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Research Report

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Placing stones in an n-dimensional space

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Placing stones in an n-dimensional space

CHEN Xi Yue

Abstract

This paper investigates a stone-placing game on an n-dimnensional lattice graphs, generalizing an Olympiad problem. Amy and Ben place stones on unoccupied vertices alternately under an independence constraint for Amy and no restrictions for Ben, aiming to maximize and minimize Amy's total placed stones, respectively. We formalize the asymptotic ratio $\ell(n,d) = \lim_{m\to\infty} K(S^n_{m,d})/m^n$ and reduce the problem to torus graphs $T^n_{m,d}$. Using a tiling lemma on k-response blocks, we establish exact values and tight bounds across various dimensions. In two dimensions, we completely resolve the problem, showing $\ell(2,d) = \frac{1}{4}$ for sums of two squares and $\frac{1}{2}$ otherwise. For higher dimensions, we prove $\ell(n,d) \le \frac{1}{4}$ when d is a sum of n squares, establish an upper bound $\ell(n,2d^2) \le \frac{1}{2n}$, and provide a non-trivial lower bound $\ell(3,2) \ge \frac{1}{7}$.

Keywords: Lattice Graphs, Combinatorial Games

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1 Introduction

This paper is inspired by a two-player game, appeared in IMO2018 Question 4.

Problem ([1])**.** A site is any point (x, y) in the plane such that x and y are both positive integers less than or equal to 20.

Initially, each of the 400 sites is unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Ben places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone.

Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

The answer of this question is $K = 20^2/4 = 100$. The full solution is included in the appendix. In this paper, we aim to generalise this game in the setting on lattice graphs.

In Section 2, we state the definitions needed to generalise the problem for n-dimensions with arbitrary distance $\sqrt{d}$. We show that the asymptotic ratio $\ell(n, d)$ as the size of the lattice graph tends to infinity is well-defined. We find that the problem can also be reduced to torus graphs and provides a tiling lemma as a technique to show that a chosen subset partitions the graph.

In Section 3, we solve the problem in two-dimensions. Using the developed tools in the previous section, the lower and upper bound of $\ell(2, d)$ can be shown to be $\frac{1}{4}$ for the non-trivial cases, whereas $\ell(2, d) = \frac{1}{2}$ if the lattice graph has no edges.

In Section 4, we attempt to solve the problem in n-dimensions. Firstly, following similar techniques as the two-dimensional case, we are able to obtain an upper bound of $\frac{1}{4}$ for all $\ell(n, d)$ and the exact value of $\ell(n, d) = \frac{1}{4}$ when d is an odd integer which can be written as the sum of n squares. However, we face difficulties when attempting to compute the exact value of the first nontrivial case when d is even, i.e. $\ell(3, 2)$. Instead, we provide results to improve the upper and lower bounds for $\ell(3, 2)$ and show that $\frac{1}{7} \leq \ell(3, 2) \leq \frac{1}{6}$.

2 Problem statement

We generalise the IMO problem to the setting of graphs. Let $G = (V, E)$ be a finite simple graph. The game is defined as follows:

Initially, the vertices of the graph are unoccupied. Amy and Ben take turns placing stones with Amy going first. On her turn, Amy places a red stone on an unoccupied vertex such that no two red stones are adjacent. On his turn, Ben places a new blue stone on any unoccupied vertex. They stop as soon as a player cannot place a stone. Then:

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Definition 2.1. *We define $K(G)$ to be the maximum number of stones Amy can place in G, regardless of the strategy of Ben.*

The focus of our paper is the game above on the following graph:

Definition 2.2. *Let n, m, d be positive integers. The **box graph** $S^n_{m,d}$ is a graph whose vertices are the lattice points $\{0, 1, ..., m-1\}^n$, and x and y are adjacent if and only if $\|x-y\|^2 = d$.*

Indeed, the IMO problem can be realised as the game on the graph $S^2_{20,5}$, and $K(S^2_{20,5}) = 100$.

Note that if $E = \emptyset$, then both players have no restrictions, so Amy can place exactly $\left\lceil \frac{|V|}{2} \right\rceil$ stones, i.e. $K(G) = \left\lceil \frac{|V|}{2} \right\rceil$. In the case of $S^n_{m,d}$, it is easy to see that this is the case if and only if d is not a sum of n squares (which is only possible for $n = 2, 3$, by Lagrange's four-square theorem).

We also have:

Proposition 2.3. *If G is bipartite, then*

$$K(G) \geq \frac{|V|}{4}.$$

Proof. Let $V = V_1 \sqcup V_2$ be a bipartition of G. One of V_1 and V_2 has size at least $|V|/2$, WLOG it is V_1. As V_1 is an independent set, Amy might restrict all her moves to V_1. During each round, Amy occupies one vertex of V_1, while Ben can occupy at most one vertex of V_1, so Amy can occupy at least half the vertices of V_1. Therefore

$$K(G) \geq \left\lceil \frac{|V_1|}{2} \right\rceil \geq \frac{|V|}{4}.$$ □

Recall the following standard notation:

Definition 2.4 (Induced subgraph)**.** *Let $U \subseteq V$ be a subset of vertices of G. The **induced subgraph** $G[U]$ is the graph whose vertex set is U and whose edge set is the edges in E that have both endpoints in U.*

Two key properties of K are the following:

Lemma 2.5. *If $U \subseteq V$, then $K(G[U]) \leq K(G)$.*

Proof. By always placing stones in vertices of U, Amy can place at least $K(G[U])$ stones. □

Lemma 2.6 (Subadditivity)**.** *Suppose V has the disjoint partition $V = \bigsqcup_{i=1}^n V_i$. Then*

$$K(G) \leq \sum_{i=1}^{n} K(G[V_i]).$$

Proof. We provide the following strategy for Ben to give an upper bound to $K(G)$: treat each $G[V_i]$ as sub-games; if Amy places a stone in V_i, Ben responds in V_i accordingly with his strategy. In the case that all vertices in V_i are occupied in Ben's turn, he may place a stone at any unoccupied vertex in G instead. This guarantees Amy placing at most $K(G[V_i])$ stones in V_i for all i, and the result follows. □

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Fixing the dimension n and distance d, we are interested in the following asymptotic ratio:

Definition 2.7. *The **asymptotic ratio** is the limit*

$$\ell(n,d) := \lim_{m\to\infty} \frac{K(S^n_{m,d})}{m^n}.$$

This is well-defined because:

Theorem 2.8. *$\ell(n,d)$ exists for all positive integers n and d.*

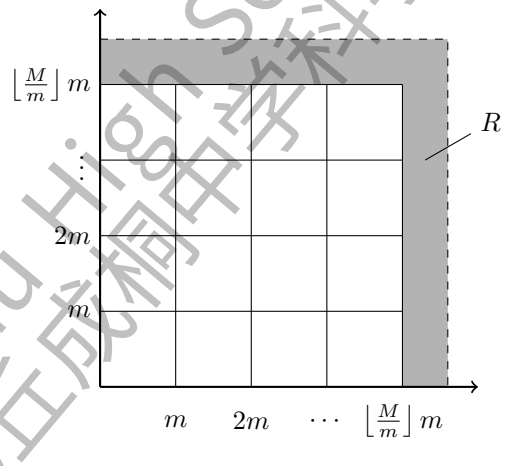
Proof. Fix m and $M \geq m$, we claim that:

$$K(S^n_{M,d}) \leq \left\lfloor \frac{M}{m} \right\rfloor^n K(S^n_{m,d}) + mnM^{n-1}.$$

Partition $\{0,\dots,M-1\}^n$ into $\left\lfloor \frac{M}{m} \right\rfloor^n$ disjoint copies of the hypercube $\{0,\dots,m-1\}^n$ (by suitable translations) and a remainder

$$R = \{0,\dots,M-1\}^n \setminus \left\{0,\dots,\left\lfloor \frac{M}{m} \right\rfloor m - 1\right\}^n.$$

Note that all copies of the hypercube have the same induced subgraph up to a translation, and thus gives the same value of K.



Hence, by Lemma 2.6,

$$K(S^n_{M,d}) \leq \left\lfloor \frac{M}{m} \right\rfloor^n K(S^n_{m,d}) + K(S^n_{M,d}[R]).$$

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The claim follows by:

$$\begin{aligned} K(S^n_{M,d}[R]) \le |R| &= M^n - \left\lfloor \frac{M}{m} \right\rfloor^n m^n \\ &\le M^n - (M-m)^n \\ &= (M-(M-m)) \cdot \sum_{i=0}^{n-1} M^i (M-m)^{n-1-i} \\ &\le mnM^{n-1}. \end{aligned}$$

Now divide both sides of the claim by M^n, we have

$$\frac{K(S^n_{M,d})}{M^n} \le \left(\frac{1}{M}\left\lfloor \frac{M}{m} \right\rfloor\right)^n K(S^n_{m,d}) + \frac{m}{M} \cdot n \le \frac{1}{m^n} K(S^n_{m,d}) + \frac{m}{M} \cdot n.$$

This shows that $\dfrac{K(S^n_{M,d})}{M^n}$ is bounded and

$$\limsup_{M\to\infty} \frac{K(S^n_{M,d})}{M^n} \le \frac{K(S^n_{m,d})}{m^n}.$$

Since m is arbitrary,

$$\limsup_{M\to\infty} \frac{K(S^n_{M,d})}{M^n} \le \liminf_{m\to\infty} \frac{K(S^n_{m,d})}{m^n}. \qquad \square$$

2.1 Reduction to tori

Our goal now is to reduce the problem to a more symmetrical setting; first, we would like the graph to be "without boundary", so that we don't need to deal with the boundaries of the box separately. Hence, we consider the set $\mathbb{Z}/m\mathbb{Z}$ instead of $\{0,\dots,m-1\}$, so that the edges of the graph wraps around.

Given fixed n, d, we let

$$D_{n,d} = \{z = (z_1, ..., z_n) \in \mathbb{Z}^n : z_1^2 + \cdots + z_n^2 = d\},$$

and $\rho_m : \mathbb{Z}^n \to (\mathbb{Z}/m\mathbb{Z})^n$ be reduction modulo m. Then:

Definition 2.9. *The **torus graph** $T^n_{m,d}$ is the graph with vertex set $(\mathbb{Z}/m\mathbb{Z})^n$, and two distinct vertices x and y are adjacent if and only if $y - x = \rho_m(z)$ for some $z \in D_{n,d}$.*

Equivalently, we may identify the vertices of $T^n_{m,d}$ by $\{0,1,\dots,m-1\}^n$, then $x, y \in \{0,1,\dots,m-1\}^n$ are adjacent if $\rho_m(y-x) = \rho_m(z)$ for some $z \in D_{n,d}$. Note that this is an undirected graph as $z \in D_{n,d}$ implies $-z \in D_{n,d}$.

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Theorem 2.10. *For all positive integers n and d, there an integer c with $c^2 \le d$ such that for every $m > c$,*

$$K(S^n_{m-c,d}) \le K(T^n_{m,d}) \le K(S^n_{m,d}).$$

Proof. If $D_{n,d}$ is empty, we may take $c = 0$, then all three terms are $\lceil \frac{1}{2}m^n \rceil$. Otherwise, we identify the vertices of both $S^n_{m,d}$ and $T^n_{m,d}$ with $\{0, \dots, m-1\}^n$, and take

$$c = \max_{z \in D_{n,d}} \max_i |z_i|.$$

If x and y are adjacent in $S^n_{m,d}$, then $y - x \in D_{n,d}$, so they are also adjacent in $T^n_{m,d}$. Thus $T^n_{m,d}$ contains all edges of $S^n_{m,d}$, and $K(T^n_{m,d}) \le K(S^n_{m,d})$, as adding edges further restricts Amy's legal moves.

Now let $R = \{0, 1, 2, ..., m-c-1\}^n$. We claim that $T^n_{m,d}[R]$ is isomorphic to $S^n_{m-c,d}$ as graphs. Suppose $x, y \in R$ are adjacent in $T^n_{m,d}$. Then, for some $z \in D_{n,d}$,

$$y_i - x_i \equiv z_i \pmod m \implies (y_i - x_i) - z_i \in m\mathbb{Z}.$$

But by triangle inequality,

$$|(y_i - x_i) - z_i| \le |y_i - x_i| + |z_i| \le (m - c - 1) + c < m,$$

so we must have $y_i - x_i = z_i$ for all i. Then

$$\|x - y\|^2 = \sum_{i=1}^n (x_i - y_i)^2 = \sum_{i=1}^n z_i^2 = d,$$

i.e. x, y are adjacent in $S^n_{m-c,d}$. Conversely, if x, y are adjacent in $S^n_{m-c,d}$, then $y - x \in D_{n,d}$, so x, y are again adjacent in $T^n_{m,d}[R]$ by definition. Therefore by Lemma 2.5,

$$K(S^n_{m-c,d}) = K(T^n_{m,d}[R]) \le K(T^n_{m,d}). \qquad \square$$

Corollary 2.11. *For all positive integers n and d, we have*

$$\ell(n,d) = \lim_{m \to \infty} \frac{K(T^n_{m,d})}{m^n}.$$

Proof. By Theorem 2.10, we have

$$\left(\frac{m-c}{m}\right)^n \frac{K(S^n_{m-c,d})}{(m-c)^n} = \frac{K(S^n_{m-c,d})}{m^n} \le \frac{K(T^n_{m,d})}{m^n} \le \frac{K(S^n_{m,d})}{m^n}.$$

The result follows as both the upper and lower bounds tend to $\ell(n,d)$ as $m \to \infty$. $\square$

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2.2 Tiling

We have the following group theoretic result:

Lemma 2.12 (Tiling lemma)**.** *Let G and A be abelian groups and $S \subseteq G$. Suppose there is a group homomorphism $\pi : G \to A$ and a subset $B \subseteq A$ such that every $a \in \operatorname{im} \pi$ can be uniquely written as*

$$a = b + \pi(s), \qquad b \in B,\ s \in S.$$

Then the sets

$$\{c + S : c \in \pi^{-1}(B)\}$$

form a disjoint partition of G.

Proof. Take $x \in G$. By assumption, we may write $\pi(x) = b + \pi(s)$ for some $b \in B, s \in S$. Let $c = x - s$, then

$$\pi(c) = \pi(x) - \pi(s) = b \implies c \in \pi^{-1}(B),$$

and $x = c + s \in c + S$.

To show that the sets are disjoint, suppose $x = c + s = c' + s'$ for some $c, c' \in \pi^{-1}(B)$ and $s, s' \in S$. Then

$$\pi(x) = \pi(c) + \pi(s) = \pi(c') + \pi(s'),$$

which implies $s = s'$ by assumption. Therefore $c = x - s = x - s' = c'$. □

For our setting, we often apply this result on $G = (\mathbb{Z}/m\mathbb{Z})^n$, and aim to set S as the following:

Definition 2.13 (k-response blocks)**.** *We say a subset $S \subseteq (\mathbb{Z}/m\mathbb{Z})^n$ is a k-**response block** if Amy can guarantee at most k stones in S, i.e. $K(T^n_{m,d}[S]) = k$.*

Then, if Lemma 2.12 holds, then $T^n_{m,d}$ can be partitioned into $m^n/|S|$ sets of k-response blocks, and Lemma 2.6 gives an upper bound to $\ell(n, d)$ via

$$\ell(n, d) = \lim_{m\to\infty} \frac{K(T^n_{m,d})}{m^n} \le \lim_{m\to\infty} \frac{1}{m^n} \cdot \frac{m^n}{|S|} \cdot K(T^n_{m,d}[S]) = \frac{k}{|S|}.$$

3 Two-dimensional case

In this section, we completely solve the two-dimensional case, therefore generalising the IMO problem to an arbitrary d.

Theorem 3.1. *Let a, b be positive integers and $d = a^2 + b^2$. If $4ab \mid m$, then the vertices of $T^2_{m,d}$ can be partitioned into $m^2/4$ sets of 1-response blocks with four elements.*

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Proof. We apply Lemma 2.12. Let $G = (\mathbb{Z}/m\mathbb{Z})^2$, and consider the four-point set

$$S = \{(a, 0), (0, b), (-a, 0), (0, -b)\} \subseteq G.$$

Note that S forms a cycle of length four, so it is a 1-response block; indeed, if Amy places a stone at $(a, 0)$, Ben can respond by placing at $(-a, 0)$, and vice versa for the other points.

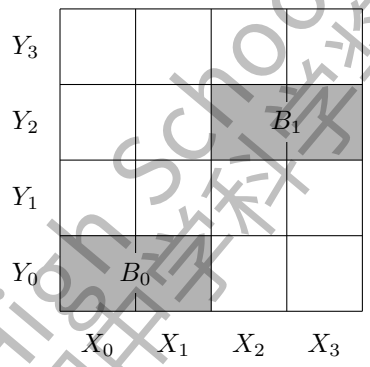
Define the abelian group $A = \mathbb{Z}/(4a)\mathbb{Z} \times \mathbb{Z}/(4b)\mathbb{Z}$ and the homomorphism $\pi : G \to A$ as the projection

$$(x, y) \mapsto (x \bmod 4a, y \bmod 4b).$$

Note that this is well-defined by $4ab \mid m$. Now for $i = 0, 1, 2, 3$, define

$$X_i = \{ia, ia + 1, \ldots, (i + 1)a - 1\} \quad \text{and} \quad Y_i = \{ib, ib + 1, \ldots, (i + 1)b - 1\}.$$

Let $B_0 = (X_0 \cup X_1) \times Y_0$, $B_1 = (X_2 \cup X_3) \times Y_2$, and $B = B_0 \cup B_1$:



We claim that any $x \in A$ can be written uniquely as $b + \pi(s)$ for some $b \in B$ and $s \in S$. Indeed,

$$B_0 + \pi(a, 0) = (X_1 \cup X_2) \times Y_0, \qquad B_0 - \pi(a, 0) = (X_0 \cup X_3) \times Y_0,$$
$$B_0 + \pi(0, b) = (X_0 \cup X_1) \times Y_1, \qquad B_0 - \pi(0, b) = (X_0 \cup X_1) \times Y_3.$$

As X_i and Y_i are pairwise disjoint, these translations of B_0 by elements of $\pi(S)$ must also be disjoint. Similarly, one can compute the the translations for B_1. This covers all possible pairs $X_i \times Y_j$, and we obtain the unique representation of $x \in A$ depending on which $X_i \times Y_j$ it belongs in. □

We have a similar result when d is a square, of which we only provide the sketch of proof as it follows the same logic as above.

Proposition 3.2. *Let a be a positive integer and $d = a^2$. If $2a \mid m$, then the vertices of $T^2_{m,d}$ can be partitioned into $m^2/4$ sets of 1-response blocks with four elements.*

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Sketch of proof. Apply Lemma 2.12 on $S = \{(0,0),(a,0),(a,a),(0,a)\}$, with $A = (\mathbb{Z}/(2a)\mathbb{Z})^2$ and $B = \{0,\dots,a-1\}^2 \subseteq A$. It is clear that $B + \pi(s)$ for $s \in S$ are pairwise disjoint and cover A. □

Note that the above two results cover all possibilities where d can be written as a sum of two squares. For the lower bound, we have the following key theorem:

Theorem 3.3. *If d is a sum of two squares, then $S^2_{m,d}$ is bipartite.*

Proof. Let $d = 4^i d_0$ where $i \geq 0$ is maximal, i.e. $4 \nmid d_0$. First, every integer solution of $a^2 + b^2 = d$ satisfies $2^i \mid a, b$, for if $4 \mid a^2 + b^2$ then a, b must be even, since the only squares modulo 4 are 0 and 1. Repeatedly applying this proves the claim.

For each lattice point (x, y) in $S^2_{m,d}$, we may uniquely write

$$x = j + 2^i X \quad \text{and} \quad y = k + 2^i Y,$$

where $0 \leq j, k < 2^i$. By above, if two points are adjacent, the differences in their coordinates must both be divisible by 2^i, so adjacent points have the same values of j and k.

Now there are two possible cases:

1). d_0 is odd.

Colour (x, y) by the parity of $X + Y$. Suppose two adjacent points differ by

$$(a, b) = 2^i(a_0, b_0).$$

Then $a_0^2 + b_0^2 = d_0$. As d_0 is odd, exactly one of a_0 and b_0 is odd, so

$$a_0 + b_0 \equiv 1 \pmod 2.$$

Thus moving along an edge changes the parity of $X + Y$, i.e. adjacent vertices have different colours.

2). $d_0 \equiv 2 \pmod 4$.

Colour (x, y) by the parity of X. Similar to above, we have $a_0^2 + b_0^2 = d_0 \equiv 2 \pmod 4$, so both a_0 and b_0 are odd. In particular, moving along an edge changes X by an odd integer, so again adjacent vertices have different colours.

Since d_0 is itself a sum of two squares and is not divisible by 4, these are the only two cases. □

Combining all results, we have:

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Corollary 3.4. *Let d be a positive integer. Then*

$$\ell(2,d) = \begin{cases} \frac{1}{4} & \text{if } d \text{ is a sum of two squares} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Proof. If d is a sum of two squares, then by Theorem 3.1 and Proposition 3.2, there is an upper bound $\ell(2,d) \le \frac{1}{4}$. By Theorem 3.3 and Proposition 2.3, we get the lower bound $\ell(2,d) \ge \frac{1}{4}$. If d is not a sum of two squares, there are no edges in $S^2_{m,d}$, so $\ell(2,d) = \frac{1}{2}$. □

4 Higher dimensional cases

In $n \ge 3$, the problem becomes much more complicated. First, we quickly cover some general results following similar techniques as the two-dimensional case.

Theorem 4.1. *If d can be represented by as a sum of n squares, then $\ell(n,d) \le \frac{1}{4}$.*

Proof. This follows the same logic as Theorem 3.1 and Proposition 3.2. Suppose

$$d = a_1^2 + a_2^2 + \cdots + a_n^2.$$

WLOG assume $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. First suppose $a_2 > 0$. It suffices to consider positive integers m with $4a_1a_2 \mid m$. Let $G = (\mathbb{Z}/m\mathbb{Z})^n$ and consider the four-point set

$$S = \{(a_1, 0, \ldots, 0), (0, a_2, \ldots, a_n), (-a_1, 0, \ldots, 0), (0, -a_2, \ldots, -a_n)\} \subseteq G.$$

Again this is a cycle of length four, so it is a 1-response block. We may use the same abelian group A as in the proof of Theorem 3.1, i.e. $A = \mathbb{Z}/(4a_1)\mathbb{Z} \times \mathbb{Z}/(4a_2)\mathbb{Z}$, and define $\pi : G \to A$ as

$$(x_1, \ldots, x_n) \mapsto (x \bmod 4a_1, y \bmod 4a_2).$$

Then $\pi(S) = \{(a_1, 0), (0, a_2), (-a_1, 0), (0, -a_2)\}$, which is exactly the image of S in the proof of Theorem 3.1. Define B similarly, then from the two-dimensional case and Lemma 2.12 we again have a partition of G into 1-response blocks, so $\ell(n,d) \le \frac{1}{4}$.

Similarly, if $a_2 = 0$ (i.e. $d = a^2$ is a square), we may follow Proposition 3.2 and apply Lemma 2.12 on

$$S = \{(0, \ldots, 0), (a, 0, 0, \ldots, 0), (a, a, 0, \ldots, 0), (0, a, 0, \ldots, 0)\}$$

together with $A = (\mathbb{Z}/(2a)\mathbb{Z})^2$ to give $\ell(n,d) \le \frac{1}{4}$. □

Proposition 4.2. *If d is odd, then $S^n_{m,d}$ is bipartite.*

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Proof. Similar to Theorem 3.3, we colour $(x_1, \ldots, x_n)$ in $S^n_{m,d}$ by the parity of $x_1 + \cdots + x_n$. If two adjacent points differ by $(a_1, \ldots, a_n)$, then $a_1^2 + \cdots + a_n^2 = d$ and

$$\sum_{i=1}^{n} a_i \equiv \sum_{i=1}^{n} a_i^2 \equiv 1 \pmod 2.$$

Thus moving along an edge changes the parity of $x_1+\cdots+x_n$, i.e. adjacent vertices have different colours. □

Again by Proposition 2.3, this gives an lower bound of $\frac{1}{4}$, thus:

Corollary 4.3. *If d is an odd integer which can be represented by a sum of n squares, then $\ell(n, d) = \frac{1}{4}$.* □

4.1 The exceptional case: $d = 2$

Unfortunately, the first even case $d = 2$ in three dimensions already shows incredible difficulty. In this section, we focus on finding bounds on $\ell(3, 2)$.

Theorem 4.4. $\ell(3, 2) \le \frac{1}{6}$.

In fact, we can show the following stronger result, based on the tiling lemma on a specific block shape that is allowed in this case:

Theorem 4.5. *For any $n \ge 3$ and $d \ge 1$, we have $\ell(n, 2d^2) \le \frac{1}{2n}$.*

Proof. It suffices to consider positive integers m divisible by $4nd$. We again apply Lemma 2.12. Let $G = (\mathbb{Z}/m\mathbb{Z})^n$ and define the $2n$-point set

$$S = \{\pm de_i : 1 \le i \le n\} \subseteq G,$$

where $e_1, \ldots, e_n$ are the standard basis vectors. We first claim that S is a 1-response block; indeed, if Amy places a stone at $\pm de_i$, Ben can respond by placing at $\mp de_i$, then all other points are blocked as their distance to $\pm de_i$ is exactly $d^2 + d^2 = 2d^2$.

Now let $A = \mathbb{Z}/(4nd)\mathbb{Z}$ and define $\pi : G \to A$ by

$$\pi(x_1, \ldots, x_n) := \sum_{i=1}^{n} (2i - 1)x_i \pmod{4nd}.$$

This is well-defined because $4nd \mid m$. Let $B = \{0, 1, \ldots, 2d - 1\} \subseteq A$, and we claim that every element of A can be written uniquely as $b + \pi(s)$ for some $b \in B$ and $s \in S$. Indeed, for $1 \le i \le n$,

$$\pi(de_i) = (2i - 1)d \quad \text{and} \quad \pi(-de_i) = -(2i - 1)d.$$

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Modulo $4nd$, these $2n$ values are $d, 3d, 5d, \ldots, (4n-1)d$, i.e. $\pi(S) = \{(2j-1)d : 1 \le j \le 2n\}$. But now

$$B + (2j-1)d = \{(2j-1)d, (2j-1)d+1, \ldots, (2j+1)d-1\}$$

are pairwise disjoint intervals and together cover all $4nd$ elements of A. The result follows. □

Note that this implies that $\lim_{n\to\infty} \ell(n, 2d^2) = 0$. For a lower bound, we have:

Theorem 4.6. $\ell(3,2) \ge \frac{1}{7}$.

Proof. It suffices to consider the case $7 \mid m$. Define the homomorphism

$$\phi : (\mathbb{Z}/m\mathbb{Z})^3 \to \mathbb{Z}/7\mathbb{Z}, \quad (x,y,z) \mapsto (x+2y) \bmod 7.$$

Consider the set

$$P = \phi^{-1}(\{0,3\}) = \{(x,y,z) : x + 2y \equiv 0, 3 \pmod 7\}.$$

We first determine the graph induced by P. Note that

$$D_{3,2} = \{\pm v_1, \pm v_2, \pm v_3\}$$

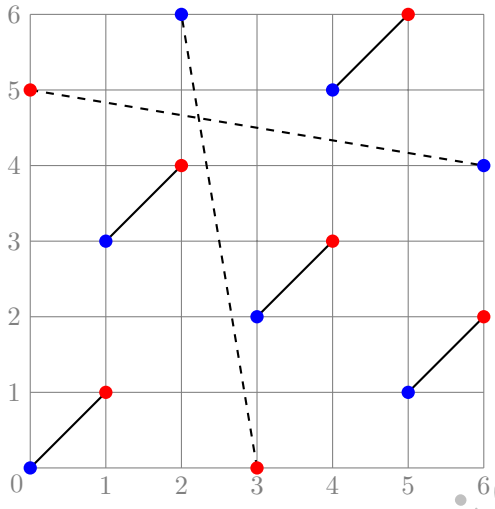
where $v_1 = (0,1,1), v_2 = (1,0,1)$ and $v_3 = (1,1,0)$. We compute:

s	v_1	$-v_1$	v_2	$-v_2$	v_3	$-v_3$
$\phi(s)$	2	5	1	6	3	4

In particular, there are no $s \in D_{3,2}$ with $\phi(s) = 0$, and $\phi(s) = \pm 3$ only if $s = \pm(1,1,0)$.

It follows that if $p \in P$ has $\phi(p) = 0$, then its only neighbour in P is $p + (1,1,0)$. Similarly, if $p \in P$ has $\phi(p) = 3$, then its only neighbour in P is $p - (1,1,0)$. Therefore, the induced graph $T^3_{m,2}[P]$ is a disjoint union of pairs (the following figure shows the case $m = 7$; $\phi^{-1}(0)$ is shown by the blue points, and $\phi^{-1}(3)$ is shown by the red points):

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Clearly ϕ is surjective, so

$$|\phi^{-1}(0)| = |\phi^{-1}(3)| = \frac{m^3}{7} \implies |P| = \frac{2m^3}{7},$$

and the number of pairs is $m^3/7$. We now provide the following strategy for Amy:

- On her first turn, she chooses either vertex of any pair.
- After each move by Ben:
  - if Ben has just played in an unoccupied pair, Amy chooses the other vertex of that pair;
  - otherwise, Amy chooses either vertex of an unoccupied pair.

This strategy guarantees that Amy can always occupy at least one vertex of each pair. Thus

$$K(T^3_{m,2}) \geq \frac{m^3}{7} \implies \ell(3,2) \geq \frac{1}{7}. \qquad \square$$

5 Further work

In this paper, we have completely solved the two-dimensional case (Theorem 3.1) and n-dimensional cases when d is odd (Theorem 4.1).

We also provide a powerful lemma to find the upper bound of $\ell(d,n)$ (Lemma 2.12). However, it is still nontrivial to find suitable sets A, B, S and the homomorphism π. To extend our work, we ought to find another way to search for these sets and homomorphisms.

It is pity that we only managed to show

$$\frac{1}{7} \leq \ell(3,2) \leq \frac{1}{6}.$$

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To obtain the exact value of $\ell(3,2)$, there are two directions:

- To find a better upper bound than $\frac{1}{6}$, one can try to find k-response sets S with $k>1$ and $\frac{k}{|S|}<\frac{1}{6}$.
- To find a better lower bound than $\frac{1}{7}$, one can try to find an induced subgraph with density greater than $\frac{2}{7}$.

Alternatively, one might need to consider a completely different response approach, such as a dynamic strategy for Amy based on which independent set Ben places in, etc.

A Solution to the IMO Problem

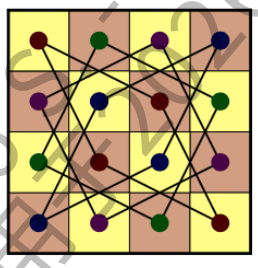
We include the solution to the original IMO problem [2]. Here, the names (Amy as Horst, Ben as Queenie) and settings (placing red stones as knights and blue stones as queens) are changed.

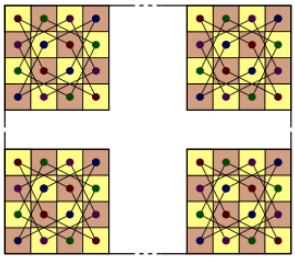
Answer. $K=20^2/4=100$. In case of a $4N\times 4M$ board, the answer is $K=4NM$.

Solution. We show two strategies, one for Amy to place at least 100 stones (knights), and another strategy for Ben that prevents Amy from putting more than 100 knights on the board.

A strategy for Amy: Put knights only on black squares, until all black squares get occupied.
Colour the squares of the board black and white in the usual way, such that the white and black squares alternate, and let Amy put her knights on black squares as long as it is possible. Two knights on squares of the same colour never attack each other. The number of black squares is $20^2/2=200$. The two players occupy the squares in turn, so Amy will surely find empty black squares in his first 100 steps.

A strategy for Ben: Group the squares into cycles of length 4, and after each step of Amy, occupy the opposite square in the same cycle.
Consider the squares of the board as vertices of a graph; let two squares be connected if two knights on those squares would attack each other. Notice that in a 4×4 board the squares can be grouped into 4 cycles of length 4, as shown in Figure 1. Divide the board into parts of size 4×4, and perform the same grouping in every part; this way we arrange the 400 squares of the board into 100 cycles (Figure 2).


Figure 1


Figure 2

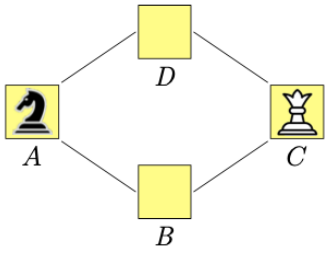


Figure 3

The strategy of Ben can be as follows: Whenever Amy puts a new knight to a certain square A, which is part of some cycle $A-B-C-D-A$, let Ben put his stone on the opposite square C in that cycle (Figure 3).

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From this point, Amy cannot put any knight on A or C because those squares are already occupied, neither on B or D because those squares are attacked by the knight standing on A. Hence, Amy can put at most one knight on each cycle, that is at most 100 knights in total.

Declaration of AI usage

We used Gemini to help us generate the base text of the abstract, with our own modifications.

References

[1] International Mathematical Olympiad. (2018) *Problem 4*, IMO Official Website. `https://www.imo-official.org/problems/2018/`

[2] International Mathematical Olympiad. (2018) *Shortlisted problems with solutions*, IMO Official Website. `https://www.imo-official.org/problems/2018/`

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