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论文题目： The Local Average Order of  
Dominating Sets in Paths

# The Local Average Order of Dominating Sets in Paths

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## Abstract

The average order of dominating sets of a graph is the mean cardinality of its dominating sets. Its local analogue is obtained by restricting attention to dominating sets containing a prescribed vertex. We determine the extremal vertices for this parameter in paths. For  $P_n = v_1v_2 \cdots v_n$  with  $n \geq 5$ , we prove that the local average order of dominating sets is minimized precisely at  $v_2$  and  $v_{n-1}$ , and maximized precisely at  $v_3$  and  $v_{n-2}$ . We also give exact expressions for the extremal values in terms of recursively defined sequences. The proof uses a one-sided domination polynomial and a uniform estimate for the associated ratio sequence. All exceptional paths of order at most four are treated separately.

**Keywords:** Dominating set; local average order; domination polynomial; path

## 1 Introduction

All graphs considered in this paper are finite, simple, and undirected. For a graph  $G$ , let  $V(G)$  and  $E(G)$  denote its vertex and edge sets. The open and closed neighborhoods of a vertex  $v$  are  $N_G(v)$  and  $N_G[v] = N_G(v) \cup \{v\}$ , respectively. For  $S \subseteq V(G)$ , put

$$N_G[S] = \bigcup_{v \in S} N_G[v].$$

A set  $S$  is a *dominating set* if  $N_G[S] = V(G)$ . We denote the family of all dominating sets by  $\mathcal{D}(G)$ , and write

$$\mathcal{D}_v(G) = \{S \in \mathcal{D}(G) : v \in S\}.$$

The path and complete graph of order  $n$  are denoted by  $P_n$  and  $K_n$ , respectively. A vertex of degree one is a leaf, and a vertex of degree at least two adjacent to a leaf is a stem.

The study of average graph parameters has a long history, beginning with the Wiener index introduced by Wiener [18]. Doyle and Graver [9] studied the mean distance of graphs and established its extremal values among connected graphs of fixed order. For example, among connected graphs of order  $n$ , the path has maximum mean distance  $(n+1)/3$ , while the complete graph has minimum mean distance 1. Jamison [13] initiated a related line of research on mean subtree order, then Wagner and Wang [16] investigated local and global means of subtree orders. Other average graph parameters include average connectivity [6], average eccentricity [1, 8], and the average size of independent sets [3].

Beaton and Brown [4] introduced the average order of dominating sets, defined by

$$\text{avd}(G) = \frac{\sum_{S \in \mathcal{D}(G)} |S|}{|\mathcal{D}(G)|}.$$

They determined extremal graphs of fixed order and proved that the star is the unique tree minimizing this parameter. They also showed that every  $n$ -vertex graph without isolated vertices satisfies  $\text{avd}(G) \leq 3n/4$ , and conjectured the sharper upper bound  $2n/3$ . Erey [10] established the latter bound for forests without isolated vertices, with equality precisely when every non-leaf

vertex has one or two leaf neighbors. Beaton and Cameron [5] subsequently extended the sharp bound to graphs without isolated vertices.

The *local average order of dominating sets* at  $v$  [7] is

$$\text{avd}_v(G) = \frac{\sum_{S \in \mathcal{D}_v(G)} |S|}{|\mathcal{D}_v(G)|}.$$

Unlike the global parameter, its value depends on the position of the prescribed vertex. Recent work has established upper bounds for specified stems and non-stem vertices. This motivates the following local extremal question: for a fixed graph  $G$ , which vertices minimize or maximize  $\text{avd}_v(G)$ ?

For completeness, let  $d_k(G)$  be the number of dominating sets of order  $k$ . The domination polynomial is

$$D(G, x) = \sum_{k=0}^{|V(G)|} d_k(G) x^k = \sum_{S \in \mathcal{D}(G)} x^{|S|}.$$

The corresponding local polynomial is

$$D_v(G, x) = \sum_{S \in \mathcal{D}_v(G)} x^{|S|}.$$

Consequently,

$$\text{avd}(G) = \frac{D'(G, 1)}{D(G, 1)}, \quad \text{avd}_v(G) = \frac{D'_v(G, 1)}{D_v(G, 1)}.$$

### 1.1 Stars

We first determine the local average order of dominating sets in stars. Let  $S_n = K_{1, n-1}$  be a star of order  $n \geq 3$ , with center  $c$  and leaves  $u_1, \dots, u_{n-1}$ . We consider the center and the leaves separately.

Since  $c$  is adjacent to every leaf, each dominating set containing  $c$  has a unique representation

$$D = \{c\} \cup A, \quad A \subseteq \{u_1, \dots, u_{n-1}\}.$$

Conversely, every set of this form dominates  $S_n$ . Hence

$$|\mathcal{D}_c(S_n)| = 2^{n-1}. \tag{1}$$

The center belongs to every member of  $\mathcal{D}_c(S_n)$ , while each leaf belongs to exactly  $2^{n-2}$  members. Thus

$$\sum_{D \in \mathcal{D}_c(S_n)} |D| = 2^{n-1} + (n-1)2^{n-2} = (n+1)2^{n-2}. \tag{2}$$

Combining (1) and (2) gives

$$\text{avd}_c(S_n) = \frac{(n+1)2^{n-2}}{2^{n-1}} = \frac{n+1}{2}. \tag{3}$$

Now fix a leaf  $u_1$ . Partition  $\mathcal{D}_{u_1}(S_n)$  according to whether the center is selected. Put

$$\mathcal{A} = \{D \in \mathcal{D}_{u_1}(S_n) : c \in D\}, \quad \mathcal{B} = \{D \in \mathcal{D}_{u_1}(S_n) : c \notin D\}.$$

Then

$$\mathcal{D}_{u_1}(S_n) = \mathcal{A} \dot{\cup} \mathcal{B}. \tag{4}$$

We consider the two cases separately.

**Case 1:**  $c \in D$ .

Every member of  $\mathcal{A}$  contains  $c$  and  $u_1$ , and any subset of the remaining  $n - 2$  leaves may be selected. Hence

$$|\mathcal{A}| = 2^{n-2}. \quad (5)$$

Each of the two prescribed vertices belongs to every member of  $\mathcal{A}$ , while each remaining leaf belongs to exactly  $2^{n-3}$  members. Consequently,

$$\sum_{D \in \mathcal{A}} |D| = 2 \cdot 2^{n-2} + (n-2)2^{n-3} = (n+2)2^{n-3}. \quad (6)$$

**Case 2:**  $c \notin D$ .

If  $c \notin D$ , every leaf must belong to  $D$ , since its only neighbor is  $c$ . Conversely, the set of all leaves dominates  $S_n$ . Thus  $\mathcal{B} = \{\{u_1, \dots, u_{n-1}\}\}$ , and hence

$$|\mathcal{B}| = 1, \quad (7)$$

and

$$\sum_{D \in \mathcal{B}} |D| = n - 1. \quad (8)$$

Combining the two cases, by (4), (5), and (7), we obtain

$$|\mathcal{D}_{u_1}(S_n)| = 2^{n-2} + 1. \quad (9)$$

Similarly, (6) and (8) give

$$\sum_{D \in \mathcal{D}_{u_1}(S_n)} |D| = (n+2)2^{n-3} + n - 1. \quad (10)$$

Dividing (10) by (9), we obtain

$$\text{avd}_{u_1}(S_n) = \frac{(n+2)2^{n-3} + n - 1}{2^{n-2} + 1}. \quad (11)$$

By symmetry, all leaves have the same local average order. Therefore, (11) yields

$$\text{avd}_{u_i}(S_n) = \frac{(n+2)2^{n-3} + n - 1}{2^{n-2} + 1}, \quad 1 \leq i \leq n - 1. \quad (12)$$

It remains to compare the two values. Subtracting (3) from (12) and simplifying gives

$$\begin{aligned} \text{avd}_{u_i}(S_n) - \text{avd}_c(S_n) &= \frac{(n+2)2^{n-3} + n - 1}{2^{n-2} + 1} - \frac{n+1}{2} \\ &= \frac{2^{n-2} + n - 3}{2(2^{n-2} + 1)}. \end{aligned} \quad (13)$$

For  $n \geq 3$ , both the numerator and the denominator in (13) are positive. Consequently,  $\text{avd}_v(S_n)$  is minimized precisely at  $c$  and maximized precisely at the leaves. More explicitly,

$$\min_{v \in V(S_n)} \text{avd}_v(S_n) = \frac{n+1}{2}, \quad \max_{v \in V(S_n)} \text{avd}_v(S_n) = \frac{(n+2)2^{n-3} + n - 1}{2^{n-2} + 1}.$$

## 1.2 Paths

The ordinary domination polynomial of a path satisfies the three-term recurrence of Alikhani and Peng [2]:

$$D(P_n, x) = x(D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x)), \quad n \geq 4.$$

However, fixing a vertex in a dominating set changes the boundary conditions on the two resulting subpaths. We therefore introduce a one-sided polynomial that incorporates an already selected external neighbor. This yields an exact decomposition of the local parameter.

Our main result is as follows. The sequences appearing in the statement are defined in Section 2.

**Theorem 1.1.** *Let  $P_n = v_1 v_2 \cdots v_n$  be a path of order  $n \geq 5$ . Then  $\text{avd}_v(P_n)$  is minimized precisely at  $v_2$  and  $v_{n-1}$ , and maximized precisely at  $v_3$  and  $v_{n-2}$ . More precisely,*

$$\min_{v \in V(P_n)} \text{avd}_v(P_n) = \text{avd}_{v_2}(P_n) = \text{avd}_{v_{n-1}}(P_n) = \frac{3}{2} + r_{n-2},$$

and

$$\max_{v \in V(P_n)} \text{avd}_v(P_n) = \text{avd}_{v_3}(P_n) = \text{avd}_{v_{n-2}}(P_n) = \frac{7}{3} + r_{n-3}.$$

When  $n = 5$ , the vertices  $v_3$  and  $v_{n-2}$  coincide, so the maximum is attained uniquely at the central vertex.

The proof is given in Section 3. Section 2 develops the one-sided recurrence and the required fixed-sum inequality. Section 4 records the small-order exceptions and some consequences. A detailed proof of the uniform spectral estimate is included in Appendix A.

## 2 Preliminaries

**Lemma 2.1.** *For every graph  $G$  and every  $v \in V(G)$ ,*

$$\text{avd}_v(G) = \frac{D'_v(G, 1)}{D_v(G, 1)}.$$

*Proof.* By definition,

$$D_v(G, x) = \sum_{S \in \mathcal{D}_v(G)} x^{|S|}.$$

Thus  $D_v(G, 1) = |\mathcal{D}_v(G)|$ , while differentiation gives

$$D'_v(G, 1) = \sum_{S \in \mathcal{D}_v(G)} |S|.$$

The assertion follows by division. □

### 2.1 A one-sided path with a selected external neighbor

For  $m \geq 0$ , let  $Q_m = u_1 u_2 \cdots u_m$ . For  $m \geq 1$ , attach an additional vertex  $z$  adjacent to  $u_m$ , and suppose that  $z$  has already been selected. Define

$$\mathcal{A}_m = \{S \subseteq V(Q_m) : S \cup \{z\} \text{ dominates } Q_m\}, \quad A_m(x) = \sum_{S \in \mathcal{A}_m} x^{|S|}.$$

The external vertex is not counted in the size of  $S$ . For  $m = 0$ , we use the convention  $\mathcal{A}_0 = \{\emptyset\}$  and  $A_0(x) = 1$ . Put

$$a_m = A_m(1), \quad s_m = A'_m(1), \quad r_m = \frac{s_m}{a_m}.$$

Thus  $a_m$  counts the admissible subsets,  $s_m$  is the sum of their cardinalities, and  $r_m$  is their average cardinality. In particular,  $a_m > 0$  for every  $m$ .

**Lemma 2.2.** *The initial polynomials are*

$$A_0(x) = 1, \quad A_1(x) = 1 + x, \quad A_2(x) = 2x + x^2.$$

For  $m \geq 3$ ,

$$A_m(x) = x(A_{m-1}(x) + A_{m-2}(x) + A_{m-3}(x)). \quad (14)$$

Consequently,

$$a_m = a_{m-1} + a_{m-2} + a_{m-3}, \quad (15)$$

$$s_m = a_m + s_{m-1} + s_{m-2} + s_{m-3}, \quad (16)$$

$$r_m = 1 + \frac{a_{m-1}r_{m-1} + a_{m-2}r_{m-2} + a_{m-3}r_{m-3}}{a_m}. \quad (17)$$

The initial values are

$$(a_0, a_1, a_2) = (1, 2, 3), \quad (s_0, s_1, s_2) = (0, 1, 4),$$

and hence

$$r_0 = 0, \quad r_1 = \frac{1}{2}, \quad r_2 = \frac{4}{3}.$$

*Proof.* The initial values follow from

$$\mathcal{A}_1 = \{\emptyset, \{u_1\}\}, \quad \mathcal{A}_2 = \{\{u_1\}, \{u_2\}, \{u_1, u_2\}\}.$$

For  $m \geq 3$ , represent an admissible set by its binary incidence sequence along  $u_1, \dots, u_m$ . Since  $u_{m-1}$  must be dominated, the final three entries cannot all be zero. The position of the rightmost selected vertex therefore gives exactly three possible terminal blocks:

$$1, \quad 10, \quad 100.$$

Deleting the appropriate block gives an admissible sequence of length  $m-1$ ,  $m-2$ , or  $m-3$ , respectively. Conversely, appending any of these blocks to a sequence from the corresponding family gives an admissible sequence of length  $m$ . The correspondence is bijective: the selected vertex at the beginning of the appended block dominates the boundary vertex of the remaining prefix, and the externally selected vertex dominates the terminal vertex when necessary.

Each block contributes exactly one selected vertex. Hence

$$A_m(x) = xA_{m-1}(x) + xA_{m-2}(x) + xA_{m-3}(x),$$

which proves (14). Evaluating at  $x = 1$  gives (15), and differentiating gives (16). Finally,  $s_j = a_j r_j$ , so (17) follows from (16).  $\square$

**Lemma 2.3.** *For  $P_n = v_1 v_2 \cdots v_n$  and  $1 \leq i \leq n$ ,*

$$D_{v_i}(P_n, x) = xA_{i-1}(x)A_{n-i}(x). \quad (18)$$

Consequently,

$$\text{avd}_{v_i}(P_n) = 1 + r_{i-1} + r_{n-i}. \quad (19)$$

*Proof.* Fix  $v_i$  and write a dominating set containing  $v_i$  uniquely as

$$S = S_L \cup \{v_i\} \cup S_R,$$

where  $S_L \subseteq \{v_1, \dots, v_{i-1}\}$  and  $S_R \subseteq \{v_{i+1}, \dots, v_n\}$ . Since  $v_i$  is selected, the two sides are independent. The left-hand selection is an element of  $\mathcal{A}_{i-1}$ , while the right-hand selection, after

reversing the vertex order, is an element of  $\mathcal{A}_{n-i}$ . The fixed vertex contributes one factor of  $x$ , proving (18).

Taking the logarithmic derivative of (18) at  $x = 1$  and applying Lemma 2.1, we obtain

$$\text{avd}_{v_i}(P_n) = 1 + \frac{A'_{i-1}(1)}{A_{i-1}(1)} + \frac{A'_{n-i}(1)}{A_{n-i}(1)} = 1 + r_{i-1} + r_{n-i}.$$

□

The next lemma is the key extremal comparison. Its proof relies on the uniform estimate established in Lemma A.1 of Appendix A.

**Lemma 2.4.** *Let  $N \geq 4$ , and let  $a, b$  be nonnegative integers with  $a + b = N$ . Then*

$$r_1 + r_{N-1} \leq r_a + r_b \leq r_2 + r_{N-2}. \quad (20)$$

*Equality on the left holds if and only if  $\{a, b\} = \{1, N-1\}$ , and equality on the right holds if and only if  $\{a, b\} = \{2, N-2\}$ .*

*Proof.* Let  $c, d$  and  $e_m$  be as in Lemma A.1. Since  $a + b = N$ ,

$$r_a + r_b = cN + 2d + e_a + e_b.$$

Thus it suffices to compare the error sums.

By Lemma A.1,

$$0 < e_0 < 0.005, \quad e_1 < -0.11, \quad e_2 > 0.10, \quad |e_m| < 0.02 \quad (m \geq 3).$$

For the proposed minimum,  $N-1 \geq 3$ , so

$$e_1 + e_{N-1} < -0.11 + 0.02 = -0.09.$$

For any other unordered pair, either one index is 0, both indices are at least 3, or one index is 2. In these cases the corresponding error sums are, respectively, greater than  $-0.02$ , greater than  $-0.04$ , or greater than  $0.08$ . Hence the minimum is attained only when  $\{a, b\} = \{1, N-1\}$ .

For the maximum, suppose first that  $N \geq 5$ . Then  $N-2 \geq 3$ , and

$$e_2 + e_{N-2} > 0.10 - 0.02 = 0.08.$$

For any other pair, an index equal to 0 gives an error sum less than  $0.025$ , an index equal to 1 gives a sum less than  $-0.09$ , and two indices at least 3 give a sum less than  $0.04$ . All are strictly smaller than  $0.08$ .

When  $N = 4$ , the only unordered pairs are  $\{0, 4\}$ ,  $\{1, 3\}$ , and  $\{2, 2\}$ . Their error sums are less than  $0.025$ , less than  $-0.09$ , and greater than  $0.20$ , respectively. Thus  $\{2, 2\}$  is the unique maximizing pair. This proves both inequalities and their equality conditions. □

### 3 Extremal vertices of paths

*Proof of Theorem 1.1.* By Lemma 2.3,

$$\text{avd}_{v_i}(P_n) = 1 + r_{i-1} + r_{n-i}.$$

Set  $a = i-1$  and  $b = n-i$ . Then  $a + b = n-1 \geq 4$ . Lemma 2.4 gives

$$1 + r_1 + r_{n-2} \leq \text{avd}_{v_i}(P_n) \leq 1 + r_2 + r_{n-3}.$$

The lower equality holds if and only if

$$\{i-1, n-i\} = \{1, n-2\},$$

equivalently,  $i = 2$  or  $i = n-1$ . The upper equality holds if and only if

$$\{i-1, n-i\} = \{2, n-3\},$$

equivalently,  $i = 3$  or  $i = n-2$ .

Finally,  $r_1 = 1/2$  and  $r_2 = 4/3$ , so the two extremal values are

$$1 + r_1 + r_{n-2} = \frac{3}{2} + r_{n-2}, \quad 1 + r_2 + r_{n-3} = \frac{7}{3} + r_{n-3}.$$

The asserted equality conditions follow, including the coincidence of the two maximizing vertices when  $n = 5$ .  $\square$

## 4 Small paths and concluding remarks

The paths of order at most four can be checked directly using (19). Their extremal values and locations are listed in Table 1.

Table 1: Extremal local average orders in paths of order at most four.

| $n$ | Minimum (attaining vertices) | Maximum (attaining vertices) |
|-----|------------------------------|------------------------------|
| 1   | 1 (all)                      | 1 (all)                      |
| 2   | $\frac{3}{2}$ (all)          | $\frac{3}{2}$ (all)          |
| 3   | 2 ( $v_2$ )                  | $\frac{7}{3}$ ( $v_1, v_3$ ) |
| 4   | $\frac{17}{6}$ (all)         | $\frac{17}{6}$ (all)         |

Thus  $P_4$  is the only nontrivial path, apart from  $P_2$ , for which the local average order is independent of the prescribed vertex.

Theorem 1.1 shows that the extremal positions are determined by the boundary of the path rather than its center: for every  $n \geq 5$ , the minimizing vertices are at distance one from an end, while the maximizing vertices are at distance two from an end. The underlying reason is that  $r_m$  is close to an affine function of  $m$ , with the first two nontrivial terms producing the dominant deviations. This suggests studying analogous boundary effects in other graph families, such as cycles, caterpillars, and trees. Such extensions would require additional structural arguments and are not addressed here.

## A A uniform estimate for the ratio sequence

We give a self-contained proof of the estimate used in Lemma 2.4. In particular, the numerical constants are consequences of the recurrence and are not assumptions based on a finite plot.

**Lemma A.1.** *Let  $\alpha$  be the unique real root of*

$$p(t) = t^3 - t^2 - t - 1 = 0,$$

*and define*

$$c = \frac{\alpha^2}{3\alpha^2 - 2\alpha - 1}, \quad d = c \left( 2 + \frac{\alpha}{\alpha + 1} - \frac{\alpha(2\alpha + 2)}{\alpha^2 + 2\alpha + 3} \right) - 1.$$

*For  $m \geq 0$ , put  $e_m = r_m - cm - d$ . Then*

$$0 < e_0 < 0.005, \quad e_1 < -0.11, \quad e_2 > 0.10,$$

*and*

$$|e_m| < 0.02 \quad (m \geq 3).$$

*Proof.* The recurrence in Lemma 2.2 gives the bivariate generating function

$$F(z, x) = \sum_{m \geq 0} A_m(x) z^m = \frac{1+z}{1-x(z+z^2+z^3)}. \quad (21)$$

Let  $\lambda_j(x)$ ,  $j = 1, 2, 3$ , be the roots of

$$\lambda^3 = x(\lambda^2 + \lambda + 1)$$

in a neighborhood of  $x = 1$ . These roots are distinct at  $x = 1$ , so they may be chosen analytically in a sufficiently small neighborhood. Partial fractions in (21) give

$$A_m(x) = \sum_{j=1}^3 C_j(x) \lambda_j(x)^m,$$

where

$$C_j(x) = \frac{\lambda_j(x)^2(\lambda_j(x) + 1)}{x(\lambda_j(x)^2 + 2\lambda_j(x) + 3)}.$$

For a root  $\lambda$  of  $p$ , implicit differentiation gives

$$\frac{\lambda'(1)}{\lambda} = \rho(\lambda) := \frac{\lambda^2}{3\lambda^2 - 2\lambda - 1}.$$

Logarithmic differentiation of  $C_j(x)$  yields

$$\frac{C'_j(1)}{C_j(1)} = \eta(\lambda),$$

where

$$\eta(\lambda) = \rho(\lambda) \left( 2 + \frac{\lambda}{\lambda+1} - \frac{\lambda(2\lambda+2)}{\lambda^2+2\lambda+3} \right) - 1.$$

Consequently,

$$s_m = \sum_{p(\lambda)=0} C(\lambda) \lambda^m (m\rho(\lambda) + \eta(\lambda)), \quad (22)$$

where

$$C(\lambda) = \frac{\lambda^2(\lambda+1)}{\lambda^2+2\lambda+3}.$$

Let  $\alpha$  be the real root and let  $\beta, \bar{\beta}$  be the nonreal roots. Write

$$A = C(\alpha) > 0, \quad B = C(\beta).$$

The product and sum of the roots give

$$|\beta| = \alpha^{-1/2}, \quad \Re \beta = \frac{1-\alpha}{2}.$$

Moreover, direct simplification gives

$$|B|^2 = \frac{-\alpha^2 + 2\alpha + 1}{2(2\alpha^2 + \alpha + 5)}.$$

Thus

$$a_m = A\alpha^m + B\beta^m + \bar{B}\bar{\beta}^m.$$

Since  $c = \rho(\alpha)$  and  $d = \eta(\alpha)$ , (22) implies

$$s_m - (cm + d)a_m = 2\Re[B\beta^m(m(\rho(\beta) - c) + \eta(\beta) - d)].$$

For clarity, we record explicit rational bounds sufficient for the argument. Substitution into  $p$  gives

$$\frac{1839286}{10^6} < \alpha < \frac{1839287}{10^6}.$$

Using this interval, the formulas above, and

$$\beta = \frac{1-\alpha}{2} + \frac{i}{2} \sqrt{\frac{4}{\alpha} - (\alpha-1)^2}$$

(with either choice of the sign of the imaginary part), gives

$$\frac{2|B|}{A} < 0.46, \quad q := \frac{|\beta|}{\alpha} < 0.401, \quad (23)$$

$$|\rho(\beta) - c| < 0.43, \quad |\eta(\beta) - d| < 0.48. \quad (24)$$

These inequalities can be verified by rational interval arithmetic; all the displayed decimals are terminating rational numbers. The same substitution gives

$$0.61841 < c < 0.61843, \quad -0.00452 < d < -0.00449.$$

For example, the first two bounds in (23) follow directly from the displayed formulas for  $A$ ,  $|B|^2$ , and  $q = \alpha^{-3/2}$ . The remaining bounds follow by substituting the expression for  $\beta$  into the rational functions  $\rho$  and  $\eta$  and bounding their real and imaginary parts.

The triangle inequality now gives

$$a_m \geq A\alpha^m \left(1 - \frac{2|B|}{A} q^m\right) > 0.$$

Consequently,

$$|e_m| \leq \frac{0.46(0.401)^m(0.43m + 0.48)}{1 - 0.46(0.401)^m}. \quad (25)$$

For  $m \geq 5$ , the right-hand side is decreasing: the ratio of its successive numerators is at most

$$0.401 \left(1 + \frac{0.43}{0.43m + 0.48}\right) < 0.401 \left(1 + \frac{1}{5}\right) < 1,$$

and the denominator is increasing. At  $m = 5$ , the right-hand side of (25) is less than 0.013. Hence

$$|e_m| < 0.02 \quad (m \geq 5).$$

It remains to check  $m = 0, 1, 2, 3, 4$ . The recurrence gives

$$r_0 = 0, \quad r_1 = \frac{1}{2}, \quad r_2 = \frac{4}{3}, \quad r_3 = \frac{11}{6}, \quad r_4 = \frac{27}{11}.$$

Using the above intervals for  $c$  and  $d$ , we obtain

$$0.00449 < e_0 < 0.00452,$$

$$-0.11394 < e_1 < -0.11389, \quad 0.10096 < e_2 < 0.10103,$$

and

$$-0.01746 < e_3 < -0.01737, \quad -0.01468 < e_4 < -0.01458.$$

These estimates establish all the asserted inequalities.  $\square$

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I chose the field of graph theory because the first time I read it and saw the images in the book, I felt that graph theory was more visual than other fields of mathematics, and it verified the view I had discussed with my mother since childhood that the essence of mathematics is a bridge between the concrete and the abstract. Then I looked up the content about average order written by Beaton & Brown, which has been widely discussed in recent years. The content argued in this article made me feel close because I usually like flying drones, and this article is precisely the underlying logic of efficiency issues such as how many times the battery needs to be changed at least during the flight of an aircraft. At the same time, I consulted my supervisors and believed that the difficulty of the relevant papers was a problem that I could solve by applying the graph theory concepts and mathematical proofs I had learned.

The issue I am researching is the application of Beaton and Brown on global average order and an expansion of Erey's paper on local average order. When I focused my thesis topic on the symmetrical and distinctive image of path, I originally thought that path would be easy to prove. However, when I initially attempted to categorize the different types of vertices in path, I encountered great difficulties. Therefore, I spent a considerable amount of time trying to classify the points in the early stage. And by looking up relevant literature and discussing possible inspirations with the supervisor. During the process of argumentation, I found that some of my intuitive conjectures about the image were wrong, but some were eventually confirmed. For instance, in the path image, I initially thought that classifying path based on the number of vertices being odd or even would have different extreme values. However, after conducting arguments and enumerations, I found that when the number of vertices is greater than 5, the pattern of path is common. Throughout the entire research process, the greatest difficulty I encountered was not the calculation itself, but how to transform intuition into strict proof. At the very beginning, I was not familiar with the sources of linear recursion, dominating polynomials and recursion expressions in other references. Therefore, although many formulas could be

calculated, I could not truly understand them. Every time I encounter a solution that I don't understand, I consult my supervisor or use online materials to self-study the formula. This process gradually made me realize that a research paper not only needs to yield correct results but also must ensure that every step of reasoning can be truly understood and clearly explained by oneself. In terms of writing the thesis, I used Overleaf and LaTeX to organize formulas, images and proofs, and repeatedly readjust the sequence of lemma, proof and introduction to make the structure of the article more coherent. Through this research, I have progressed from not being very familiar with graph theory to conducting a topic selection that combines my own areas of interest. From not knowing where to start writing the first paragraph or even the first sentence, to delving into how to create the first LaTeX documentarticle using overleaf, and then to completing this paper. With the firm support of my mentors and parents, I overcame the difficulties that I once thought were unattainable. This experience also made me understand that the process of resolutely devoting oneself to the right thing is the harvest.

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