2026 S.T. Yau High School Science Award (Asia)

Research Report

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Title of Research Report

Size of Graph Restricted Direction Set on Polynomial Parametric Curves

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Size of Graph Restricted Direction Set on Polynomial Parametric Curves

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Abstract

This study is about the size of set of distinct directions generated between two finite point sets $A, B \subset \gamma$, where γ is a polynomial parametric curve. This problem is also related to Elekes–Rónyai problem for rational functions, since when $\gamma : u \mapsto (x(u), y(u))$, the direction can be thought as the secant slope function

$$F(u, v) = \frac{y(u) - y(v)}{x(u) - x(v)}.$$

So the problem actually becomes the bound of $F(S \times T)$ when $|S| = |T| = n$ and $S, T \subset \mathbb{R}$. This paper provides a novel and elementary combinatorial approach to handle these kind of problems, by proving that small graph restricted direction set forces repeated slope matrices. Also, employing algebraic geometric foundations, we prove that repeated slope matrices force γ to be contained in a conic. Combining these results, we obtain that, if γ is not contained in a conic, then there exists $\varepsilon > 0$ such that for every bipartite graph G with $\|G\| = \Omega(n^{2-\varepsilon})$ satisfies $|S_G(A, B)| = \omega(n)$. Moreover, this result implies that the direction set between A and B has size $\omega(n^{1+c_d})$ for some $c_d > 0$ depending on $d = \max(\deg x, \deg y)$. In particular, this proves an Elekes-Rónyai-type superlinear expansion result for rational functions represented as slopes of polynomially parametrized curves, including the graph-restricted setting. Moreover, the proof methods provide a new combinatorial framework that can be adapted in various similar problems.

Keywords: Direction sets; Elekes-Rónyai expansion; Polynomial parametrized curves; Bipartite graphs; Affine algebraic geometry.

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Disclosure of AI Usage

AI models were not used during the research and writings.

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1 Introduction

1.1 Research background

In 1946, Erdős [1] asked two questions for a set of n points: the number of distinct distances and the number of distinct directions determined by pairs of points. These became central problems in combinatorial geometry, and it became a significant topic to give a lower bounds for the distance set and the direction set. While there were some breakthroughs for the unit distance problems recently, this paper only focuses on the problem about directions. The original problem of directions, for n noncollinear points, was resolved by Peter Ungar [2] in 1982, by an elementary combinatorial geometric argument. Later, the problem when n noncollinear points are not general, especially when it lies on a given curve, was considered crucial. Elekes first solved the case when the points lie on the curve $y - P(x) = 0$ for some polynomial P. This was generalized by Elekes and Rónyai in [3], proving that for a bivariate polynomial $f(u, v) \in \mathbb{R}[u, v]$, the size of a set

$$f(S \times T) = \{f(s, t) : s \in S, t \in T\}$$

where $|S| = |T| = n$ is $\omega(n)$, or f satisfies particular additive or multiplicative form. Nowadays, there are many stronger results, including the result that the size of $|f(S \times T)|$ is $\Omega(n^{4/3})$ in [5]. However, the same problem for rational function f is currently open. Combining this unsolved Elekes–Rónyai type problem for rational functions and the Erdős direction set problem, we are interested in the problem regarding the direction set between points on polynomial parametric curves. This problem lies in between the two important problems, since the direction can be viewed also as the value of the secant slope function

$$F(u, v) = \frac{y(u) - y(v)}{x(u) - x(v)}$$

where $\gamma : u \mapsto (x(u), y(u))$ is a polynomial parametric curve. That is, bounding the size of direction set between sets $A = \gamma(S)$ and $B = \gamma(T)$ is closely related to bounding the Elekes–Rónyai expansion $|F(S \times T)|$ for such rational function F. In this paper, we prove that there is a constant $c_d > 0$ only depending on $d = \max(\deg x, \deg y)$ such that either γ is contained in a conic, or $|F(S \times T)| = \omega(n^{1+c_d})$. The exclusion of conics is natural from the aspect of the Elekes–Rónyai theorem. The secant slope function of a polynomial parameterized conic is a function of the form $\varphi(u) + \varphi(v)$. Thus, conic is the geometric counterpart of the exceptional forms of Elekes–Rónyai theorem.

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1.2 Research Methods

We view the graph restricted direction set $S_G(A, B)$, only counting distinct directions among $(a, b) \in A \times B$ that has corresponding edge in a bipartite graph G. This paper suggests a novel two step approach for these kinds of problems. We use geometrical intuitions based on newly defined concepts called point lists, and a relation called translation between point lists. This proves that from only elementary methods, small $|S_G(A, B)|$ implies that there is a structural constraints on γ. Then, we prove that such structural constraints force γ to be conic or a line, using algebraic geometrical foundations and Heintz's Bezout type bounds in [13].

Especially, the first step, forcing a structural constraint, can be widely adapted to many different mathematical situations. We predict that this result can be generalized over polynomial parametric curves to a general situation.

1.3 Logical Dependency Diagram

The following diagram shows the logical dependencies among the core results. Some frequently used preliminaries or minor results are omitted. This diagram is included so that the readers can understand the paper and the results more easily.

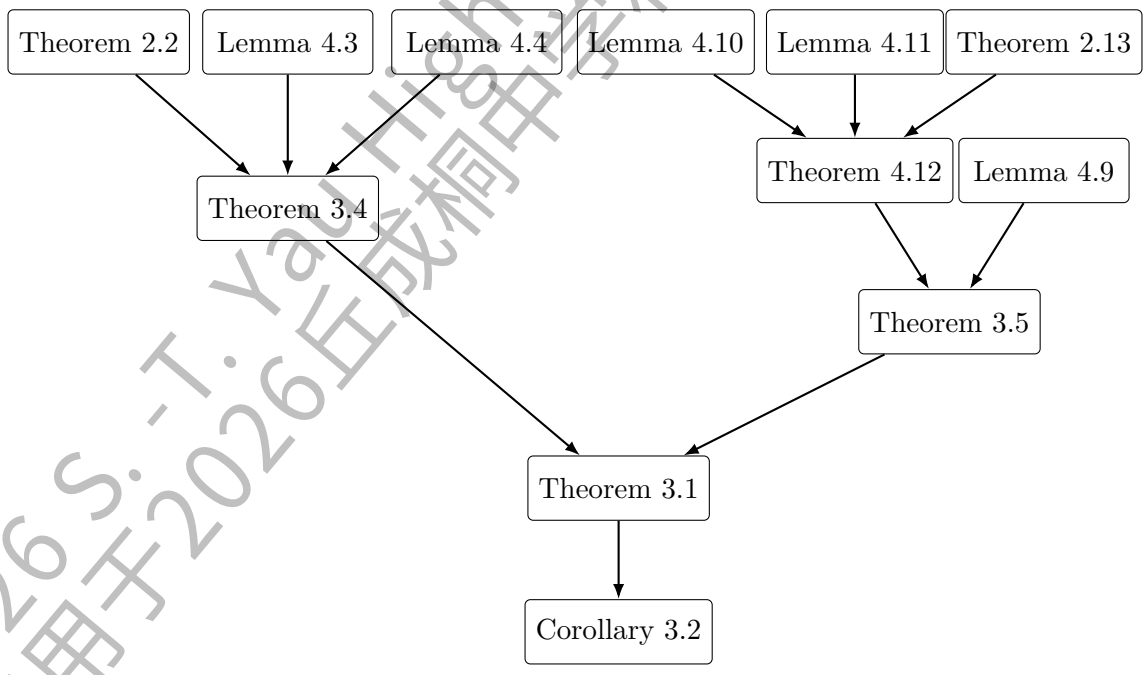


Figure 1: Logical dependencies among the core results.

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2 Preliminaries and Related Work

2.1 Algebraic and combinatorial preliminaries

Definition 2.1. Zarankiewicz function
Zarankiewicz function, first used in the Zarankiewicz problem [7] was defined as below following the notation of recent work such as [9]. $z(m, n; s, t)$ is the maximum possible number of edges $\|G\|$ of a bipartite graph $G = (U \sqcup V, E)$ with $|U| = m$, $|V| = n$, and no $K_{s,t}$ as a subgraph. Here, the s points must lie in U and the t points must lie in V.

Theorem 2.2. Kovari–Sos–Turan Theorem [8]
The best known upper bound for the Zarankiewicz function is given by the Kovari–Sos–Turan theorem [8]. This was strengthened by Stefan Znám [9], and the result is

$$z(m, n; s, t) < (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m$$

Theorem 2.3. Combinatorial Nullstellensatz
Let k be a field, and $f \in k[x_1, x_2, \cdots, x_n]$ be a polynomial of $\deg f = \sum_{1=1}^{n} t_i$ *where the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. For any n sets $S_1, S_2, \cdots, S_n \subset k$ where $|S_i| > t_i$ for $1 \le i \le n$, there exists a point $p \in S_1 \times S_2 \times \cdots \times S_n$ such that $f(p) \neq 0$.*

Definition 2.4. Algebraic sets and Zariski topology
For a field k, $\mathbb{A}_k^n$ is the affine space over k. A set X is called an algebraic set when $X = V(\mathcal{F})$ for some set of polynomials $\mathcal{F} \subset k[x_1, x_2, \cdots, x_n]$. Here,

$$V(\mathcal{F}) = \{p \in \mathbb{A}_k^n : f(p) = 0 \quad \forall f \in \mathcal{F}\}$$

The topology on $\mathbb{A}_k^n$ in which the algebraic sets are the closed sets is called the Zariski topology. That is, a set is Zariski closed if and only if the set is an algebraic set.

Throughout this paper, we work over $k = \mathbb{C}$, and write $\mathbb{A}^n = \mathbb{A}_{\mathbb{C}}^n = \mathbb{C}^n$. Unless otherwise specified, all topological notions refer to the Zariski topology. When the usual topology on $\mathbb{A}_k^n$ is intended, we explicitly refer to it as the Euclidean topology.

Definition 2.5. Vanishing ideal
For a set $X \subseteq \mathbb{A}_k^n$, define the vanishing ideal $I(X)$ of X as follows.

$$I(X) := \{f \in k[x_1, x_2, \cdots, x_n] : f(p) = 0 \quad \forall p \in X\}.$$

Definition 2.6. Irreducibility
Let Y be a nonempty subset of a topological space X. The set Y is called irreducible if,

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whenever $Y = Y_1 \cup Y_2$ for closed sets Y_1, Y_2, one has $Y_1 = Y$ or $Y_2 = Y$. Note that if Y is an irreducible and closed in Zariski topology on $\mathbb{A}_k^n$, Y is called an affine variety.

Definition 2.7. Irreducible components
For an algebraic set X, an irreducible component of X is a maximal irreducible closed subset of X. In other words, an irreducible subset $W \subseteq X$ is an irreducible component of X if for any $W \subseteq Y \subseteq X$, Y is irreducible then $Y = W$.

Theorem 2.8. Irreducible decompsition
For an algebraic set X, there exists a set of irreducible components $\{W_1, W_2, \cdots, W_r\}$ such that

$$X = W_1 \cup W_2 \cup \cdots \cup W_r$$

and such decomposition is unique. Note that W_i are all irreducible components, so $W_i \nsubseteq W_j$ for any $i \neq j$.

Definition 2.9. Locally closed set and Closure
For an ambient space X, a subset $S \subseteq X$ is called locally closed if it can be written as $S = V \cap U$, where V is closed and U is open. Also, a closure of S on X, denoted by $\overline{S}^X$, is defined as

$$\overline{S}^X = \bigcap_{\substack{S \subseteq V \subseteq X \\ V \text{ is closed}}} V.$$

Thus, for any closed subset of X containing S also contains $\overline{S}^X$. In Zariski topology, the usual ambient space is $\mathbb{A}_k^n$ and Zariski closure will be simply denoted as $\overline{S}$.

Definition 2.10. Dimensions
For a nonempty topological space X, define the dimension $\dim X$ of X as

$$\dim X := \sup\{n : \emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ are irreducible closed subsets of } X\}$$

Definition 2.11. Algebraic curves
An affine variety $C \subseteq \mathbb{A}_k^n$ is an affine algebraic curve when $\dim C = 1$.

Theorem 2.12. Dimension of a locally closed set and its closure
For any locally closed set S in Zariski topology,

$$\dim S = \dim \overline{S}.$$

Theorem 2.13. Heintz's Bezout type bound [13]
For an algebraic set $X = V(\mathcal{F}) \subseteq \mathbb{A}_k^n$, suppose $\mathcal{F} = \{f_1, f_2, \cdots, f_m\} \subset k[x_1, x_2, \cdots, x_n]$ and $m \geq n$. If $\deg f_i = d_i$ and $d_1 \geq d_2 \geq \cdots \geq d_m$, the number of all zero dimensional irreducible componenets of X is at most $d_1 d_2 \cdots d_n$.

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Theorem 2.14. Dimension of hypersurface sections
Let $X \subseteq \mathbb{A}_k^n$ be an irreducible locally closed set, and let $H \subseteq \mathbb{A}_k^n$ be a hypersurface defined as the zero set of a single nonconstant polynomial. If $d = \dim X \geq 1$, $X \not\subseteq H$, and $X \cap H \neq \emptyset$, every irreducible component Y of $X \cap H$ satisfies $\dim Y = d - 1$.

Corollary 2.15. Existence of an affine algebraic curve through a point
Let $W \subseteq \mathbb{A}_k^n$ be an affine variety with $\dim W \geq 1$. For every point $p \in W$, there exists an affine algebraic curve $C \subseteq W$ such that $p \in C$.

Proof. For the sake of contradiction, choose a counterexample W of minimum dimension, and write $d = \dim W$. If $d = 1$, it is enough to take $C = W$, which is a contradiction. Thus, $d \geq 2$. Suppose that for every $a_1, a_2, \cdots, a_n \in \mathbb{C}$, the hyperplane through $p = (p_1, p_2, \cdots, p_n)$ given by $H(a_1, a_2, \cdots, a_n) = V(a_1(x_1 - p_1) + a_2(x_2 - p_2) + \cdots + a_n(x_n - p_n))$ contains W. Then $W \subseteq H$ for all such H, and hence

$$W \subseteq \bigcap_{(a_1, a_2, \cdots, a_k)} H(a_1, a_2, \cdots, a_k) = \{p\},$$

but this is a clear contradiction. Therefore, there exists $(a_1, a_2, \cdots, a_k)$ such that $W \not\subseteq H(a_1, a_2, \cdots, a_k) = H$. Since W is Zariski closed and so is H, $W \cap H$ is Zariski closed. Hence, from theorem 2.8, there exists and irreducible component W_1 of $W \cap H$ containing p. By theorem 2.14, $\dim W_1 = d - 1$. Since W was chosen to have minimum Krull dimension among the counterexamples, there exists an algebraic curve C in W_1 containing p. Since $C \subseteq W_1 \subsetneq W$, this contradicts the assumption that W does not satisfy the conclusion. Therefore, for any W, there exists an affine algebraic curve C satisfying the conditions. □

Definition 2.16. Tangent spaces
Let $V = V(\mathcal{F}) \subseteq \mathbb{A}_k^n$ be an affine variety. For $p \in V$, the tangent space T_pV is defined as

$$T_pV = \left\{ v \in k^n : \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(p) v_j = 0 \quad \forall f \in I(V) \right\}.$$

The dimension $\dim T_pV$ or the tangent space is defined as the vector space dimension of T_pV. One can easily see that T_pV is a vector space.

Definition 2.17. Smooth points and Singular points
For an affine variety V, a point $p \in V$ is called smooth if $\dim V = \dim T_pV$, and otherwise singular. Let V_{sm} be the set of all smooth points on V, and V_{sing} be the set of all singular points on V.

Theorem 2.18. Openness of locus of Smooth points
Let C be an affine algebraic curve over $\mathbb{A}_k^n$. Then C_{sm} is nonempty Zariski open subset of C. Actually, C_{sing} is a finite Zariski closed set.

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Theorem 2.19. Existence of Inverse functions
Let $U \subset \mathbb{C}$ be open in the usual Euclidean topology, and $f : U \to \mathbb{C}$ be a holomorphic function. If z_0 satisfies $f'(z_0) \neq 0$, then there exist $\rho > 0$ and $\delta > 0$ such that

$$\forall w \in D_\delta(w_0), \text{ there exists a unique } z_w \in D_\rho(z_0) \text{ such that } f(z_w) = w.$$

For such w, write $z_w = f^{-1}(w)$. The function f^{-1} is holomorphic on its domain of definition.

2.2 Setup and notation

Definition 2.20. Direction and slope
To avoid confusion in terminology, direction of a line refers to its angle respect to the axis, while slope refers to its gradient or inclination. To be precise, for two points $a = (a_x, a_y)$ and $b = (b_x, b_y)$ on $\mathbb{R}^2$, the direction $\mathrm{dir}(ab)$ of line ab is the only value $\theta \in [0, \pi)$ that satisfies $[a_x - b_x : a_y - b_y] = [\cos\theta : \sin\theta] \in \mathbb{RP}^1$, while the slope is the value $\frac{a_y - b_y}{a_x - b_x}$, which is not defined when $a_x = b_x$.

Definition 2.21. Indexing map on point set
For a parametric curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, a set A of n distinct points on γ has an indexing map $\mathrm{idx} : A \to \{1, 2, \cdots, n\}$. The indexing map is chosen so that, if $a = \gamma(t_{\mathrm{idx}(a)})$, then the sequence t_i is increasing, and this choice is unique.

Definition 2.22. Direction set determined by G
Let A and B be sets of points on a curve γ, and let $G(U, V, E)$ be a bipartite graph, where $U = \{u_1, u_2, \cdots, u_n\}$ and $V = \{v_1, v_2, \cdots, v_n\}$. For a bipartite graph $G'(A, B, E')$ whose edges are $E' = \{(a, b) : a \in A, b \in B, \{u_{\mathrm{idx}(a)}, v_{\mathrm{idx}(b)}\} \in E\}$, the graph-restricted direction set $S_G(A, B)$ is defined as

$$S_G(A, B) := \{\mathrm{dir}(ab) : (a, b) \in E'\}$$

Definition 2.23. Point list sets
Let A be a set of points on a curve γ, and let $\delta \geq 0$. For a given k, define the set of δ-short point lists of A by

$$\overline{A^k}_\delta := \{[a_1, a_2, \cdots, a_k] \in \overline{A^k} : \mathrm{idx}(a_k) - \mathrm{idx}(a_1) \leq |A|^\delta \text{ and } \mathrm{idx}(a_i) \text{ is strictly increasing}\}$$

The lists of k distinct elements of A with strictly increasing indices is called point lists. $\overline{A^k} = \overline{A^k}_1$ is the set of all point lists. Additionally, let $\ell(\alpha) = [\mathrm{idx}(a_1), \mathrm{idx}(a_k)]$ be the corresponding interval of point list α.

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Definition 2.24. Translation
Two k-point lists $\alpha = [a_1, a_2, \cdots, a_k]$ and $\beta = [b_1, b_2, \cdots, b_k]$ are translates of each other with direction θ if, for every $1 \leq i \leq k$, the line $a_i b_i$ has direction θ. If there exists such a direction θ, then it is said as follows: α is a translate of β with direction θ or shortly, α is a translate of β.

Definition 2.25. (G, C, n)-restricted
Suppose a curve γ is given. If there exist two point sets A, B of the same size n on γ such that

$$|S_G(A, B)| \leq Cn$$

then we say that γ is (G, C, n)-restricted.

Definition 2.26. Contained in a conic
Let $x, y \in \mathbb{R}[u]$ be polynomials, and let $\gamma(u) = (x(u), y(u))$ be a polynomial parametric curve. We say that γ is contained in a conic if there exists a nonzero polynomial

$$Q(X, Y) \in \mathbb{C}[X, Y] \quad \text{s.t.} \quad Q(x(u), y(u)) = 0 \quad \forall u$$

with $\deg Q \leq 2$. Note that here, conic possibly refers to a degerate one.

2.3 Known results

Theorem 2.27 (Ungar's theorem for distinct directions [2])**.** *Let P be a set of $n \geq 3$ points in $\mathbb{R}^2$ that are not all collinear. Then the lines determined by pairs of distinct points of P have at least $2\lfloor n/2 \rfloor$ distinct directions. In other words, P determines at least n distinct directions when n is even and at least $n - 1$ distinct directions when n is odd.*

Theorem 2.28 (Elekes–Rónyai theorem [3])**.** *For a given bivariate polynomial $f(u, v) \in \mathbb{R}[u, v]$, for any sets A and B with $|A| = |B| = n$,*

$$|f(A \times B)| = \omega(n)$$

provided that f is not of the special form $f(x, y) = h(\varphi(x)+\psi(y))$ or $f(x, y) = h(\varphi(x)\cdot\psi(y))$ for some one-variable polynomials h, φ, ψ.

Theorem 2.29 (Improved bound of the Elekes–Rónyai problem [5])**.** *For a bivariate polynomial $f(x, y)$ that is not of the special form in Theorem 2.1, for any sets A and B with $|A| = |B| = n$,*

$$|f(A \times B)| = \Omega(n^{4/3})$$

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3 Main Results

Throughout the paper, $\gamma : u \mapsto (x(u), y(u))$ is a polynomial parametric curve, where $x, y \in \mathbb{R}[u]$. Let $d = \max(\deg x, \deg y)$. Also, A and B are sets of n distinct points on γ, i.e. $A = \gamma(S)$ and $B = \gamma(T)$, where S and T are finite subsets of $\mathbb{R}$ where $|S| = |T| = n$. Let $F(u, v) = F_\gamma(u, v) = \frac{y(u)-y(v)}{x(u)-x(v)} \in \mathbb{R}(u, v)$ be the secant slope function of γ, which is a rational function.

Theorem 3.1. *There exists a constant $\varepsilon > 0$, only depending on d, that satisfies the following. For any constants $c, C > 0$, there exists a constant $n_0 = n_0(C, c, \gamma)$ such that for any $n \geq n_0$ and every bipartite graph $G(A, B, E)$ satisfying $\|G\| \geq cn^{2-\varepsilon}$, either γ is not $(G, C, n)-$restricted or γ is contained in a conic.*

The exclusion of conics in theorem 3.1 is necessary, since for every positive integer n, there exist n points on a conic that determine only $\mathcal{O}(n)$ distinct directions.

Corollary 3.2. *If γ is not contained in a conic, there exists a positive constant $\varepsilon > 0$ only depending on d, that satisfies*

$$|F_\gamma(W)| = \omega(n)$$

for any $W \subseteq S \times T$ where $|W| = \Omega(n^{2-\varepsilon})$.

Corollary 3.3. *If γ is not contained in a conic, there exists a positive constant $c_d > 0$ only depending on d, that satisfies*

$$F_\gamma(S \times T) = \omega(n^{1+c_d})$$

for any $S, T \subset \mathbb{R}$ where $|S| = |T| = n$.

The main theorem theorem 3.1 is the result of two following sub theorems.

Theorem 3.4. *m and k are given positive integers. There exists a positive constant $\varepsilon > 0$ depending on m, k, d such that if γ is $(G, C, n)-$restricted for some sufficiently large n and some bipartite graph G with $\|G\| = \Omega(n^{2-\varepsilon})$, there exists point list sets $\mathcal{A}' \subseteq \overline{A^k}$ and $\mathcal{B}' \subseteq \overline{B^k}$ such that $|\mathcal{A}'| = |\mathcal{B}'| = m$ and for any $(\alpha', \beta') \in \mathcal{A}' \times \mathcal{B}'$, α' is a translate of β'.*

Theorem 3.5. *$m > 6d^2$ is a given positive integer, and suppose that γ is not contained in a conic. There exists a positive integer $K = K(m)$ such that there does not exist K distinct $2m-$dimensional points $(s_1^\ell, s_2^\ell, \cdots, s_m^\ell, t_1^\ell, t_2^\ell, \cdots, t_m^\ell) \in \mathbb{R}^{2m} \quad (\ell = 1, 2, \cdots, K)$ that satisfies $\gamma(s_i^{\ell_1})\gamma(t_j^{\ell_1}) \parallel \gamma(s_i^{\ell_2})\gamma(t_j^{\ell_2})$ for all $1 \leq i, j \leq m$ and $1 \leq \ell_1, \ell_2 \leq K$.*

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4 Proofs

Definition 4.1. $\theta-$direction subgraph of G
Given a bipartite graph $G(A, B, E)$ where A and B are point sets on $\mathbb{R}^2$, let $G_\theta(A, B, E_\theta)$ be a spanning subgraph of G that satisfies

$$\{a, b\} \in E_\theta \iff \{a, b\} \in E \quad \text{and} \quad \text{dir}(ab) = \theta$$

Definition 4.2. Translation graph
Given a positive integer k and subsets $\mathcal{A}_0 \subseteq \overline{A^k}$, $\mathcal{B}_0 \subseteq \overline{B^k}$, direction $\theta-$translation graph on $\mathcal{A}_0$ and $\mathcal{B}_0$ is defined as the bipartite graph $H_\theta(\mathcal{A}_0, \mathcal{B}_0)$ such that

$$E(H_\theta) = \{\{\alpha, \beta\} : \alpha \in \mathcal{A}_0, \beta \in \mathcal{B}_0, \alpha \text{ is a tranlate of } \beta \text{ with direction } \theta\}.$$

Additionally, define translation graph on $\mathcal{A}_0$ and $\mathcal{B}_0$ as the bipartite graph $H(\mathcal{A}_0, \mathcal{B}_0)$ such that $e \in E(H)$ whenever there exists a direction θ such that $e \in E(H_\theta)$.

Lemma 4.3. *A positive integer k and δ_1, ε satisfying $1 - \varepsilon > \delta_1 > \varepsilon > 0$ are given. If $\|G\| \geq cn^{2-\varepsilon}$ and $|S_G(A, B)| \leq Cn$, for translation graph H on $\overline{A^k}_{\delta_1}$ and $\overline{B^k}$ satisfies*

$$\|H\| = \Omega_{c,C}(n^{2-\varepsilon+(k-1)(\delta_1-\varepsilon)})$$

Proof. For each $1 \leq p \leq n$ and $\theta \in S_G(A, B)$, define $\tau_p(\theta)$ be the number of $(a, b) \in A \times B$ such that $\text{dir}(ab) = \theta$ and $\text{idx}(a) \in [p, p + n^{\delta_1}]$. For each $a \in A$ such that $\deg_{G_\theta}(a) \geq 1$, there are at most n^{δ_1} number of p where a contributed for counting $\tau_p(\theta)$, since that is the number of p such that $p \leq \text{idx}(a) \leq p + n^{\delta_1}$. Also, for each $a \in A$ satisfying $n^{\delta_1} < \text{idx}(a) < n - n^{\delta_1}$, there are $\lfloor n^{\delta_1} \rfloor$ such p. Therefore,

$$n^{\delta_1} \cdot (\|G_\theta\| - 2dn^{\delta_1}) \sum_{p=1}^{n} \tau_p(\theta) = \left|\left\{(a, b, p) : \text{dir}(ab) = \theta, p \leq \text{idx}(a) \leq p + n^{\delta_1}\right\}\right| \leq n^{\delta_1} \cdot \|G_\theta\|$$

because for any $a \in A$, $\deg_{G_\theta}(a) \leq d$ and there are at most $2n^{\delta_1}$ number of $a \in A$ satisfying $\text{idx}(a) \leq n^{\delta_1}$ or $\text{idx}(a) \geq n - n^{\delta_1}$. Now, for some translation (α, β), since α is a δ_1-short point list, there exists some p such that $\ell(\alpha) \subseteq [p, p + n^{\delta_1}]$. For fixed p, the number of such translations is $\binom{\tau_p(\theta)}{k}$. One translation (α, β) can be counted in at most n^{δ_1} distinct p, so,

$$\begin{aligned} \|H_\theta\| &\geq \frac{1}{n^{\delta_1}} \cdot \sum_{p=1}^{n} \binom{\tau_p(\theta)}{k} \\ &= \frac{1}{n^{\delta_1}} \cdot \frac{1}{k!} \cdot \sum_{p=1}^{n} \left(\tau_p(\theta)^k - \mathcal{O}(\tau_p(\theta)^{k-1})\right) \end{aligned}$$

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Here, from Jensen's Inequality, the following holds.

$$\sum_{p=1}^{n}\left(\tau_p(\theta)^k-\mathcal{O}(\tau_p(\theta)^{k-1})\right)\geq n\left(\frac{\sum_{p=1}^{n}\tau_p(\theta)}{n}\right)^k-\mathcal{O}\left(n\left(\frac{\sum_{p=1}^{n}\tau_p(\theta)}{n}\right)^{k-1}\right)$$

Now, using the bounds established above, we can obtain that

$$\begin{aligned}\|H_\theta\|&\geq\frac{1}{k!\cdot n^{\delta_1}}\left(n^{1-k}\left(\sum_{p=1}^{n}\tau_p(\theta)\right)^k-\mathcal{O}\left(n^{2-k}\left(\sum_{p=1}^{n}\tau_p(\theta)\right)^{k-1}\right)\right)\\&\geq\frac{n^{1-k(1-\delta_1)}}{k!\cdot n^{\delta_1}}(\|G_\theta\|-\mathcal{O}(n^{\delta_1}))^k-\mathcal{O}\left(n^{2-k}(n^{\delta_1}\|G_\theta\|)^{k-1}\right)\end{aligned}$$

Now, as $E(H)=\bigsqcup_{\theta\in S_G(A,B)}E(H_\theta)$, we can sum the above inequality to get

$$\begin{aligned}\|H\|&=\sum_{\theta\in S_G(A,B)}\|H_\theta\|\\&\geq\frac{1}{n^{(k-1)(1-\delta_1)}k!}\cdot\sum_{\theta\in S_G(A,B)}\left((\|G_\theta\|-\mathcal{O}(n^{\delta_1}))^k-\mathcal{O}\left(n^{1-\delta_1}\|G_\theta\|^{k-1}\right)\right)\end{aligned}$$

Here, applying Jensen's Inequality again for θ gives us

$$\begin{aligned}&\sum_{\theta\in S_G(A,B)}\left((\|G_\theta\|-\mathcal{O}(n^{\delta_1}))^k-\mathcal{O}\left(n^{1-\delta_1}\|G_\theta\|^{k-1}\right)\right)\\&\geq|S_G(A,B)|\cdot\left(\left(\frac{\sum_{\theta\in S_G(A,B)}(\|G_\theta\|-\mathcal{O}(n^{\delta_1}))}{|S_G(A,B)|}\right)^k-\mathcal{O}\left(n^{1-\delta_1}\frac{\|G\|^{k-1}}{|S_G(A,B)|^{k-1}}\right)\right)\\&\geq|S_G(A,B)|\cdot\left(\left(\frac{\|G\|-|S_G(A,B)|\cdot\mathcal{O}(n^{\delta_1}))}{|S_G(A,B)|}\right)^k-\mathcal{O}\left(n^{1-\delta_1}\frac{\|G\|^{k-1}}{|S_G(A,B)|^{k-1}}\right)\right)\end{aligned}$$

Since we have assumed $\|G\|\geq cn^{2-\varepsilon}$ and $|S_G(A,B)|\leq Cn$, it follows that from $\delta_1<1-\varepsilon$, $\|G\|-|S_G(A,B)|\cdot\mathcal{O}(n^{\delta_1})=\Omega(n^{2-\varepsilon})$. Moreover, from $\delta_1>\varepsilon$, we have that

$$n^{1-\delta_1}\frac{\|G\|^{k-1}}{|S_G(A,B)|^{k-1}}=o\left(\frac{\|G\|^k}{|S_G(A,B)|^k}\right)$$

Therefore, the dominating term is can be easily known, and combining with $\|G\|=\Omega(n^{2-\varepsilon})$ and $|S_G(A,B)|=\mathcal{O}(n)$, we have that

$$\|H\|=\Omega\left(\frac{1}{n^{(k-1)(1-\delta_1)}k!}\cdot\frac{\|G\|^k}{|S_G(A,B)|^k}\right)=\Omega_{c,C}(n^{2-\varepsilon+(k-1)(\delta_1-\varepsilon)})$$

□

Lemma 4.4. *A positive integer k and δ_1,δ_2 satisfying $(k+2)\delta_1>\delta_2>(k+1)\delta_1>0$ is*

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given. H is the translation graph on $\overline{A^k}_{\delta_1}$ and $\overline{B^k}$. Define H' be the bipartite subgraph of H induced between $\overline{A^k}_{\delta_1}$ and $\overline{B^k} \setminus \overline{B^k}_{\delta_2}$. $\|H'\|$ satisfies

$$\|H'\| = \mathcal{O}(n^{2-\delta_2+k\delta_1})$$

Proof. Consider the direction $\theta-$translation graph H'_θ on $\overline{A^k}_{\delta_1}$ and $\overline{B^k} \setminus \overline{B^k}_{\delta_2}$ for each $\theta \in S_G(A,B)$. Here, $E(H') = \bigsqcup_{\theta \in S_G(A,B)} E(H'_\theta)$, so it suffices to give a upper bound for $\|H'_\theta\|$. Now, fix one $\theta \in S_G(A,B)$. We will prove that $\|H'_\theta\| = \mathcal{O}(n^{1-\delta_2+k\delta_1})$.

First, delete every point list $\beta \in \overline{B^k} \setminus \overline{B^k}_{\delta_2}$ with degree zero, or, $\deg_{H'_\theta}(\beta) = 0$. Denote by $\overline{B^k}' = \overline{B^k}'(\theta)$ the resulting set. Note that this set can change for each θ we have fixed. Now, consider the simple graph $X(\overline{B^k}', E_X)$ defined on the point lists $\overline{B^k}'$, where

$$\{\beta, \beta'\} \in E_X \iff \ell(\beta) \cap \ell(\beta') \neq \emptyset$$

Let $I = I(\theta)$ be the maximum independent set satisfying the following property.

$$\text{(property) The following sum is minimal: } \sum_{\beta \in I} |\ell(\beta)|$$

For convenience, let $I = \{\beta_1, \beta_2, \cdots, \beta_{|I|}\}$. For each $i \neq j$, since I is an independent set, $\ell(\beta_i) \cap \ell(\beta_j) = \emptyset$. This implies that $|I| \leq n^{1-\delta_2}$, because from $|\ell(\beta_i)| \geq n^{\delta_2}$,

$$|I| \cdot n^{\delta_2} \leq \sum_{\beta \in I} |\ell(\beta)| \leq \left| \bigsqcup_{\beta \in I} \ell(\beta) \right| \leq |B| = n.$$

If there exists $\beta' \in \overline{B^k}'$ such that for some $1 \leq j \leq |I|$, $\ell(\beta') \subsetneq \ell(\beta_j)$, then considering a new independent set $I' = I \setminus \{\beta_j\} \cup \{\beta'\}$ gives a contradiction to the minimality since $\sum_{\beta \in I'} |\ell(\beta)| < \sum_{\beta \in I} |\ell(\beta)|$. Also, if there exists $\beta' \in \overline{B^k}'$ such that for all $1 \leq j \leq |I|$, $\ell(\beta') \cap \ell(\beta_j) = \emptyset$, this means that $I \cup \{\beta'\}$ is also an independent set, which is impossible as I was the maximum independent set. Therefore, we can conclude that for any $\beta' \in \overline{B^k}'$, there exists $1 \leq j \leq |I|$ such that when $\beta_j = [\beta_j[1], \beta_j[2], \cdots, \beta_j[k]]$, we have either $\mathrm{idx}(\beta_j[1]) \in \ell(\beta')$ or $\mathrm{idx}(\beta_j[k]) \in \ell(\beta')$.

Now, define the function $\phi_\theta : \overline{B^k}' \to \{1, 2, \cdots, |I|\}$ where

$$\beta' \xmapsto{\phi_\theta} \min\{i : \ell(\beta_i) \cap \ell(\beta') \neq \emptyset\}.$$

This function is well defined for each θ, since the set $\{i : \ell(\beta_i) \cap \ell(\beta') \neq \emptyset\}$ is nonempty,

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as aforementioned. Here, define $\phi_\theta^{-1}(i)$ as the fiber of i under f, which is the following set $\{\beta' \in \overline{B^k}' : \phi(\beta') = i\}$. Now, we will prove that for each $1 \le i \le |I|$,

$$|\phi_\theta^{-1}(i)| = \mathcal{O}(n^{k\delta_1}).$$

For any $\beta' \in \phi_\theta^{-1}(i)$, either $\mathrm{idx}(\beta_i[1]) \in \ell(\beta')$ or $\mathrm{idx}(\beta_i[k]) \in \ell(\beta')$. Hence, if we denote by $D_{\theta,1}$ the set of β' that $\mathrm{idx}(\beta_i[1]) \in \ell(\beta')$, and $D_{\theta,2}$ as the set of β' that $\mathrm{idx}(\beta_i[k]) \in \ell(\beta')$, we have that $|\phi_\theta^{-1}(i)| \le |D_{\theta,1}| + |D_{\theta,2}|$. It suffices to prove that $|D_{\theta,1}| = \mathcal{O}(n^{k\delta_1})$, as analogous bound holds for $|D_{\theta,2}|$.

For each $\theta \in S_G(A, B)$ we have fixed, we defined the set $D_{\theta,1} \subseteq \overline{A^k}_{\delta_1}$. To be clear, $D_{\theta,1}$ is the following set.

$$D_{\theta,1} := \left\{\alpha \in \overline{A^k}_{\delta_1} : \alpha \text{ is a translate of } \beta' \text{ with direction } \theta \text{ where } \beta' \in C_1\right\}$$

Since for every $\beta' \in \overline{B^k}'$, we have that $\deg_{H'_\theta}(\beta') \ge 1$, for every $\beta' \in D_{\theta,1}$, there exists at least one corresponding $\alpha \in \overline{A^k}_{\delta_1}$ which is a translate of β' with direction θ.

Now let $N_{G_\theta}(\beta_i[1]) = \{a_1, a_2, \cdots, a_t\}$. Since γ is a polynomial parametric curve with $d = \max(\deg x, \deg y)$, and $a_1, a_2, \cdots, a_t$ are the intersections of γ and a line passing through $\beta_i[1]$ with direction θ, we know that $t \le d$. By the definition of $D_{\theta,1}$, we have that

$$D_{\theta,1} = \left\{\alpha \in \overline{A^k}_{\delta_1} : \exists h \text{ s.t. } \mathrm{idx}(a_h) \in \ell(\alpha)\right\}$$

For every $1 \le h \le t$, the number of point lists $\alpha \in \overline{A^k}_{\delta_1}$ such that $\ell(\alpha) = [p, p + |\ell(\alpha)|]$ and $\mathrm{idx}(a_h) \in \ell(\alpha)$ is at most $(n^{\delta_1})^{k-1}$ since number of ways selecting $k-1$ more indices, distinct with p, in the interval $[p, p + n^{\delta_1}]$ is at most $(n^{\delta_1})^{k-1}$. Since the number of possible p is $\lfloor n^{\delta_1} \rfloor$, we have that

$$\#\left\{\alpha \in \overline{A^k}_{\delta_1} : \mathrm{idx}(a_h) \in \ell(\alpha)\right\} \le \sum_p n^{(k-1)\delta_1} \le n^{k\delta_1}.$$

Finally, since $D_{\theta,1} = \bigcup_{1 \le h \le t} \left\{\alpha \in \overline{A^k}_{\delta_1} : \mathrm{idx}(a_h) \in \ell(\alpha)\right\}$, we have that

$$|D_{\theta,1}| \le \sum_{1 \le h \le t} \#\left\{\alpha \in \overline{A^k}_{\delta_1} : \mathrm{idx}(a_h) \in \ell(\alpha)\right\} \le t n^{k\delta_1} \le d n^{k\delta_1}.$$

Therefore, $|D_{\theta,1}| = \mathcal{O}(n^{k\delta_1})$, so $|\phi_\theta^{-1}(i)| = \mathcal{O}(n^{k\delta_1})$, as desired.

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Since for each θ, H'_θ is a bipartite graph,

$$\|H'\| = \sum_{\theta \in S_G(A,B)} \|H'_\theta\| = \sum_{\theta \in S_G(A,B)} \sum_{\beta' \in \overline{B^k}'(\theta)} \deg_{H'_\theta}(\beta')$$

Here, consider the neighborhood $N_{H'_\theta}(\beta')$ of $\beta' = [\beta'[1], \beta'[2], \cdots, \beta'[k]]$ in the graph H'_θ. Since for each θ, every $\alpha \in N_{H'_\theta}(\beta')$ has to satisfy that for each $1 \le i \le k$, line $\alpha[i]\beta[i]$ has direction θ. Such choice of $\alpha[i]$ is finite, at most d possible choices, so the total number of α is bounded by at most d^k. Hence, $\deg_{H'_\theta}(\beta') = |N_{H'_\theta}(\beta')| \le d^k$. This implies that

$$\|H'\| \le \sum_{\theta \in S_G(A,B)} d^k \cdot |\overline{B^k}'(\theta)|$$

Recall that $|I(\theta)| = \mathcal{O}(n^{1-\delta_2})$. As every single element in $\overline{B^k}'(\theta)$ must be corresponding in some $i \in \{1, 2, \cdots, |I(\theta)|\}$ by ϕ_θ, we have that

$$|\overline{B^k}'(\theta)| \le |I_\theta| \cdot \mathcal{O}(n^{k\delta_1}) = \mathcal{O}(n^{1-\delta_2+k\delta_1}).$$

Therefore,

$$\|H'\| \le \sum_{\theta \in S_G(A,B)} d^k \cdot |\overline{B^k}'(\theta)| \le |S_G(A,B)| \cdot d^k \cdot \mathcal{O}(n^{1-\delta_2+k\delta_1}) = \mathcal{O}(n^{2-\delta_2+k\delta_1})$$

□

We proved two crucial lemmas to prove theorem 3.4. Notice that what theorem 3.4 is actually stating is, using the definitions of translation graph, the translation graph between $\overline{A^k}$ and $\overline{B^k}$ contains a bipartite complete graph $K_{m,m}$ as an induced subgraph.

Proof of theorem 3.4. Assume the contrary that $K_{m,m}$ is never a subgraph of H. Set $\frac{1}{3mk^2} > \varepsilon > 0$, $2\varepsilon > \delta_1 > \varepsilon > 0$, and $(k+2)\delta_1 > \delta_2 > (k+1)\delta_1$. For the translation graph H on $\overline{A^k}_{\delta_1}$ and $\overline{B^k}$, let $\mathcal{B}$ be the set of point lists $\beta \in \overline{B^k} \setminus \overline{B^k}_{\delta_2}$ where $\deg_H(\beta) \neq 0$. Now, let H'' be the induced subgraph of H between $\mathcal{A} = \overline{A^k}_{\delta_1}$ and $\mathcal{B}$. It means that $E(H') \sqcup E(H'') \subseteq E(H)$. This gives that

$$\|H''\| = |E(H')| = |E(H) \setminus E(H')| = |E(H)| - |E(H')| = \|H\| - \|H'\|.$$

Using lemma 4.3, we know that $\|H\| = \Omega\left(n^{2-\varepsilon+(k-1)(\delta_1-\varepsilon)}\right)$. Also, from lemma 4.4, we have that $\|H'\| = \mathcal{O}\left(n^{2-\delta_2+k\delta_1}\right)$. Here, since $\delta_2 > (k+1)\delta_1 > \delta_1 + k\varepsilon$, we have that $n^{2-\delta_2+k\delta_1} = o(n^{2-\varepsilon+(k-1)(\delta_1-\varepsilon)})$, implying the following result.

$$\|H''\| = \Omega\left(n^{2-\varepsilon+(k-1)(\delta_1-\varepsilon)}\right)$$

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The bipartite graph H'' was defined on $\mathcal{A}$ and $\mathcal{B}$. From theorem 2.2, since we assumed that H does not contain $K_{m,m}$ as a subgraph, H'' also does not contain $K_{m,m}$ as a subgraph. Hence, we have that

$$\|H''\| < (m-1)^{1/m}(|\mathcal{A}| - m + 1)|\mathcal{B}|^{1-1/m} + (m-1)|\mathcal{B}|$$

However, $|\overline{A^k}_{\delta_1}| = \mathcal{O}(n^{1+(k-1)\delta_1})$ and $|\overline{B^k}_{\delta_2}| = \mathcal{O}(n^{1+(k-1)\delta_2})$ from the definition of $\delta-$short point lists. Therefore,

$$\|H''\| = \mathcal{O}\left(n^{1+(k-1)\delta_1+(1-1/m)(1+(k-1)\delta_2)}\right).$$

From the set up $\delta_2 < (k+2)\delta_1 < 2(k+2)\varepsilon$, we have

$$1+(k-1)\delta_1+(1-1/m)(1+(k-1)\delta_2) < 2+(k-1)\delta_1+(1-1/m)2(k-1)(k+2)\varepsilon - 1/m$$

and since $1 > 3k^2m\varepsilon$, we have $1/m > (k+(1-1/m)2(k-1)(k+2))\varepsilon$, we have that

$$\|H''\| = o\left(n^{2-\varepsilon+(k-1)(\delta_1-\varepsilon)}\right),$$

which is a clear contradiction to the lower bound of $\|H''\|$ we have obtained earlier. Therefore, we can conclude that H has to contain $K_{m,m}$ as a subgraph, completing the theorem. □

Definition 4.5. Slope Matrix.
A secant slope function $F(u,v)$ of a polynomial parametric curve γ is given. For a pair of m tuples $s = (s_1, s_2, \cdots, s_{m_0})$ and $t = (t_1, t_2, \cdots, t_{m_0})$, define the slope matrix $M(s,t)$ as

$$M(s,t) = (F(s_i, t_j))_{ij}$$

Definition 4.6. Slope Fiber
For a given slope matrix M and polynomial parametric curve γ, define the slope fiber X_M as follows.

$$X_M := \{(s,t) \in \mathbb{C}^m \times \mathbb{C}^m : M(s,t) = M, \quad x(s_i) \neq x(t_j), s_i \neq s_j, t_i \neq t_j \quad \forall i \neq j\}$$

Now, define a extended slope fiber Y_M as follows.

$$Y_M := \{(s,t) \in \mathbb{C}^m \times \mathbb{C}^m : y(s_i) - y(t_j) = M_{ij}(x(s_i) - x(t_j)) \quad \forall i,j\}$$

Note that X_M is a locally closed set, Y_M is a algebraic set, and $\overline{X_M} \subseteq Y_M$.

Definition 4.7. Coordinate projection
For each $(s,t) \in \mathbb{C}^m \times \mathbb{C}^m$, the s_i-coordinate projection $\pi_{s_i} : \mathbb{C}^{2m} \to \mathbb{C}$ is defined as

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follows when $s = (s_1, s_2, \cdots, s_m)$ and $t = (t_1, t_2, \cdots, t_m)$.

$$(s,t) \overset{\pi_{s_i}}{\longmapsto} s_i$$

Also, similarly define $\pi_{t_j} : \mathbb{C}^{2m} \to \mathbb{C}$ as $(s,t) \overset{\pi_{t_j}}{\longmapsto} t_j$ for each $1 \le j \le m$.

Lemma 4.8. *γ is a given polynomial parametric curve that is not contained in a conic. For each constant α, the number of points $(s,t) \in Y_M$ such that $\pi_{s_i}(s,t) = \alpha$ is finite. Similarly, the number of (s,t) such that $\pi_{t_j}(s,t) = \alpha$ is finite. Here, finite refers to less than a constant only depending on γ, and M is a $m_0 \times m_0$ slope matrix where m_0 is a given positive integer.*

Proof. Since $(s,t) \in Y_M$, we have that $y(s_i) - y(t_j) = M_{ij}(x(s_i) - s(t_j))$ for all $1 \le i, j \le m_0$. Assume that the number of (s,t) with $\pi_{s_i}(s,t) = \alpha$ is more than dm_0. This implies that there exists some $1 \le j \le m0$ such that at least $d+1$ distinct values of t satisfies

$$y(\alpha) - y(t) = M_{ij}(x(\alpha) - x(t))$$

However, this is a polynomial with degree at most d, so it has to be an identity. This implies that $y(t) - M_{ij}x(t)$ is constant, or, γ is contained in a conic. Hence this is a clear contradiction and the number of (s,t) satisfying $\pi_{s_i}(s,t) = \alpha$ is at most dm_0. Similarly, we can prove this for π_{t_j}. □

Lemma 4.9. *Positive integers m and m_0 satisfies $m > 2dm_0$, where $d \ge 1$ is a degree of a nonconstant polynomial $x(u) \in \mathbb{R}[u]$ that is given. For any pair of m tuples $(s_1, s_2, \cdots, s_m)$ and $(t_1, t_2, \cdots, t_m)$, there exists sets $I, J \subset \{1, 2, \cdots, m\}$ such that $|I| = |J| = m_0$ and $x(s_i) \ne x(t_j)$ for any $i \in I, j \in J$.*

Proof. Let $I_0 = J_0 = \{1, 2, \cdots, m\}$, and Y be a bipartite graph on I_0 and J_0 such that

$$(i,j) \in E(Y) \iff x(s_i) = x(t_j).$$

Since $\deg x \le d$, we have that for each i, $\deg_Y(i) \le d$. Now, let $I \subset I_0$ be an arbitrary set of size m_0. Here,

$$|N_Y(I)| \le \sum_{i \in I} \deg_Y(i) \le d|I| = dm_0,$$

so there exists at least dm_0 $j \in J_0$ satisfying $x(t_j) \ne x(s_i)$ for any $i \in I$. This is because $|J_0 \setminus N_Y(I)| = |J_0| - |N_Y(I)| > dm_0$. Therefore, select arbitrarily m_0 among them to obtain the set J satisfying $x(s_i) \ne x(t_j)$ for any $i \in I, j \in J$. □

Lemma 4.10. *γ is a given polynomial parametric curve that is not contained in a conic. For some positive integer $m_0 \ge 3d$ and a $m_0 \times m_0$ slope matrix M, if $\dim X_M \ge 1$, there*

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exists an affine algebraic curve C with pair of m_0 tuples $(s,t) \in \mathbb{C}^{m_0} \times \mathbb{C}^{m_0}$ such that when $s = (s_1, s_2, \cdots, s_{m_0})$ and $t = (t_1, t_2, \cdots, t_{m_0})$,

- $(s,t) \in C_{\mathrm{sm}} \cap X_M$,
- *$F_u(u,v)$ and $F_v(u,v)$ takes a nonzero value for $(u,v) = (s_i, t_j)$ for any $1 \le i, j \le m_0$,*
- *the components of the tangent direction ξ of T_pC is always nonzero.*

Proof. Let $\overline{X_M}$ be the Zariski closure of X_M. From theorem 2.12, we have that $\dim X_M = \dim \overline{X_M} \ge 1$. Also, note that $\overline{X_M}$ is an algebraic set, so there exists finite irreducible components $W_1, W_2, \cdots, W_r$ of $\overline{X_M}$ such that

$$\overline{X_M} = W_1 \cup W_2 \cup \cdots \cup W_r.$$

Now, let U be a Zariski open set defined as

$$U = \left\{ (s,t) : \prod_{i,j} (x(s_i) - x(t_j)) \prod_{k<\ell} (s_k - s_\ell)(t_k - t_\ell) \neq 0 \right\}$$

so that $X_M = \overline{X_M} \cap U$. Now, assume that there exists an irreducible component W of $\overline{X_M}$ such that $W \neq W_i$ for all $1 \le i \le r$. Then

$$W \subseteq \bigcup_{1 \le i \le r} W_i \Longrightarrow W = \bigcup_{1 \le i \le r} (W \cap W_i).$$

Note that W and W_i are always Zariski closed, so $W \cap W_i$ is also Zariski closed. This implies that from irreducibility of W, that $W = W \cap W_i$ for some $1 \le i \le r$. However, if $W \subsetneq W_i$, W cannot be an irreducible component since it is not maximal. Therefore, $W = W_i$ for some i, which is a contradiction.

Now, since $\dim \overline{X_M} = \max \dim W$ among all irreducible component W, there exists $1 \le i \le r$ such that $\dim W_i = \dim \overline{X_M} = \dim X_M \ge 1$. We will then prove that $W_i \cap X_M \neq \emptyset$. Assume otherwise, that $W_i \cap X_M = \emptyset$. This means that from $X_M = \overline{X_M} \cap U$ and $W_i \subseteq \overline{X_M}$, $W_i \cap U = \emptyset$. This implies that $X_M \subseteq \bigcup_{j \neq i} W_j$, since $X_M = \overline{X_M} \cap U$. However, since the union of Zariski closed sets also is Zariski closed, we have that

$$\overline{X_M} \subseteq \bigcup_{j \neq i} W_j \subsetneq \overline{X_M}$$

which is a clear contradiction. Hence, $W_i \cap X_M \neq \emptyset$.

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Now, since $\dim W_i \geq 1$ and $W_i \cap X_M \neq \emptyset$, for some point $p \in W_i \cap X_M$, by applying corollary 2.15, there exists an affine algebraic curve $C \subseteq W_i$ such that $p \in C$. This implies that $p \in X_M \cap C$, so $C \cap X_M$ is nonempty. Note that $C \subseteq \overline{X_M}$ so $C \cap X_M = C \cap U = C \setminus (C \setminus U)$, and $C \setminus U$ is Zariski closed. Also, since C is irreducible and C_{sing} is Zariski closed, we have that C_{sm} is irreducible and closed.

For each $1 \leq i, j \leq m_0$, define polynomials $A_{ij}^s, A_{ij}^t \in \mathbb{C}[u_1, u_2, \cdots, u_{m_0}, v_1, v_2, \cdots, v_{m_0}]$ as

$$A_{ij}^s = y'(u_i) - M_{ij}x'(u_i) \quad \text{and} \quad A_{ij}^t = y'(v_j) - M_{ij}x'(v_j).$$

Note that $F_u(u,v) = \frac{A_{ij}^s(u,v)}{x(u)-x(v)}$ and $F_v(u,v) = \frac{A_{ij}^t(u,v)}{x(u)-x(v)}$. For each point $p \in C_{\text{sm}}$, we have that $\dim T_p C = 1$, or, $T_p C = \xi\mathbb{C}$ for some $\xi \in \mathbb{C}^{2m_0}$. Now, consider the following set

$$C^o := C \setminus \left(C_{\text{sing}} \cup (C \setminus U) \cup \bigcup_{i,j} \left(V(A_{ij}^s) \cup V(A_{ij}^t) \right) \right).$$

Here, C^o is obtained only by reducing finite number of Zariski closed sets from an irreducible set C, so this is either nonempty, or there exists one of the sets that equals C. Since $C \setminus U \neq C$ and $C_{sing} \neq C$, if C^o is empty, there exists some $1 \leq i, j \leq m_0$ such that $C \subseteq V(A_{ij}^s)$ or $C \subseteq V(A_{ij}^t)$. Assume that $\forall p \in C$, $A_{ij}^s(p) = 0$. Since $\dim C \geq 1$ and by lemma 4.8, there are only finite number of $p \in C$ satisfying $\pi_{s_i}(p) = \alpha$ for each α. However, since γ is not contained in a conic, A_{ij}^s is not a zero polynomial, which implies that there are only a finite number of $p \in Y_M$ such that $A_{ij}^s(p) = 0$. However, since $\dim C = 1$, C is an infinite set, a contradiction. Therefore, $C \subseteq V(A_{ij}^s)$ cannot hold, and analogously, $C \subsetneq V(A_{ij}^t)$. Hence, we can conclude that C^o is a nonempty set.

Since C^o is nonempty, select one point $p \in C^o$. The point p satsifies

- $p \in C$,
- $p \notin C_{\text{sing}}$,
- $p \notin C \setminus U$,
- and for all $1 \leq i, j \leq m_0$, $p \notin V(A_{ij}^s)$ and $p \notin V(A_{ij}^t)$.

This implies that $p \in C_{\text{sm}} \cap X_M$. Also, as $p \in X_M$, we have that $x(\pi_{s_i} p) \neq x(\pi_{t_j} p)$ for all $1 \leq i, j \leq m_0$, which means that $F_u(u,v)$ and $F_v(u,v)$ is nonzero for all $(u,v) = (\pi_{s_i} p, \pi_{t_j} p)$. Now, let $\xi = (\sigma_1, \sigma_2, \cdots, \sigma_{m_0}, \tau_1, \tau_2, \cdots, \tau_{m_0}) \neq \mathbf{0}$ be the tangent direction satisfying

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$T_pC = \xi\mathbb{C}$. For each i, j, from $F(\pi_{s_i}p, \pi_{t_j}p) = 0$, we have

$$\sigma_i F_u(\pi_{s_i}p, \pi_{t_j}p) + \tau_j F_v(\pi_{s_i}p, \pi_{t_j}p) = 0.$$

Now assume that $\sigma_i = 0$ for some $1 \le i \le m_0$. This means that from $\tau_j = 0$ for all $1 \le j \le m_0$ since $F_v(\pi_{s_i}p, \pi_{t_j}p) \neq 0$. Also, fix one $\tau_j = 0$ implies that $\sigma_i = 0$ for all $1 \le i \le m_0$. Hence, from $\xi \neq \mathbf{0}$, we have $\sigma_i \neq 0$ and $\tau_j \neq 0$ for all $1 \le i, j \le m_0$. Therefore, this point p and an affine algebraic curve C satisfies all the conditions we wanted, so lemma 4.10 is proved. □

Lemma 4.11. *For a given polynomial paramerized curve γ and its secant slope function $F(u, v)$, if there exists a nonzero rational function $U \in \mathbb{C}(u)$ that satisfies*

$$U(u)F_u(u, v) - U(v)F_v(u, v) = 0$$

then γ is contained in a conic.

Proof. Let $\gamma : u \mapsto (x(u), y(u))$ where $x, y \in \mathbb{R}[u]$. Since U is a rational function, write $U = P/Q$ for polynomials $P, Q \in \mathbb{C}[u]$. Since x is nonconstant, x' is a nonzero polynomial. Thus, the zero sets of x', P, Q is finite, so, by theorem 2.19, we can choose $z_0 \in \mathbb{C}$ and $\eta_1, \eta_2 > 0$ that satisfies follwing three conditions,

1. $x'(z) \neq 0, P(z) \neq 0, Q(z) \neq 0 \quad \forall z \in D_{\eta_1}(z_0)$
2. $D_{\eta_2}(w_0) \subseteq x(D_{\eta_1}(z_0))$
3. there exists a unique $z_w \in D_{\eta_1}(z_0)$ such that $f(z_w) = w, \quad \forall w \in D_{\eta_2}(w_0)$

For this z_0, η_1, η_2, let disc Ω be $D_{\eta_2}(w_0)$. By the third condition, $D \subseteq \mathbb{C}$ be open set such that, restriction of x, $x|_D : D \longrightarrow \Omega$ and its inverse $\sigma : \Omega \longrightarrow D$ is holomorphic and bijective. Then, define two functions g, a by

$$g(u) := y(\sigma|_D(u)), \quad a(u) := \frac{U(\sigma|_D(u))}{x'(\sigma|_D(u))}$$

These two functions are holomorphic on Ω from the choice of D and Ω. For $s, t \in D$, let $u = x(s)$ and $v = x(t)$. Then $y(s) = g(u)$ and $y(t) = g(v)$. Define function $K : \Omega \times \Omega \to \mathbb{C}$ as the following,

$$K(u, v) = \begin{cases} \dfrac{g(v) - g(u)}{v - u} & \text{if } v \neq u, \\ g'(u) & \text{if } v = u. \end{cases}$$

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When converted into integral form, we also have

$$K(u,v) = \int_0^1 g'(u+\theta(v-u))\, d\theta.$$

Therefore, K is also holomorphic for $u, v \in \Omega$. In particular, when $u \neq v$, then $K(x(s), x(t)) = K(u,v) = F(s,t)$ from definition. It follows that $F_s(s,t) = x'(s)K_u(u,v)$ and $F_t(s,t) = x'(t)K_v(u,v)$. Applying these identities to $U(t)F_s(s,t) - U(s)F_t(s,t) = 0$ and dividing both sides by $x'(s)x'(t)$, we obtain

$$a(v)K_u(u,v) - a(u)K_v(u,v) = 0.$$

This identity also holds for $u = v$, that is, $x(s) = x(t)$. Expanding this from the defintion of K, we obtain

$$a(v)(g(v) - g(u) - (v-u)g'(u)) = a(u)((v-u)g'(v) - g(v) + g(u)).$$

Which, also holds for $u = v$. Substitute $v = u + \varepsilon$, and denote the resulting expression by $E(u,\varepsilon)$

$$E(u,\varepsilon) := a(u+\varepsilon)(g(u+\varepsilon) - g(u) - \varepsilon g'(u)) - a(u)(\varepsilon g'(u+\varepsilon) - g(u+\varepsilon) + g(u))$$

If $u \in \Omega$ and ε is sufficiently small so that $D_\varepsilon(u) \subseteq \Omega$, then $E(u,\varepsilon) = 0$. Fix $u \in \Omega$. Since $E(u,\varepsilon)$ is holomorphic in a neighborhood of $\varepsilon = 0$, expand it in a Taylor series at $\varepsilon = 0$

1. $\varepsilon g'(u+\varepsilon) - g(u+\varepsilon) + g(u) = \dfrac{g''}{2}\varepsilon^2 + \dfrac{g'''}{3}\varepsilon^3 + \dfrac{g^{(4)}}{8}\varepsilon^4 + \dfrac{g^{(5)}}{30}\varepsilon^5 + \mathcal{O}(\varepsilon^6)$
2. $g(u+\varepsilon) - g(u) - \varepsilon g'(u) = \dfrac{g''}{2}\varepsilon^2 + \dfrac{g'''}{6}\varepsilon^3 + \dfrac{g^{(4)}}{24}\varepsilon^4 + \dfrac{g^{(5)}}{120}\varepsilon^5 + \mathcal{O}(\varepsilon^6)$,
3. $a(u+\varepsilon) = a + a'\varepsilon + \dfrac{a''}{2}\varepsilon^2 + \dfrac{a'''}{6}\varepsilon^3 + \mathcal{O}(\varepsilon^4)$.

Since $E(u,\varepsilon) = 0$, the coefficient of ε^3 is zero. Comparing the coefficients of ε^3 of $E(u,\varepsilon)$, we obtain

$$-a\frac{g'''}{6} + a'\frac{g''}{2} = 0 \Longleftrightarrow \left(\frac{g''}{a^3}\right)' = 0.$$

Therefore, $\dfrac{g''}{a^3} = C$, where C is a constant depending on γ, D. Similarly, compare the coefficients of ε^5 in $E(u,\varepsilon)$ we obtain

$$\frac{ag^{(5)}}{120} + \frac{a'g^{(4)}}{24} + \frac{a''g'''}{12} + \frac{a'''g''}{12} - \frac{ag^{(5)}}{30} = 0.$$

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Substituting the previously obtained equality $g'' = Ca^3$, we obtain

$$0 = \frac{Ca}{120}(a^2a''' - 9aa'a'' + 12(a')^3).$$

Since D was chosen as a neighborhood on which $U(z_0) \neq 0$ and $x'(z_0) \neq 0$, we have $a \neq 0$. If $C = 0$, then $g'' = 0$, so g is linear. Since $g(x(z)) = y(z)$, the curve $\gamma(x(t), y(t))$ is contained in a line, which is a contradiction. Thus, it is enough to consider the case $C \neq 0$. In this case, $a^2a''' - 9aa'a'' + 12(a')^3 = 0$. Define the holomorphic function $q : \Omega \to \mathbb{C}$ by $q(u) := 1/a(u)^2$. Direct differentiation gives $q''' = -2a^{-5}(a^2a''' - 9aa'a'' + 12(a')^3) = 0$. Therefore, q is a polynomial of degree at most two. Write $q(u) = \alpha u^2 + \beta u + \eta$, and divide the proof into cases according to the form of q.

Case 1. $\alpha = \beta = 0$

Since $q(u)$ is constant, the function a is also constant. Therefore, $g'' = Ca^3$ is constant. Hence, g is a polynomial of degree at most two. Since for every $z \in D$, $g(x(z)) = y(z)$. So $g(x(z)) = y(z)$ for every $z \in \mathbb{C}$, inducing that the set γ is contained in a conic.

Case 2. $4\alpha\eta - \beta^2 \neq 0$

Define the holomorphic function $r : \Omega \to \mathbb{C}$ by $r(u) := 1/a(u)$. Then $r(u)^2 = q(u)$. Substituting this into $r(u)^2 = \alpha u^2 + \beta u + \eta$ and simplifying, we obtain

$$2rr' = 2\alpha u + \beta, \qquad rr'' = \alpha - (r')^2$$

Eliminating r' from these two equations gives

$$r'' = \frac{4\alpha\eta - \beta^2}{4r^3} = \frac{4\alpha\eta - \beta^2}{4}a^3.$$

Using $g'' = Ca^3$, we obtain $g'' = \dfrac{4C}{\Delta}r''$. Therefore, there exists a linear function ℓ such that, $g(u) - \dfrac{4Cc}{4\alpha\eta - \beta^2}r(u) = \ell(u)$. Squaring this equality and using $r(u)^2 = q(u)$, we obtain the following.

$$(g(u) - \ell(u))^2 - \left(\frac{4C}{4\alpha\eta - \beta^2}\right)^2 (\alpha u^2 + \beta u + \eta) = 0$$

This is an equality of rational functions holding at infinitely many points in a neighborhood of z_0, so it is an identity resulting the following equation.

$$\gamma'(X, Y) = (Y - \ell(X))^2 - \left(\frac{4C}{4\alpha\eta - \beta^2}\right)^2 (\alpha X^2 + \beta X + \eta).$$

Thus, $\gamma(C) \subseteq \gamma'$ is contained in a conic.

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Case 3. $4\alpha\eta - \beta^2 = 0$

There exist $\rho, \sigma \in \mathbb{C}$ such that $\rho \neq 0$ and $q(u) = (\rho u + \sigma)^2$. Then, define $r : \Omega \to \mathbb{C}$ by $r(u) := 1/a(u)$. Since $r(u)^2 = q(u)$, r can be written as $r(u) = \rho u + \sigma$. Then,

$$g'' = C(\rho u + \sigma)^{-3} = \frac{C}{2\rho^2}\left(\frac{1}{\rho u + \sigma}\right)''.$$

Therefore, there exists a linear function ℓ such that $g(u) = \ell(u) + \dfrac{C}{2\rho^2} \cdot \dfrac{1}{\rho u + \sigma}$.

$$(g(u) - \ell(u))(\rho u + \sigma) - \frac{C}{2\rho^2} = 0$$

This is an equality of rational functions and holds at infinitely many points in a neighborhood of z_0, so it induces the following

$$\gamma'(X, Y) = (Y - \ell(X))(\rho X + \sigma) - \frac{C}{2\rho^2}$$

Thus, $\gamma(C) \subseteq \gamma'$ is contained in a conic. This completes all cases and proves lemma 4.11

□

Theorem 4.12. *$m_0 \geq 3d$ is a given positive integer, and suppose that γ is not contained in a conic. For any fixed $m_0 \times m_0$ slope matrix M, $|X_M| \leq d^{2m_0}$.*

Proof. For the sake of contradiction, assume that $|X_M| > d^{2m_0}$ for some slope matrix M. This means that either X_M is an infinite set, or $|X_M| > d^{2m_0}$. Here, we have that $X_M \subset \overline{X_M} \subseteq Y_M$ for any M. We divide the situation into two distinct cases.

Case 1. $\dim X_M = 0$.

Here, we have that $X_M = Y_M \cap U$ where

$$U = \left\{(s,t) : \prod_{i,j}(x(s_i) - x(t_j)) \prod_{k<\ell}(s_k - s_\ell)(t_k - t_\ell) \neq 0\right\}$$

Note that Y_M is an algebraic set, defined as

$$Y_M = V\left(\{(s,t) \in C^{2m_0} : y(s_i) - y(t_j) = M_{ij}(x(s_i) - x(t_j))\}\right),$$

so from theorem 2.8, there exists irreducible components $Z_1, Z_2, \cdots, Z_\rho$ such that $Z_i \nsubseteq Z_j$ for any $1 \leq i \neq j \leq \rho$, and

$$Y_M = Z_1 \cup Z_2 \cup \cdots \cup Z_\rho.$$

For any $p \in X_M$, we have that $p \in Y_M$, so there exists $1 \leq i \leq \rho$ such that $p \in Z_i$. Now,

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assume that there exists Zariski closed set Z_i' such that

$$Z_i \cap U \subseteq Z_i' \subsetneq Z_i.$$

However, this implies that $Z_i = Z_i' \cup (Z_i \setminus U)$, and since U is Zariski open, we have that $Z \setminus U$ is Zariski closed. Therefore, since Z_i is irreducible, we must have either $Z_i = Z_i'$ or $Z_i = Z_i \setminus U$. The first equality cannot hold since $Z_i' \subsetneq Z_i$. If $Z_i = Z_i \setminus U$, we have that $Z_i \cap U = \emptyset$. However, $p \in X_M \subseteq U$, and $p \in Z_i$, we know that $p \in Z_i \cap U$, which is a contradiction. Hence, there are no such Z_i'. This is equivalent to $\overline{Z_i \cap U} = Z_i$ in the ambient space Z_i, so we have that $\dim(Z_i \cap U) = \dim Z_i$. However, since $Z_i \cap U \subseteq Y_M \cap U = X_M$, we have that

$$\dim Z_i = \dim(Z_i \cap U) \leq \dim X_M = 0.$$

If $\dim Z_i = 0$, we must have that $Z_i = \{p\}$, since if not, $\{p\} \subsetneq Z_i$ and $\{p\}$ is Zariski closed, so $\dim Z_i \geq 1$. Therefore,

$$|X_M| = \#\{p \in X_M\} \leq \#\{\text{zero dimensional irreducible components of } Y_M\}.$$

Now, let $P_{ij} = y(s_i) - y(t_j) = M_{ij}(x(s_i) - x(t_j))$. We always have that $\deg P_{ij} \leq d$, and the total number of such equations is $m_0^2 \geq 2m_0$. Applying theorem 2.13,

$$\#\{\text{zero dimensional irreducible components of } Y_M\} \leq d^{2m_0}$$

which implies that $|X_M| \leq d^{2m_0}$, a contradiction. Therefore, the first case is completely resolved.

Case 2. $\dim X_M \geq 1$.
From lemma 4.10, there exists an affine algebraic curve C with a pair of m_0 tuples $p = (s, t) \in C^{2m_0}$ where $s = (s_1, s_2, \cdots, s_{m_0})$ and $t = (t_1, t_2, \cdots, t_{m_0})$, satisfying the conditions in lemma 4.10. Now, since $p \in Y_M$, we have that $F(s_i, t_j) = M_{ij}$ for all $1 \leq i, j \leq m_0$. When the tangent direction ξ of T_pC is $\xi = (\sigma_1, \sigma_2, \cdots, \sigma_{m_0}, \tau_1, \tau_2, \cdots, \tau_{m_0})$, we have

$$\sigma_i F_u(s_i, t_j) + \tau_j F_v(s_i, t_j) = 0.$$

Define $G(u, v) \in \mathbb{C}(u, v)$ a rational function

$$G(u, v) = \frac{F_u(u, v)}{F_v(u, v)}.$$

Under this definition, $G(s_i, t_j) = -\frac{\tau_j}{\sigma_i}$ for all $1 \leq i, j \leq m_0$. For any $1 \leq i, j, i', j' \leq m_0$,

$$G(s_i, t_j)G(s_{i'}, t_{j'}) = \frac{\tau_j \tau_{j'}}{\sigma_i \sigma_{i'}} = G(s_i, t_{j'})G(s_{i'}, t_j)$$

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so plugging in $i' = j' = 1$ gives

$$G(u,v)G(s_1,t_1) - G(u,t_1)G(s_1,v) = 0$$

for all $(u,v) = (s_i,t_j)$. Define polynomials $A, B \in \mathbb{C}[u,v]$ as

$$\begin{cases} A(u,v) = x'(u)(y(v)-y(u)) - y'(u)(x(v)-x(u)) \\ B(u,v) = y'(v)(x(v)-x(u)) - x'(v)(y(v)-y(u)) \end{cases}$$

Basic calculation shows that $G(u,v) = \frac{A(u,v)}{B(u,v)}$ and $\deg A, \deg B < 2d$. Define $R \in \mathbb{C}[u,v]$ as

$$R(u,v) = (G(u,v)G(s_1,t_1) - G(u,t_1)G(s_1,v)) \cdot B(u,v)B(s_1,t_1)B(u,t_1)B(s_1,v)$$

As $\deg_v A, \deg_u B \le d$, we have that $\deg_u R < 3d \le m_0$ and $\deg_v R < 3d \le m_0$. Therefore, if R is not a zero polynomial, by theorem 2.3, among $\{s_1, s_2, \cdots, s_{m_0}\} \times \{t_1, t_2, \cdots, t_{m_0}\}$, there exists a pair (s_a, t_b) satisfying $R(s_a,t_b) \neq 0$. However, this is contradiction to $R(s_i,t_j) = 0$ for all $1 \le i,j \le m_0$. Therefore, R must be a zero polynomial. Also, note that $B(s_i,t_j) \neq 0$ so B cannot be a zero polynomial. This means that

$$G(u,v) = \frac{G(u,t_1)G(s_1,v)}{G(s_1,t_1)}$$

is an identity. Since F is symmetric, we have $F(u,v) = F(v,u)$ and $G(u,v) \cdot G(v,u) = 1$. Also, we define a rational function $\overline{F}$ as

$$\overline{F}(u,v) = \begin{cases} F(u,v) & (u \neq v) \\ \frac{y'(u)}{x'(u)} & (u = v) \end{cases}$$

Here, we have that $\overline{F}_u(u,u) = \overline{F}_v(u,u)$. Since $G(u,v) = \frac{\overline{F}_u(u,v)}{\overline{F}_v(u,v)}$,

$$\lim_{v \to u} G(u,v) = \lim_{v \to u} \frac{\overline{F}_u(u,v)}{\overline{F}_v(u,v)} = 1.$$

Hence considering the rational extension of G, we can say that $G(u,u) = 1$. Hence,

$$1 = G(u,u) = \frac{G(u,t_1) \cdot G(s_1,u)}{G(s_1,t_1)}$$

This implies that

$$G(u,v) = G(u,t_1) \cdot \frac{G(s_1,v)}{G(s_1,v)} = \frac{G(u,t_1)}{G(v,t_1)}$$

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which means that there exists a rational function $U(u) \in \mathbb{C}[u]$ satisfying

$$\frac{F_u(u,v)}{F_v(u,v)} = \frac{U(u)}{U(v)}$$

for all (u,v) in the domain of $\frac{F_u(u,v)}{F_v(u,v)}$. This implies that $U(u)F_u(u,v) - U(v)F_v(u,v) = 0$ is an identity, since expanding this gives a polynomial and there are a infinite set of U and V where $(u,v) \in U \times V$ satisfies the equation. Applying lemma 4.11, we can obtain a contradiction to the assumption that γ is not contained in a conic. Therefore, this case is also resolved, finishing theorem 4.12. □

Proof of theorem 3.5. For given $m > 6d^2$, set m_0 be a positive integer satisfying $m > 2dm_0$ and $m_0 \geq 3d$. Set $K = (d^{2m_0}+1) \cdot \binom{m}{m_0}^2$. Assume the contrary that such $2m-$dimensional points exists. From lemma 4.9, for each $1 \leq \ell \leq K$, we have a set I_ℓ and J_ℓ, each one of which has size m_0, such that $x(s_i^\ell) \neq x(t_j^\ell)$ for any $i \in I_\ell, j \in J_\ell$. The number of distinct (I,J) we can select is exactly $\binom{m}{m_0}^2$, so by pigeonhole principle, there exists a pair of size m_0 sets (I,J) such that the number of $1 \leq \ell \leq K$ satisfying $(I_\ell, J_\ell) = (I,J)$ is at least $K/\binom{m}{m_0}^2 \geq d^{2m_0}+1$. This implies that for fixed slope matrix M between $I \times J$, $|X_M| \geq d^{2m_0}+1$. However, since $m_0 \geq 3d$, from theorem 4.12, this cannot happen, which deduces that for such K, there exists no such distinct $2m-$dimensional points. □

Proof of theorem 3.1. Set $m > 6d^2$, $k \geq K(m)$, and $n \geq n_0$ where $K(m)$ is the bound from theorem 3.5 and n_0 is the bound from *theorem* 3.4. If γ is not contained in a conic, and is $(G,C,n)-$restricted for a graph G where $\|G\| \geq cn^{2-\varepsilon}$, by theorem 3.4, there exists point list sets $\mathcal{A}'$ and $\mathcal{B}'$ such that $|\mathcal{A}'| = |\mathcal{B}'| = m$ and for any $(\alpha', \beta') \in \mathcal{A}' \times \mathcal{B}'$, we have that α' is a translate of β'. From theorem 3.5, we have that such distinct m point lists cannot exist when $k \geq K(m)$ when γ is not contained in a conic. Therefore, theorem 3.1 is proven. □

Proof of corollary 3.2. Selecting G as $(S \sqcup T, W)$ gives the desired bound when applying theorem 3.1. □

Proof of corollary 3.3. First, set S' and T' be a set of size n', were $S \subset S'$, $T \subset T'$, and $n'^{2-\varepsilon} \leq n^2$. Here, considering $W = S \times T \subseteq S' \times T'$ and $|W| = n^2 \geq n'^{2-\varepsilon}$, we have that $|F(W)| = \omega(n')$ from corollary 3.2. Hence, we have that for $c_d = \varepsilon/2$,

$$|F(S \times T)| = |F(W)| = \omega(n') = \omega(n^{1+c_d})$$

□

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5 Conclusion

In this paper, we proved a graph-restricted expansion bound for secant slope functions. Let $\gamma : u \mapsto (x(u), y(u))$ be a polynomial parametric curve not contained in a conic, let $d = \max \deg x, \deg y$, and let

$$F(u, v) = \frac{y(u) - y(v)}{x(u) - x(v)}.$$

There exists $\varepsilon > 0$, depending only on d, such that, for all $S, T \subset \mathbb{R}$ with $|S| = |T| = n$ and every $W \subseteq S \times T$ with $|W| = \Omega(n^{2-\varepsilon})$ on which F is defined, one has

$$|F(W)| = \omega(n).$$

Thus, even when the pairs are restricted to the edges of a sufficiently dense bipartite graph, they determine a superlinear number of directions. As a consequence, there exists $c_d > 0$, depending only on d, such that

$$|F(S \times T)| = \omega\left(n^{1+c_d}\right).$$

The graph-restricted statement establishes an Elekes–R'onyai-type expansion result for secant slope functions and provides a direct connection between the Erdős direction problem and the Elekes–Rónyai problem.

A principal contribution of this work is the elementary combinatorial framework developed in the proof. The notions of δ-short point lists and translations allow us to convert the assumption of few directions into a dense translation graph, from which repeated finite direction patterns can be extracted using extremal graph theory. This part of the argument does not depend on the specific algebraic form of the secant slope function; it uses only the ordering of the point sets and a uniform bound on intersections with lines of a fixed direction. It may therefore be adaptable to other graph-restricted incidence problems with similar bounded-multiplicity properties.

The proof also separates the combinatorial and algebraic aspects of the problem. The elementary combinatorial argument first forces repeated direction patterns, while algebraic geometry is used only afterward to show that sufficiently many repetitions force the curve to lie on a conic. This separation provides a flexible strategy for studying other geometric expansion problems. Possible further directions include more general curves, other geometric relations in place of directions, or improvements of the constants ε, c_d.

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A AI Usage Records

The following table records all AI use connected with the project and report.

Date and frequency	Tool and version	Stage and purpose	Human verification and effect on the report

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