i+1} + v_i + \left(0, \ldots, 0, 1, -t^{-1}, 0, \ldots, 0\right), \\ v_j \mapsto v_j, \quad j \neq i, i+1. \end{cases}$$

(There are $n-1$ zeros preceding the -1 in the vector $\left(0, \ldots, 0, -1, t^{-1}, 0, \ldots, 0\right)$ and the 1 in the vector $\left(0, \ldots, 0, 1, -t^{-1}, 0, \ldots, 0\right)$.)
Hence, the corresponding matrix transformation can be constructed as:

$$\sigma_i'(A)$$

$$= \begin{pmatrix} I_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 0 & t^{-1} \\ 1 & 1-t^{-1} \end{pmatrix} & \vdots \\ 0 & \cdots & I_{n-i-1} \end{pmatrix} \left(A + \begin{pmatrix} 0_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \vdots \\ 0 & \cdots & 0_{n-i-1} \end{pmatrix} \right) - \begin{pmatrix} 0_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \vdots \\ 0 & \cdots & 0_{n-i-1} \end{pmatrix}.$$

□

Here we notice that:

$$\begin{pmatrix} I_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 0 & t^{-1} \\ 1 & 1-t^{-1} \end{pmatrix} & \vdots \\ 0 & \cdots & I_{n-i-1} \end{pmatrix} = \begin{pmatrix} I_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix} & \vdots \\ 0 & \cdots & I_{n-i-1} \end{pmatrix}^{-1}$$

The matrix on the right is the isomorphism corresponding to σ_i in the Burau representation. Here, we examine the relationship between the Burau representation and the Alexander matrix.

Proposition 4.3.3 (Relation between Alexander matrix and Burau representation)**.** *Define*

$$\sigma_i \mapsto A_i = \begin{pmatrix} I_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix} & \vdots \\ 0 & \cdots & I_{n-i-1} \end{pmatrix}.$$

If

$$\prod_{j=1}^{k} \sigma_{i_j} \mapsto \prod_{j=1}^{k} A_{i_j}$$

and

$$\prod_{j=1}^{k} \sigma_{i_j} \mapsto \sigma'_{i_1}\left(\sigma'_{i_2}\left(\ldots\left(\sigma'_{i_k}(0)\right)\ldots\right)\right),$$

then we have

$$\sigma'_{i_1}\left(\sigma'_{i_2}\left(\ldots\left(\sigma'_{i_k}(0)\right)\ldots\right)\right) + I_n = \left(\prod_{j=1}^{k} A_{i_{k+1-j}}\right)^{-1}.$$

Proof. Define

$$S_k = \left(\prod_{j=1}^{k} A_{i_{k+1-j}}\right)^{-1}, \quad S_0 = I_n, \quad D_i = \begin{pmatrix} 0_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \vdots \\ 0 & \cdots & 0_{n-i-1} \end{pmatrix}$$

Note that:

$$\sigma'_{i_1}\left(\sigma'_{i_2}\left(\ldots\left(\sigma'_{i_k}(0)\right)\ldots\right)\right) = \sum_{j=1}^{k} \left(S_j D_{i_j} - S_{j-1} D_{i_j}\right)$$

and

$$A_i^{-1}(I_n - D_i) = I_n - D_i.$$

We prove this by mathematical induction:

- For $k = 1$,

$$S_1 D_{i_1} - D_{i_1} + I_n = S_1,$$

which holds.

- Assume that for $k = m$,

$$\left(\sum_{j=1}^{m} S_j D_{i_j} - S_{j-1} D_{i_j}\right) + I_n = S_m$$

holds.

- For $k = m+1$, we have:

$$\sum_{j=1}^{m} S_j D_{i_j} - S_{j-1} D_{i_j} + I_n = S_m,$$

which is equivalent to

$$-S_m D_{i_{m+1}} + \sum_{j=1}^{m} S_j D_{i_j} - S_{j-1} D_{i_j} + I_n = S_m (I_n - D_{i_{m+1}}),$$

equivalently,

$$-S_m D_{i_{m+1}} + \sum_{j=1}^{m} S_j D_{i_j} - S_{j-1} D_{i_j} + I_n = S_m A_{i_{m+1}}^{-1} (I_n - D_{i_{m+1}}),$$

which implies

$$-S_m D_{i_{m+1}} + \sum_{j=1}^{m} S_j D_{i_j} - S_{j-1} D_{i_j} + I_n = S_{m+1} (I_n - D_{i_{m+1}}),$$

and so

$$\sum_{j=1}^{m+1} S_j D_{i_j} - S_{j-1} D_{i_j} + I_n = S_{m+1}.$$

Thus, by mathematical induction, for all $k \in \mathbb{N}$,

$$\sigma'_{i_1}\left(\sigma'_{i_2}\left(\ldots\left(\sigma'_{i_k}(0)\right)\ldots\right)\right) = \sum_{j=1}^{k} \left(S_j D_{i_j} - S_{j-1} D_{i_j}\right) = S_k = \left(\prod_{j=1}^{k} A_{i_{k+1-j}}\right)^{-1}.$$

□

Next, we extend to the case of 2-knots. We know that the Artin representation of the loop braid group of 2-knots is given by:

$$\sigma_i : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j \mapsto x_j, \quad \text{for } j \neq i, i+1, \end{cases}$$

$$\rho_i : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \quad \text{for } j \neq i, i+1, \end{cases}$$

$$\tau_i : \begin{cases} x_i \mapsto x_i^{-1}, \\ x_j \mapsto x_j, \quad \text{for } j \neq i. \end{cases}$$

It is easy to see that ρ_i corresponds to a permutation matrix, which does not affect our previous proofs. Moreover, if $f(P(t)) = kP(t)$, then f is an involution if and only if $k = \pm 1$, and clearly neither satisfies this condition. Therefore, it is impossible to represent τ_i linearly.

4.4 Situation with multiple knots(link)

Previously, we discussed examples involving a single knot or 2-knot. Here, we aim to study the way multiple knots or 2-knots are linked together, and connect this to the theories introduced above.

First, we note that after abelianization, the conditions in the previously stated theorem no longer hold. As a result, we obtain a multivariable Laurent polynomial with integer coefficients, which makes the computations more complicated.

Below, we consider the effect of the linking of two 2-knots on the Alexander matrix. We know this will be a function of two variables, and thus we need to redefine the function P:

Definition 4.4.1. Define a map $P' : \mathbb{R}\big(\{\alpha_1, \dots, \alpha_n\} \cup \{\beta_1, \dots, \beta_m\}\big) \to \mathbb{Z}[t, t^{-1}, u, u^{-1}]$ satisfying:

$$P'(\alpha_i) = t, \quad P'(\beta_j) = u, \quad P'(1) = 1,$$

and $\forall w, v \in \mathbb{R}\big(\{\alpha_1, \dots, \alpha_n\} \cup \{\beta_1, \dots, \beta_m\}\big)$,

$$P'(wv) = P'(vw) = P'(w) \cdot P'(v),$$

$$P'(w + v) = P'(w) + P'(v).$$

We denote the elements generated by the two knots as $\langle\alpha\rangle_{i=1}^n$, $\langle\beta\rangle_{i=1}^m$, and consider the commutation relation in the fundamental group of the complement:

$$s_q = \alpha_i \beta_k \alpha_j^{-1} \beta_k^{-1}.$$

Then the Fox derivatives evaluated by P' give:

$$P'\left(\frac{\partial s_q}{\partial \alpha_i}\right) = 1,$$

$$P'\left(\frac{\partial s_q}{\partial \beta_k}\right) = u - 1,$$

$$P'\left(\frac{\partial s_q}{\partial \alpha_j}\right) = -t.$$

Since the variables differ, the previous proof for the uniqueness of the Alexander polynomial no longer holds, requiring separate treatment. Moreover, to preserve the independence of the two knots, the skein form of the Alexander polynomial becomes more complicated and difficult to handle.

We want to determine, purely from the braid group, which element belongs to which knot. We pose the following problem:

Problem 4.4.2. *Define the permutation $\rho_i = (1, 2, \dots, i+1, i, \dots, n)$. how can we calculate the number of cyclic permutations in $\prod_{j=1}^{k} \rho_{i_j}$?*

This problem, according to [14], does not have a closed-form formula.

Next, we discuss the case where we know to which knot each element belongs: Let x_i, x_{i+1}

satisfy $P'(x_i) = t$, $P'(x_{i+1}) = u$. Note that

$$\varphi : B_n \to GL_n\big(\mathbb{Z}[t, t^{-1}, u, u^{-1}]\big)$$

and

$$\varphi(\sigma_i) \to \begin{cases} v_i \mapsto u^{-1}v_i + 0 + \cdots + 0 - 1 \cdot v_{i+1} + (u^{-1})v_{i+1} + 0 + \cdots + 0, \\ v_{i+1} \mapsto 1 - t^{-1}v_{i+1} + v_i + 0 + \cdots + 0 + 1 - t^{-1} + 0 + \cdots + 0, \\ v_j \mapsto v_j, \quad \text{for } j \neq i, i+1, \end{cases}$$

so we have

$$\sigma_i'(A)$$

$$= \begin{pmatrix} I_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 0 & u^{-1} \\ 1 & 1-t^{-1} \end{pmatrix} & \vdots \\ 0 & \cdots & I_{n-i-1} \end{pmatrix} \left(A + \begin{pmatrix} 0_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \vdots \\ 0 & \cdots & 0_{n-i-1} \end{pmatrix} \right) - \begin{pmatrix} 0_{i-1} & \cdots & 0 \\ \vdots & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \vdots \\ 0 & \cdots & 0_{n-i-1} \end{pmatrix}.$$

However, since $P'(x_i) = u$, $P'(x_{i+1}) = t$, it is not so straightforward to derive the Alexander matrix directly from the braid group.

5 Applications

Knot theory, although an abstract field, has a variety of applications in the real world. These include medical imaging, such as constructing dynamic surfaces from MRI data, high-dimensional data analysis in data science, and the study of molecular structures in materials science.One of the hottest research areas is undoubtedly the analysis of quantum computation in topological quantum field theory.

In the literature [6], it is mentioned that pairs of Fibonacci anyons can be created in vacuum. When two anyons fuse, the outcomes may be:

- annihilation back to the vacuum,
- combination into a new anyon.

When two anyons annihilate, we denote the state as $|0\rangle$, and when they combine, as $|1\rangle$.

To compute the positions after anyon exchanges in a 2+1 dimensional world, we can represent the exchange paths by braid groups, as illustrated in the figure. The article mentions that we

Figure 21: Illustration adapted from [6].

can construct braids to simulate arbitrary quantum circuits. Compared to other computational methods that require local data, topological quantum computation depends only on the topology of the braids, making the computation results less susceptible to local disturbances.

At the end of the article, the authors also propose the idea of replacing knots with 2-knots, and mention that the computations involved are related to the fundamental group.

Besides the above, we introduce another application: cryptography.
In [5, 11], two different encryption methods were proposed based on distinct invariants and problems. Here, we present a different encryption approach.

In the previous theory, we discussed the Alexander matrix of knot braid groups and the general calculation of the Alexander matrix. However, as mentioned in [10], it is impossible to derive the braid group of a knot solely from its fundamental group or quandle. We propose to develop this problem as the core of an encryption scheme.

6 Conclusion

In this work, we examined the relationship between 2-knot diagrams and the fundamental groups of their complements, emphasizing the role of associated invariants and properties. We extended our discussion to the Alexander matrix, braid groups, and loop braid groups, establishing connections among different invariants and interpreting these connections from both geometric and algebraic perspectives.

Our approach aims to provide a more fundamental and intuitive framework—via 2-knot diagrams and graph-theoretic methods—for interpreting invariants and properties of 2-knots. This framework is intended to serve as a teachable and simulatable mathematical model, thereby lowering the entry barrier for understanding 2-knots and their invariants.

Future directions include the study of multiple interlinked knots and the exploration of further relationships between invariants, such as those involving the Hecke algebra and the Jones polynomial, together with their geometric interpretations. These developments will contribute to a more comprehensive and accessible understanding of knot theory.

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